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TOPOLOGICAL AND GEOMETRIC ASPECTS OF YAMABE-TYPE EQUATIONS

T E S I S

Que para obtener el grado de

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To my family, for their constant support and motivation.

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Abstract

The main subject of this thesis is devoted to studying the multiplicity and uniqueness of solutions for the Yamabe-type equations, for that, we explore the geometric and topological properties of the equation. Our most important assumption is the existence of an isoparametric function on a Riemannian manifold. Indeed, we classify the isoparametric functions on $\mathbb{R}^n \times M^m$, $n, m \geq 2$, with compact level sets, where M^m is a connected, closed Riemannian manifold of dimension m. Also, we classify the isoparametric hypersurfaces in $\mathbb{S}^2 \times \mathbb{R}^2$ with constant principal curvatures.

On the other hand, we study positive solutions of the equation $-\Delta_g u + \lambda u = \lambda u^q$, with $\lambda > 0$, $q > 1$. If M supports a proper isoparametric function with focal varieties M_1, M_2 of dimension $d_1 \geq d_2$ we show that for any $q < \frac{n-d_2+2}{n-d_2-2}$ the number of positive solutions of the equation $-\Delta_g u + \lambda u = \lambda u^q$ tends to ∞ as $\lambda \to +\infty$. When $d_2 > 0$, this result implies multiplicity for positive solutions of critical and supercritical equations.

Keywords:
Isoparametric functions
Yamabe equation
Bifurcation theory
Focal submanifolds

Contents

Chapter 1

Introduction

In the early of 20th century, H. Poincaré [36] and P. Koebe [28] gave a proof of the celebrated Uniformization Theorem of Riemann Surface which, in particular, implies that every closed Riemannian 2-manifold has a conformal metric of constant Gauss curvature. In the subsequence years, different proofs appeared for this theorem both for its original formulation and for generalizations of it. But all of them in the case of dimension two. It was not until 1960 when H. Yamabe claimed in [49] the proof of a sort of generalization for the last statement in all dimensions; every closed Riemannian n-manifold is conformally equivalent to one of constant scalar curvature. Let us describe the main idea of Yamabe's approach.

Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$. We denote by $[g]$ the conformal class of the metric g. Writing a conformal metric as $h = u^{p_n-2}g \in [g]$ (for a positive function u) we have that h has constant scalar curvature λ if and only if u is a positive solution of the Yamabe equation

$$-\frac{4(n-1)}{n-2}\Delta_g u + s_g u = \lambda u^{p_n-1}, \tag{1.0.1}$$

where $p_n = \frac{2n}{n-2}$ and s_g is the scalar curvature of metric g.

In the case of $s_g = \alpha \in \mathbb{R}$ one can normalize the solution u by an appropriate constant l such that $v = lu$ is a solution of

$$-\Delta_g v + \beta v = \beta v^{p_n-1}, \quad \beta \in \mathbb{R}.$$

Yamabe considered the total scalar curvature functional

$$S(h) = \frac{\int_M s_h \, dvol_h}{vol(M,h)^{(n-2)/n}},$$

where $dvol_h$ is the volume element of h.

Expressing the metric h as $u^{p_n-2}g$ the functional $S(h)$ take the following form:

$$S(h) = Y_g(u) := \frac{\int_M a_n \|\nabla u\|^2 + s_g u^2 \, dvol_g}{\|u\|_{p_n}^2},$$

where $a_n = 4(n-1)/(n-2)$. The functional Y_g (which can be defined in the Sobolev space $W^{1,2}(M)$) is called the Yamabe functional.
Yamabe noted that the critical points u of this functional are solutions of equation (1.0.1) for $\lambda = Y_g(u)\|u\|_{p_n}^{2-p_n}$. Therefore solutions of equation (1.0.1) correspond to critical points of S restricted to the conformal class $[g]$.
It is not hard to see that the Yamabe functional satisfy that $Y_g \geq (\inf s_g)Vol(M,g)^{2/n}$ if $\inf s_g \leq 0$, otherwise $Y_g \geq 0$. So we can define:

$$Y(M,[g]) := \inf_{u \in C^\infty} Y_g(u) = \inf_{h \in [g]} S(h).$$

This invariant of the conformal class $[g]$ is called Yamabe constant and play an important role in the analysis of equation (1.0.1).
The goal of Yamabe was to prove that this constant is realized. But his proof contained a subtle error discovered by N. Trudinger in [44] given rise to the known nowadays as the Yamabe problem. The solution of this problem was completed in several steps beginning with Neil Trudinger in [44], Thierry Aubin [4] and finally by Richard Schoen in [40].
Guaranteed the existence of the problem it is very interesting to try to understand the set of all solutions. In the case that $Y(M,[g]) \leq 0$ the solutions are unique up to a constant. Thereby the only appealing case is $Y(M,[g]) > 0$. By a result of M. Obata [34] the uniqueness holds in the positive case when the Riemannian manifolds is Einstein different from the round sphere $(\mathbb{S}^n, g_0)$. The trivial example with $Y(M,[g]) > 0$ completely understood is $(\mathbb{S}^n, g_0)$, where the set of solutions is a noncompact family in C^2 topology but all of them produce isometric metrics of constant scalar curvature $n(n-1)$. In [37] D. Pollack proved that every conformal class with positive Yamabe constant can be C^0 approximated by a conformal class with an arbitrary number of (non-isometric) metrics of constant scalar curvature.
Another feature investigated per years in the positive case it is the compactness of solutions for manifolds not conformally diffeomorphic to $(\mathbb{S}^n, g_0)$. In [42] Schoen proved the compactness for manifolds locally conformally flat and conjectured that it is true for the general setting. However, in [8] S. Brendle showed a metric in $\mathbb{S}^n$, $52 \leq n$ for which the compactness fails. After that, Brendle and Marques in [9] constructed examples in dimension $25 \leq n \leq 51$ where again the compactness fails. Surprisingly, the compactness holds for all remaining dimensions (Brendle-Marques-Schoen [25]).
In general, it is a very difficult task to describe the space of solutions of the Yamabe equation. Only in some cases it is possible, one of them is the case of cylinders $(\mathbb{S}^n \times \mathbb{S}^1, [g_0 + T^2dt^2])$ for $T \in \mathbb{R}$. For instance in [26], [27] O. Kobayashi and Schoen in [41] pointed out that all solutions are constant along the spheres $\mathbb{S}^n$ and therefore the Yamabe equation reduces to an ordinary differential equation. They finally conclude that the number of solutions increases with respect to T.
In [23] Q. Jin, YY. Li and H. Xu considered in $(\mathbb{S}^n, g_0)$ the *Yamabe type* equation

$$-\Delta_{\mathbb{S}^n} u = \lambda(u^q - u),$$

for $\lambda \in \mathbb{R}_{>0}$ and $q < p_n - 1$ (subcritical exponent). They were concerned about multiplicity result of this equation restricted to the space of radial functions on $\mathbb{S}^n$ respect to a fixed axis, which again, it turns out to be an ordinary differential equation.
In [22] G. Henry and J. Petean studied multiplicity results for solutions of the Yamabe equation on the products $(\mathbb{S}^n \times \mathbb{S}^k, g_0^n + T^2 g_0^k)$ for $2 \leq n, k$. First, they considered the Yamabe equation for functions that only depends on the factor $(\mathbb{S}^n, g_0)$. Thus, if $u : \mathbb{S}^n \times \mathbb{S}^k \to \mathbb{R}$ is a smooth function constant along $\mathbb{S}^k$ then solutions of the equation

$$-\Delta_{\mathbb{S}^n} u = \frac{1}{a_{n+k}}(n(n-1) + (1/T^2)k(k-1))(u^{p_{n+k}-1} - u),$$

correspond to solutions of the Yamabe equation on $(\mathbb{S}^n \times \mathbb{S}^k, g_0^n + T^2 g_0^k)$ constant along $\mathbb{S}^k$.
Since $p_{n+k} < p_n$ the last equation is subcritical on $\mathbb{S}^n$. So, similar to [23] the authors in [22] studied the ordinary differential equation associated to the Yamabe equation restricted to special space of functions. In order to precise their statement, we need to introduce the following definition.
Let (N, h) be a complete connected Riemannian manifold. A non-constant smooth function $f : N \to \mathbb{R}$ is called *isoparametric* if there exist smooth functions $a, b : \mathbb{R} \to \mathbb{R}$ such that

$$(1)\ |\nabla f|^2 = a(f) \quad \text{and} \quad (2)\ \Delta f = b(f).$$

The smooth hypersurfaces $M_t = f^{-1}(t)$ for t regular value of f are called *isoparametric hypersurfaces*.
Suppose now a positive function $\varphi : \mathbb{R} \to \mathbb{R}$ and function u of the form $u = \varphi \circ f$ for an isoparametric function on $\mathbb{S}^n$. By chain rule, $-\Delta_{\mathbb{S}^n} u$ is a function of f so that the Yamabe equation for functions $u = \varphi \circ f$ become in an ordinary differential equation.

Theorem 1.0.1 ([22])**.** *Let f be an isoparametric function on $(\mathbb{S}^n, g_0)$. If*

$$\frac{1}{T^2} > \frac{6(n+5)(n+k-1) - n(n-1)}{k(k-1)},$$

then there exists a solution u of the Yamabe equation on $(\mathbb{S}^n \times \mathbb{S}^k, g_0^n + T^2 g_0^k)$ with level sets $u^{-1}(t) = (\varphi \circ f)^{-1}(t) \times \mathbb{S}^k$.

In particular, this theorem produce infinitely many solutions for each isoparametric hypersurface.
For this fact, it seems to be reasonable to look at solutions of this type in other manifolds. In this direction, would be interesting to investigate the existence of isoparametric functions in some family of manifolds. For example, in the case of a Riemannian product $(M \times N, g + h)$ an isoparametric function on any of the factors gives an isoparametric function in the product. However, there are examples of products with mixed isoparametric functions. A trivial family are the radial functions on $\mathbb{R}^n$, but there are also examples for instance in $\mathbb{S}^2 \times \mathbb{S}^2$ (see [45]).

In this work, we first study isoparametric functions on Riemannian products with Euclidean space, namely $(M \times \mathbb{R}^n, g+dx^2)$. The motivation to understand these are into study of the finite energy solutions to the Yamabe equation in such products (see for instance [1, 3]). Note that positive finite energy solutions (which have to vanish at infinity) must have compact level sets. There is a well-known such solution which is a radial function on $\mathbb{R}^n$ (see [1]). Are there other solutions? It is conjectured that the answer is NO under certain conditions, for instance if g is Yamabe (the metric g realize the Yamabe constant $Y(M,[g])$). This conjecture implies that

$$Y(M \times \mathbb{R}^n, [g+dx^2]) := \inf_{\substack{u \in C_0^\infty(M\times\mathbb{R}^n)\\ u\neq 0}} Y_{g+dx^2}(u) = \inf_{u \in C_0^\infty(\mathbb{R}^n)-\{0\}} Y_{g+dx^2}(u).$$

The advantage of this equality is that the Yamabe invariant of $M \times N$ is bounded below by $Y(M\times\mathbb{R}^n, [g+dx^2])$ for any closed Riemaniann $n-$manifolds N ([1]), and the constant

$$Y_{\mathbb{R}^n}(M \times \mathbb{R}^n, g+dx^2) := \inf_{\substack{u \in C_0^\infty(\mathbb{R}^n)\\ u\neq 0}} Y_{g+dx^2}(u)$$

can be computed numerically in some cases.
The case when $(M,g) = (\mathbb{S}^m, g_0)$ is particularly important. For instance, when $n = m = 2$ if the conjecture is true then one would prove that the Yamabe invariant of $\mathbb{S}^2 \times \mathbb{S}^2$ is strictly greater than the one of $\mathbb{CP}^2$.

Our first result says that such solutions could not be built by an isoparametric function:

Theorem 1.0.2. *An isoparametric function on $\mathbb{R}^n \times \mathbb{M}^m$, $n, m \geq 2$, with compact level sets (where $\mathbb{M}^m$ is a closed Riemannian manifold) is a radial function of $\mathbb{R}^n$.*

Without the compactness condition on the level sets of the isoparametric function one would still like to know if there could be examples which do not come from isoparametric functions on M. We will only consider the case of $\mathbb{S}^2 \times \mathbb{R}^2$. Using the ideas developed by F. Urbano in [45] for the case $\mathbb{S}^2 \times \mathbb{S}^2$, we will prove

Theorem 1.0.3. *The isoparametric hypersurfaces with constant principal curvatures in $\mathbb{S}^2 \times \mathbb{R}^2$ are of the form $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{S}^1(r)$ (for $r \in \mathbb{R}^+$) or $\mathbb{S}^1(t) \times \mathbb{R}^2$ (for $t \in (0,1)$).*

Note that in general, there are examples of isoparametric hypersurfaces with non-constant principal curvatures, as in the examples in [46] for certain complex projective spaces.

Another remark in the Henry-Petean results (Theorem 1.0.1) is about of the tools employing in the study for subcritical Yamabe equation of $(\mathbb{S}^n, g_0)$. They used the bifurcation theory and properties of isoparametric hypersurfaces in the round sphere so as to describe the global behaviour of nontrivial solution branches emanating from bifurcation points.
In this thesis, we generalize this description to closed Riemannian manifolds of constant scalar curvature with an isoparametric function.

We take a closed Riemannian manifold (M^n, g) of dimension $n \geq 3$ and consider the following *Yamabe-type* equation

$$-\Delta_g u + \lambda u = \lambda u^q, \tag{1.0.2}$$

with $\lambda > 0$ and $q > 1$. Also we will assume that there is an isoparametric function $f : M \to [t_0, t_1]$ and look for solutions of equation (1.0.2) of the form $u = \varphi \circ f$, where $\varphi : [t_0, t_1] \to \mathbb{R}_{>0}$. It is known from the general theory of isoparametric functions that the only zeros of the function $b : [t_0, t_1] \to \mathbb{R}_{\geq 0}$ are t_0 and t_1. Moreover $M_1 = f^{-1}(t_0)$ and $M_2 = f^{-1}(t_1)$ are smooth submanifolds and are called the *focal submanifolds* of f. We call d_i the dimension of M_i. If $d_1, d_2 \leq n - 2$ we call f a *proper* isoparametric function, as in [19]. We will assume that f is proper. The most familiar case of isoparametric functions comes from cohomogeneity one isometric actions. Assume that G acts isometrically on (M, g) with regular orbits of codimension one and that the orbit space is an interval. If f is a smooth function which is G-invariant and its only critical points are the two singular orbits, then f is isoparametric. Note that in this situation the singular orbits have codimension at least 2, and therefore the isoparametric function f is proper. In [22] this situation was considered when (M, g) is the round sphere (S^n, g_0^n) and q is subcritical, and multiplicity results were obtained in this case for f-invariant solutions of equation (1.0.2). But there are proper isoparametric in much more general situations than the one of cohomogeneity one isometric actions. For instance in [38] C. Qian and Z. Tang proved that given a Morse-Bott function f on a closed manifold M (with appropriate conditions on its critical set) there is a Riemannian metric g on M so that f is proper isoparametric for (M, g).

We consider the space $C_f^{2,\alpha}$ of $C^{2,\alpha}$ functions on M which are f-invariant and we consider equation (1.0.2) as an operator equation on $(u, \lambda) \in C_f^{2,\alpha} \times (0, \infty)$. We study solutions bifurcating from the family of trivial solutions $\lambda \mapsto (1, \lambda)$. Using the well-known theory of local bifurcation for simple eigenvalues [16], we prove :

Theorem 1.0.4. *For any $q > 1$ there is a sequence of values $\lambda_m(q) \to \infty$ and branches $t \mapsto (u(t), \lambda(t))$, $t \in (-\varepsilon, \varepsilon)$, of f-invariant solutions of (1.0.2) so that $\lambda(0) = \lambda_m$, $u(0) = 1$ and $u(t) \neq 1$ if $t \neq 0$.*

Next we study the behavior of the local branches appearing at the bifurcation points. We will apply the global bifurcation theorem of P. Rabinowitz [39]. To do so we will need to impose conditions on q as it the analytical properties of the equation 1.0.2 depends drastically on the value of exponent q. For instance, when $q < p_n - 1$ (subcritical case) the equation is easy to solve and it might be impossible to solve in the case $q > p_n - 1$ (supercritical case). Recall that d_i is the dimension of the focal submanifold M_i of the proper isoparametric function and let $d = \min\{d_1, d_2\} \leq n - 2$. Then we let $p_f = \frac{n-d+2}{n-d-2}$, $p_f = \infty$ in case $d = n - 2$. Note that if $d > 0$ then $p_f > p_n - 1$. For the next results we will ask that $q < p_f$. If $d > 0$ the results apply to some supercritical equations.

An interesting question that was raised for instance in [6, 10, 30] is to find conditions under which for λ small the only solution of (1.0.2) is the trivial solution $u = 1$. In fact, in the subcritical case J.R. Licois and L. Veron proved in [30] that there exists some positive constant $\lambda_0 = \lambda_0(M, g, q)$ for which the equation (1.0.2) admits only the constant solution for all $0 < \lambda < \lambda_0$. We will prove a similar result, restricting to f-invariant solutions but allowing q to be supercritical:

Theorem 1.0.5. *If $q < p_f$ there exists $\lambda_0 > 0$ such that if $\lambda \in (0, \lambda_0)$ and u is a positive f-invariant solution of (1.0.2) then $u = 1$.*

Theorem 1.0.4 says that at each bifurcation point $(1, \lambda_m)$ appears a branch B_m of nontrivial solutions. Explicitly, let B_m be the connected component containing the nontrivial solutions appearing close to $(1, \lambda_m)$, in the space of nontrivial solutions of (1.0.2). Theorem 1.0.5 says in particular that if $(u, \lambda) \in B_m$ then $\lambda \in [\lambda_0, \infty)$. This will allow us to apply the global bifurcation theorem to prove:

Theorem 1.0.6. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$, and $f : M \to \mathbb{R}$ a proper isoparametric function. For any $q \in (1, p_f)$ and any positive integer k there exists $\lambda_k > 0$ such that the sequence (λ_k) is increasing, $\lambda_k \to \infty$, and for any $\lambda \in (\lambda_k, \lambda_{k+1}]$ equation (1.0.2) has at least k different positive solutions.*

The value of λ_k in Theorem 1.0.6 is the same as $\lambda_k(q)$ appearing in Theorem 1.0.4. The theorem is proved by showing that the branches B_m are disjoint to each other and "bend to the right to ∞", meaning that for any $\lambda > \lambda_k$ there exists a solution $(u, \lambda) \in B_k$. We will prove in this thesis that for any fixed $\lambda > 0$ only a finite number of the branches cut the "vertical" line $C^{2,\alpha}_f \times \{\lambda\}$. This implies that for any $K > 0$ there exists k_0 such that if $k \geq k_0$ the branch $B_k \subset C^{2,\alpha}_f \times [K, \infty)$.

Consider the isometric $O(n)$-action on the curvature one metric on the sphere, $(\mathbb{S}^n, g_0)$, fixing an axis. A linear function invariant by the action gives a proper isoparametric function. In this case $d = 0$ and Theorem 1.0.6 applies to the subcritical case $q < p_n - 1$. In this case the theorem was proved by Jin - Li - Xu in [23] (note that in this case the invariant functions are precisely the radial functions, with respect to the invariant axis). In this case Theorem 1.0.5 was proved by M-F. Bidaut-Veron and L. Veron in [6], and the constant λ_0 is explicit: $\lambda_0 = \frac{n}{q-1}$. In [22] it is considered the case of any isoparametric function on the sphere, but again only for the subcritical case.

The simplest example to apply Theorem 1.0.6 is to consider an isoparametric function f on $(\mathbb{S}^3, g_0)$ invariant by the natural isometric cohomogeneity one action of $S^1 \times S^1$. Both singular orbits have dimension 1, so f is proper and $p_f = \infty$. Moreover the values of λ_k in Theorem 1.0.4 and Theorem 1.0.6 are $\lambda_k = \frac{\mu_k}{q-1}$, where μ_k are the eigenvalues of $-\Delta_g$ restricted to f-invariant functions (and are easily computed in the case of torus invariant functions on the round $\mathbb{S}^3$). Therefore Theorem 1.0.6 says

Corollary 1.0.7. *For any $q > 1$ the equation (1.0.2) on $(\mathbb{S}^3, g_0)$ has at least k positive different torus invariant solutions if $\lambda \in (\frac{4k(k+1)}{q-1}, \frac{4(k+1)(k+2)}{q-1})$.*

Note that as $q \to \infty$ we obtain solutions with λ very close to 0. In $(\mathbb{S}^4, g_0)$ one could consider an isoparametric function f invariant by the isometric cohomogeneity one action of $O(3)\times O(2)$. In this case the singular orbits have dimensions 1 and 2, respectively. Then f is proper and $p_f = 5$ (note that $p_4 = 3$) and we obtain:

Corollary 1.0.8. *For any $q \in (1,5)$ the equation (1.0.2) on $(\mathbb{S}^4, g_0)$ has at least k positive different $O(3)\times O(2)$-invariant solutions if $\lambda \in (\frac{2k(2k+3)}{q-1}, \frac{2(k+1)(2k+5)}{q-1})$.*

One can consider of course more general spaces. For instance one has isoparametric functions invariant by cohomogeneity one actions on complex projective space $(\mathbf{CP}^n, g_{FS})$, where g_{FS} is the Fubini-Study metric. The simplest is the action by $U(n)$ for which the singular orbits are a point and $\mathbf{CP}^{n-1}$. Theorem 1.0.6 then gives solutions in the subcritical case. But one can consider other cohomogeneity one actions. For example in the case of $(\mathbf{CP}^2, g_{FS})$ there is a natural cohomogeneity one action by $SO(3)$ which has singular orbits of dimension 2: the real points $\mathbf{RP}^2 \subset \mathbf{CP}^2$ and $\{[z_0, z_1, z_2] \in \mathbf{CP}^2 : z_0^2 + z_1^2 + z_2^2 = 0\} = \mathbb{S}^2$. Then an invariant isoparametric function f is proper and $p_f = \infty$. It is well-known that the eigenvalues of $-\Delta_{\mathbf{CP}^2}$ restricted to $SO(3)$-invariant functions are $16k(k+1)$. Therefore Theorem 1.0.6 says

Corollary 1.0.9. *For any $q > 1$ the equation (1.0.2) on $(\mathbf{CP}^2, g_{FS})$ has at least k positive different torus invariant solutions if $\lambda \in (\frac{16k(k+1)}{q-1}, \frac{16(k+1)(k+2)}{q-1})$.*

There is also an action on $(\mathbf{CP}^n, g_{FS})$ by $U(m)\times U(l)$ where $m + l = n + 1$ and $m \geq l \geq 2$. We will see in section 3.4 that for an invariant isoparametric function f one has $p_f = \frac{2n-2l+4}{2n-2l} > p_{2n}$ and that the f-invariant eigenvalues are the same as the eigenvalues of the full Laplacian, then applying Theorem 1.0.6 we obtain:

Corollary 1.0.10. *For any $q \in (1, \frac{2n-2l+4}{2n-2l})$ the equation (1.0.2) on $(\mathbf{CP}^n, g_{FS})$ has at least k positive different $U(m)\times U(l)$-invariant solutions if $\lambda \in \left(\frac{4k(k+n)}{q-1}, \frac{4(k+1)(k+1+n)}{q-1}\right]$.*

Finally, we give an application for $SO(n+1)$-invariant solutions in $(\mathbf{CP}^n, g_{FS})$ (by cohomogeneity one isometric action of $SO(n+1) \subset U(n+1)$)

Corollary 1.0.11. *Let $q \in (1, \frac{n+2}{n-2})$. Equation (1.0.2) on $(\mathbf{CP}^n, g_{FS})$ has at least k positive different $SO(n+1)$-invariant solutions if $\lambda \in \left(\frac{4k(4k+2n)}{q-1}, \frac{4(k+1)(4(k+1)+2n)}{q-1}\right]$.*

This thesis is organized in the following way. In chapter 2 we introduce the principal tools from the nonlinear analysis that we will use in the next chapters. More precisely, we recall the implicit function theorem and some consequences from it. Also, we explain the Lyapunov-Schmidt reduction in order to prove the local bifurcation theorem for simple eigenvalues. In chapter 3 the theory of isoparametric functions is explored. Using structural results of Wang [47] and minimal immersion into product $\mathbb{M}^m \times \mathbb{R}^n$, we will show the Theorem 1.0.2. Furthermore, we consider the Kähler structures on $\mathbb{S}^2$, $\mathbb{R}^2$ so as to describe the behavior of isoparametric hypersurfaces with constant principal curvature in $\mathbb{S}^2 \times \mathbb{R}^2$ with respect to product structure on $\mathbb{S}^2 \times \mathbb{R}^2$. This approach was made by Urbano [45] in the case of $\mathbb{S}^2 \times \mathbb{S}^2$ and will allow us to prove Theorem 1.0.3. In chapter 4 we shall see the proof of Theorem 1.0.4. Through of asymptotic behavior of mean curvature function associated to isoparametric function f we obtain the conditions in order to apply Theorem 2.3.2 and get the local bifurcation result for Yamabe-type equation. Moreover, we built a barrier for non-trivial $f-$invariant solution of equation (1.0.2) showing a priori estimate for this solutions (Theorem 1.0.5). Finally, we use the Global Bifurcation Theorem so as to prove theorem 1.0.6.

Chapter 2

Elements of nonlinear functional analysis

In this chapter, we give a brief review of some basic tool from nonlinear functional analysis needed for our treatment of Yamabe-type equation in later chapters. For more details, we recommend the books [2], [33] and references therein.

2.1 Calculus in Banach space

Let X, Y be Banach spaces, $u \in X$ and consider a map $F : X \to Y$. In the particular case that $Y = \mathbb{R}$, F is called a functional. The map F is called Fréchet differentiable at $u \in X$ if there exists a linear continuous map $L : X \to Y$ such that

$$\frac{\|F(u+v) - F(u) - L(v)\|_Y}{\|v\|_X} \to 0 \quad \text{as} \quad \|v\|_X \to 0.$$

The linear map L is uniquely determined by F and u, so that we will denote $dF_u := L$. When X, Y are Euclidean spaces the Fréchet derivative coincides with the usual notion of differential.
We say that F is continuously differentiable at u if F is Fréchet differentiable in an open neighbourhood of u in X and $w \mapsto dF_w \in L(X, Y)$ is continuous at $w = u$.
Now we state two classical properties of Fréchet derivative.

Theorem 2.1.1 (Chain rule)**.** *Let X, Y, Z be Banach spaces. If $F : X \to Y$ and $G : Y \to Z$ are Fréchet differentiable at u and $F(u)$, respectively, then $G \circ F$ is differentiable at u and the following holds:*

$$d(G \circ F)_u = dG_{F(u)} \circ dF_u.$$

The Fréchet derivative has also a version of implicit function theorem

Theorem 2.1.2. *Let X, Y, T be Banach spaces. We consider a map $F : T \times X \to Y$ and a fix point $(t_0, u_0) \in T \times X$ which satisfies:*

1. $F(t_0, u_0) = 0$

2. *F continuously differentiable at (t_0, u_0)*

3. *Partial Fréchet derivative $d_u F_{(t_0,u_0)}$ is invertible.*

Then there exists a neighbourhood U of u_0 in X and a neighbourhood N of t_0, such that the equation $F(t, u) = 0$ has a unique solution $u(t)$ for all $t \in N$. Moreover, the function $u(t)$ is continuously differentiable and

$$du_{t_0} = -(d_u F_{(t_0,u_0)})^{-1} \circ d_t F_{(t_0,u_0)}.$$

In the particular case that $X = T = Y = \mathbb{R}$, for each real-valued function $F \in C^1(\mathbb{R}^2)$ with $F(t_0, 0) = 0$ if $\nabla F(t_0, 0) \neq 0$ then F satisfies the last theorem at $(t_0, 0)$. However, when $\nabla F(t_0, 0) = 0$ it is possible to give conditions on F in order to guarantee the existence of an implicit curve $t(u)$ of solutions for the equation $F(t, u) = 0$ in a neighbourhood of $(t_0, 0)$. Indeed, we suppose that $F \in C^2(\mathbb{R}^2)$ and $\nabla F(t_0, 0) = 0$. If $F(t, 0) = 0$ for all t, we can define a function $h(t, u)$ of class C^1 in a neighbourhood of $(t_0, 0)$ by $h(t, u) = \frac{F(t,u)}{u}$ for $u \neq 0$ and $h(t, u) = \partial_u F(t, 0)$ if $u = 0$.
Clearly $h(t_0, 0) = 0$. On the other hand, one has

$$\partial_t h(t_0, 0) = \lim_{u \to 0} \frac{\partial_t F(t_0, u)}{u} = \partial^2_{t,u} F(t_0, 0).$$

Therefore, $\partial^2_{t,u} F(t_0, 0) \neq 0$ implies that there exists $\epsilon > 0$ and function $t(u)$ defined in $u \in (-\epsilon, \epsilon)$ such that

$$t(0) = t_0, \quad h(t(u), u) = 0, \quad u \in (-\epsilon, \epsilon).$$

Thus, $h(t(u), u) = 0$ implies $F(t(u), u) = 0$.
We summarize this argument in the following statement.

Proposition 1. *Let F be a real-valued function of class C^2 in a neighbourhood of $(t_0, 0) \in \mathbb{R}^2$. We assume that*

1. $F(t, 0) = 0$ *for all* t,

2. $\nabla F(t_0, 0) = 0$,

3. $\partial^2_{t,u} F(t_0, 0) \neq 0$.

Then there exists $\epsilon > 0$ and curve $t : (-\epsilon, \epsilon) \to \mathbb{R}$ such that $F(t(u), u) = 0$ for $u \in (-\epsilon, \epsilon)$.

Remark 1. *This proposition gives us two curves of solutions for $F(t,u) = 0$ in a neighbourhood of $(t_0, 0)$; the curve $(t(u), u)$ and trivial one $(t, 0)$. In virtue of Morse Lemma we can see that these are the only ones. We consider the assumptions of proposition 1. The Hessian of F at $(t_0, 0)$ has the form*

$$HessF(t_0, 0) = \begin{pmatrix} 0 & \partial^2_{t,u} F(t_0, 0) \\ \partial^2_{t,u} F(t_0, 0) & * \end{pmatrix}.$$

From $\partial^2_{t,u} F(t_0, 0) \neq 0$ we have that $det(HessF(t_0, 0)) < 0$, so the critical point $(t_0, 0)$ is non-degenerate with index 1. By Morse Lemma there exists a diffeomorphism φ on a neighbourhood of $(t_0, 0)$ such that

$$F(\varphi^{-1}(x, y)) = x^2 - y^2 = (x - y)(x + y).$$

Thus the function F (up to diffeomorphisms) has only two curves of zeroes in a small enough neighbourhood of $(t_0, 0)$.

2.2 Compact Operators

In this section we will recall some results about linear compact operators in Banach spaces. We begin with the definition of compact operators.

Definition 2.2.1. *A continuous map $K : X \to Y$ is compact if $K(B)$ is a relatively compact in Y for all $B \subset X$ bounded.*

In other words, the compact operators send bounded sequences to sequences with a converging subsequence.
Compact operators are, in some sense, the generalization of the class of finite-rank operators in an infinite-dimensional setting. In fact, the class of these operators (defined from a bounded subset Ω of Banach space X into X) are characterized by the following result.

Theorem 2.2.2. *Let Ω be any closed, bounded subset of X. Then $K : \Omega \to X$ is compact if and only if K is a uniform limit of mappings whose ranges lie in finite-dimensional subspaces.*

Sketch of proof. If K is compact then $\overline{K(\Omega)}$ is compact in X. For $\epsilon > 0$, there exist $n(\epsilon)$ open balls $B_\epsilon(x_j)$ of radius ϵ and centers x_j with $j \in \{1, \ldots, n(\epsilon)\}$ such that

$$\overline{K(\Omega)} \subset \bigcup_{j=1}^{n(\epsilon)} B_\epsilon(x_j).$$

We take nonnegative $\varphi_j(x)$ such that $\{\varphi_j\}$ is a partition on unity on $\overline{K(\Omega)}$ subordinate by $B_\epsilon(x_j)$. Since $\sum \varphi_j = 1$ we get,

$$\|K - \sum \varphi_j(K(x))x_j\| = \| \sum \varphi_j(K(x))(k(x) - x_j)\|.$$

If $\varphi_j(K(x)) \neq 0$ then $K(x) \in B_\epsilon(x_j)$. Thereby $\|K - \sum \varphi_j(K(x))x_j\| < \epsilon$.

Also, we note that the function $K_\epsilon := \sum \varphi_j(K(x))x_j$ belong to the convex hull of finite set in X, i.e. the range of K_ϵ is finite-dimensional. $\square$

Definition 2.2.3. *A linear operator $L : X \to X$ is called a Fredholm operator if $dimKer[L] < \infty$ and $Range[L]$ is closed and has finite codimension.*

In this case, the index of L is $dimKer[L] - codimRange[L]$.

Compact linear operators provide a especial family of Fredholm operators in the following way

Theorem 2.2.4. *Let X be a Banach space and let $K : X \to X$ be linear and compact. Then:*

1. *$Ker[I - K]$ is finite dimensional,*
2. *$Range[I - K]$ is closed, has finite codimension and $Range[I - K] = Ker[I - K^*]^\perp$, where K^* denotes the adjoint of K ,*
3. *$Ker[I - K] = 0 \Longleftrightarrow Range[I - K] = X$.*

We remark some consequences from this theorem. One of them is that the index of operator $I - K$ is zero. The other is that either for all $v \in X$, there exist a unique solution in X for the equation $(I - K)u = v$ or else, the homogeneous equation $(I - K)u = 0$ has nontrivial solution. This dichotomy is called the Fredholm alternative.

Another important subject in nonlinear analysis is devoted to the spectrum of compact operators , we recall this definition.

Definition 2.2.5. *Let K be a compact operator. The resolvent of K is the set*

$$\rho(K) = \{\lambda \in \mathbb{R}/K - \lambda I \text{ is bijective from } X \text{ to itself}\}.$$

The spectrum $\lambda(K)$ of K is defined as $\lambda(K) = \mathbb{R} \setminus \rho(K)$ and λ is called an eigenvalue of K if $K - \lambda I$ has non-trivial kernel.
The Riesz–Fredholm theory allows obtaining the next interesting assertion.

Theorem 2.2.6. *Let K be linear and compact. Then $\lambda(K)$ spectrum of K is compact and $\lambda(K) \subset [-\|K\|, \|K\|]$. Additionally, if X is infinite dimensional, it follows that:*

1. *$0 \in \lambda(K)$;*
2. *Every $\lambda_* \in \lambda(K) \setminus \{0\}$ is an eigenvalue of K ;*
3. *Either $\lambda(K) = \{0\}$, or $\lambda(K)$ is finite, or $\lambda(K) \setminus \{0\}$ is a sequence which tends to zero.*

Moreover, for every $\lambda_* \in \lambda(K) \setminus \{0\}$, there exists $1 \leq m$ such that

$$Ker[(K - \lambda_* I)^l] = Ker[(K - \lambda_* I)^{l+1}], \quad \forall m \leq l$$

2.3 Bifurcation from simple eigenvalues

We now turn to local bifurcation theory which is a prototype for nonuniqueness in non-linear analysis. We will discuss the simplest situation, bifurcation points from a simple eigenvalue by analytical methods.

Let $S : \mathbb{R} \times X \to Y$ be a continuously differentiable map.

Definition 2.3.1. *A point $(\lambda_*, 0)$ is called a bifurcation point for the equation $S(\lambda, u) = 0$ if $S(\lambda_*, 0) = 0$ and there exist sequences $\lambda_n \in \mathbb{R}$, $u_n \in X - \{0\}$ such that*

1. $S(\lambda_n, u_n) = 0,$
2. $(\lambda_n, u_n) \longrightarrow (\lambda_*, 0)$ *as* $n \to \infty.$

In other words, bifurcation points are accumulation points of the set of non-trivial solutions.
From now on we assume that the line $\mathbb{R}$ is a trivial solution, i.e. $S(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. The first neccesary condition for the existence of bifurcation points is due to the implicit function theorem

Proposition 2. *If $(\lambda_*, 0)$ is a bifurcation point then $d_u S_{(\lambda_*,0)} \in L(X, Y)$ is not invertible.*

Proof. If $d_u S_{(\lambda_*,0)}$ is invertible, by implicit function theorem, there exists $\epsilon > 0$ and neighbourhood U_0 of 0 in X such that

$$\forall (\lambda, u) \in (\lambda_* - \epsilon, \lambda_* + \epsilon) \times U_0, \quad S(\lambda, u) = 0 \quad \text{if and only if} \quad u = u(\lambda).$$

But $S(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$. Hence $u = 0$, and $(\lambda_*, 0)$ can not be bifurcation point. □

The rest of this section is devoted to give sufficient condition for existence bifurcation points. We restrict our attention in the case $L := d_u S_{(\lambda_*,0)}$ not invertible. Additionally, assume that $S \in C^2(\mathbb{R} \times X, Y)$ and

- $Ker(L)$ has topological complement W in X.
- $Range(L)$ is closed and has a topological complement Z in Y.

For any $u \in X$ there exist unique $v \in Ker(L)$ and $w \in W$ such that $u = v + w$. Also, we can define projections P, Q on Y onto Z and $Range(L)$, respectively. Writting $u = v + w$ and applying P and Q to $S(\lambda, u) = 0$ we get the following equations:

$$PS(\lambda, v + w) = 0, \tag{2.3.1}$$
$$QS(\lambda, v + w) = 0. \tag{2.3.2}$$

Near to $(\lambda_*, 0)$ equation (2.3.2) satisfies the implict function theorem:

Proposition 3. *There exist neighbourhoods V_0 of $v = 0$ in $Ker(L)$, W_0 of $w = 0$ in W, $\epsilon > 0$, and a function $w(\lambda, v) \in C^2((\lambda - \epsilon, \lambda + \epsilon) \times V_0, W_0)$ such that $\forall(\lambda, v, w) \in (\lambda_* - \epsilon, \lambda_* + \epsilon) \times V_0 \times W_0$, we have*

$$QS(\lambda, v + w) = 0 \quad \text{if and only if} \quad w = w(\lambda, v).$$

Moreover

$$\forall \lambda \in (\lambda_* - \epsilon, \lambda_* + \epsilon), \quad w(\lambda, 0) = 0;$$

and

$$d_v w_{(\lambda_*, 0)} = 0.$$

Proof. One has that $QS \in C^1(\mathbb{R} \times Ker(L) \times W, Range(L))$, and

$$T := d_w QS_{(\lambda_*, 0, 0)} = Q d_u S_{(\lambda_*, 0)} = QL$$

is a linear map from W to $Range(L)$. Therefore T is the restriction of L to W (so T is bijective). Since that $Range(L)$ is closed we conclude, by bounded inverse theorem, that T is invertible. The proposition follows from implicit function theorem. □

If we replace $w = w(\lambda, v)$ into equation (2.3.1) the initial problem to find solutions of $S(\lambda, v + w) = 0$ is reduced to the problem

$$PS(\lambda, v + w(\lambda, v)) = 0.$$

The procedure described above is known as Lyapunov-Schmidt reduction and the latter equation is called the *bifurcation equation*.

Remark 2. *The Lyapunov-Schmidt reduction implies that if $(\lambda_*, v = 0)$ is a bifurcation point for the bifurcation equation then $(\lambda_*, u = 0)$ is of bifurcation for $S(\lambda, u) = 0$. More precise, if a sequence of solutions for bifurcation equation satisfying that $(\lambda_n, v_n) \to (\lambda_*, 0)$ with $v_n \neq 0$. Then $u_n = v_n + w(\lambda_n, v_n) \neq 0$; $u_n \to 0$ and $S(\lambda_n, u_n) = 0$ for n large enough.*

Roughly speaking, the Lyapunov-Schmidt procedure allows us to splits the initial equation $S(\lambda, u) = 0$ in a system of two equations, one of them is uniquely solved (by implicit function theorem), while the other one (bifurcation equation) possibly has no non-trivial solutions. In the next theorem, we will impose some additional conditions on operator S so as to get bifurcation points for the bifurcation equation.

Theorem 2.3.2. *Let $S \in C^2(\mathbb{R} \times X, Y)$ and $\lambda_* \in \mathbb{R}$ such that*

1. *$S(\lambda, 0) = 0$ for all λ;*
2. *$L := d_u S_{(\lambda_*, 0)}$ is not invertible;*
3. *$Ker(L)$ is one dimensional ($span(u^*) = Ker(L)$) and has topological complement W in X.*

4. *Range*(L) *is closed and has a topological complement* Z *in* Y.

5. *codimRange*$(L) = 1$ *and there exists* $\phi \in Y^* - \{0\}$ *for which Range*$(L) = \{y \in Y/\langle \phi, y\rangle = 0\}$.

The value λ_* *is of bifurcation, provided that*

$$d^2_{u,\lambda}S_{(\lambda_*,0)}[u^*] \notin Range(L)$$

Proof. By conditions 1 to 4 and last Remark we have that if $(\lambda_*, t = 0)$ is a bifurcation point of bifurcation equation

$$PS(\lambda, tu^* + w(\lambda, tu^*)) = 0,$$

then $(\lambda_*, u = 0)$ is also bifurcation for $S(\lambda, u) = 0$.
On the other hand, condition 5 implies a correspondence between solutions of

$$PS(\lambda, tu^* + w(\lambda, tu^*)) = 0$$

and solutions of,

$$\beta(\lambda, t) := \langle \phi, S(\lambda, tu^* + w(\lambda, tu^*))\rangle = 0.$$

First, we note that $\beta(\lambda, 0) = 0$ since $w(\lambda, 0) = 0$ (proposition 3).
Now, we calculate the first derivative of β respect to t:

$$\partial_t \beta(\lambda, t) = \langle \phi, d_u S_{(\lambda,tu^*+w(\lambda,tu^*))}[u^* + d_v w_{(\lambda,tu^*)}u^*]\rangle.$$

Again proposition 3 says that $d_v w_{(\lambda^*,0)} = 0$ so that $\partial_t \beta(\lambda^*, 0) = 0$.
Finally we compute the second mixed derivative $\partial^2_{t,\lambda}\beta(\lambda, t)$

$$\begin{aligned}
\partial^2_{t,\lambda}\beta(\lambda^*, 0) &= \langle \phi, d^2_{u,\lambda}S_{(\lambda_*,0)}[u^* + d_v w_{(\lambda_*,0)}u^*]\rangle \\
&\quad + \langle \phi, d_u S_{(\lambda_*,0)}[d^2_{v,\lambda}w_{(\lambda_*,0)}u^*]\rangle \\
&= \langle \phi, d^2_{u,\lambda}S_{(\lambda_*,0)}[u^*]\rangle + \langle \phi, L[d^2_{v,\lambda}w_{(\lambda_*,0)}u^*]\rangle.
\end{aligned}$$

In virtue of the condition 5 we get

$$\langle \phi, L[d^2_{v,\lambda}w_{(\lambda_*,0)}u^*]\rangle = 0.$$

Also we have that $\langle \phi, d^2_{u,\lambda}S_{(\lambda_*,0)}[u^*]\rangle \neq 0$ by hypothesis.

From Proposition 1 we obtain $\epsilon > 0$ and curve $\lambda(t)$ such that $\beta(\lambda(t), t) = 0$ for $t \in (-\epsilon, \epsilon)$.
Therefore the pair $(\lambda(t), tu^* + w(\lambda(t), tu^*))$ are non-trivial solutions of $PS = 0$ such that converge to $(\lambda^*, 0)$.
Hence $(\lambda^*, 0)$ is a bifurcation point of $S(\lambda, u) = 0$. □

Finally, we conclude this section with the celebrated global bifurcation theorem due to Rabinowitz. For this, we consider equations of the form:

$$S(\lambda, u) := u - \lambda Au - T(u) = 0, \tag{2.3.3}$$

where A is a linear compact operator from X to in itself and $T \in C^1(X, X)$ is compact such that $T(0) = 0$, $dT_0 = 0$.
We denote by Σ the set of non-trivial solutions of 2.3.3.

Theorem 2.3.3 (Global bifurcation)**.** *Let $1/\lambda_*$ be an eigenvalue of A with odd multiplicity. Then $(\lambda_*, 0)$ is a bifurcation point of (2.3.3) (i.e $(\lambda_*, 0) \in \overline{\Sigma}$).*
Furthermore, let Σ_{λ_*} be the connected component of $\overline{\Sigma}$ containing $(\lambda_*, 0)$. We have the following dichotomy:

1. *Σ_{λ_*} is non-compact in the domain of S, or*

2. *Σ_{λ_*} contain other bifurcation point $\lambda_+ \neq \lambda_*$.*

Our main purpose in Chapter 4 is to give, via this theorem, a global bifurcation description of Yamabe-type equations for simple eigenvalues. In order to do that we rule out the option 2. Explicitly, we will show that all bifurcation branches associated to different bifurcation points are disjoint, so by an argument similar to Remark 1 we will get only two curves of solutions in some neighbourhood of each bifurcation point, the trivial one $(\lambda, 0)$ and the respective bifurcation branch. Thus, Global bifurcation theorem will imply that all bifurcation branch are unbounded.

Chapter 3

Isoparametric functions

This chapter is devoted to the theory of isoparametric functions on general Riemannian manifolds. We divided the chapter into four sections. In the first one, we begin with the classical definition of isoparametric functions (as far as we know) and we describe the main properties from it.
In the second section, we use some structural result of section 3.1 and classify the isoparametric functions on $\mathbb{R}^n \times \mathbb{M}^m$, $n, m \geq 2$, with compact level sets, where $\mathbb{M}^m$ is a connected, closed Riemannian manifold of dimension m.
In the third section, we turn to the isoparametric hypersurfaces in $\mathbb{S}^2 \times \mathbb{R}^2$ with constant principal curvatures. Through the product structure of $\mathbb{S}^2 \times \mathbb{R}^2$ we prove that the unit normal vector field of such isoparametric hypersurfaces has a constant position in the tangent space $T(\mathbb{S}^2 \times \mathbb{R}^2)$ and therefore the only families obtained are $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{S}^1(r)$ (for $r \in \mathbb{R}^+$) or $\mathbb{S}^1(t) \times \mathbb{R}^2$ (for $t \in (0, 1)$).
Finally, in section 3.4 we will calculate the eigenvalues of Laplacian $-\Delta_g$ respect to some isoparametric functions on a compact Riemannian manifold (M, g)

3.1 Background of isoparametric hypersurfaces

The first notion of isoparametric functions can be founded in the works of Carlo Somigliana [43] about to the relations between the Huygens principle and geometric optics, in particular he claimed the following:
"According to the Huygens principle, one of the most simple models of what wave propagation in an isotropic media should be consists in a family of parallel surfaces that are intersected perpendicularly at every point by a set of straight lines. The sequence of parallel surfaces, each one of which can be considered as the envelope of a set of spheres of radius equal to the distance between the surface and one of the previous ones, constitutes the family of wavefronts."

This work leads the following definition introduced (possibly) by T. Levi-Civita in [29] for the case of Euclidean space $\mathbb{R}^3$. But for our purpose we will present it in the general

Riemannian manifolds.

Definition 3.1.1. *Let (N, h) be a connected Riemannian manifold. A non-constant smooth function $f : N \to \mathbb{R}$ is called isoparametric if there exist smooth functions $a, b : \mathbb{R} \to \mathbb{R}$ such that*

$$(1)\ \|\nabla f\|^2 = b(f) \quad \textit{and} \quad (2)\ \Delta f = a(f).$$

The smooth hypersurfaces $M_t = f^{-1}(t)$ for t regular value of f are called *isoparametric hypersurfaces*. The preimage of the maximum and minimum of the isoparametric function f are denoted by M_+ and M_- (resp.); they are called *focal varieties* of f.
The main research line on this subject has been the problem of classification in Riemannian manifolds, which started with the works of É. Cartan who proved in [11] that, when the ambient manifold has constant sectional curvature (space form), a hypersurface is isoparametric if and only if has constant principal curvatures. A complete classification in Euclidean and real hyperbolic spaces followed, but the case of the spheres was much more difficult, and only recently a complete classification was obtained [15]. Cartan classified the isoparametric hypersurfaces in the sphere with $l \in \{1, 2, 3\}$ different principal curvatures. Later, H.F. Münzner [32] proved that, an isoparametric hypersurface in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with l distinct principal curvatures is contained in a level set of a homogeneous polynomial of degree l on $\mathbb{R}^{n+1}$ satisfying certain equations known now as the Cartan–Münzner differential equations. He used this to prove that the number l of distinct principal curvatures can only be 1, 2, 3, 4, or 6. Then several authors worked on the difficult cases of $l = 4$ or 6 distinct principal curvatures: see for instance [14, 12, 15]. For a more detailed study of isoparametric hypersurfaces in space forms see for example [13].

Now, we will explain some basic properties of isoparametric functions.

Proposition 4. *The isoparametric hypersurfaces of an isoparametric function are parallel hypersurfaces of constant mean curvature.*

Proof. Let $f : N \to \mathbb{R}$ isoparametric and $L = \nabla f / \|\nabla f\|$ a unit normal vector field to the hypersurface $M = f^{-1}(c)$ (c regular value of f).
We suppose $S(X) = -\nabla_X L$ the shape operator with respect to L.
If $\{E_i\}$ is an orthonormal frame on M, then the mean curvature H of M is given by

$$\begin{aligned} H = trS = \sum_{i=1}^{n-1} \langle SE_i, E_i\rangle = -\sum_{i=1}^{n-1} \frac{1}{\sqrt{b}} \langle \nabla_{E_i} \nabla f, E_i\rangle = -\frac{1}{\sqrt{b}} \sum_{i=0}^{n-1} Hess_f(E_i, E_i) \\ = -\frac{1}{\sqrt{b}}(\Delta f - Hess_f(L, L)) = -\frac{1}{\sqrt{b}}(\Delta f - \frac{1}{2b}\nabla f(b)) = -\frac{1}{2\sqrt{b}}(2a - b'). \end{aligned}$$

Hence each regular hypersurface $f^{-1}(c)$ has constant mean curvature.

The proposition follows if we show that the regular hypersurface $M_t = exp_M(tL)$ parallel to M is a regular level set of f. It is enough to show that integral curves of L are geodesics.

Since $\|L\| = 1$ we get $\langle \nabla_L L, L\rangle = 0$.
On the other hand, for $X \in \Gamma(TM)$ we have

$$-\langle \nabla_L L, X\rangle = \langle \nabla_L X, L\rangle = \langle \nabla_L X - \nabla_X L, L\rangle = \langle [L,X], L\rangle = \frac{1}{\sqrt{b}}[L,X](f).$$

Therefore $X(f) = 0$ and $X(L(f)) = X(\sqrt{b}) = 0$ imply that $\nabla_L L = 0$. □

Remark 3. *The converse of this proposition is also true, that means, family of parallel hypersurfaces of constant mean curvatures define an isoparametric function.*

A natural family of examples of isoparametric hypersurfaces and hypersurfaces with constant principal curvatures is provided by cohomogeneity one isometric actions on a Riemannian manifold. These are the so-called homogeneous hypersurfaces:
We consider $M = G\cdot p$ a codimension-one orbit of an isometric action $G \times N \to N$ through a point $p \in N$. For any two points $q, x \in M$ there exists $h \in G$ such that $h(M) = M$ and $h(q) = x$. Then the shape operators of M at q and x are conjugated $S_x = h_* S_q h_*^{-1}$ so that M has constant principal curvature.
Furthermore, Let σ be a geodesic normal to M at some point $p \in M$. The tangent space to any orbit of G is generated by Killing vector fields induced by G. If X is a Killing vector field induced by G then $\langle \nabla_{\dot\sigma} X, \dot\sigma\rangle = 0$ (skew-symmetry). Thus $\langle X, \dot\sigma\rangle$ is constant along to σ and vanish at p. We conclude that σ is perpendicular to the other orbits intersect it.

In the setting of Riemannian submersions we have the following elementary construction:

Claim 1. *Let $\pi : (E, \hat g) \to (B, g)$ be a Riemannian submersion such that each fiber is minimal. If f is an isoparametric function on (B, g) then $F = f \circ \pi$ is isoparametric on $(E, \hat g)$.*

Proof. By definition F is a horizontal function on E, i.e. $\nabla^{\hat g} F$ is orthogonal to the fibers. In particular this implies that

$$\|\nabla^{\hat g} F\|^2 = \|\nabla^g f\|^2.$$

Assume that L is a fiber of the submersion. In the case that $X, Y \in \Gamma(TL)$ it follows that

$$Hess_{\hat g} F(X,Y) = \hat g(\nabla^{\hat g}_X \nabla^{\hat g} F, Y) = -\hat g(\nabla^{\hat g} F, \nabla^{\hat g}_X Y).$$

Henceforth

$$\Delta^{\hat g} F = tr(HessF)|_{(TL)^\perp} + tr(HessF)|_{(TL)} = \Delta^g f \circ \pi - \langle H, \nabla^{\hat g} F\rangle,$$

where H is the mean curvature vector of L.
From our hypothesis, the function F is isoparametric. □

Isoparametric hypersurfaces allow reducing certain systems partial differential equations to ordinary differential equations, which can help to find explicit solutions. This is one of the reasons for which it is important to investigate the existence of such functions. For instance it has been applied to the study of the multiplicity of solutions to the Yamabe problem on the Riemannian manifold (M^m, g) (see [22]), which consists of finding metrics of constant scalar curvature conformal to g. If the scalar curvature of g (denote s_g) is constant, then writing a conformal metric as $h = u^{\frac{4}{m-2}} g$ (for a positive function u) we have that h has constant scalar curvature λ if and only if u is a positive solution of the Yamabe equation

$$-\frac{4(m-1)}{m-2}\Delta_g u + s_g u = \lambda u^{\frac{m+2}{m-2}}.$$

If there is an isoparametric function f on (M, g) then one can look for solutions of the form $u = \varphi \circ f$, where $\varphi : \mathbb{R} \to \mathbb{R}^+$ is a positive smooth function. It follows that u solves the Yamabe equation if φ solves the ordinary differential equation

$$-\frac{4(m-1)}{m-2}(\varphi'' a + \varphi' b) + s_g \varphi = \lambda \varphi^{\frac{m+2}{m-2}} \quad \text{for } a, b \text{ given by } (1), (2) \text{ in } 3.1.1.$$

In this thesis, we are especially interested in the relations between isoparametric functions and Yamabe-type equations which, in turn, is related to the conformal geometry. For this reason we would like to see, for instance, about isoparametricity of functions under special conformal deformation.
By direct computation we have the following proposition

Proposition 5. *Let f be an isoparametric function on (N, g). Then f is isoparametric on $(N, \hat{g})$, where $\hat{g} = e^{2u(f)} g$ for smooth function $u : \mathbb{R} \to \mathbb{R}$.*

Proof. Since $\nabla^{\hat{g}} f = e^{-2u(f)} \nabla^g f$,

$$\|\nabla^{\hat{g}} f\|^2 = e^{-2u(f)} b(f).$$

On the other hand, for $X, Y \in \Gamma(TN)$

$$Hess_{\hat{g}} f(X,Y) = \hat{g}(\nabla^{\hat{g}}_X \nabla^{\hat{g}} f, Y) = -2u'(f) g(\nabla^g f, X) g(\nabla^g f, Y) + e^{-2u(f)} \hat{g}(\nabla^{\hat{g}}_X \nabla^g f, Y).$$

The koszul formula for ∇^g tell us that,

$$\begin{aligned} 2g(\nabla^g_X Z, Y) &= X(g(Z,Y)) + Z(g(X,Y)) - Y(g(X,Z)) \\ &\quad - \{g(Z,[X,Y]) + g(X,[Z,Y]) + g(Y,[Z,X])\}. \end{aligned}$$

Hence, taking $Z = \nabla^g f$ and the Koszul formula for $\nabla^{\hat{g}}$ we get

$$\begin{aligned} 2\hat{g}(\nabla^{\hat{g}}_X \nabla^g f, Y) &= 2e^{2u(f)} g(\nabla^g_X \nabla^g f, Y) + \nabla^g f(e^{2u(f)}) g(X,Y) \\ &\quad + X(e^{2u(f)}) g(\nabla^g f, Y) - Y(e^{2u(f)}) g(\nabla^g f, X) \end{aligned}$$

$$
\begin{aligned}
&= 2e^{2u(f)}g(\nabla^g_X \nabla^g f, Y) + \nabla^g f(e^{2u(f)})g(X,Y) \\
&= 2e^{2u(f)}\{g(\nabla^g_X \nabla^g f, Y) + u'(f)b(f)g(X,Y)\}.
\end{aligned}
$$

Thereby

$$
\begin{aligned}
Hess_{\hat{g}} f(X,Y) &= -2u'(f)g(\nabla^g f, X)g(\nabla^g f, Y) + g(\nabla^g_X \nabla^g f, Y) + u'(f)b(f)g(X,Y) \\
&= -2u'(f)X(f)Y(f) + Hess_g f(X,Y) + u'(f)b(f)g(X,Y).
\end{aligned}
$$

If we take an $\hat{g}-$ orthonormal frame $\{E_i\}$ then $\{e^u(f)E_i\}$ is $g-$orthonormal, so that

$$
\Delta^{\hat{g}} f = e^{-2u(f)}\{a(f) + (n-2)u'(f)b(f)\},
$$

and f is isoparametric on $(N, \hat{g})$. □

Remark 4. *There always exists infinite Riemannian metrics admitting isoparametric functions on a fixed manifold once there exists one.*

In general setting Q.M. Wang in [47] proved and conjectured several beautiful structural properties about isoparametric hypersurfaces. Here we remember the main results

Theorem 3.1.2 (Wang,[47])**.** *Let M be a connected, complete, smooth Riemannian manifold and f an isoparametric function on M. Then*

1. The focal varieties of f are smooth submanifolds of M;
2. Each regular level set of f is a tube over either of the focal varieties (the dimensions of the fibers may differ on different connected components).

Remark 5. *The theorem is valid for a more general space of functions, i.e. functions that only satisfy (1) in definition 3.1.1. These functions are known as transnormal.*

Lemma 3.1.3. *Let d the only critical value of f in $[c,d] \subset f(M)$. Then the improper integral*

$$
\int_c^d \frac{dt}{\sqrt{b(t)}}
$$

converges.

As a consequence of this lemma, we can obtain that: The interior of $f(M)$ only has regular values.
If we suppose that there exists d critical value of f such that $[d-\epsilon, d] \subset Int f(M)$ then, $b(d) = 0$. Since $0 \leq b$ in $f(M)$ we have that $b'(d) = 0$. Thus there exists a constant C such that $0 \leq b(t) \leq C(t-d)^2$ for $t \in [d-\epsilon, d]$.
Therefore

$$
\int_c^d \frac{dt}{d-t} \leq \int_c^d \frac{\sqrt{C}}{\sqrt{b(t)}} dt
$$

diverges for $c \in [d-\epsilon, d]$ and $\epsilon > 0$ small enough. This is a contradiction according to the lemma.

Another simple consequence of lemma and the argument above is: At the maximum (resp. minimum) of f we get $b' < 0$ ($b' > 0$) if it exists.

Now, we recall two remarkable facts. Let $f : N \to \mathbb{R}$ be an isoparametric function on a complete connected Riemannian manifold N. It follows that

1. The focal varieties of an isoparametric function on a complete Riemannian manifold are minimal submanifolds (Theorem 1.1, [20]);

2. The Hessian of f restricted to the focal variety M_- (resp. M_+) has only two eigenvalues: zero in direction of TM_- (TM_+) and $b'(f(t_-))/2$ in direction $(TM_-)^\perp$ for any $t_- \in M_-$ (Lemma 6, [47]).

Thanks to Wang's result, the isoparametric hypersurfaces have a simple structure in general Riemannian manifolds (are tubes over focal submanifolds) but this hypersurfaces may be disconnected, which could lead to undesired behaviour. We conclude this section with a brief description of a natural family of isoparametric functions with connected level sets.

Definition 3.1.4. *An isoparametric function is called proper if each component of focal varieties does not have codimension less than 2.*

This definition was introduced by J. Ge and Z. Tang in [19]. In this work, Ge - Tang deduced very interesting properties of this special type of functions.

Proposition 6. *If $f : N \to \mathbb{R}$ is a proper isoparametric function then each level set $M_t = f^{-1}(t)$ is connected.*
Furthermore, when the ambient manifolds N is closed there exists at least one minimal isoparametric hypersurface M_{t_0}.

Proof. First, let $[c, d] = f(N)$. The values c, d may be infinity.
If each component of M_- (and M_+) does not have codimension less than 2, then we have that $N - M_- \cup M_+ = M_{t_0} \times (c, d)$ is connected (for some $c < t_0 < d$) and thus each regular level hypersurface is connected, therefore M_- and M_+ are connected. This implies the first assertion.
We assume now that N is closed. As in the proof of proposition 4 we can obtain the mean curvature function $h(t)$ associated to hypersurfaces M_t by formula

$$h(t) = \frac{1}{2\sqrt{b}}(-2a(t) + b'(t)).$$

From the last result of latter section

$$a(c) = \Delta f|_{M_-} = tr(Hess_g f)|_{M_-} = \frac{1}{2}b'(c)codim(M_-).$$

Hence

$$-2a(c) + b'(c) = b'(c)(1 - codim(M_-)).$$

We know that $b'(c) > 0$, so that $\lim_{t\to c} h(t) < 0$.
By analogous argument, $\lim_{t\to d} h(t) > 0$.
Continuity of h implies that there exists $t_0 \in (c, d)$ such that $h(t_0) = 0$. □

Additionally, when the closed ambient manifold has positive Ricci curvature the minimal isoparametric hypersurface given by the above proposition is unique. More precisely, let $L(t) := \nabla f(t)/|\nabla f(t)|$ be the normal vector fields of isoparametric hypersurfaces M_t. By radial curvature equation we have

$$\nabla_L S_t = S_t^2 + R_L,$$

where S_t is the shape operator of M_t and $R_L(\cdot) = R(\cdot, L)L$ the curvature tensor of N. The trace of this equation allow us to describe the monotonicity of h:

$$(n-1)h'(t) = tr(S_t^2) + Ricc(L, L) \geq Ricc(L, L) > 0.$$

Thereby h is strictly increasing so that there exists a unique minimal isoparametric hypersurface associated to f.

Remark 6. *We consider a simple example of non-proper isoparametric function. Let $f : \mathbb{S}^3 \to [0, 1]$ defined by $f(x_0, x_1, x_2, x_3) = x_0^2$. The function f is isoparametric since $\|\nabla f(x)\|^2 = 4f$ and $\Delta f = 2$; and non-proper because $f^{-1}(0) = \mathbb{S}^2$. For $t \in (0, 1)$ we note that the regular level hypersurfaces $f^{-1}(t)$ are disconnected and no minimal.*

3.2 Compact isoparametric hypersurfaces in $\mathbb{R}^n \times \mathbb{M}^m$

In this section, we will prove Theorem 1.0.2.
First, we start stating a structural result of R. Miyaoka, Theorem 1.1 in [31].

Theorem 3.2.1 (Miyaoka)**.** *Let M be a complete connected Riemannian manifold which admits a transnormal function f. Then either one of the following holds:*

1. *M is diffeomorphic to a vector bundle over a submanifold Q of M.*
2. *M is diffeomorphic to a union of two disk bundles over two submanifolds Q and Q' of M, where Q and/or Q' may be hypersurface(s).*

Now, Let $\mathbb{M}^m$ be a connected, closed Riemannian manifold. We consider a family of compact isoparametric hypersurfaces M_t in $\mathbb{R}^n \times \mathbb{M}^m$ with $n, m \geq 2$, i.e. exist an isoparametric function $f : \mathbb{R}^n \times \mathbb{M}^m \to \mathbb{R}$ such that each $M_t = f^{-1}(t)$ is compact.

If the focal varieties of f are empty (i.e. $M_- = M_+ = \phi$) then, from Miyaoka's Theorem and the fact that $\mathbb{R}^n \times \mathbb{M}^m$ can not be an $\mathbb{S}^1$ bundle over some M_t (since each M_t are compact), $\mathbb{R}^n \times \mathbb{M}^m$ is a rank one vector bundle over some M_t regular hypersurface.
It is well-known that exits a deformation retract of the total space over the base space of a

vector bundle. This implies that the homology group of M_t and $\mathbb{R}^n \times \mathbb{M}^m$ are equivalent. In particular, we obtain $0 = H_{m+n-1}(\mathbb{M}^m) = H_{m+n-1}(M_t)$ since $n-1>0$, which is a contradiction. Therefore there is a non-empty focal variety.
In the case that f has $M_- \neq \phi$ and $M_+ \neq \phi$, again by Miyaoka's Theorem we have that $\mathbb{M}^m \times \mathbb{R}^n$ is diffeomorphic to a union of two disk bundles over M_+ and M_-. Since that M_- and M_+ are compact, $\mathbb{R}^n \times \mathbb{M}^m$ would be compact.
Without loss of generality, we can assume that the set M_- of minimum points of f it is non-empty and $M_+ = \phi$.
Since $\mathbb{R}^n \times \mathbb{M}^m$ cannot be the union of two disk bundles over compact submanifolds, we have that $\mathbb{R}^n \times \mathbb{M}^m$ is a vector bundle over M_-.

Now, we point out a fact about minimal submanifolds. See [48] for more details.

Lemma 3.2.2. *Let $\Phi : L^n \to \mathbb{R}^k$ be an isometric immersion with the mean curvature vector H, then*

$$\Delta\Phi = nH,$$

where $\Delta\Phi = (\Delta\Phi^1, \cdots, \Delta\Phi^k)$.

Proof. Let $\{e_i\}$ be a local orthonormal frame field of L. Then

$$\begin{aligned}\Delta\Phi &= \sum_i \nabla^{\mathbb{R}^k}_{\Phi_* e_i}\Phi_* e_i - \Phi_*\nabla^L_{e_i} e_i \\ &= \sum_i (\nabla^{\mathbb{R}^k}_{\Phi_* e_i}\Phi_* e_i)^\perp = nH.\end{aligned}$$

□

On the other hand, let $L \to \bar{L} \subset \bar{\bar{L}}$ be isometric immersions with connections $\nabla, \bar{\nabla}$ and $\bar{\bar{\nabla}}$ respectively. Denote H and $\bar{H}$ to be the mean curvatures of L in $\bar{L}$ and L in $\bar{\bar{L}}$ respectively. Then

$$\begin{aligned} nH &= \sum (\bar{\nabla}_{e_i} e_i)^\perp \\ &= (\sum (\bar{\bar{\nabla}}_{e_i} e_i)^{T\bar{L}})^\perp \\ &= (\sum (\bar{\bar{\nabla}}_{e_i} e_i)^\perp)^{T\bar{L}} = n\bar{H}^{T\bar{L}}. \end{aligned}$$

In our situation, $\Phi : M_- \to \mathbb{R}^n \times \mathbb{M}^m \subset \mathbb{R}^{n+k}$ is minimal for some k. Thus

$$\Delta\Phi \perp T(\mathbb{R}^n \times \mathbb{M}^m).$$

Hence, $(\Delta\Phi^1, \cdots, \Delta\Phi^n) = 0$.
Since M_- is compact it follows that the functions Φ^j are constant for all $j = 1, \ldots, n$. Thus, the focal variety of f is of the form $M_- = \{p\} \times V$, where $V \subseteq \mathbb{M}^m$ is a submanifold and $p \in \mathbb{R}^n$.

Since the submanifold V and $\mathbb{R}^n \times \mathbb{M}^m$ are homotopy equivalent, we have

$$H_m(\mathbb{M}^m) = H_m(V).$$

Therefore, $dim(V) = m$ and

$$M_- = \{p\} \times \mathbb{M}^m.$$

But since the level sets M_t are tubes over M_- this, of course, implies Theorem 1.0.2.

3.3 Isoparametric hypersurfaces with C.P.C. in $\mathbb{S}^2 \times \mathbb{R}^2$

In this section, we will prove Theorem 1.0.3. For this, we follow some ideas of F. Urbano [45] used into the study of homogeneous and isoparametric hypersurfaces in $\mathbb{S}^2 \times \mathbb{S}^2$. Since that the curvature tensor of $\mathbb{S}^2 \times \mathbb{S}^2$ depends on product structure on it, the fundamental equations (Gauss and Codazzi) for the theory of submanifolds might reflect on these hypersurfaces some relations with such product structure. Indeed, Urbano proved that the behavior of these types of hypersurfaces respect to product structure is very rigid, i.e. the function associated to this behavior is constant. We will employ this program in the case of isoparametric hypersurfaces with constant principal curvatures in $\mathbb{S}^2 \times \mathbb{R}^2$.

Notation and background will be the same as in [45].
Let $\mathbb{S}^2$, $\mathbb{R}^2$ be space forms with curvatures 1 and 0 respectively. We define the complex structures L_1 and L_2 by:

$$\begin{aligned} L_1 : T\mathbb{S}^2 &\to T\mathbb{S}^2 \\ v &\mapsto L_1(v) := p \wedge v \quad \text{for } p \in \mathbb{S}^2, \quad v \in T_p\mathbb{S}^2; \\ L_2 : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (q_1, q_2) &\mapsto L_2((q_1, q_2)) := (-q_2, q_1). \end{aligned}$$

We consider $\mathbb{S}^2 \times \mathbb{R}^2$ with the product metric and the complex structures $J_1 = (L_1, L_2)$, $J_2 = (L_1, -L_2)$. We notice that the product structure P in $\mathbb{S}^2 \times \mathbb{R}^2$ defined by $P(v_1, v_2) = (v_1, -v_2)$ satisfies that $P = -J_1 J_2 = -J_2 J_1$, moreover, P is parallel with respect of the Levi-Civita connection of $\mathbb{S}^2 \times \mathbb{R}^2$.

Let $M^3 \subset \mathbb{S}^2 \times \mathbb{R}^2$ be an oriented hypersurface with $N = (N_1, N_2)$ a unit normal vector field to M^3. We consider the function C and vector field X tangent to M^3 given by:

$$C := \langle PN, N \rangle \ \text{ and } \ X := PN - CN.$$

Lemma 3.3.1. *Let $f : \mathbb{S}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be an isoparametric function. If each regular hypersurface $M_t = f^{-1}(t)$ has constant principal curvatures, then the function C_t corresponding to each M_t is constant.*

Proof. By condition (1) of the definition of isoparametric function the unit vector field $N = \frac{\nabla f}{|\nabla f|}$ is a geodesic field. Since the product structure P is parallel, the function C_t is independent of the regular hypersurfaces M_t (since $N(C_t) = \langle \nabla_N N, PN \rangle + \langle N, P\nabla_N N \rangle = 0$).
Now, we consider the open set $U = \{p \in M_t / C^2(p) < 1\}$. If U is not empty then we can consider on U the local orthonormal frame field,

$$B = \left\{ B_1 = \frac{X}{\sqrt{1-C^2}}, B_2 = \frac{J_1N + J_2N}{\sqrt{2(1+C)}}, B_3 = \frac{J_1N - J_2N}{\sqrt{2(1-C)}} \right\}.$$

From the radial curvature equation (also called the Riccati equation) we have

$$-\nabla_N S_t + S_t^2 = -R_N,$$

where S_t is the shape operator of M_t corresponding to N and $R_N(\cdot) = R(\cdot, N)N$.
Taking trace we obtain that

$$-tr(R_N) = -tr(\nabla_N S_t) + tr(S_t^2) = -\nabla_N tr S_t + tr(S_t^2) = -3H'(t) + tr(S_t^2).$$

We are assuming that the principal curvatures $\mu_1(t), \mu_2(t)$ and $\mu_3(t)$ of M_t are constant, then we have that $tr(S_t^2) = \mu_1^2 + \mu_2^2 + \mu_3^2$ is constant. Therefore $tr(R_N)$ is constant in M_t. Now we compute $tr(R_N)$ in the frame B. Using the formula of the curvature tensor of $\mathbb{S}^2$ and $\mathbb{R}^2$ we obtain

$$\begin{aligned}
R_N(B_1) &= \frac{1}{8\sqrt{1-C^2}} R^{\mathbb{S}^2}(PX + X, PN + N)(PN + N) \\
&= 0 \qquad \text{(since } PX + X = (1-C)(PN+N));\\
R_N(B_2) &= \frac{|PN + N|^2}{4} B_2;\\
R_N(B_3) &= R^{\mathbb{R}^2}(B_3, N_2)N = 0.
\end{aligned}$$

Thus $tr(R_N) = \frac{1+C}{2}$ and the lemma follows. □

Theorem 1.0.3 is equivalent to the following:

Claim 2. *The isoparametric functions on $\mathbb{S}^2 \times \mathbb{R}^2$ with regular level sets of constant principal curvatures only depend on one factor, i.e. $C^2 = 1$.*

Proof. Let f be an isoparametric function on $\mathbb{S}^2 \times \mathbb{R}^2$ with $M_t = f^{-1}(t)$ of constant principal curvatures. The Lemma 3.3.1 implies that the function C is constant.
Assume that $C \in (-1, 1)$.
We are going to express the shape operator $S_0 = S$ and the tangential component of the product structure P^T in the orthonormal frame field B considered inside the proof of Lemma 3.3.1.

Note that

$$\begin{aligned}\langle \nabla C, Y\rangle = \nabla_Y \langle N, PN\rangle &= \langle \nabla_Y N, PN\rangle + \langle N, P\nabla_Y N\rangle \\ &= 2\langle PN, -S(Y)\rangle = \langle -2S(X), Y\rangle\end{aligned}$$

for $Y \in \Gamma(TM)$. Then $S(X) = -\nabla C/2 = 0$ and we can write

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{23} & \sigma_{33} \end{pmatrix}.$$

On the other hand,

$$PB_1 = \frac{1}{\sqrt{1-C^2}}(-CX + N(1-C^2))\ ,\ PB_2 = B_2\ ,\ PB_3 = -B_3,$$

thus

$$P^T = \begin{pmatrix} -C & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

By a direct computation, we obtain the Hessian of the function C:

$$\nabla^2 C(V,W) = -2\nabla S(V,X,W) - 2C\langle SV, SW\rangle + 2\langle PSV, SW\rangle.$$

Using the Codazzi equation of $M = M_0$:

$$\nabla S(V,W,Z) - \nabla S(W,V,Z) = \frac{1}{4}\langle V,X\rangle\langle PW + W, Z\rangle - \frac{1}{4}\langle W,X\rangle\langle PV + V, Z\rangle,$$

we compute the Laplacian of the function C in the frame field B.

$$\Delta C = tr(\nabla^2 C(\cdot,\cdot)) = -2tr\nabla S(\cdot, X, \cdot) - 2Ctr(S^2) + 2tr(P^T S^2)$$

but

$$\begin{aligned}-2tr\nabla S(\cdot,X,\cdot) &= -2tr\nabla S(X,\cdot,\cdot) - \frac{1}{2}tr\{\langle\cdot,X\rangle\langle PX + X,\cdot\rangle - \langle X,X\rangle\langle P\cdot + \cdot,\cdot\rangle\} \\ &= -2\nabla_X trS(\cdot,\cdot) - \frac{1}{2}\{\langle PX + X, X\rangle - \langle X,X\rangle(3-C)\} \\ &= -6\langle X, \nabla H\rangle + \langle X, X\rangle.\end{aligned}$$

Hence

$$\Delta C = -6\langle X,\nabla H\rangle + |X|^2 - 2Ctr(S^2) + 2tr(P^T S^2).$$

Then we have

$$tr(S^2)C - |X|^2/2 = tr(P^T S^2) = \sigma_{22}^2 - \sigma_{33}^2 = 3H(\sigma_{22} - \sigma_{33}).$$

Assume now that $H \neq 0$.
From the expression

$$\sigma_{22} - \sigma_{33} = \frac{1}{3H}\{tr(S^2)C - |X|^2/2\} = \frac{1}{3H}\{tr(S^2)C - (1-C^2)/2\}$$

and the fact that $tr(S^2)$ is constant (M has constant principal curvatures) we see that $\sigma_{22} - \sigma_{33}$ must be constant.

Then $\sigma_{22}^2 + \sigma_{23}^2$ is also constant since $tr(S^2) + 3H\{\sigma_{22} - \sigma_{33}\} = 2\sigma_{22}^2 + 2\sigma_{23}^2$.
And since

$$tr(S^2) + 9H^2 = 2\sigma_{22}^2 + 2\sigma_{23}^2 + 2\sigma_{33}^2 + 2\sigma_{22}\sigma_{33},$$

we have that $2\sigma_{33}(\sigma_{33} + \sigma_{22})$ is constant.
Since $H \neq 0$, σ_{33} must be constant, and hence σ_{22} is also constant. It follows that each σ_{ij} is constant.

Now, we compute $X(\sigma_{22})$, $X(\sigma_{33})$ and $X(\sigma_{23})$.
Since J_i are parallel and $S(X) = 0$, we have $\nabla_X B_j = 0$ for $j = 1, 2, 3$.
Thus,

$$\begin{aligned}
X(\sigma_{22}) &= \nabla S(X, B_2, B_2) \\
&= \nabla S(B_2, X, B_2) + \frac{|X|^2}{2} \\
&= \frac{|X|^2}{2} + \langle PSB_2, SB_2\rangle - C\langle SB_2, SB_2\rangle \\
&= \frac{1-C^2}{2} + (1-C)\sigma_{22}^2 - (1+C)\sigma_{23}^2,
\end{aligned}$$

$$\begin{aligned}
X(\sigma_{33}) &= \nabla S(X, B_3, B_3) \\
&= \nabla S(B_3, X, B_3) \\
&= \langle PSB_3, SB_3\rangle - C\langle SB_3, SB_3\rangle \\
&= (\sigma_{23}^2 - \sigma_{33}^2) - C(\sigma_{23}^2 + \sigma_{33}^2) \\
&= (1-C)\sigma_{23}^2 - (1+C)\sigma_{33}^2,
\end{aligned}$$

$$\begin{aligned}
X(\sigma_{23}) &= \nabla S(X, B_2, B_3) \\
&= \nabla S(B_2, X, B_3) \\
&= \langle PSB_2, SB_3\rangle - C\langle SB_2, SB_3\rangle \\
&= (1-C)\sigma_{22}\sigma_{23} - (1+C)\sigma_{23}\sigma_{33}.
\end{aligned}$$

Therefore we get

$$\frac{1-C^2}{2} + (1-C)\sigma_{22}^2 - (1+C)\sigma_{23}^2 = 0, \tag{3.3.1}$$

$$(1-C)\sigma_{23}^2 - (1+C)\sigma_{33}^2 = 0, \tag{3.3.2}$$
$$(1-C)\sigma_{22}\sigma_{23} - (1+C)\sigma_{23}\sigma_{33} = 0. \tag{3.3.3}$$

Now, from equation

$$(\sigma_{22}^2 + 2\sigma_{23}^2 + \sigma_{33}^2)C - (1-C^2)/2 = tr(S^2)C - |X|^2/2 = 3H(\sigma_{22} - \sigma_{33}) = \sigma_{22}^2 - \sigma_{33}^2,$$

we obtain

$$-2C\sigma_{23}^2 = (C-1)\sigma_{22}^2 + (C+1)\sigma_{33}^2 - |X|^2/2 = -X(\sigma_{22}) = 0.$$

If $\sigma_{23} = 0$ then $\sigma_{22}^2 = \frac{1-C^2}{2(C-1)}$ (from the formula (3.3.1)) but this is a contradiction since $C^2 < 1$.
Therefore $\sigma_{23} \neq 0$ and $C = 0$. But the formula (3.3.3) gives then that $\sigma_{22} = \sigma_{33}$. Then $\sigma_{23}^2 = \sigma_{33}^2$ (from (3.3.2)). Hence $\sigma_{22}^2 = \sigma_{23}^2$ and replacing these value in (3.3.1) gives a contradiction.

The above argument means that the family of isoparametric hypersurfaces with constant function $C \in (-1,1)$ are all minimal.
But, if we assume that the family M_t are all minimal, then from the trace of the radial curvature equation (Riccati equation) we have

$$0 = 3H'(t) = tr(S_t^2) + tr(R_N) = tr(S_t^2) + \frac{1+C}{2} > 0.$$

This argument allow us to conclude that $C^2 = 1$. □

In contrast to Urbano's work we obtain only two family of hypersurfaces $C = 1$ or $C = -1$ while Urbano's classification three family $C = 1, C = -1$ and $C = 0$.

3.4 Eigenvalues of restricted Laplacians

In order to apply our global Theorem in the next chapter we need to understand the eigenvalues of Δ_g restricted to f-invariant functions and the dimension of the focal submanifolds. While the dimension of the focal submanifolds is usually simple to understand, to compute the eigenvalues of the restricted Laplacian might be lengthy.

The situation we will consider is a Riemannian submersion with totally geodesic fibers $\pi : (M_1, g_1) \to (M_2, g_2)$. In this situation the corresponding Laplacians commute: for any function $f : M_2 \to \mathbb{R}$, $\Delta_{g_2}(f) \circ \pi = \Delta_{g_1}(f \circ \pi)$. And it is easy to check that f is isoparametric for (M_1, g_1) if and only if $f \circ \pi$ is isoparametric for (M_2, g_2). Actually $\|\nabla f\|^2 = a \circ f$ and $\Delta f = b \circ f$ if and only if $\|\nabla(f \circ \pi)\|^2 = a \circ (f \circ \pi)$ and $\Delta(f \circ \pi) = b \circ (f \circ \pi)$. Then it follows easily that $h = \alpha \circ f$ is an eigenfunction of Δ_{g_2} with eigenvalue λ if and only if $h \circ \pi$ is an eigenfunction of Δ_{g_1} with eigenvalue λ. So it is equivalent to study f-invariant eigenfunctions or $(f \circ \pi)$-invariant eigenfunctions.

Another fact we will use is that the problem is easy to solve in the case of the round sphere. In general if f is an isoparametric function then one can consider the family of isoparametric functions of the form $\alpha \circ f$, where α is a monotone function. These isoparametric functions are in certain sense equivalent: they have the same level sets and the spaces of f-invariant functions and $(\alpha \circ f)$-invariant functions are the same. In the case of the round sphere, $(\mathbb{S}^n, g_0)$, there is a canonical way to pick a representative of these families of *equivalent* isoparametric functions. Namely, in any such family H. F. Münzner ([32]) proved that there is a *Cartan-Münzner polynomial*. This is a homogeneous harmonic polynomial F (in $\mathbb{R}^{n+1}$) of degree k which solves the Cartan-Münzner equations:

$$\|\nabla F(x)\|^2 = k^2 \|x\|^{2k-2}$$

$$\Delta F(x) = \frac{1}{2} c k^2 \|x\|^{k-2},$$

for some integer c. But then one can easily see by studying the resulting linear ordinary differential equation that the F-invariant eigenvalues are exactly $\mu_i = \lambda_{ki}$, $i \geq 1$, where $\lambda_j = j(n+j-1)$ are the eigenvalues of $-\Delta_{(\mathbb{S}^n, g_0)}$ (see [22, Lemma 3.4]).

Let us now consider the case of the complex projective spaces with the Fubini-Study metric $(\mathbf{CP}^n, g_{FS})$. Recall that the positive eigenvalues of $-\Delta_{g_{FS}}$ are $2i(2i+2n)$, $i \geq 1$. There is a Riemannian submersion (the Hopf fibration) $\mathbb{S}^{2n+1} \to (\mathbf{CP}^n, g_{FS})$, obtained by considering the canonical diagonal $\mathbb{S}^1$-action on $\mathbb{S}^{2n+1}$. It has totally geodesic fibers (which are circles, the orbits of the $\mathbb{S}^1$-action) so we can apply the previous ideas. An isoparametric function f on $(\mathbf{CP}^n, g_{FS})$ lifts to an isoparametric function $\overline{f} : (\mathbb{S}^{2n+1}, g_0) \to [t_0, t_1]$. And we can look for the corresponding Cartan-Münzner polynomial.

We will consider the three simplest examples of isoparametric functions on $(\mathbf{CP}^n, g_{FS})$. These are given by cohomogeneity one actions.

1) Let us consider first the action of $U(n) \subset SO(2n)$. This action lifts to a cohomogeneity one action on $\mathbb{S}^{2n+1}$ which commutes with the diagonal $\mathbb{S}^1$-action (the action on $S^{2n+1} \subset \mathbf{R}^{2n+2}$ is given by $A.(x_1, x_2, y_1, ...y_{2n}) = (x, Ay)$. We consider on $\mathbb{R}^{2n+2}$ the homogeneous harmonic polynomial $F(x,y) = \|x\|^2 - \|y\|^2$. It is invariant by the action of $\mathbb{S}^1 \times O(2n)$ and therefore projects to an isoparametric function f on $(\mathbf{CP}^n, g_{FS})$ invariant by the $U(n)$-action. F is a Cartan-Münzner polynomial of degree 2. Then it follows that the f-invariant eigenvalues of $-\Delta_{g_{FS}}$ are $\lambda_{2i} = 2i(2i+2n)$. Note that these are actually the eigenvalues of the full Laplacian $-\Delta_{g_{FS}}$. Also note that the action of $U(n)$ on $(\mathbf{CP}^n, g_{FS})$ has a fixed point.

2) Let us now consider the action of $U(m) \times U(l) \subset U(n+1)$ on $(\mathbf{CP}^n, g_{FS})$, where we ask $n \geq 3$, $m + l = n + 1$ and $m \geq l \geq 2$. Similarly to the previous case we can easily lift the action to $\mathbb{S}^{2n+1} \subset \mathbf{R}^{2n+2}$, commuting with the diagonal $\mathbb{S}^1$-action. The action looks like $(A, B).(x, y) = (Ax, By)$. Again $F(x,y) = \|x\|^2 - \|y\|^2$ is an invariant

Cartan-Münzner polynomial of degree 2 which projects to an isoparametric function f on $(\mathbf{CP}^n, g_{FS})$. It follows that the f-invariant eigenvalues of $-\Delta_{g_{FS}}$ are $\lambda_{2i} = 2i(2i+2n)$. But now note that the critical orbits are $\mathbf{CP}^{m-1}$ and $\mathbf{CP}^{l-1}$.

3) There is a cohomogeneity one isometric action of $SO(n+1) \subset U(n+1)$ in $(\mathbf{CP}^n, g_{FS})$ given by considering the natural action on $\mathbb{C}^{n+1}$(in the introduction we considered the case $n = 2$). This action can be lifted to $\mathbb{S}^{2n+1} \subseteq \mathbb{R}^{2n+2}$. The corresponding isoparametric polynomial on the sphere $\mathbb{S}^{2n+1}$ is given by $F(x,y) = (\|x\|^2 - \|y\|^2)^2 + 4\langle x, y\rangle^2$, which is invariant under the action of $SO(n+1)$. It follows that the f-invariant eigenvalues of $-\Delta_{g_{FS}}$ are $\lambda_{4i} = 4i(4i+2n)$. The singular orbits for this action are $\mathbf{RP}^n$ and the Grassmanian of oriented two-planes $\widetilde{\mathrm{Gr}}(2, \mathbb{R}^{n+1})$, which have dimensions n and $2n-2$, respectively.

Chapter 4

Global bifurcation technique for Yamabe-type equation

In this chapter, we consider a closed Riemannian manifold (M^n, g) of dimension $n \geq 3$ and study positive solutions of the equation $-\Delta_g u + \lambda u = \lambda u^q$, with $\lambda > 0$, $q > 1$. In the case that M admits a proper isoparametric function with focal varieties M_1, M_2 of dimension $d_1 \geq d_2$ we show that for any $q < \frac{n-d_2+2}{n-d_2-2}$ the number of positive solutions of the equation $-\Delta_g u + \lambda u = \lambda u^q$ tends to ∞ as $\lambda \to +\infty$. We apply this result to prove multiplicity results for solutions of the Yamabe equations.

4.1 Yamabe-type equations for f-invariant functions

Next, we consider geodesics which are transversal to the level sets of f:

Definition 4.1.1. *A geodesic $\gamma : [l_1, l_2] \to M$ is called an $f-$segment if $f(\gamma(l))$ is an increasing function of l and $\gamma'(l) = \nabla f / \sqrt{b}$ wherever $\nabla f \neq 0$.*

Note that f-segments are parametrized by arc-length. It is also easy to see that the integral curves of ∇f (parametrized by arc-length) are f-segments, and that f-segments realize the distance between the level sets M_s, M_t (see [47, Lemma 1]). If $\gamma : [0, s] \to M$ is an f segment then $s = \text{length}(\gamma) = d(M_{f(\gamma(0))}, M_{f(\gamma s)})$, and $(f \circ \gamma)'(t) = \sqrt{b(f(\gamma(t))}$. By reparametrizing γ by $l = (f \circ \gamma)^{-1}(s)$ for $s \in [f(\gamma(0), f(\gamma(s))]$ is easy to obtain the formula for $d_g(M_c, M_d)$ for any $t_0 \leq c < d \leq t_1$:

$$d_g(M_c, M_d) = \int_c^d \frac{1}{\sqrt{b(t)}} dt,$$

Let $t^* = d_g(M_{t_0}, M_{t_1})$ and $\mathbf{d} : M \to [0, t^*]$, $\mathbf{d}(x) = d_g(M_{t_0}, x)$.

We will consider functions which are constant on the level sets of f:

Definition 4.1.2. *A function $u : M \to \mathbb{R}$ is called f-invariant if $u(x) = \phi(\mathbf{d}(x))$ for some function $\phi : [0, t^*] \to \mathbb{R}$.*

We will denote by $\mathcal{B} = \{\phi \in C^{2,\alpha}([0,t^*]) : \phi'(0) = 0 = \phi'(t^*)\}$.

Lemma 4.1.3. *If we denote by $C^{2,\alpha}_f(M)$ the set of $C^{2,\alpha}$ functions on M which are f-invariant, then the application $\phi \mapsto u(x) = \phi(\mathbf{d}(x))$ identifies $\mathcal{B}$ with $C^{2,\alpha}_f(M)$.*

Proof. Let $\phi \in \mathcal{B}$ and $u(x) = \phi(\mathbf{d}(x))$. By a direct computation we get

$$\nabla u(x) = \frac{\phi'(\mathbf{d}(x))}{\sqrt{b(f(x))}} \nabla f(x),$$

and

$$\nabla^2 u(X,Y) = \frac{\phi'(\mathbf{d})}{\sqrt{b}} \nabla^2 f(X,Y) + \frac{1}{b}\langle \nabla f, X\rangle\langle \nabla f, Y\rangle \left(\phi''(\mathbf{d}) - \frac{\phi'(\mathbf{d})}{\sqrt{b}} \cdot \frac{b'}{2}\right),$$

for X, Y vector fields on M.
It is clear that the function $u \in C^{2,\alpha}(f^{-1}(c,e))$ for $t_0 < c < e < t_1$.

Now, we choose a curve $\gamma : [0,l] \to M$ such that $\gamma(0) \in M_{t_0}$ and $\gamma(l) \in M_{f(x)}$. By mean value theorem there exists $t_{l_1} \in (0,l)$ that satisfies

$$\phi'(\mathbf{d}(\gamma(l))) = \phi'(\mathbf{d}(\gamma(l))) - \phi'(\mathbf{d}(\gamma(0))) = \langle \nabla(\phi' \circ \mathbf{d}), \gamma'(t_{l_1})\rangle (l-0),$$

thus

$$\phi'(\mathbf{d}(\gamma(l))) = \phi'' \circ \mathbf{d}(\gamma(t_{l_1}))\langle \nabla \mathbf{d}(\gamma(t_{l_1})), \gamma'(t_{l_1})\rangle l.$$

If γ is an $f-$segment, we conclude that

$$\phi' \circ \mathbf{d}(\gamma(l)) = \phi'' \circ \mathbf{d}(\gamma(t_{l_1}))l.$$

Similary, there exists $t_{l_2} \in (0,l)$ such

$$\sqrt{b(f(\gamma(l)))} = \frac{b'(f(\gamma(t_{l_2})))}{2} l.$$

By taking l close to 0 we prove that

$$\lim_{x \to M_{t_0}} \frac{\phi'(\mathbf{d})}{\sqrt{b(f)}} = \frac{2\phi''(0)}{b'(t_0)}.$$

This fact allow us to show that the function $u \in C^{2,\alpha}(f^{-1}([t_0, e)) :$
Since $\phi'' \in C^{0,\alpha}$ and $\phi'' - \frac{\phi'(\mathbf{d})}{\sqrt{b}} \cdot \frac{b'}{2} \to 0$ when $x \to M_{t_0}$,

$$\phi'' - \frac{\phi'(\mathbf{d})}{\sqrt{b}} \cdot \frac{b'}{2} \in C^{0,\alpha}$$

in a neighborhood of M_{t_0}.
Using that

$$\frac{1}{b}\langle \nabla f, X\rangle\langle \nabla f, Y\rangle$$

is bounded for x close to M_{t_0},

$$\frac{1}{b}\langle \nabla f, X\rangle\langle \nabla f, Y\rangle \left(\phi''(\mathbf{d}) - \frac{\phi'(\mathbf{d})}{\sqrt{b}} \cdot \frac{b'}{2}\right) \in C^{0,\alpha}(f^{-1}([t_0, e)).$$

Therefore $u \in C^{2,\alpha}(f^{-1}([t_0, e))$.
A similar argument in the other focal variety M_{t_1} shows that

$$u \in C^{2,\alpha}(M).$$

□

Now we obtain the expression of the Yamabe-type equations of (M, g) for f-invariant functions.

Lemma 4.1.4. *Let $u \in C^{2,\alpha}_f(M)$, $u(x) = \phi(\mathbf{d}(x))$, with $\phi \in \mathcal{B}$. Then u is a solution of equation (1.0.2) if and only if the function ϕ satisfies*

$$-\left(\phi'' + (-h)\phi'\right) + \lambda\phi = \lambda\phi^q. \tag{4.1.1}$$

on $[0, t^*]$. For $t \in (0, t^*)$ $h(t)$ is the mean curvature of M_t.

Proof. With the notation $f(x) = t_x$ we have

$$\begin{aligned}\Delta(u(x)) &= \Delta(\phi(\mathbf{d}(x)) \\ &= \phi''(|\nabla\mathbf{d}(x))|^2) + \phi'(\Delta(\mathbf{d}(x))).\end{aligned}$$

But

$$\begin{aligned}|\nabla\mathbf{d}|^2 &= 1 \quad \text{and,} \\ \Delta\mathbf{d}(x) &= \Delta\left(\int_{t_0}^{t_x} \frac{df}{\sqrt{b(f)}}\right) = \frac{-b'}{2b\sqrt{b}}|\nabla f|^2 + \frac{\Delta f}{\sqrt{b}} \\ &= \frac{1}{2\sqrt{b}}(-b' + 2a) = -h(t_x),\end{aligned}$$

where $h(t_x)$ is the mean curvature of the hypersurface M_{t_x}.

Hence ϕ is a solution of the ordinary differential equation

$$-(\phi'' + (-h)\phi') + \lambda\phi = \lambda\phi^q.$$

□

The function h is smooth away from the focal varieties. However, the well-known fact about the structure of regular hypersurfaces M_t over M_{t_0} allows getting the asymptotic behaviour of h close to M_{t_0}. Since each M_t is a tube over either M_{t_0,t_1} a coordinate system centered in M_{t_0} or M_{t_1} appropriate for this study are Fermi coordinates, which are the generalization of normal coordinates that arises when the center of the normal neighborhood is replaced by a submanifold.
Indeed, in [20] the authors used this coordinates to compute the power series expansion formula for the shape operator of M_t with respect to the distance to M_{t_0}.
More explicitly, corollary 2.2 in [20] implies that,

$$h(t) = -\frac{codim(M_{t_0}) - 1}{t} + t(trace(A) + trace(B)) + o(t^2),$$

where A, B are matrices independent of t. In particular we have the following asymptotic behaviour of h close to the focal varieties, which we will need later:

Lemma 4.1.5.

$$\lim_{t \to 0} -t\ h(t) = n - d_1 - 1 \quad , \quad \lim_{t \to t^*} -t\ h(t) = n - d_2 - 1.$$

4.2 Bifurcation points

In this section we will use local bifurcation theory (as can be found for instance in [2, 33]) to prove Theorem 1.0.4. We fix a proper isoparametric function f on the closed Riemannian manifold (M, g). As in the previous section, we denoted by t^* the distance between the two focal varieties. It follows from the previous section that positive f-invariant solutions of (1.0.2) are given by positive solutions of the problem

$$\phi'' - h\phi' + \lambda(\phi^q - \phi) = 0,$$
$$\phi'(0) = 0 = \phi'(t^*).$$

For all positive constant λ the function $\phi \equiv 1$ is a *trivial* solution of the equation. We will prove Theorem 1.0.4 by studying bifurcation from this path of trivial solutions $\lambda \mapsto (\lambda, 1)$.

Proof. (Theorem 1.0.4) We define $S : \mathbb{R}^+ \times \mathcal{B} \to C^{0,\alpha}([0, t^*])$ by

$$S(\lambda, \phi) = \phi'' - h\phi' + \lambda(\phi^q - \phi).$$

Then we are trying to solve the operator equation $S(\lambda, u) = 0$.

It is easy to compute that

$$d_\phi S_{(\lambda,\phi)}(u) = u'' - hu' + \lambda(q\phi^{q-1}u - u),$$

and in particular

$$d_\phi S_{(1,\lambda)}(u) = u'' - hu' + \lambda(q-1)u.$$

Bifurcating branches will appear at the values of λ for which the kernel of linear operator $d_\phi S_{(\lambda,1)}$ is nontrivial. Note that since the asymptotic behavior of h in 0 is $-\frac{n-d_1-1}{t}$, the standard contraction map argument implies that the following initial value problem has a unique solution

$$\begin{aligned} u'' - hu' + \lambda(q-1)u &= 0, \\ u(0) &= 1, \\ u'(0) &= 0. \end{aligned}$$

The equation $d_\phi S_{(\lambda,1)}(u) = 0$ for $u \in \mathcal{B}$, is of course the eigenvalue equation for the Laplacian restricted to f-invariant functions. One can see (for instance in [22, Proposition 3.2]) the existence of infinite eigenvalues of the Laplacian operator in the set of f-invariant functions. This means that there is a sequence $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots < \infty$ and solutions u_k of the above initial value problem (with $\lambda = \lambda_k$) such that $u_k'(t^*) = 0$. Therefore $ker\ d_\phi S_{(\lambda_k(q-1),1)} \neq 0$, and it has dimension one.

We can normalize u_k so that $\int_M u_k^2 = 1$. Since the operator $L = d_\phi S_{(\lambda_k(q-1),1)}$ is self-adjoint we have

$$Range(L) = \{u \in C^{0,\alpha}(0,t^*)/ \int_M uu_k = 0\}.$$

On the other hand we have that $d^2_{\phi,\lambda} S_{(\lambda_k(q-1),1)}[u_k] = (q-1)u_k \notin Range(L)$.

Therefore from the well-known theory of local bifurcation for simple eigenvalues (see for instance Theorem 2.3.2, [2, Theorem 2.8], or the original article by M. G. Crandall and P. H. Rabinowitz [16]) we can see that for all $k \geq 1$, $(\lambda_k(q-1), 1)$ is a bifurcation point and moreover all nontrivial solutions in a neighborhood of $(\lambda_k(q-1), 1)$ are given a branch $t \mapsto (\lambda(t), u(t))$, $t \in (-\varepsilon, \varepsilon)$, such that $\lambda(0) = \lambda_k(q-1)$, $u(0) \equiv 1$, and $u(t) \neq 1$ if $t \neq 0$. This proves Theorem 1.0.4.

□

Later we will need the following result about the number of simple zeros n_k of the functions u_k:

Lemma 4.2.1. *The sequence n_k is strictly increasing.*

Proof. Let u_k, u_{k+1} be solutions (resp.) of

$$\begin{aligned} u_k'' - hu_k' + \lambda_k(q-1)u_k &= 0, \\ u_{k+1}'' - hu_{k+1}' + \lambda_{k+1}(q-1)u_{k+1} &= 0. \end{aligned}$$

Recall that we have set $u_k(0) = 1 = u_{k+1}(0)$. In particular these solutions are non-trivial and therefore if for any $t \in (0, t^*)$, $u_k(t) = 0$ then $u_k'(t) \neq 0$.

Let $0 < t_1 < t_2 < \cdots < t_{n_k} < t^*$, be the points such that $u_k(t_i) = 0$, $i = 1, \ldots, n_k$. Since

$$\frac{u'_{k+1}(0)}{u_{k+1}(0)} = 0 = \frac{u'_k(0)}{u_k(0)},$$

and $\lambda_{k+1} > \lambda_k$, if u_{k+1} does not have zeros in $(0, t_1)$ then by Sturm's comparison theorem (and since the functions u_k and u_{k+1} are linearly independent in any open interval)

$$\frac{u'_{k+1}}{u_{k+1}} < \frac{u'_k}{u_k} \quad \text{in } (0, t_1).$$

Therefore there must be at least one value $s \in (0, t_1)$ such that $u_{k+1}(s) = 0$. We can also apply the same argument to show that u_{k+1} must have another zero in (t_{n_k}, t^*). Also by standard Sturm's comparison we can see that the function u_{k+1} has at least one zero in the interval (t_i, t_{i+1}), for each $i = 1, \ldots, n_k - 1$. Therefore u_{k+1} has at least $n_k + 1$ zeroes in $(0, t^*)$, proving the lemma. □

4.3 Auxiliary results

We assume we have a proper isoparametric function f on a closed Riemannian manifold (M, g). The dimension of the focal submanifolds are $d_1, d_2 \leq n - 2$: we call $d = \min\{d_1, d_2\} \leq n - 2$. And we let $p_f = \frac{n-d+2}{n-d-2}$, $p_f = \infty$ in case $d = n - 2$. We consider equation (4.1.1) with $q < p_f$. The main goal of this section is to prove the next proposition which we will need in the following sections.

Proposition 7. *Let $q \in (1, p_f)$ and fix positive numbers $\varepsilon < \lambda^*$. The set $C = \{\phi \in \mathcal{B} : \phi$ is positive and solves equation (4.1.1) with $\lambda \in [\varepsilon, \lambda^*]\}$ is compact in $\mathcal{B}$.*

We first consider the case when λ is fixed:

Lemma 4.3.1. *Consider equation (4.1.1) with $q \in (1, p_f)$ and $\lambda = \lambda_0 > 0$ fixed. If ϕ_α is the solution of the initial value problem with $\phi'(0) = 0$, $\phi(0) = \alpha$, then there exists $A = A(\lambda_0) > 0$ such that if $\alpha \geq A$ then there exists $t \in (0, t^*)$ such that $\phi_\alpha(t) = 0$. Similarly, if φ_α is the solution of (4.1.1) with $\varphi'(t^*) = 0$, $\varphi(t^*) = \alpha$, then there exists $B > 0$ such that if $\alpha \geq B$ then there exists $t \in (0, t^*)$ such that $\varphi_\alpha(t) = 0$.*

Proof. We consider the first statement, the proof of the second statement is similar. Equation (4.1.1) can be written as

$$\phi''(r) + \frac{H(r)}{r}\phi'(r) + \lambda\phi^q(r) - \lambda\phi(r) = 0. \tag{4.3.1}$$

where $H(r) = -rh(r)$ and $H(0) = n - d_1 - 1$ by Lemma (4.1.5). If $d_1 = n - 2$ then $H(0) = 1$. If $d_1 < n - 2$ then

$$\frac{H(0)+1}{2} = \frac{n-d_1}{2} \le \frac{n-\mathbf{d}}{2} = \frac{p_f+1}{p_f-1} < \frac{q+1}{q-1}.$$

Then we can apply [18, Theorem 3.1] which says that under the previous conditions on $H(0)$ and q there exists $A>0$ such that if $\alpha \ge A$ then the solution ϕ_α has a zero.

□

Corollary 4.3.2. *Fix $\lambda = \lambda_0 > 0$ in equation (4.1.1). There exits $A>0$ such that if $\phi \in \mathcal{B}$ is a positive solution of equation (4.1.1) then $\phi \le A$.*

Proof. It follows from the previous lemma that we can find $C>0$ such that any positive solution in $\mathcal{B}$ is bounded by C in 0 and in t^*. But then for any $\epsilon > 0$ there exists a constant $A>0$ such that the solution must be bounded by A in $[0, t^*-\epsilon]$ and in $[\epsilon, t^*]$. □

Corollary 4.3.3. *Fix $\lambda = \lambda_0 > 0$ in equation (4.1.1). The set of $\phi \in \mathcal{B}$ such that ϕ is a positive solution of equation (4.1.1) is compact.*

Proof. Let $\phi_i \in \mathcal{B}$ be sequence of positive solutions of equation (4.1.1). Then it follows from the lemma that we can take a subsequence so that the sequences $\phi_i(0)$, $\phi_i(t^*)$ are convergent. Let $\lim_{i\to\infty} \phi_i(0) = \mathbf{a}$, $\lim_{i\to\infty} \phi_i(t^*) = \mathbf{b}$. But then we consider the solutions ϕ^1, ϕ^2 of equation (4.1.1) with $\phi^1(0) = \mathbf{a}$, ${\phi^1}'(0) = 0$, $\phi^2(t^*) = \mathbf{b}$, ${\phi^2}'(t^*) = 0$. Then for any $\varepsilon > 0$ small, ϕ_i converges in $[0, t^*-\varepsilon]$ to ϕ^1 and ϕ_i converges in $[\varepsilon, t^*]$ to ϕ^2. Then $\phi^1 = \phi^2$ and give a function in $\mathcal{B}$ which is a positive solution of equation (4.1.1). And (for the subsequence) $\lim_{i\to\infty} \phi_i = \phi^1 = \phi^2$. □

Remark 7. *If $\phi \in \mathcal{B}$ is a non-trivial positive solution of equation (4.1.1), then $\#\{t : \phi(t) = 1\} < \infty$, and there is an open neighborhood U of $\phi \in \mathcal{B}$ such that for any $\varphi \in U$, $\#\{t : \varphi(t) = 1\} = \#\{t : \phi(t) = 1\}$. Also if $\phi \in \mathcal{B}$ is a non-trivial positive solution of equation (4.1.1) close to the trivial solution then $\#\{t : \varphi(t) = 1\}$ is equal to the number of zeroes of the linearized equation at the trivial solution (which is finite).*

Corollary 4.3.4. *Fix $\lambda = \lambda_0 > 0$ in equation (4.1.1). There exits $k_0 > 0$ such that if $\phi \in \mathcal{B}$ is a positive solution of equation (4.1.1) then $\#\{t : \phi(t) = 1\} \le k_0$.*

Proof. Any sequence of positive solutions $\phi_i \in \mathcal{B}$ of equation (4.1.1) must have a convergent subsequence. Then by the remark $\#\{t : \phi_i(t) = 1\}$ is bounded (independently of i). □

Now as in the proposition we will fix positive numbers $\varepsilon < \lambda^*$ and consider the equation (4.1.1) with $\lambda \in [\varepsilon, \lambda^*]$.

Lemma 4.3.5. *For any $0 < \varepsilon < \lambda^*$ there exists $A > 0$ such that if $\lambda \in [\varepsilon, \lambda^*]$ and ϕ_α is the solution of equation (4.1.1) with $\phi_\alpha'(0) = 0$, $\phi_\alpha(0) = \alpha > A$ then ϕ_α has a zero in $(0, t^*)$. Similarly, there exists $B > 0$ such that if φ_α is the solution of equation (4.1.1) with $\varphi_\alpha'(t^*) = 0$, $\varphi_\alpha(t^*) = \alpha > B$, then φ_α has a zero in $(0, t^*)$.*

Proof. We will prove the first statement, the proof of the second statement is similar. For each $\lambda \in [\varepsilon, \lambda^*]$ let $A_\lambda = \inf\{A : \phi_\alpha$ has a zero in $(0, t^*)$ for $\alpha > A\}$. It follows from the previous lemma that $A_\lambda < \infty$ for any $\lambda \in [\varepsilon, \lambda^*]$. Assume that there exists a sequence $\lambda_i \in [\varepsilon, \lambda^*]$ such that $A_{\lambda_i} \to \infty$. We can asume that $\lambda_i \to \lambda_0 \in [\varepsilon, \lambda^*]$. Then we have a solution ϕ_i of equation (4.1.1) with $\lambda = \lambda_i$, such that $\phi_i(0) = \alpha_i^{\frac{2}{q-1}} \to \infty$, and ϕ_i is positive in $[0, t^*)$. Then we argue as in [18, Theorem 3.1] (see also [21, Proposition 3.8]): We let

$$w_i(t) = \alpha_i^{\frac{2}{1-q}} \phi_i \left(\frac{t}{\alpha_i \sqrt{\lambda_i}} \right).$$

Then w_i solves

$$w_i''(t) + \frac{H\left(\frac{t}{\alpha_i\sqrt{\lambda_i}}\right)}{t} w_i'(t) + w_i^q(t) - \frac{w_i(t)}{\alpha_i^2} = 0,$$

where $H(r) = -rh(r)$, $w_i(0) = 1$ and $w_i'(0) = 0$. Note that w_i is defined in $[0, \alpha_i\sqrt{\lambda_i}t^*)$, and $\lim_{i\to\infty} \alpha_i\sqrt{\lambda_i}t^* = \infty$.

Then one can see that for any fixed $K > 0$ w_i converges uniformly on $[0, K]$ to the solution w of

$$w'' + \frac{H(0)}{t} w' + w^q = 0,$$

with $w(0) = 1$, $w'(0) = 0$. The proof is the same as in [18, Lemma 3.2]), where the proof is detailed in the case $\lambda_i = \lambda > 0$, instead of $\lambda_i \to \lambda$ as in our case (but this does not affect the proof of the statement given in [18]).

It is proved in [21, Proposition 3.9] that by picking K large we can assume that w has any number of zeroes in $[0, K]$ and by the uniform convergence it follows that w_i must have a zero in $[0, K]$ and therefore ϕ_i has in zero in $(0, \frac{K}{\alpha_i\sqrt{\lambda_i}})$. This is a contradiction, and therefore $A = \sup_{\lambda\in[\varepsilon,\lambda^*]} A_\lambda < \infty$, proving the lemma.

□

We can now prove Proposition 4.1:

Proof. Let $\phi_j \in \mathcal{B}$ be a sequence of solutions of equation(4.1.1) with $\lambda = \lambda_j \in [\varepsilon, \lambda^*]$. ϕ_j is determined by $\alpha_j = \phi_j(0)$ and by $\beta_j = \phi_j(t^*)$. From the previous lemma we know that we can take a subsequence and assume that $(\lambda_j, \alpha_j, \beta_j) \to (\lambda_0, \alpha_0, \beta_0)$, where α_0, $\beta_0 > 0$. Let ϕ^1 be the solution of equation (4.1.1) with $\lambda = \lambda_0$ such that $\phi^1(0) = \alpha_0$ and ${\phi^1}'(0) = 0$. Let ϕ^2 be the solution of equation (4.1.1) with $\lambda = \lambda_0$ such that $\phi^2(t^*) = \beta_0$ and ${\phi^2}'(t^*) = 0$. Then for any $\delta > 0$ ϕ_j converges on $[0, t^* - \delta]$ to ϕ^1 and on $[\varepsilon, t^*]$ to ϕ^2. It follows that on $(0, t^*)$ $\phi^1 = \phi^2$ and therefore they define a function $\phi \in \mathcal{B}$ which is positive, solves equation (4.1.1), and verfies $\phi(0) = \alpha_0$, $\phi(t^*) = \beta_0$. And $\phi_i \to \phi$.

□

4.4 f-invariant solutions for parameter close to zero

In this section, we will prove Theorem 1.0.5 i.e. we will show that all non-negative $f-$invariant solutions of 1.0.2 are constant for λ close to zero. In order to get this result, we will give a priori estimate for $f-$invariant solutions of the equation. First, consider the equivalent equation:

$$-\Delta_g w = w^q - \lambda w. \tag{4.4.1}$$

Note that u is a solution of equation (4.4.1) if and only if $\lambda^{\frac{-1}{q-1}}u$ is a solution of equation (1.0.2). Also as in section 4.1, $u \in C^{2,\alpha}_f(M)$, $u(x) = \phi(\mathbf{d}(x))$, with $\phi \in \mathcal{B}$ is a solution of equation (4.4.1) if and only if the function ϕ satisfies

$$-(\phi'' + (-h)\phi') + \lambda\phi = \phi^q. \tag{4.4.2}$$

on $[0, t^*]$.

Similar problems have been considered before, for instance in [30]. We denote λ_1 the first non-zero eigenvalue of $-\Delta_g$. We will make use of the following result from [30, Theorem 2.2]

Theorem 4.4.1 ([30]). *Assume $0 < \lambda$ and $q > 1$. If w is a solution of (4.4.1) which satisfies*

$$q\|w\|_{L^\infty}^{\frac{1}{q-1}} \leq \lambda + \lambda_1,$$

then $w = \lambda^{\frac{1}{q-1}}$.

Similarly, if u is a solution of (1.0.2) which satisfies

$$q\|u\|_{L^\infty}^{\frac{1}{q-1}} \leq \frac{\lambda + \lambda_1}{\lambda^{\frac{1}{(q-1)^2}}},$$

then $u = 1$.

We first find an appropriate bound for positive f-invariant solutions:

Lemma 4.4.2. *There exist constants $\varepsilon, c > 0$ such that for $\lambda \in (0, \epsilon]$ any positive $f-$invariant solution w of (4.4.1) satisfies*

$$w \leq c\lambda^{\frac{1}{q-1}}.$$

If u is a positive solution of equation (1.0.2) then $u \leq c$.

Proof. We follow a similar treatment of [30, Theorem 2.3]. Suppose that the lemma is not true. Then we have a sequence of positive numbers $\lambda_m \to 0$, a sequence of positive numbers $c_m \to +\infty$, and a sequence $p_m \in M$, such that there exists a positive solution w_m of (4.4.1) which satisfies that

$$\max_M w_m = w_m(p_m) = c_m \lambda_m^{\frac{1}{q-1}}. \tag{4.4.3}$$

By taking a subsequence we can assume that $w_m(p_m) \to \mathbf{a} \in [0, \infty]$.

Note that if $\mathbf{a} = 0$ then the solutions w_m satisfies the conditions on Theorem 4.4.1 and therefore we would have $w_m = \lambda_m^{\frac{1}{q-1}}$ for m large. This would say that $c_m = 1$, which is a contradiction (we were assuming that $c_m \to \infty$).

Assume now that $\mathbf{a} \in (0, \infty)$. Let $v_m = \frac{w_m}{w_m(p_m)}$. Note that v_m solves the equation

$$\Delta_g v_m - \lambda_m v_m + w_m(p_m)^{q-1} v_m^q = 0,$$

with $\|v_m\|_{L^\infty} = 1$. Recall that $\lambda_m \to 0$ and $w_m(p_m) \to \mathbf{a}$. Therefore, from the theory of elliptic operators we get that the sequence v_m converges to a function v which is a non-negative solution of

$$\Delta_g v + \mathbf{a}^{q-1} v^q = 0$$

on M. Since M is closed, v is equal to zero, but this is a contradiction since $\|v_m\|_{L^\infty} = 1$.

Therefore we can assume that $w_m(p_m) \to \infty$. In this case we will need to use our hypothesis that the functions w_m are f-invariant. Then w_m is determined by a function $\phi_m \in \mathcal{B}$ which solves equation (4.4.2). The function ϕ_m is determined by $\alpha_m = \phi_m(0)$ and by $\beta_m = \phi_m(t^*)$. If the sequences α_m and β_m are bounded then ϕ_m would be uniformly bounded, which is not the case. We can therefore assume for instance that $\alpha_m \to \infty$.

We call $H(t) = -th(t)$. Then

$$\phi_m''(t) + \frac{H(t)}{t}\phi_m'(t) + \phi_m^q(t) - \lambda_m \phi_m(t) = 0.$$

We let $\delta_m = \alpha_m^{\frac{2}{q-1}}$ and

$$\varphi_m(t) := \delta_m^{\frac{2}{1-q}} \phi_m\left(\frac{t}{\delta_m}\right).$$

Then φ_m solves

$$\varphi_m''(t) + \frac{H(\frac{t}{\delta_m})}{t}\varphi_m'(t) + \varphi_m^q(t) - \frac{\lambda_m}{\delta_m^2}\varphi_m(t) = 0,$$

and satisfies $\varphi_m(0) = 1$ and $\varphi_m'(0) = 0$.
Since $\delta_m \to \infty$ and $\lambda_m \to 0$ we can argue as in the proof of Lemma 4.3.5, or [18, Lemma 3.2], to prove that the sequence φ_m converges uniformly on any compact interval $[0, K]$ to the solution w of

$$w'' + \frac{H(0)}{t}w' + w^q = 0,$$

with $w(0) = 1$, $w'(0) = 0$. We recall that it is proved in [21, Proposition 3.9] that by picking K large we can assume that w has any number of zeroes in $[0, K]$ and by the uniform convergence it follows that for m large enough φ_m must have a zero in $[0, K]$. This implies that ϕ_m has in zero in $(0, \frac{K}{\delta_m})$. This contradicts our assumption that the solution $w_m(x) = \phi_m(\mathbf{d}(x))$ was positive, finishing the proof of the lemma.

□

We are now ready to prove Theorem 1.0.5

Proof. By the previous lemma there exists $\varepsilon > 0$ such that if $\lambda \in (0, \varepsilon]$ then any positive $f-$invariant solution w of (4.4.1) satisfies

$$w \leq c\lambda^{\frac{1}{q-1}},$$

for some positive constant c independent of λ. Then there exists $\lambda_0 \in (0, \varepsilon)$ such that if $\lambda \in (0, \lambda_0]$ then $q(c\lambda^{\frac{1}{q-1}})^{\frac{1}{q-1}} \leq \lambda + \lambda_1$. Then it follows from Theorem 4.4.1 that w must be constant.

□

4.5 Global bifurcation

In this section, we will prove Theorem 1.0.6.

Let $D = \{(\phi, \lambda) \in (\mathcal{B} - \{1\} \times (0, \infty) : \phi \in \mathcal{B}$ is a positive nontrivial solution of (4.1.1)$\}$. Let $\overline{D}$ be the closure of D in $\mathcal{B}$ and D_i the connected component of $\overline{D}$ containing the bifurcation point $(1, \lambda_i(q-1))$ (as in Section 4.2).

It follows from Theorem 1.0.5 that there exists $\varepsilon > 0$ such that for any i D_i is contained in $\{\mathcal{B} \times [\varepsilon, \infty)\}$.

Now we shall see that each D_i is not compact, using the global bifurcation theorem of P. Rabinowitz (Theorem 2.3.2, see also [33, Theorem 3.4.1], [2, Theorem 4.8] or [39]) .

It follows from Rabinowitz's theorem that either

a) D_i is not compact in $O = \{(\phi, \lambda) \in C^{2,\alpha}_f \times \mathbb{R}^+ / \ \phi > 0\}$

or

b) D_i contains a point $(1, \lambda_j(q-1))$ for $j \neq i$.

For each $1 \leq i$, we let

$$Z_i := \{\phi \in \mathcal{B} \ / \phi - 1 \text{ has exactly } \ n_i \text{ simple zeros in } (0, t^*) \ \},$$

where we recall from Section 4.2 that n_i is the number of zeroes of the solution of the linearized equation at $(1, \lambda_i(q-1))$.

Each Z_i is an open set in $\mathcal{B}$. Note also that if $\phi \in \mathcal{B}$ is a nontrivial solution of (4.1.1) then the zeros of $\phi - 1$ are simple (since it solves a second order ordinary differential equation for which the constant function 1 is a solution).

Recall from Section 4.2 (the proof of Theorem 1.0.4, using [16]) that the points in D near to $(1, \lambda_i(q-1))$ can be parametrized by a curve $s \mapsto (k_i(s), \mu_i(s))$, $|s| < a_i$ with a_i sufficiently small, where $\mu_i(0) = \lambda_i(q-1)$. The map k_i is of the form $k_i(s) = su_i + sQ(s)$ and $Q(0) = 0$, where, as in Section 4.2, $u_i \in Z_i$ is an eigenfunction of $-\Delta$ associated to $\lambda_i(q-1)$ (see again [33, Theorem 3.2.2], for instance).

Therefore, $k_i(s) \in Z_i$ for s sufficiently small, $s \neq 0$. Then it follows that $D_i - \{(1, \lambda_i(q-1))\} \subseteq Z_i$. And in particular it follows that $D_i \cap D_j = \emptyset$ if $i \neq j$. This says that alternative (b) in the global Theorem of Rabinowitz does not happen and therefore D_i is not compact, for any i.

If there exists a constant $\lambda_0 > \lambda_i(q-1)$ such that for any $(\phi, \lambda) \in D_i$ we have $\lambda \leq \lambda_0$. Then D_i would be a closed set of $\{(\phi, \lambda) \in \mathcal{B} \times [\varepsilon, \lambda_0] : \phi$ is a positive solution of (4.1.1) $\}$. Then it follows from Proposition 7 that D_i is compact. Therefore such an λ_0 does not exist and since D_i is connected it follows that for any $\lambda \geq \lambda_i(q-1)$ there exists $(\phi, \lambda) \in D_i$. Then for $\lambda \in [\lambda_i(q-1), \lambda_{i+1}(q-1))$ and for each $k \leq i$ there exists $(\phi, \lambda) \in D_k$ and this proves Theorem 1.0.6.

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