CIMAT Centro de Investigación en Matemáticas, A.C.

STANLEY-REISNER THEORY: SIMPLICIAL COMPLEXES, HOCHSTER'S FORMULA AND TERAI'S THEOREM

T E S I S

Que para obtener el grado de

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Introduction

Algebra, topology and combinatorics are fundamental areas of mathematics. Although the objects of study in each area seems different, historically they benefit from each other. For example, homology and homotopy theory both define functors from topological spaces (and continuous maps) to groups (and group homomorphisms). Furthermore, many concepts of the algebra are inspired by the combinatorics: permutation groups. In this way, the study of algebra is better understood when associating it with topology and combinatorics, and vice-versa.

In the beginning, the algebra was focus in the study of polynomial roots. Between the XVII century and the XVIII century, the work of mathematicians such as Galois, Cauchy, Gauss, Jordan and Grassmann focused in the study of permutations groups associated to roots of polynomials equations.

On 1871, Felix Klein and Sophus Lie published in Mathematische Annalen their famous article *"Ueber diejenigen ebenen Curven, welche durch ein geschlossenes System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich bergehen"* in which developed the concept of closed system. The goal of the work of Klein and Lie is study the intrinsic properties of the objects by studying of their automorphism. In this way the properties that do not change under an automorphism are called invariants of the object. These invariants are used to distinguish objects. The following questions arise naturally: Do algebraic objects such as groups, rings and fields have invariants that characterize their? Given a simplicial complex, does their exists some invariant that characterize it?

Stanley-Reisner theory provides the central link between combinatorics and commutative algebra. Pioneered in the 1970s, the correspondence between simplicial complexes and square-free monomial ideals has been responsible for substantial progress in both fields. Among the most celebrated results are Reisner's criterion for Cohen-Macaulayness [Rei76], Stanley's proof of the Upper Bound Conjecture for simplicial spheres [Sta96], and Hochster's formula for computing multi-graded Betti numbers of square-free monomial ideals via simplicial homology [Hoc77]. The last statement is one of the goals of this work and it is proven in the Chapter 4.

Chapter 1 consist of background. We present the basic constructions, definitions and results of associated primes, square-free monomial ideals, free resolutions, Betti numbers, depth, Cohen-Macaulay rings and local cohomology. Furthermore, we establish notation that we use in all this manuscript.

In Chapter 2 we give one of the most important definition for our work: simplicial complex. In addition, we study basic invariants of simplicial complexes such as their f-vector and their h-vector. Moreover, we start developing the Stanley-Reisner theory which relates simplicial complexes with square-free monomial ideals. Afterwards, we discuss the Alexander duality. In its original form, it formulates a relation between the Betti numbers and torsion coefficients of a subcomplex A of the n-sphere S^n, and the Betti numbers and torsion coefficients of the complement $S^n - A$. We use this theory to compute Betti numbers in Chapter 4.

Throughout Chapter 3 we study the Hilbert series and the Hilbert polynomial for simplicial complexes. The degree of the Hilbert polynomial is one of the most important invariants in commutative algebra, because it gives the dimension of a module and the degree of a projective variety. Although we introduce the Hilbert series and the Hilbert polynomial for any finitely generated module, we focus only in study the Hilbert series for Stanley-Reisner rings. Our goal is compute its Hilbert series and its Hilbert polynomial using the f-vector and the h-vector of the correspondent simplicial complex. Furthermore, we study simplicial complexes from the topological point of view using shelling order and minimal faces.

In Chapter 4 we define the Koszul complex. It was introduced by Jean-Louis Koszul on 1950 [Kos50]. Its homology can be used to characterize when a set of elements of a ring is a regular sequence. It is used to prove basic facts about the depth of a module. We employ this homology to study minimal free resolution for the residual field (Proposition 4.4). We also compute Betti numbers and relate them with the reduced homology of a simplicial complex (Theorem 4.12). These results are used to prove the dual form of the Hochster's formula (Theorem 4.16). This formula shows that the multi-graded Betti numbers of a square-free monomial ideal I are encoded in the homology of simplicial complexes.

Finally, in Chapter 5, we prove the Terai's Theorem (Theorem 5.8). This result establishes the equality between the regularity of the Stanley-Reisner ideal and the projective dimension of its Alexander dual. For that, it is necessary study more extensively the Betti numbers and its relate with local cohomology. Moreover, in this chapter we use detph, regularity and projective dimension.

Chapter 1

Background

In this chapter we establish basic definitions and results about associated primes ideals, chain complex, depth, Cohen-Macaulay rings, Betti numbers, projective modules and local cohomology. We also establish notation that we use throughout the text. The acquainted reader with those topics can skip this chapter.

Throughout the text $\mathbb{K}$ denotes an arbitrary field, $S = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring in n indeterminates over $\mathbb{K}$ and $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle \subsetneq S$ its homogeneous maximal ideal. Some definitions and theorems are written in general form, in those statements R is an arbitrary commutative Noetherian ring with unit, and $(R, \mathfrak{m}, K)$ (or simply $(R, \mathfrak{m})$) is a local Noetherian ring with unit, $\mathfrak{m}$ its maximal ideal and $K = R/\mathfrak{m}$ its residual field. The reason why we use R instead of S is because S has stronger assumption that R, for example S has no zero-divisors, and it is Cohen-Macaulay. Therefore, some definitions and results are satisfied by S but no by R.

Definition 1.1. A **monomial** in S is defined by $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and is denoted by $\mathbf{x}^{\mathbf{a}}$ where $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$. An ideal in S is called a **monomial ideal** if it is generated by monomials. Furthermore, a monomial $\mathbf{x}^{\mathbf{a}}$ is **square-free** if $\mathbf{a} \in \{0, 1\}^n$. An ideal is **square-free** if it is generated by square-free monomials.

Proposition 1.2. A square-free monomial ideal is a radical ideal.

Proof. Let I be a square-free monomial ideal such that $I = \langle m_1, \ldots, m_r \rangle$. Suppose that $m_i = x_{j(i)_1} \cdots x_{j(i)_{l(i)}}$, for all $i \in [r]$. Then,

$$\begin{aligned} I &= \textstyle\bigcap_{k_1=1}^{l(1)} \langle x_{j(1)_{k_1}}, m_2, \ldots, m_r \rangle \\ &= \textstyle\bigcap_{k_1=1}^{l(1)} \bigcap_{k_2=1}^{l(2)} \langle x_{j(1)_{k_1}}, x_{j(2)_{k_2}}, \ldots, m_r \rangle \\ &\vdots \\ &= \textstyle\bigcap_{k_1=1}^{l(1)} \bigcap_{k_2=1}^{l(2)} \cdots \bigcap_{k_r=1}^{l(r)} \langle x_{j(1)_{k_1}}, x_{j(2)_{k_2}}, \ldots, x_{j(r)_{k_r}} \rangle. \end{aligned}$$

Therefore, I is a finite intersection of prime ideals of type

$$P_k = \langle x_{k_1}, \ldots, x_{k_t} \rangle.$$

We write $I = P_1 \cap \cdots \cap P_N$ for some prime ideals P_i. Therefore,

$$\operatorname{rad}(I) = \operatorname{rad}(P_1 \cap \cdots \cap P_N) = \operatorname{rad}(P_1) \cap \cdots \cap \operatorname{rad}(P_N) = P_1 \cap \cdots \cap P_N = I. \qquad \blacksquare$$

Definition 1.3. Let M be an S-module. A prime ideal P of S is **associated** to M if P is the annihilator of an element of M. The set of all associated primes of M is written $\mathrm{Ass}_S(M)$ or simply $\mathrm{Ass}(M)$ when there is no confusion.

Let $I \subseteq S$ be an ideal. If $P \in \mathrm{Ass}(I)$ and there is no $Q \in \mathrm{Ass}(I)$ such that $I \subseteq Q \subsetneq P$, then P is a **minimal prime** of I. We write $\mathrm{Min}(I)$ for the set of all minimal primes of I.

Theorem 1.4. Let I be a radical ideal. Then $\mathrm{Ass}(I) = \mathrm{Min}(I)$.

Since, we focus in study square-free monomial ideals, from Theorem 1.4 and Proposition 1.2 we have the following properties for their associated prime ideals.

Theorem 1.5. Let I be a square-free monomial ideal. For a monomial prime ideal P the following are equivalent:

(a) P is a minimal prime of I;

(b) I has a primary descomposition, $\bigcap_{i=1}^{n} Q_i$, and $P = \mathrm{rad}(Q_i)$ for some i;

(c) There is a monomial $m \notin I$ such that $mx \in I$ if and only if $x \in P$, for some $x \in S$.

1.1 Chain Complex

Now, we study free resolutions and Tor_i^R modules. The goal of this section is give us the tools to study the Koszul complex and the Betti numbers in Chapter 4.

Definition 1.6. A **complex** (or **chain complex**) is a sequence of modules and maps

$$\cdots \longrightarrow C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \longrightarrow \cdots,$$

such that $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. We denote the whole complex as $C_\bullet$ or $(C_\bullet, d_\bullet)$. The i**-th homology module** of $C_\bullet$ is

$$H_i(C_\bullet) = \ker d_i / \operatorname{im} d_{i+1}$$

Definition 1.7. If $(C_\bullet, d_\bullet)$ and $(C'_\bullet, d'_\bullet)$ are complexes, a **chain map** $f : (C_\bullet, d_\bullet) \longrightarrow (C'_\bullet, d'_\bullet)$ is a sequence of maps $f_i : C_i \longrightarrow C'_i$ for all $i \in \mathbb{Z}$, such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"'] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & C'_{i+1} \arrow[r, "d'_{i+1}"'] & C'_i \arrow[r, "d'_i"'] & C'_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

Furthermore, we define $H_i(f) : H_i(C_\bullet) \longrightarrow H_i(C'_\bullet)$ by $H_i(f)([x]) = [f_i(x)]$, which it is called the map **induced** by f, and it is usually denoted by f_*.

Theorem 1.8. Let $0 \longrightarrow C'_\bullet \xrightarrow{f} C_\bullet \xrightarrow{g} C''_\bullet \longrightarrow 0$ be a short exact sequence of complexes. For each i, there is a homomorphism $\partial_i : H_i(C''_\bullet) \longrightarrow H_{i-1}(C'_\bullet)$ defined by $\partial_i([x]) = [f_{i-1}^{-1} \circ d_i \circ g_i^{-1}(x)]$, which is called **connecting homomorphism**.

From Theorem 1.8, we have the next result for study homology modules through exact sequence.

Theorem 1.9. Let $0 \longrightarrow C'_\bullet \xrightarrow{f} C_\bullet \xrightarrow{g} C''_\bullet \longrightarrow 0$ be a short exact sequence of complexes. Then we have a long exact sequence on homology:

$$\cdots \longrightarrow H_{i+1}(C''_\bullet) \xrightarrow{\partial} H_i(C'_\bullet) \xrightarrow{f_*} H_i(C_\bullet) \xrightarrow{g_*} H_i(C''_\bullet) \xrightarrow{\partial} H_{i-1}(C'_\bullet) \xrightarrow{f_*} H_{i-1}(C_\bullet) \longrightarrow \cdots$$

Definition 1.10. Let M be an R-module. A **free resolution** of M is a complex

$$\cdots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow 0,$$

where each F_i is a free module over R, and $(F_\bullet, d_\bullet)$ is exact, i.e., $\operatorname{im} d_{i+1} = \ker d_i$ for all i. We say that $F_\bullet$ is **minimal** if each of the modules F_i has minimum possible rank.

Suppose that M is a graded S-module. Then $F_\bullet$ is a **graded free resolution** of M if each F_i is a twisted free graded module and each d_i is a graded homomorphism of degree zero. A graded free resolution of a graded finitely generated S-module $F_\bullet$ is **minimal** if $d_i(F_i) \subseteq \mathfrak{m}F_{i-1}$ for all i.

Theorem 1.11. Every finitely generated S-module has a finite free resolution of lenght at most n.

Proposition 1.12. Let M be a finitely generated graded S-module. Then, any two minimal free resolutions of M are isomorphic.

Definition 1.13. Let M, N be R-modules, and let $F_\bullet$ be a free resolution of M. We define

$$\operatorname{Tor}_i^R(M, N) \coloneqq H_i(F_\bullet \otimes_R N).$$

Theorem 1.14. Let R be a commutative ring and let M and N be R-modules. Then for all $i \in \mathbb{Z}$, $\operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_i^R(N, M)$.

1.2 Betti numbers

The minimal free resolutions are characterized by having the ranks of their free modules all simultaneously minimized, those ranks are called the Betti numbers of M.

Definition 1.15. Given a minimal free resolution of a graded module M as

$$0 \longrightarrow F_l \xrightarrow{d_l} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0, \tag{1.15.1}$$

We write,

$$F_i = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \qquad \text{and} \qquad F_i = \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{i,j}}$$

in the $\mathbb{N}^n$-graded case and in the $\mathbb{N}$-graded case, respectively.

The ***i*-th Betti number** of M in degree $\mathbf{a}$ and in degree j are the numbers $\beta_{i,\mathbf{a}}(M) = \beta_{i,\mathbf{a}}$ and $\beta_{i,j}(M) = \beta_{i,j}$, respectively. For $j \in \mathbb{N}$ we observe that

$$\beta_{i,j}(M) = \sum_{|\mathbf{a}|=j} \beta_{i,\mathbf{a}}(M),$$

where $|\mathbf{a}| = a_1 + a_2 + \cdots + a_n$. In particular, we have $\beta_{i,\mathbf{a}}(M) = 0$ for all $i > n$. The ***i*-th total Betti number** of M, denoted by $\beta_i(M)$, is simply the rank of F_i, in other words

$$\beta_i(M) = \sum_{j \in \mathbb{N}} \beta_{i,j}(M).$$

Definition 1.16. The **Castelnuovo-Mumford regularity** of M is defined by

$$\operatorname{reg}(M) = \max\{j - i : \beta_{i,j}(M) \neq 0\}.$$

The **initial degree** of M is defined by

$$\operatorname{indeg}(M) = \min\{j : \beta_{0,j}(M) \neq 0\}.$$

We now characterize the Betti numbers in terms of Tor_i^R modules.

Proposition 1.17. Let M be a $\mathbb{N}^n$-graded finitely generated S-module and $\mathbf{a} \in \mathbb{N}^n$. Then we have $\beta_{i,\mathbf{a}}(M) = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(M, \mathbb{K})_{\mathbf{a}}$.

Proof. Let $(F_\bullet, d_\bullet)$ be a minimal free resolution of M and let $\mathbb{K} = S/\mathfrak{m}$. Since $F_\bullet$ is minimal, we have that the induced maps in the complex $F_\bullet \otimes \mathbb{K}$ are identically $\mathbf{0}$, i.e., we have the following complex

$$\cdots \longrightarrow \left(\bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i+1,a}}\right) \otimes \mathbb{K} \xrightarrow{\mathbf{0}} \left(\bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,a}}\right) \otimes \mathbb{K} \xrightarrow{\mathbf{0}} \left(\bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i-1,a}}\right) \otimes \mathbb{K} \longrightarrow \cdots,$$

and since $\left(\bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,a}}\right) \otimes \mathbb{K} \cong \bigoplus_{\mathbf{a}\in\mathbb{N}^n} \mathbb{K}(-\mathbf{a})^{\beta_{i,a}}$, we have that

$$\operatorname{Tor}_i^S(M, \mathbb{K})_{\mathbf{a}} = H_i(F_\bullet \otimes \mathbb{K})_{\mathbf{a}} \cong \mathbb{K}(-\mathbf{a})^{\beta_{i,a}}.$$

Therefore, $\beta_{i,\mathbf{a}}(M) = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(M, \mathbb{K})_{\mathbf{a}}$. ∎

Let $I = \langle m_1, \ldots, m_r \rangle$ be a monomial ideal. A way to compute a free resolution of I is using the Taylor resolution. The Taylor resolution of I is constructed as follows.

For any subset σ of $\{1, \ldots, r\}$, set $m_\sigma = \operatorname{lcm}\{m_i : i \in \sigma\}$. For each such σ, define a basis vector $\mathbf{e}_\sigma$ in $\mathbb{N}^n$-graded degree $\deg(m_\sigma)$. For each i, set T_i equal to the free S-module with basis $\{\mathbf{e}_\sigma : |\sigma| = i\}$. Note that $T_0 = S[\varnothing]$ is a free module of rank one.

Define $\phi_{-1} : T_0 \longrightarrow S/I$ by $\phi_{-1}(f[\varnothing]) = f$; otherwise, we construct $\phi_i : T_{i+1} \longrightarrow T_i$ as follows. Given σ with $|\sigma| = i + 1$ and written in increasing order, take

$$\phi_\sigma = \sum_{i\in\sigma} \operatorname{sign}(i, \sigma) \frac{m_\sigma}{m_{\sigma-\{i\}}} \mathbf{e}_{\sigma-\{i\}},$$

where $\operatorname{sign}(i, \sigma) = (-1)^{j-1}$ if i is the j-th element of σ. Define $\phi_i : T_{i+1} \longrightarrow T_i$ by extending the various ϕ_σ. The **Taylor resolution** of I is the complex

$$\mathbb{T}_I : \quad 0 \longrightarrow T_r \xrightarrow{\phi_{r-1}} T_{r-1} \longrightarrow \cdots \longrightarrow T_1 \xrightarrow{\phi_0} T_0 \xrightarrow{\phi_{-1}} S/I \longrightarrow 0.$$

Theorem 1.18. The Taylor resolution of I is a resolution of I.

Theorem 1.19. The Taylor resolution is minimal if and only if for all $\sigma \subseteq \{1, \ldots, r\}$ and all indices $i \in \sigma$, the monomials m_σ and $m_{\sigma-\{i\}}$ are different.

For the reader who is interested on the Taylor resolution, we recommend the classic readings on this subjects: [Pee11], [MS05], and [Mer12].

Remark 1.20. *Let $I = \langle m_1, \ldots, m_r \rangle$ be a square-free monomial ideal. Using the notation from the Taylor resolution of I we have that $\deg(m_\sigma) \leq n$ for all $\sigma \subseteq \{1, \ldots, r\}$. Thus, all the entries of the associated matrix to each ϕ_i are square-free monomials. This implies that $\beta_{i,j}(I) = 0$ for all $j > n$.*

1.3 Depth

In this section we introduce the depth, which it is a tool to explore some basic facts about Cohen-Macaulay rings in the next section. But first, we give the definition of regular sequence.

Definition 1.21. Let R be a ring and let M be an R-module. A sequence of elements $f_1, \dots, f_l \in R$ is called a **regular sequence** in M (or an M**-sequence**) if

(a) $(f_1, \dots, f_l)M \neq M$, and

(b) f_i is a non-zero divisor on $M/(f_1, \dots, f_{i-1})M$ for all $i \in [l]$.

By definition, we have the following two observations

(a) $l \leq \dim(M)$, and

(b) x is an M-sequence if and only if $M \xrightarrow{x} M$ is an exact sequence.

Proposition 1.22. Let $(R, \mathfrak{m}, K)$ be a local ring and let $f_1, \dots, f_l \in \mathfrak{m}$ be an M-sequence. Then, $f_{\sigma(1)}, \dots, f_{\sigma(l)}$ is an M-sequence, for every permutation σ on $[l]$.

Corollary 1.23. Let $(R, \mathfrak{m})$ be a local ring. If $f_1, \dots, f_l \in \mathfrak{m}$ is an M-sequence, then $f_1^{\alpha_1}, \dots, f_l^{\alpha_l} \in \mathfrak{m}$ is an M-sequence for all $\alpha_i \geq 1$.

Definition 1.24. Let $I \subseteq R$ be an ideal and let M be a finitely generated R-module. The I**-depth** of M, denoted by $\operatorname{depth}_I(M)$, is defined as follows

(a) If $IM \neq M$, then $\operatorname{depth}_I(M) = \sup\{l \in \mathbb{N} : \text{exists } f_1, \dots, f_l \in I \text{ such that is an } M\text{-sequence}\}$,

(b) if $IM = M$ we set $\operatorname{depth}_I(M) = \infty$.

If $(R, \mathfrak{m})$ is a local ring we call $\operatorname{depth}_{\mathfrak{m}}(M)$ simply the **depth** of M and write $\operatorname{depth}(M)$.

Lemma 1.25. Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring. Let M be a non-zero finitely generated R-module. Then, $\operatorname{depth}(M) \leq \dim(\operatorname{Supp}(M))$.

Lemma 1.26. Let R be a Noetherian ring, $I \subseteq R$ an ideal, and M a finitely generated R-module such that $IM \neq M$. Then, $\operatorname{depth}_I(M) < \infty$.

1.4 Cohen-Macaulay rings

The Cohen-Macaulay rings are one of the central definitions in commutative algebra. In this section, we only give the definition and some important facts. For details, we refer the interested reader to the book by Bruns and Herzog [BH93].

Theorem 1.27. Let R be a ring such that $\operatorname{depth}(P) = \operatorname{ht}(P)$ for every maximal ideal $P \subsetneq R$. If $I \subsetneq R$ is a proper ideal, then $\operatorname{depth}(I) = \operatorname{ht}(I)$.

Definition 1.28. Let $(R, \mathfrak{m}, K)$ be a local ring. R is called a **Cohen-Macaulay** ring if $\operatorname{depth}(R) = \dim(R)$.

Let $I \subseteq (R, \mathfrak{m})$ be an ideal, then

$$\operatorname{depth}_I(R) \leq \operatorname{ht}(I) \leq \operatorname{ht}(\mathfrak{m}) = \dim(R).$$

Definition 1.29. Let R be a Noetherian ring. R is called a **Cohen-Macaulay** ring if R_Q is Cohen-Macaulay for every prime ideal $Q \subseteq R$.

Theorem 1.30. Let R be a Noetherian ring. Then, R is Cohen-Macaulay if and only if $\operatorname{depth}_I(R) = \operatorname{ht}(I)$ for every ideal $I \subseteq R$.

Proposition 1.31. Let R be a Noetherian ring. R is Cohen-Macaulay if and only if the polynomial ring $R[x]$ is Cohen-Macaulay.

1.5 Projective dimension

Definition 1.32. A module W is **projective** if for every epimorphism of modules $f : M \longrightarrow N$ and every map $g : W \longrightarrow N$, there exists a map $h : W \longrightarrow M$ such that $g = f \circ h$.

Free modules are projective because if W is free on a set of generators p_i, then we choose elements q_i of M that map to the elements $g(p_i) \in N$, and take h to be the map sending p_i to q_i.

The definition of projectivity has several useful reformulations.

Proposition 1.33. Let W be an R-module. The following are equivalent:

(a) W is projective;

(b) For every epimorphism of modules $f : M \longrightarrow N$, the induce map $\operatorname{Hom}(W, M) \longrightarrow \operatorname{Hom}(W, N)$ is an epimorphism;

(c) For some epimorphism $F \longrightarrow W$, where F is free, the induced map $\operatorname{Hom}(W, F) \longrightarrow \operatorname{Hom}(W, W)$ is an epimorphism;

(d) P is a direct summand of a free module;

(e) Every epimorphism $f : M \longrightarrow W$ splits: That is, there is a map $g : W \longrightarrow M$ such that $f \circ g = 1_W$.

Lemma 1.34. Let $0 \longrightarrow C'_\bullet \longrightarrow C_\bullet \longrightarrow C''_\bullet \longrightarrow 0$ be a short exact sequence of complexes. If all modules in $C'_\bullet$ and $C''_\bullet$ are projective, so are all the modules in $C_\bullet$.

Definition 1.35. An R-module W is **finitely presented** if it is finitely generated and there exists a surjection of some finitely generated free module onto W.

Proposition 1.36. Let W be a finitely presented R-mdoule. Then W is projective if and only if W_Q is projective for every prime ideal $Q \subseteq R$, and this holds if and only if $W_{\mathfrak{m}}$ is projective for every maximal ideal $\mathfrak{m} \subseteq R$.

Definition 1.37. Let M be an R-module. A **projective resolution** of M is a complex

$$\cdots \longrightarrow W_{i+1} \xrightarrow{d_{i+1}} W_i \xrightarrow{d_i} W_{i-1} \longrightarrow \cdots \longrightarrow W_1 \xrightarrow{d_1} W_0 \longrightarrow 0,$$

where each W_i is a projective module over R, and $(W_\bullet, d_\bullet)$ is exact, i.e., $\operatorname{im} d_{i+1} = \ker d_i$ for all i.

We say that M has **finite projective dimension** if there exists a projective resolution

$$0 \longrightarrow W_i \longrightarrow W_{i-1} \longrightarrow \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow 0$$

of M. The minimum such i for a given M is called the **projective dimension** of M, and is denoted by $\operatorname{pd}_R(M)$, or simply by $\operatorname{pd}(M)$ if the context is clear.

Theorem 1.38 (Auslander-Buchsbaum Formula)**.** Let R be a local Noetherian ring and M a finitely generated R-module with $\operatorname{pd}(M) < \infty$. Then

$$\operatorname{pd}(M) + \operatorname{depth}(M) = \operatorname{depth}(R).$$

Corollary 1.39. If $\operatorname{depth}(R) = 0$ and $\operatorname{pd}(M) < \infty$, then $\operatorname{pd}(M) = 0$.

1.6 Local Cohomology

In order to prove one of ours main results (Terai's Theorem) we need to study local cohomology, and for that its necessary introduce the definition of injective module.

Definition 1.40. Let R be a ring and E be an R-module. We say that E is **injective** if for every monomorphism of R-modules $\alpha : N \longrightarrow M$ and every homomorphism of R-modules $\beta : N \longrightarrow E$, there exists a homomorphism of R-modules $\gamma : M \longrightarrow E$ such that $\beta = \gamma \circ \alpha$.

Proposition 1.41. Let R be a ring and let E be an R-module. Then the following are equivalent:

(a) E is injective;

(b) (Baer's Criterion) Let $I \subseteq R$ be an ideal. Every homomorphism from I to E extends to a homomorphism from R to E;

(c) $\operatorname{Hom}_R(-, E)$ preserves short exact sequences (contravariantly).

Definition 1.42. If $N \subseteq M$ are R-modules, then M is said to be **essential** over N if every non-zero submodule T of M has a non-zero intersection with N.

Proposition 1.43. Let R be a ring and $M \subseteq E$ be R-modules. The following conditions are equivalent:

(a) E is a **maximal essential extension** of M, i.e., if $E \subseteq F$ and F is also essential over M, then $E = F$;

(b) E is a **minimal injective** containing M, i.e., if $M \subseteq F \subseteq E$ and F is injective, then $F = E$;

(c) E is an injective module and is an essential extension of M.

Definition 1.44. A module E with any of the properties of Proposition 1.43 is called an **injective hull** of M and is denoted by $E_R(M)$.

Definition 1.45. An **injective resolution** $E^\bullet$ of an R-module M is an exact sequence:

$$0 \longrightarrow M \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \longrightarrow E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} \cdots,$$

where each E^i is an injective R-module. An injective resolution is called a **minimal injective resolution** if E^0 is an injective hull of M, E^{i+1} is an injective hull of $\ker(d^{i+1}) = \operatorname{im}(d^i)$ for all $i \geq 0$.

Definition 1.46. Let $I \subseteq R$ be an ideal and let M be an R-module. Set $\Gamma_I(M) = \{x \in M : xI^n = 0 \text{ for some } n \in \mathbb{N}\}$. For a homomorphism $f : M \longrightarrow N$ of R-modules, there is a mapping $\Gamma_I(f) : \Gamma_I(M) \longrightarrow \Gamma_I(N)$ which agrees with f on each element of $\Gamma_I(M)$. We call Γ_I the **I-torsion functor**.

By definition, we have the following result.

Lemma 1.47. Let I, J be ideals of R and M be an R-module. Then,

(a) $\Gamma_I(\Gamma_J(M)) = \Gamma_{I+J}(M)$;

(b) $\Gamma_I(M) = \Gamma_J(M)$ if and only if $\operatorname{rad}(I) = \operatorname{rad}(J)$;

(c) The I-torsion functor Γ_I is left exact.

We now come to the basic definition of this section. We use it and its properties on the proof of one of the most important result of the thesis: Theorem 5.7, which implies Terai's Theorem.

Definition 1.48. For $i \in \mathbb{N}$, the i-th right derivated functor of Γ_I is denoted by H_I^i and it is referred to as the **i-th local cohomology functor** with respect to I.

For an R-module M, we refer to $H_I^i(M)$, as the **i-th local cohomology module** of M with respect to I, and to $\Gamma_I(M)$ as the **I-torsion submodule** of M. We say that M is **I-torsion-free** precisely when $\Gamma_I(M) = 0$, and that M is **I-torsion** when $\Gamma_I(M) = M$.

To compute $H_I^i(M)$, one proceeds as follows. Take an injective resolution

$$E^\bullet : \quad 0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \longrightarrow E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} \cdots$$

of M, so that there is an R-homomorphism $\alpha : M \longrightarrow E^0$ such that the sequence

$$0 \longrightarrow M \xrightarrow{\alpha} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots \longrightarrow E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} \cdots,$$

is exact. Apply the functor Γ_I to the complex $E^\bullet$ to obtain

$$0 \longrightarrow \Gamma_I(E^0) \xrightarrow{\Gamma_I(d^0)} \Gamma_I(E^1) \xrightarrow{\Gamma_I(d^1)} \cdots \longrightarrow \Gamma_I(E^i) \xrightarrow{\Gamma_I(d^i)} \Gamma_I(E^{i+1}) \xrightarrow{\Gamma_I(d^{i+1})} \cdots$$

and take the i-th cohomology module of this complex,

$$H_I^i(M) = \ker(\Gamma_I(d^i))/\operatorname{im}(\Gamma_I(d^{i-1})).$$

Since Γ_I is left exact, we have that $H_I^0(M) = \Gamma_I(M)$.

Lemma 1.49. Let M be an R-module and I, J be ideals of R such that $\operatorname{rad}(I) = \operatorname{rad}(J)$. Then $H_I^i(M) = H_J^i(M)$ for all $i \in \mathbb{N}$.

Theorem 1.50. Let $(R, \mathfrak{m})$ be a local ring, and let M be a finitely generated R-module. Let $d = \dim(M)$, and let $\delta = \operatorname{depth}(M)$. We have:

(a) $H_{\mathfrak{m}}^i(M) = 0$ for $i < \delta$ and for $i > d$;

(b) $H_{\mathfrak{m}}^i(M) \neq 0$ for $i = \delta$ and for $i = d$.

Of course it follows that $\delta \leq d$.

Definition 1.51. Let $\underline{x} = x_1, \ldots, x_l \in R$. Define the **Čech complex** on R with respect to $x_1, \ldots, x_l$ by

$$C^\bullet(x_1; R) : \quad 0 \longrightarrow R \longrightarrow R_{x_1} \longrightarrow 0,$$

where $r \mapsto \frac{r}{1}$, and

$$\begin{aligned} C^\bullet(x_1, \ldots, x_l; R) &\coloneqq C^\bullet(x_1, \ldots, x_{l-1}; R) \otimes_R C^\bullet(x_l; R) \\ &= \oplus_{i=1}^l C^\bullet(x_i; R) \end{aligned}$$

Example 1.52. Lets compute $C^\bullet(x, y; R)$. We get the sequence

$$0 \longrightarrow R \otimes R \xrightarrow{f} R_x \otimes R \oplus R \otimes R_y \xrightarrow{g} R_x \otimes R_y \longrightarrow 0,$$

where $f(1 \otimes 1) = \frac{1}{1} \otimes 1 \oplus 1 \otimes \frac{1}{1}$, $g(\frac{1}{1} \otimes 1, 0) = (-1)\frac{1}{1} \otimes \frac{1}{1}$, and $g(0, 1 \otimes \frac{1}{1}) = \frac{1}{1} \otimes \frac{1}{1}$. Simplifying this, we get

$$0 \longrightarrow R \longrightarrow \oplus_{i=1}^l R_{x_i} \longrightarrow \oplus_{i<j} R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_1 \cdots x_l} \longrightarrow 0,$$

where the differentials are the same as the maps in the Koszul co-complex with 1's in the place of the x_i's. $\Diamond$

Definition 1.53. If M is an R-module, we define $C^\bullet(\underline{x}; M) := C^\bullet(\underline{x}; R) \otimes_R M$. The i-th **Čech cohomology** of M is $H^i(C^\bullet(\underline{x}; M))$.

Remark 1.54. *Let M be an R-module, $\underline{x} = x_1, \ldots, x_l \in R$ and $I = \langle \underline{x} \rangle$. From the above, $C^\bullet(\underline{x}; M)$ starts out as $0 \longrightarrow M \xrightarrow{\partial_0} \oplus_{i=1}^l M_{x_i}$. Now*

$$\begin{aligned} m \in H^0_{\underline{x}}(M) &\iff m \in \ker \partial_0 \\ &\iff \frac{m}{1} = 0 \text{ in } M \text{ for all } i \\ &\iff \text{there exists } t \geq 0 \text{ such that } x_i^t m \text{ for all } i \\ &\iff \text{there exists } t \geq 0 \text{ such that } I^t m = 0 \\ &\iff m \in H^0_I(M). \end{aligned}$$

Then, $H^0_{\underline{x}}(M) \cong H^0_I(M)$.

Theorem 1.55. Let M be an R-module and $\underline{x} = x_1, \ldots, x_l \in R$. Then, $H^i_{\langle \underline{x} \rangle}(M) = H^i(C^\bullet(\underline{x}; M))$.

Chapter 2

Simplicial Complexes

In this chapter we introduce the concept of simplicial complex which is a tool in algebra, topology and combinatorics. The information carried by square-free monomial ideals can be characterized in many ways, for example, using simplicial complexes. The main results of this chapter are Theorem 2.10 and Proposition 2.20.

Definition 2.1. A **simplicial complex** Δ on a vertex set $X = \{x_1, x_2, \dots, x_n\}$ is a collection of subsets of X, called **faces** or **simplices**, satisfaying that $\{x_i\} \in \Delta$ for every $i \in [n]$ and, if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$. A face of Δ not properly contained in another face of Δ is called a **facet**.

A face $F \in \Delta$ of cardinality $|F| = i + 1$ has **dimension** i and is called an i-**face** of Δ. We denote f_i as the number of i-faces of Δ. The **dimension** of Δ is $\dim \Delta = \max\{\dim F : F \in \Delta\}$, or $-\infty$ if $\Delta = \{\}$ is the **void complex**, which has no faces. We say that Δ is **pure** if all its facets have the same dimension. The **boundary** of a simplex F is the union of its faces with dimension $|F| - 1$ and it is denoted by ∂F.

Definition 2.2. For a simplicial complex Δ, its f-**vector** is $f(\Delta) \coloneqq (f_{-1}, f_0, f_1, \dots, f_{d-1})$, where $\dim \Delta = d - 1$ and $f_{-1} \coloneqq 1$ correspond to the empty face, whenever $\Delta \neq \{\}$. The f-**polynomial** is the generating function of the f-vector,

$$f_\Delta(t) = f_{-1}t^d + f_0 t^{d-1} + \cdots + f_{d-2}t + f_{d-1}.$$

The h-**polynomial** of Δ is the polynomial

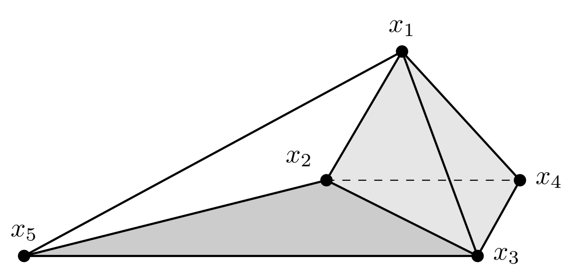
$$h_\Delta(t) = f_\Delta(t-1) = h_0 t^d + h_1 t^{d-1} + \cdots + h_{d-1}t + h_d,$$

the h-**vector** of Δ is the sequence $h(\Delta) = (h_0, h_1, \dots, h_d)$.

Example 2.3. Consider the following simplicial complex on the vertex set $X = \{x_i : 1 \le i \le 5\}$

$$\Delta = \{\{x_1, x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_4\}, \{x_2, x_3, x_4\}, \{x_2, x_3, x_5\}, \{x_1, x_5\}\},$$

then $\dim \Delta = 2$, $f(\Delta) = (1, 5, 9, 5)$ and $h(\Delta) = (1, 2, 2)$. We represent Δ visually as show the following figure



$\Diamond$

Definition 2.4. Let Δ be a simplicial complex and let $Y \subseteq X$. The **induced subcomplex** of Δ with vertex set on Y is defined by

$$\Delta[Y] \coloneqq \{F \in \Delta : F \subseteq Y\}.$$

Example 2.5. Let Δ be the simplicial complex from Example 2.3 and $Y = \{x_1, x_2, x_3, x_4\}$. Then,

$$\Delta[Y] = \{\{x_1, x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_4\}, \{x_2, x_3, x_4\}\}. \qquad \diamondsuit$$

2.1 Stanley-Reisner theory

The Stanley-Reisner correspondence arises from observations connecting the information in simplices to that in square-free monomials. One observation is that these are in natural bijection.

Definition 2.6. Let $A \subseteq X$ be a subset. Then the **monomial supported** on A is the square-free monomial $\mathbf{x}^A = \prod_{x_i \in A} x_i$. Conversaly, if m is a square-free monomial, then its **support** is $\operatorname{supp} m \coloneqq \{x_i : x_i \mid m\}$.

Definition 2.7. Let I be a square-free monomial ideal, then the **Stanley-Reisner complex** of I is the simplicial complex consisting of the monomials not in I,

$$\Delta_I \coloneqq \{F \subseteq X : \mathbf{x}^F \notin I\}.$$

Similarly, for a simplicial complex Δ the **Stanley-Reisner ideal** of Δ is the square-free monomial ideal generated by monomials corresponding to **non-faces** of Δ

$$I_\Delta \coloneqq \langle \mathbf{x}^F : F \notin \Delta \rangle.$$

The **Stanley-Reisner ring** is the quotient ring S/I_Δ.

Proposition 2.8. Let I be a square-free monomial ideal and let Δ be a simplicial complex. Then, $I_{\Delta_I} = I$ and $\Delta_{I_\Delta} = \Delta$.

Proof. Let $F \subseteq X$ such that $\mathbf{x}^F \in I_{\Delta_I}$. Observe that

$$\mathbf{x}^F \in I_{\Delta_I} \iff F \notin \Delta_I \iff \mathbf{x}^F \in I \tag{2.8.1}$$

hence, $I_{\Delta_I} \subseteq I$. Now, let $m \in \operatorname{gens}(I)$ and $G \subseteq X$ such that $\mathbf{x}^G = m$. Then by (2.8.1) we have $m \in I_{\Delta_I}$, hence $I_{\Delta_I} \supseteq I$. Therefore, $I_{\Delta_I} = I$. Analogously, $\Delta_{I_\Delta} = \Delta$. ∎

Notation 2.9. If $A \subseteq X$ is a non-empty subset, then write P_A for the prime ideal generated by the elements of A, i.e., $P_A = \langle x_i : x_i \in A \rangle$. If m is a monomial, write P_m for $P_{\operatorname{supp} m}$.

Now, we establish the following correspondence between simplicial complexes and square-free monomial ideals.

Theorem 2.10. There is a bijection between simplicial complexes on X and square-free monomial ideals on S. Furthermore,

$$I_\Delta = \bigcap_{F \in \Delta} P_{F^c}.$$

Proof. The bijection is the one from Proposition 2.8. For the second claim, we prove the double containment.

Suppose that $\bigcap_{F\in\Delta} P_{F^c} = \langle m_1,\dots,m_k\rangle$ with m_i be a square-free monomial for all $i\in[k]$. Fix $i\in[k]$, then $F^c\cap(\operatorname{supp} m_i)\neq\varnothing$ for all $F\in\Delta$. Thus, $(\operatorname{supp} m_i)\notin\Delta$ and $\mathbf{x}^{\operatorname{supp} m_i} = m_i\in I_\Delta$. Therefore, $I_\Delta\supseteq\bigcap_{F\in\Delta} P_{F^c}$.

Conversely, let $\mathbf{x}^F\in I_\Delta$ with $F\notin\Delta$. Then, $F\cap G^c\neq\varnothing$ for all $G\in\Delta$. This implies that $\mathbf{x}^F\in P_{G^c}$ for all $G\in\Delta$. Hence, $I_\Delta\subseteq\bigcap_{F\in\Delta} P_{F^c}$. ∎

Example 2.11. Consider the simplicial complex from Example 2.3, then we have

$$\begin{aligned} I_\Delta &= \langle x_4,x_5\rangle\cap\langle x_2,x_5\rangle\cap\langle x_1,x_5\rangle\cap\langle x_1,x_4\rangle\cap\langle x_2,x_3,x_4\rangle \\ &= \langle x_4x_5, x_1x_2x_5, x_1x_3x_5, x_1x_2x_3x_4\rangle. \end{aligned}$$ ◊

Proposition 2.12. Let I be a square-free monomial ideal and let m be a square-free monomial. Then,

(a) $I\subseteq P_m$ if and only if $(\operatorname{supp} m)^c\in\Delta_I$.

(b) $P_m\in\operatorname{Ass}(I)$ if and only if $(\operatorname{supp} m)^c$ is a facet of Δ_I.

Proof.

(a) Observe that

$$\begin{aligned} I\subseteq P_m &\iff (\operatorname{supp} m)\cap(\operatorname{supp}\mu)\neq\varnothing,\ \text{ for every } \mu\in\operatorname{gens}(I) \\ &\iff \mu\nmid\mathbf{x}^{(\operatorname{supp} m)^c},\ \text{ for every } \mu\in\operatorname{gens}(I) \\ &\iff \mathbf{x}^{(\operatorname{supp} m)^c}\notin I \\ &\iff (\operatorname{supp} m)^c\in\Delta_I. \end{aligned}$$

(b) First, suppose that $P_m\in\operatorname{Ass}(I)$ and let $F\in\Delta_I$ such that $(\operatorname{supp} m)^c\subsetneq F$. Thus $F^c\subsetneq\operatorname{supp} m$ and from part (a), we have $I\subseteq P_{F^c}\subsetneq P_m$, which is a contradiction. Therefore, $(\operatorname{supp} m)^c$ is a facet.

Conversely, suppose that $(\operatorname{supp} m)^c$ is a facet and suppose there exits a prime ideal P such that $I\subseteq P\subsetneq P_m$. Since I and P_m are square-free monomial ideals, then $P=P_F$ for some $F\subseteq X$. By part (a), $F^c\in\Delta_I$. Furthermore, $(\operatorname{supp} m)^c\subsetneq F^c$, which is a contradiction. Therefore, $P_m\in\operatorname{Ass}(I)$. ∎

2.2 Alexander Dual theory

The notion of Alexaner duality comes from algebraic topology (cf. Munkres, [Mun84]). The combinatorial way of Alexander duality, which we discuss here, produces a dual complex $\Delta^\vee$ from a simplicial complex Δ, and relates this complex with the associated prime ideals of I_Δ (Proposition 2.20).

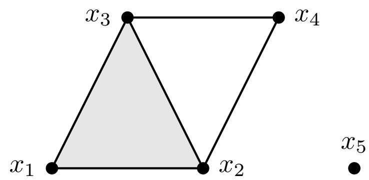
Definition 2.13. For a square-free monomial ideal I, the **Alexander dual** of I is

$$I^\vee := \bigcap_{m\in\operatorname{gens}(I)} P_m.$$

If Δ is a simplicial complex, then its **Alexander dual** is defined by

$$\Delta^\vee := \{F^c : F\notin\Delta\}.$$

Example 2.14. Let Δ be the simplicial complex from Example 2.3, its Alexander dual is $\Delta^\vee = \{\{x_1, x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_5\}\}$, that has the following geometric realization



$\Diamond$

Lemma 2.15. Let Δ be a simplicial complex on the vertex set X. If $\dim \Delta \leq n-3$, then $\Delta^\vee$ is also a simplicial complex on the vertex set X.

Proof. By definition, $\Delta^\vee$ is a simplicial complex. We need to prove that $\{y\} \in \Delta^\vee$ for all $y \in X$. Fix $y \in X$ and set $G = X - \{y\}$. Hence $|G| = n-1$. Then $G \notin \Delta$, because $\dim \Delta \leq n-3$ implies that $|F| \leq n-2$ for all $F \in \Delta$. Thus, $G^c \in \Delta^\vee$. Since $\{y\} = G^c$ we conclude that $\{y\} \in \Delta^\vee$. Therefore, $\Delta^\vee$ is a simplicial complex on the vertex set X. ∎

Example 2.16. Let $\Gamma = \{\{x_1, x_2, x_5\}, \{x_3, x_4, x_5\}\}$ be a simplical complex on the vertex set $X = \{x_i : 1 \leq i \leq 5\}$ such that $\dim \Gamma = 2$. Then, $\Gamma^\vee = \{\{x_2, x_3, x_5\}, \{x_1, x_4, x_5\}, \{x_2, x_5, x_4\}, \{x_1, x_3, x_5\}\}$ is a simplicial complex on the vertex set X. Their have the following geometric realization

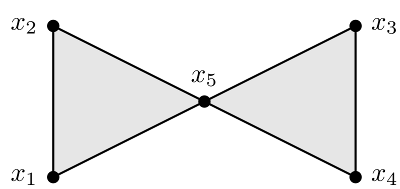


Figure 2.1: Simplicial complex Γ

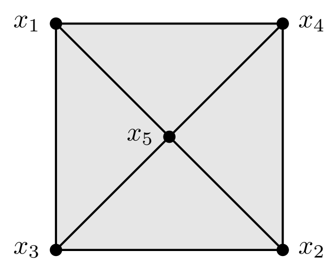


Figure 2.2: Simplicial complex $\Gamma^\vee$

$\Diamond$

Proposition 2.17. Let Δ be a simplicial complex and I be a square-free monomial ideal. Then,

(a) $(\Delta^\vee)^\vee = \Delta$;

(b) $(I_\Delta)^\vee = I_{\Delta^\vee}$;

(c) $(I^\vee)^\vee = I$.

Proof.

(a) By definition,

$$\begin{aligned} F \in (\Delta^\vee)^\vee &\Longleftrightarrow F^c \notin \Delta^\vee \\ &\Longleftrightarrow F^c \neq G^c \text{ for all } G \notin \Delta \\ &\Longleftrightarrow F \neq G \text{ for all } G \notin \Delta \\ &\Longleftrightarrow F \in \Delta. \end{aligned}$$

(b) By Theorem 2.10, we have

$$(I_\Delta)^\vee = \bigcap_{F \notin \Delta} P_F = \bigcap_{F \in \Delta^\vee} P_{F^c} = I_{\Delta^\vee}.$$

(c) By Theorem 2.10, there exists a simplicial complex $\Delta \subseteq X$ such that $I = I_\Delta$. By part (a) and part (b) we have

$$(I^\vee)^\vee = ((I_\Delta)^\vee)^\vee = (I_{\Delta^\vee})^\vee = I_{(\Delta^\vee)^\vee} = I_\Delta = I. \qquad \blacksquare$$

Example 2.18. Let Δ be the simplicial complex from Example 2.3. The Alexander dual of its Stanley-Reisner ideal is give by

$$(I_\Delta)^\vee = I_{\Delta^\vee} = \langle x_1x_4, x_2x_3x_4, x_1x_5, x_2x_5, x_3x_5, x_4x_5 \rangle. \qquad \Diamond$$

Furthermore, we have the following properties relates to Alexander duality.

Proposition 2.19. Let Δ_1, Δ_2 be simplicial complexes and I, J be square-free monomial ideals. Then,

(a) $(\Delta_1 \cap \Delta_2)^\vee = \Delta_1^\vee \cup \Delta_2^\vee$;

(b) $(\Delta_1 \cup \Delta_2)^\vee = \Delta_1^\vee \cap \Delta_2^\vee$;

(c) $I_{\Delta_1 \cup \Delta_2} = I_{\Delta_1} \cap I_{\Delta_2}$;

(d) $I_{\Delta_1 \cap \Delta_2} = I_{\Delta_1} + I_{\Delta_2}$;

(e) $(I + J)^\vee = I^\vee \cap J^\vee$;

(f) $(I \cap J)^\vee = I^\vee + J^\vee$;

(g) If $\Delta_1 \subseteq \Delta_2$, then $\Delta_2^\vee \subseteq \Delta_1^\vee$.

Proof.

(a) By definition,

$$\begin{aligned} F \in (\Delta_1 \cap \Delta_2)^\vee &\iff F^c \notin \Delta_1 \cap \Delta_2 \\ &\iff F^c \notin \Delta_1 \text{ or } F^c \notin \Delta_2 \\ &\iff F \in \Delta_1^\vee \text{ or } F \in \Delta_2^\vee \\ &\iff F \in \Delta_1^\vee \cup \Delta_2^\vee \end{aligned}$$

(b) Observe that,

$$\begin{aligned} F \in \Delta_1^\vee \cap \Delta_2^\vee &\iff F \in \Delta_1^\vee \text{ and } F \in \Delta_2^\vee \\ &\iff F^c \notin \Delta_1 \text{ and } F^c \notin \Delta_2 \\ &\iff F^c \notin \Delta_1 \cup \Delta_2 \\ &\iff F \in (\Delta_1 \cup \Delta_2)^\vee. \end{aligned}$$

(c) By definition,

$$\begin{aligned} \mathbf{x}^F \in I_{\Delta_1 \cup \Delta_2} &\iff F \notin \Delta_1 \cup \Delta_2 \\ &\iff F \notin \Delta_1 \text{ and } F \notin \Delta_2 \\ &\iff \mathbf{x}^F \in I_{\Delta_1} \text{ and } \mathbf{x}^F \in I_{\Delta_2} \\ &\iff \mathbf{x}^F \in I_{\Delta_1} \cap I_{\Delta_2}. \end{aligned}$$

(d) We prove the double containment.

Let $\mathbf{x}^F \in I_{\Delta_1 \cap \Delta_2}$ with $F \notin \Delta_1 \cap \Delta_2$. Then $F \notin \Delta_1$ or $F \notin \Delta_2$. Without loss of generality, suppose $F \notin \Delta_1$. Then $\mathbf{x}^F \in I_{\Delta_1}$, which implies $\mathbf{x}^F \in I_{\Delta_1} + I_{\Delta_2}$. Therefore, $I_{\Delta_1 \cap \Delta_2} \subseteq I_{\Delta_1} + I_{\Delta_2}$.

Conversely, without loss of generality, let $\mathbf{x}^F \in I_{\Delta_1}$ with $F \notin \Delta_1$. Then, $F \notin \Delta_1 \cap \Delta_2$ which implies that $\mathbf{x}^F \in I_{\Delta_1 \cap \Delta_2}$ and $I_{\Delta_1} \subseteq I_{\Delta_1 \cap \Delta_2}$. Analogously, $I_{\Delta_2} \subseteq I_{\Delta_1 \cap \Delta_2}$. Therefore $I_{\Delta_1} + I_{\Delta_1} \subseteq I_{\Delta_1 \cap \Delta_2}$.

(e) Let $\Gamma_1, \Gamma_2 \subseteq X$ be simplicial complexes such that $I = I_{\Gamma_1}$ and $J = I_{\Gamma_2}$. Then,

$$I^\vee \cap J^\vee = I_{\Gamma_1^\vee} \cap I_{\Gamma_2^\vee} = I_{\Gamma_1^\vee \cup \Gamma_2^\vee} = I_{(\Gamma_1 \cap \Gamma_2)^\vee} = (I_{\Gamma_1} + I_{\Gamma_2})^\vee = (I + J)^\vee.$$

(f) Let $\Gamma_1, \Gamma_2 \subseteq X$ be simplicial complexes such that $I = I_{\Gamma_1}$ and $J = I_{\Gamma_2}$. Then,

$$(I \cap J)^\vee = (I_{\Gamma_1} \cap I_{\Gamma_2})^\vee = I^\vee_{\Gamma_1 \cup \Gamma_2} = I_{(\Gamma_1 \cup \Gamma_2)^\vee} = I_{\Gamma_1^\vee \cap \Gamma_2^\vee} = I_{\Gamma_1^\vee} + I_{\Gamma_2^\vee} = I^\vee + J^\vee.$$

(g) Observe that $F \in \Delta_2^\vee$ if and only if $F^c \notin \Delta_2$. Hence, $F^c \notin \Delta_1$. This implies that $F \in \Delta_1^\vee$. Therefore, $\Delta_2^\vee \subseteq \Delta_1^\vee$. ∎

Now, we have enough tools to prove the following theorem. With it, we compute the Alexander dual of an ideal using its associted prime ideals, and vice-versa.

Proposition 2.20. Let I be a square-free monomial ideal. Then $I^\vee = \langle m : P_m \in \mathrm{Ass}(I) \rangle$. Furthermore, $\mathrm{Ass}(I) = \{P_\mu : \mu \in \mathrm{gens}(I^\vee)\}$.

Proof. First, suppose that $I = \langle m_1, \ldots, m_k \rangle$. Then,

$$\begin{aligned} I^\vee &= \cap_{m \in \mathrm{gens}(I)} P_m = \cap_{m \in \mathrm{gens}(I)} \langle x_i : x_i \mid m \rangle = \langle m : (\forall j)(\exists x_{i_{(j)}} \in \operatorname{supp} m)(x_{i_{(j)}} \mid m_j) \rangle \\ &= \langle m : P_m \in \mathrm{Ass}(I) \rangle. \end{aligned}$$

Therefore, $I^\vee = \langle m : P_m \in \mathrm{Ass}(I) \rangle$.

Now, we prove the second claim. Let $\mu \in \mathrm{gens}(I^\vee)$. By the first claim, we have that $P_\mu \in \mathrm{Ass}(I)$. Therefore, $\mathrm{Ass}(I) \supseteq \{P_\mu : \mu \in \mathrm{gens}(I^\vee)\}$.

Conversely, let $P \in \mathrm{Ass}(I)$. Since P is a monomial prime ideal, there exists a square-free monomial m such that $P = P_m$. By the first claim $m \in \mathrm{gens}(I^\vee)$. Therefore, $\mathrm{Ass}(I) \subseteq \{P_\mu : \mu \in \mathrm{gens}(I^\vee)\}$. ∎

Corollary 2.21. If I is a square-free monomial ideal, then $\Delta_{I^\vee} = (\Delta_I)^\vee$.

Proof. By definition,

$$F \in \Delta_{I^\vee} \iff \mathbf{x}^F \notin I^\vee \iff P_F \notin \mathrm{Ass}(I) \iff \mathbf{x}^{F^c} \in I \iff F^c \notin \Delta_I \iff F \in (\Delta_I)^\vee.$$ ∎

Corollary 2.22. Let Δ be a simplicial complex and let I be a square-free monomial ideal. Then,

(a) The facets of $\Delta^\vee$ are $(\operatorname{supp} m)^c$, where m ranges over the generators of I_Δ.

(b) The generators of $I^\vee$ are the monomials $\mathbf{x}^{(\operatorname{supp} m)^c}$, where m rages over the facets of Δ_I.

Proof. By Proposition 2.12 part (b) and Proposition 2.20 we have that

(a) Observe that $m \in \operatorname{gens}(I)$ if and only if $P_m \in \operatorname{Ass}(I^\vee)$ if and only if $(\operatorname{supp} m)^c \in \Delta_{I_\Delta^\vee} = \Delta^\vee$ is a facet.

(b) Observe that $m \in \operatorname{gens}(I^\vee)$ if and only if $P_m \in \operatorname{Ass}(I)$ if and only if $(\operatorname{supp} m)^c \in \Delta_I$ is a facet. ∎

Definition 2.23. A square-free monomial ideal is **equidimensional** if all its associated primes have the same height.

Corollary 2.24. A square-free monomial ideal I is equidimensional if and only if Δ_I is pure.

Proof. We know that

$$F \in \Delta_I \text{ is a facet } \iff \mathbf{x}^{F^c} \in \operatorname{gens}(I^\vee) \iff P_{F^c} \in \operatorname{Ass}(I).$$

Suppose that I is equidimensional. Let $F_1, F_2 \in \Delta_I$ be facets, then $P_{F_1{}^c}, P_{F_2{}^c} \in \operatorname{Ass}(I)$. Since

$$n = \dim(S) = \operatorname{ht}(P_{F_1{}^c}) + |F_1| = \operatorname{ht}(P_{F_2{}^c}) + |F_2| \tag{2.24.1}$$

and $\operatorname{ht}(P_{F_1{}^c}) = \operatorname{ht}(P_{F_2{}^c})$, then $\dim(F_1) = \dim(F_2)$.

Conversely, suppose that Δ_I is pure. Let $P, Q \in \operatorname{Ass}(I)$, then $P = P_{F_1{}^c}, Q = P_{F_2{}^c}$ for some facets $F_1, F_2 \in \Delta_I$. Since $\dim(F_1) = \dim(F_2)$, then by (2.24.1) we have $\operatorname{ht}(P_{F_1{}^c}) = \operatorname{ht}(P_{F_2{}^c})$. ∎

Corollary 2.25. Let I be a square-free monomial ideal. Then, the generators of I have the same degree if and only if $I^\vee$ is equidimensional.

Proof. First, suppose that the generators of I have the same degree. Let $P, Q \in \operatorname{Ass}(I^\vee)$. By Proposition 2.20 there exists $m_1, m_2 \in \operatorname{gens}(I)$ such that $P = P_{m_1}$ and $Q = P_{m_2}$. By hypothesis, $|\operatorname{supp} m_1| = |\operatorname{supp} m_2|$. Therefore, $\operatorname{ht}(P_{m_1}) = \operatorname{ht}(P_{m_2})$.

Conversely, assume that $I^\vee$ is equidimensional. Let $m_1, m_2 \in \operatorname{gens}(I)$. By hypothesis, $\operatorname{ht}(P_{m_1}) = \operatorname{ht}(P_{m_2})$, which implies that $|\operatorname{supp} m_1| = |\operatorname{supp} m_2|$. Therefore, $\deg m_1 = \deg m_2$. ∎

Chapter 3

Hilbert Series and h-vectors

In this chapter we study the Hilbert series for $\mathbb{N}^n$-graded modules, and replacing indeterminates, the Hilbert series in several variables becomes into the series only in one variable. The latter is the most common way to study the Hilbert series. Afterwards, we relate the degree of the Hilbert polynomial with the Krull dimension (Proposition 3.9). Finally, we focus on compute the Hilbert series of the Stanley-Reisner ring (Theorem 3.18), and we compute the h-vector of some simplicial complexes using shelling order (Theorem 3.23).

3.1 Hilbert Series

Definition 3.1. An S-module M is $\mathbb{N}^n$**-graded** if $M = \bigoplus_{\mathbf{b}\in\mathbb{N}^n} M_{\mathbf{b}}$ and $\mathbf{x}^{\mathbf{a}} M_{\mathbf{b}} \subseteq M_{\mathbf{a}+\mathbf{b}}$, where $M_{\mathbf{b}} \subseteq M$ is a $\mathbb{K}$-vector spaces, for all $\mathbf{b} \in \mathbb{N}^n$. If the dimension $\dim_{\mathbb{K}}(M_{\mathbf{a}})$ is finite for all $\mathbf{a} \in \mathbb{N}^n$, then the **Hilbert series** of M is

$$\mathrm{HS}(M;\mathbf{y}) \coloneqq \sum_{\mathbf{a}\in\mathbb{N}^n} \dim_{\mathbb{K}}(M_{\mathbf{a}})\mathbf{y}^{\mathbf{a}}.$$

Example 3.2. The Hilbert series of S is $\mathrm{HS}(S,\mathbf{y}) = \prod_{i=1}^n \frac{1}{(1-y_i)}$. To show this, we proceed by induction on n. For $n = 1$, then $S = \mathbb{K}[x_1]$ has the basis $\{1, x_1, {x_1}^2, \ldots\}$ and its Hilbert series is

$$\mathrm{HS}(S,\mathbf{y}) = 1 + y_1 + {y_1}^2 + \cdots = \frac{1}{1-y_1}.$$

Suppose $n > 1$. Consider the following exact sequence

$$0 \longrightarrow Sx_n \longrightarrow S \longrightarrow S/(Sx_n) \longrightarrow 0,$$

since $\dim_{\mathbb{K}}$ is an additive function we have that

$$\mathrm{HS}(S,\mathbf{y}) - \mathrm{HS}(Sx_n,\mathbf{y}) = \mathrm{HS}(\mathbb{K}[x_1,\ldots,x_{n-1}],\mathbf{y}).$$

By induction hypothesis

$$(1-y_n)\,\mathrm{HS}(S,\mathbf{y}) = \frac{1}{(1-y_1)\cdots(1-y_{n-1})} \implies \mathrm{HS}(S,\mathbf{y}) = \prod_{i=1}^n \frac{1}{(1-y_i)}.$$

$\Diamond$

If M is a finitely generated $\mathbb{N}^n$-graded S-module, then M has an $\mathbb{N}$-grading as S-module and we choose $y_i = t$ for each $i \in [n]$. In this case, the Hilbert series is

$$\begin{aligned}\mathrm{HS}(M,t) &\coloneqq \mathrm{HS}(M,t,\dots,t) = \sum_{\mathbf{a}\in\mathbb{N}^n} \dim_{\mathbb{K}}(M_{\mathbf{a}})t^{a_1+\cdots+a_n} \\ &= \sum_{l=0}^{\infty}\left(\sum_{\mathbf{a}\in\mathbb{N}^n,\,|\mathbf{a}|=l} \dim_{\mathbb{K}}(M_{\mathbf{a}})\right)t^l \\ &= \sum_{l=0}^{\infty} \dim_{\mathbb{K}}(M_l)t^l.\end{aligned}$$

Corollary 3.3. The Hilbert series of S on t is $1/(1-t)^n$.

Proof. Substituting y_i by t in Example 3.2, we have

$$\mathrm{HS}(S,t) = \prod_{i=1}^{n}\frac{1}{(1-t)} = \frac{1}{(1-t)^n}.$$

∎

The next statement, is one of the most common and basic results related to the Hilbert series.

Theorem 3.4 (Hilbert-Serre's Theorem)**.** Let M be a finitely generated $\mathbb{N}$-graded S-module. Then,

$$\mathrm{HS}(M,t) = \frac{p(t)}{(1-t)^n},$$

for some polynomial $p(t) \in \mathbb{Z}[t]$.

Corollary 3.5. Let M be a finitely generated $\mathbb{N}$-graded S-module. Suppose that,

$$\mathrm{HS}(M,t) = \frac{h(t)}{(1-t)^d},$$

where $h(1) \neq 0$. Then, there exists a polynomial $H_M(t) \in \mathbb{Q}[t]$ of degree $d-1$ such that $H_M(n) = \dim_{\mathbb{K}}(M_n)$ for all sufficiently large n.

Definition 3.6. With the notation of Corollary 3.5, $H_M(n)$ is called the **Hilbert polynomial** of M, and $h(t)$ is called the h**-polynomial** of M.

Theorem 3.7 (Krull's principal ideal theorem)**.** Let R be a Noetherian ring. If $x \in R$ and P is minimal among primes of R containing x, then $\dim R_P \leq 1$.

Remark 3.8. *Let M be a finitely generated S-module. Suppose that $u_1,\dots,u_k$ generate M. Since S is Noetherian we can find a finite filtration of M such that every M_{i+1}/M_i, $0 \leq i \leq k$ is a prime cyclic module, i.e., has the form S/P_i for some prime ideal P_i of S. One first chooses u_1 such that $\mathrm{Ann}_S(u_1) = P_1$. Let $M_1 = u_1S \subseteq M$. Proceeding recursively, suppose that $u_1,\dots,u_i$ have been chosen in M such that, $M_j = u_1S + \cdots + u_jS$ for $j \in [i]$, we have that $M_j/M_{j-1} \cong S/P_j$ with P_j prime. If $M_i = M$ we are done. If not, we can choose $u_{i+1} \in M$ such that the annihilator of its image in M/M_i is a prime ideal P_{i+1} of S. Then $M_{i+1}/M_i \cong S/P_{i+1}$ and $M_i \subsetneq M_{i+1}$. The process must terminate, since M is Noetherian. In others words, eventually we reach M_l such that $M_l = M$.*

Now, we prove a result to relate the degree of the Hilbert polynomial with the Krull dimension, for that we use a double process of induction.

Proposition 3.9. Let M be a finitely generated $\mathbb{N}$-graded S-module. Then $\deg H_M(t) = d-1$, where d is the Krull dimension of M.

Proof. We proceed by induction on $d = dim(M)$. Observe that $d = 0$ if and only if M has finite length, in which case $M_n = 0$ for all $n \gg 0$. Therefore, $H_M(t) = 0$ and its degree is -1.

Suppose $d > 0$, then by Remark 3.8 we may construct a finite filtration of M

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots M_{l-1} \subsetneq M_l = M,$$

such that $M_i/M_{i-1} \cong S/P_i$ for all $i \in [l]$, where P_i is a prime ideal. Therefore, we have that

$$\dim(M) = \max\{\dim(S/P_i) : 1 \leq i \leq l\}.$$

We study the case where M has the form S/P for some prime ideal P. Since $d > 0$, it follows that there exists x_i such that $x_i \notin P$. Then we have the following exact sequence of graded modules

$$0 \longrightarrow M(-1) \xrightarrow{x_i} M \longrightarrow M/x_iM \longrightarrow 0,$$

by the additive of $\dim_{\mathbb{K}}$ we have that

$$H_M(t) - H_M(t-1) = H_{M/x_iM}(t). \tag{3.9.1}$$

By the Krull's principal ideal theorem, $\dim(M/x_iM) = d-1$. Then, by induction hypothesis $\deg H_{M/x_iM}(t) = d-2$. Hence, by (3.9.1)

$$\deg(H_M(t)) - 1 = d - 2 \implies \deg H_M(t) = d-1 \tag{3.9.2}$$

Now, we proceed by induction on l to prove

$$H_M(t) = \sum_{k=1}^{l} H_{S/P_k}(t) \tag{3.9.3}$$

Suppose that $l = 2$, then we have the filtration

$$0 \subsetneq M_1 \subsetneq M_2 = M.$$

Consider the following exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow S/P_2 \longrightarrow 0,$$

and, since $M_1/M_0 \cong M_1 \cong S/P_1$ we have

$$H_{M_2}(t) = H_{M_1}(t) + H_{S/P_2}(t) = H_{S/P_1}(t) + H_{S/P_2}(t).$$

Suppose that it holds for $l > 2$. By the following exact sequence and induction hypothesis

$$0 \longrightarrow M_{l-1} \longrightarrow M \longrightarrow S/P_l \longrightarrow 0,$$

we have

$$\begin{aligned} H_M(t) &= H_{M_{l-1}}(t) + H_{S/P_l}(t) \\ &= \sum_{k=1}^{l-1} H_{S/P_k}(t) + H_{S/P_l}(t) \\ &= \sum_{k=1}^{l} H_{S/P_k}(t). \end{aligned}$$

Therefore, it holds (3.9.3). Let $l_0 \in [l]$ such that $\dim(M) = \dim(S/P_{l_0})$. Then, by (3.9.2) and (3.9.3)

$$\deg H_M(t) = \deg H_{S/P_{l_0}}(t) = \dim(M) - 1.$$

∎

Theorem 3.10. Let M be a finitely generated $\mathbb{N}$-graded S-module. If M is Cohen-Macaulay, then its h-polynomial has non-negative coefficients.

Proof. We proceed by induction on $d = \dim(M)$. When $d = 0$, by Hilbert-Serre's Theorem, we have

$$\operatorname{HS}(M,t) = \sum_{k=0}^{\infty} \dim_{\mathbb{K}}(M_k)t^k = h(t),$$

then $h(t) \geq 0$.

Assume $|\mathbb{K}| = \infty$ and let x be a non-zero divisor of degree 1. Consider the following exact sequence

$$0 \longrightarrow xM \longrightarrow M \longrightarrow M/(xM) \longrightarrow 0,$$

then

$$(1-t)\operatorname{HS}(M,t) = \operatorname{HS}(M/(xM),t) = \frac{h(t)}{(1-t)^{d-1}} \implies \operatorname{HS}(M,t) = \frac{h(t)}{(1-t)^d},$$

and by induction hypothesis, $h(t) \geq 0$. ∎

3.2 The square-free Hilbert Series

Our next goal is compute the Hilbert series for the Stanley-Reisner ring. For that, we need to study more the square-free monomials and the square-free monomial ideals, and related them with the Hilbert series and the square-free Hilbert series.

Definition 3.11. Let I be a square-free monomial ideal. Then the **square-free Hilbert function** of I is the function $\operatorname{HF}_I^{\text{sqfree}} : \mathbb{Z} \to \mathbb{Z}$ such that

$$\operatorname{HF}_I^{\text{sqfree}}(d) = |\{m \in I : m \text{ is a square-free monomial of degree } d\}|.$$

Similarly we define $\operatorname{HF}_{S/I}^{\text{sqfree}}$ and the **square-free Hilbert series** $\operatorname{HS}^{\text{sqfree}}(I,t)$ and $\operatorname{HS}^{\text{sqfree}}(S/I,t)$ to be the generating functions of the corresponding square-free Hilbert functions.

Definition 3.12. The **square-free part** of a monomial m is the square-free monomial defined by

$$\operatorname{sqfree}(m) \coloneqq \prod_{x_i \mid m} x_i.$$

Lemma 3.13. Let I be a square-free monomial ideal. A monomial $m \in I$ if and only if $\operatorname{sqfree}(m) \in I$.

Proof. Suppose that $m \in I$. Let $m = {x_1}^{\alpha_1}{x_2}^{\alpha_2}\cdots{x_n}^{\alpha_n} \in I$ and $I = \langle m_1, \ldots, m_k \rangle$. Since m is a mononimial, then $m = m' \cdot m_j$ for some monomial m' and some $j \in [k]$. Set $A = \operatorname{supp}(\operatorname{sqfree}(m')) - \operatorname{supp} m_j$. Therefore, $\operatorname{sqfree}(m) = \mathbf{x}^A \cdot m_j \in I$.

Conversely, suppose that $\operatorname{sqfree}(m) \in I$. Since $\operatorname{sqfree}(m) \mid m$, then $m = q \cdot \operatorname{sqfree}(m)$ for some monomial q. Therefore, $m \in I$. ∎

Notation 3.14. Let $A \subseteq X$ be a non-empty subset and suppose that $A = \{x_{i_1}, \ldots, x_{i_k}\}$, then define the polynomial ring $\mathbb{K}[A] \coloneqq \mathbb{K}[x_{i_1}, \ldots, x_{i_k}]$.

Corollary 3.15. Let I be a square-free monomial ideal. Then, as $\mathbb{K}$-vector spaces, I and S/I decompose over the set of square-free monomials as

$$I = \bigoplus_{m \in I} m \cdot \mathbb{K}[\operatorname{supp} m] \quad \text{and} \quad S/I = \bigoplus_{m \notin I} m \cdot \mathbb{K}[\operatorname{supp} m].$$

Proof. We prove the double containment.

Let $m' \in \operatorname{gens}(I)$. Since $m' \in m' \cdot \mathbb{K}[\operatorname{supp} m']$ then $m' \in \bigoplus_{m \in I} m \cdot \mathbb{K}[\operatorname{supp} m]$. This implies, $I \subseteq \bigoplus_{m \in I} m \cdot \mathbb{K}[\operatorname{supp} m]$.

On the other hand. Since $m \in I$ we have that $m \cdot \mathbb{K}[\operatorname{supp} m] \subseteq I$. Let $f \in \bigoplus_{m \in I} m \cdot \mathbb{K}[\operatorname{supp} m]$. We write f as the finite sum of elements of $\bigoplus_{m \in I} m \cdot \mathbb{K}[\operatorname{supp} m]$, say $g_1, \dots, g_r$ where $g_i \in m_i \cdot \mathbb{K}[\operatorname{supp} m_i]$ for some square-free monomial $m_i \in I$. Hence $g_i \in I$, this implies that $f \in I$. Therefore, $I \supseteq \bigoplus_{m \in I} m \cdot \mathbb{K}[\operatorname{supp} m]$.

Analogously, since $S/I = \langle m : m \notin I \rangle$ we have that $S/I = \bigoplus_{m \notin I} m \cdot \mathbb{K}[\operatorname{supp} m]$. ∎

Lemma 3.16. Let m be a square-free monomial. Then, $\operatorname{HS}(m \cdot \mathbb{K}[\operatorname{supp} m], t) = \left(\frac{t}{1-t}\right)^{\deg m}$.

Proof. Since m is a square-free monomial, we have $|\operatorname{supp} m| = \deg m$. By Corollary 3.3,

$$\operatorname{HS}(\mathbb{K}[\operatorname{supp} m], t) = \frac{1}{(1-t)^{\deg m}}.$$

Hence,

$$\operatorname{HS}(m \cdot \mathbb{K}[\operatorname{supp} m], t) = t^{\deg m} \operatorname{HS}(\mathbb{K}[\operatorname{supp} m], t) = \left(\frac{t}{1-t}\right)^{\deg m}.$$

∎

Theorem 3.17. Let I be a square-free monomial ideal. Then, $\operatorname{HS}(S/I, t) = \operatorname{HS}^{\text{sqfree}}(S/I, \frac{t}{1-t})$ and $\operatorname{HS}(I, t) = \operatorname{HS}^{\text{sqfree}}(I, \frac{t}{1-t})$.

Proof. By Corollary 3.15 and Lemma 3.16, we have

$$\begin{aligned}
\operatorname{HS}(I, t) &= \sum_{\substack{m \in I, \\ \text{square-free}}} \operatorname{HS}(m \cdot \mathbb{K}[\operatorname{supp} m], t) = \sum_{\substack{m \in I, \\ \text{square-free}}} \left(\frac{t}{1-t}\right)^{\deg m} \\
&= \sum_{k=1}^{n} \left(\sum_{m \in I,\, \deg m = k} \right) \left(\frac{t}{1-t}\right)^{k} = \sum_{k=1}^{n} \operatorname{HF}_I^{\text{sqfree}}(k) \left(\frac{t}{1-t}\right)^{k} \\
&= \operatorname{HS}^{\text{sqfree}}\left(I, \frac{t}{1-t}\right).
\end{aligned}$$

Similarly, $\operatorname{HS}(S/I, t) = \operatorname{HS}^{\text{sqfree}}(S/I, \frac{t}{1-t})$. ∎

Theorem 3.18. Let Δ be a $(d-1)$-dimensional simplicial complex and $(f_{-1}, f_0, \dots, f_{d-1})$ its f-vector. Then,

(a) $\operatorname{HS}^{\text{sqfree}}(S/I_\Delta, t) = \sum_{i=0}^{d} f_{i-1} t^i$;

(b) $\operatorname{HS}(S/I_\Delta, t) = \sum_{i=0}^{d} \frac{f_{i-1} t^i}{(1-t)^i} = \frac{1}{(1-t)^d} \sum_{i=0}^{d} h_i t^i$.

Proof.

(a) Let $k \in \mathbb{Z}$. There is a bijection between the square-free monomials of degree k in S/I_Δ and the $(k-1)$-dimensional faces of Δ. Therefore, $\mathrm{HF}^{\mathrm{sqfree}}_{S/I_\Delta}(k) = f_{k-1}$, which implies that

$$\mathrm{HS}^{\mathrm{sqfree}}(S/I_\Delta, t) = \sum_{i=0}^{d} f_{i-1} t^i.$$

(b) Observe that

$$\sum_{i=0}^{d} h_i t^{d-i} = \sum_{i=0}^{d} f_{i-1}(t-1)^{d-i}.$$

Replacing t by $1/t$, we have

$$\sum_{i=0}^{d} \frac{h_i}{t^{d-i}} = \sum_{i=0}^{d} f_{i-1}\left(\frac{1-t}{t}\right)^{d-i}.$$

Multiplying by t^d we get

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i}. \tag{3.18.1}$$

By Theorem 3.17 and part (a) we have that

$$\mathrm{HS}(S/I_\Delta, t) = \sum_{i=0}^{d} \frac{f_{i-1}t^i}{(1-t)^i} = \frac{1}{(1-t)^d}\sum_{i=0}^{d} f_{i-1}t^i(1-t)^{d-i} \overset{(3.18.1)}{=} \frac{1}{(1-t)^d}\sum_{i=0}^{d} h_i t^i.$$ ■

From Theorem 3.18, knowing $f_\Delta(t)$ is equivalent to knowing $h_\Delta(t)$.

Example 3.19. For the simplicial complex in Example 2.3 we have $f(\Delta) = (1, 5, 9, 5)$ and $d = 3$. Then by Theorem 3.18, its Hilbert series is

$$\mathrm{HS}(S/I_\Delta, t) = \frac{2t^2 + 2t + 1}{(1-t)^3}.$$ ◊

3.3 Shellable complexes

To end this chapter, we introduce the definition of shelling. Shellable complexes occur frequently throughout combinatorics. We uses the shellability condition to compute the h-vector without having to know the f-vector.

Definition 3.20. Let Δ be a simplicial complex and $F_1, F_2, \ldots, F_t$ its facets. For each $j \in [t]$, we say that $A \subseteq F_j - (F_1 \cup \cdots \cup F_{j-1})$ is the **minimal face** associated to F_j if for every face $B \subseteq F_j - (F_1 \cup \cdots \cup F_{j-1})$ we have that A is the unique element such that $A \subseteq B$.

An ordering $F_1, F_2, \ldots, F_t$ of the facets of a simplicial complex Δ is a **shelling** if, for each $j \in [t]$, the intersection

$$\left(\bigcup_{i=1}^{j-1} F_i\right) \cap F_j,$$

is a non-empty union of facets of ∂F_j. If there exists a shelling of Δ then Δ is called **shellable**.

Example 3.21. Let Δ be the simplicial complex from Example 2.3, then a shelling for Δ is $\{x_1,x_2,x_3\}$, $\{x_1,x_2,x_4\}$, $\{x_1,x_3,x_4\}$, $\{x_2,x_3,x_4\}$, $\{x_2,x_3,x_5\}$, $\{x_1,x_5\}$ and the associated minimal faces are, $\varnothing$, $\{x_4\}$, $\{x_3,x_4\}$, $\{x_2,x_3,x_4\}$, $\{x_5\}$, $\{x_1,x_5\}$, respectively. $\Diamond$

Proposition 3.22. Let Δ be a simplicial complex and $F_1,F_2,\dots,F_t$ its facets. This order is a shelling of Δ if and only if for each $j\in[t]$, F_j has a minimal face.

Proof. Suppose that $F_1,F_2,\dots,F_t$ is a shelling. Fix $j>1$ and define $A=\{a\in F_j : F_j-\{a\}\subseteq \cup_{i=1}^{j-1}F_i\}$. Let $B\subseteq F_j$ be a face. Since $F_j\cap(\cup_{i=1}^{j-1}F_i)$ is a union of facets of ∂F_j, we have $B\subseteq \cup_{i=1}^{j-1}F_i$ if and only if $B\subseteq F_j-\{a\}$, for some $a\in A$. Therefore,

$$B\not\subseteq \cup_{i=1}^{j-1}F_i \iff \nexists\, a\in A \text{ such that } B\not\subseteq F_j-\{a\}\iff A\subseteq B.$$

Therefore, A is the minimal face associated to F_j.

On the other hand, fix $j>1$. Let A be the minimal face associated to F_j. Since the empty set $\varnothing$ is a face of $\cup_{i=1}^{j-1}F_i$, we have that $A\neq\varnothing$.

Let $B\subseteq F_j$ be a face. Since A is minimal, we have that $B\subseteq F_j\cap(\cup_{i=1}^{j-1}F_i)$ if and only if $\exists\, a\in A$ such that $a\notin B$. This implies that $B\subseteq F_j-\{a\}$ for some $a\in A$. Hence, $B\subseteq \cup_{a\in A}(F_j-\{a\})$. Therefore, $\cup_{a\in A}(F_j-\{a\})\supseteq F_j\cap(\cup_{i=1}^{j-1}F_i)$.

Conversely, fix $a\in A$ and suppose that $F_j-\{a\}\not\subseteq \cup_{i=1}^{j-1}F_i$. For be minimal, $A\subseteq F_j-\{a\}$. This implies that $a\notin A$, which is a contradiction. Therefore, $F_j-\{a\}\subseteq \cup_{i=1}^{j-1}F_i$, for all $a\in A$. Then $\cup_{a\in A}(F_j-\{a\})\subseteq F_j\cap(\cup_{i=1}^{j-1}F_i)$. We conclude that $\cup_{a\in A}(F_j-\{a\})=F_j\cap(\cup_{i=1}^{j-1}F_i)$. Thus, the order is a shelling. ■

Theorem 3.23. Let Δ be a pure $(d-1)$-dimensional complex with shelling $F_1,\dots,F_r$. Then the h-vector is given as follows: For each i,

$$h_i=|\{j:\dim A_j=i-1\}|,$$

where A_j is the minimal face associated to F_j, for all $j\in[r]$.

Proof. We proceed by induction on r to prove $f_\Delta(t)=\sum_{j=1}^{r}(t+1)^{d-|A_j|}$. Since $\dim F_1=d-1$ and $A_1=\varnothing$ is the associated minimal face to F_1, we have that

$$f_{F_1}(t)=(t+1)^d=(t+1)^{d-|A_1|}.$$

Let $r=2$. Since the order is a shelling, $F_1\cap F_2\neq\varnothing$. To compute $f_{F_1\cup F_2}(t)$ we need to known the faces $B\subseteq F_2-F_1$. Let A_2 be the associated minimal face to F_2, then $B\subseteq F_2-F_1$ if and only if $A_2\subseteq B$. Observe that $|B|=k+|A_2|$ for some $0\le k\le d-|A_2|$, hence

$$\begin{aligned} f_{F_1\cup F_2}(t)&=f_{F_1}(t)+t^{d-|A_2|}+\binom{d-|A_2|}{1}t^{d-|A_2|-1}+\cdots+\binom{d-|A_2|}{d-|A_2|-1}t+1\\ &=f_{F_1}(t)+(t+1)^{d-|A_2|}.\end{aligned}$$

Suppose the result holds for $r>2$. Using that $\Delta=F_1\cup F_2\cup\cdots\cup F_r$ and the same argument for the previous case, we have that

$$\begin{aligned} f_\Delta(t)&=f_{F_1\cup F_2\cup\cdots\cup F_{r-1}}+t^{d-|A_r|}+\binom{d-|A_r|}{1}t^{d-|A_r|-1}+\cdots+\binom{d-|A_r|}{d-|A_r|-1}t+1\\ &=f_{F_1\cup F_2\cup\cdots\cup F_{r-1}}+(t+1)^{d-|A_r|}\\ &=\sum_{j=1}^{r}(t+1)^{d-|A_j|}. \end{aligned}\tag{3.23.1}$$

By definition

$$h_\Delta(t+1) = f_\Delta(t) = h_0(t+1)^d + h_1(t+1)^{d-1} + \cdots + h_{d-1}(t+1) + h_d,$$

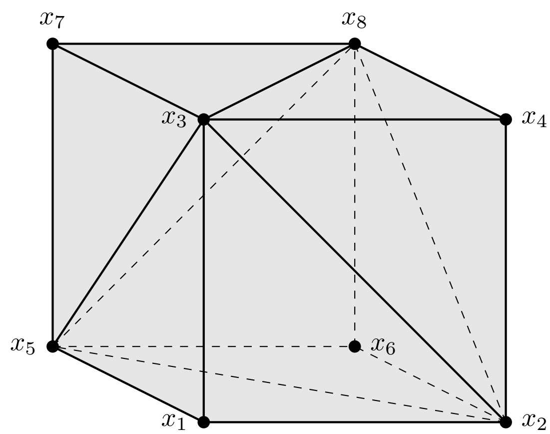
by the previous equation and (3.23.1), we have that

$$h_i = |\{j : \dim A_j = i-1\}|. \qquad \blacksquare$$

Example 3.24. Consider the following simplicial complex on the vertex set $X = \{x_i : 1 \leq i \leq 8\}$

$$\Delta = \{\{x_1,x_2,x_3\},\{x_2,x_3,x_4\},\{x_1,x_3,x_5\},\{x_3,x_5,x_7\},\{x_2,x_4,x_8\},\{x_2,x_6,x_8\},$$
$$\{x_3,x_4,x_8\},\{x_3,x_7,x_8\},\{x_5,x_6,x_8\},\{x_5,x_7,x_8\},\{x_1,x_2,x_5\},\{x_2,x_5,x_6\},\}$$

which is represent visually as show the following figure



Since $f(\Delta) = (1, 8, 18, 12)$ then $h(\Delta) = (1, 5, 5, 1)$. Now consider the following shelling for Δ

$\{x_5,x_7,x_8\},\{x_3,x_7,x_8\},\{x_5,x_6,x_8\},\{x_2,x_6,x_8\},\{x_3,x_4,x_8\},\{x_2,x_4,x_8\},\{x_3,x_5,x_7\},\{x_2,x_5,x_6\},$
$\{x_1,x_3,x_5\},\{x_1,x_2,x_5\},\{x_2,x_3,x_4\},\{x_1,x_2,x_3\}$

and the associated minimal faces are

$$\varnothing,\{x_3\},\{x_6\},\{x_2\},\{x_4\},\{x_2,x_4\},\{x_3,x_5\},\{x_2,x_5\},\{x_1\},\{x_1,x_2\},\{x_2,x_3\},\{x_1,x_2,x_3\} \qquad (3.24.1)$$

respectively. Then by Theorem 3.23 and (3.24.1) we have that $h_0 = 1$, $h_1 = 5$, $h_2 = 5$ and $h_3 = 1$. This check our previous results. $\lozenge$

Chapter 4

Hochster's formula

We start this chapter defining the Koszul complex. We use some homological tools to prove that Koszul complex is a minimal free resolution of the residual field (Proposition 4.4). After, we study the Betti numbers for a graded module. In Theorem 4.12, we compute the Betti numbers for square-free monomial ideals and related them with the homology of the Koszul complex. Using these results, we prove the main theorem of this chapter; the Hochster's formula (Theorem 4.17). This formula shows that the multi-graded Betti numbers of a square-free monomial ideal I are encoded in the homology of induced subcomplexes of Δ_I.

4.1 The Koszul complex

The goal of this section is prove that the Koszul complex is a minimal free resolution of $S/\mathfrak{m}$. With that restul, we compute homology modules and Betti numbers.

Remark 4.1. *Given two chain complexes $C_\bullet$ and $C'_\bullet$, we tensor their as follows: $(C_\bullet, d_\bullet) \otimes (C'_\bullet, d'_\bullet)$ is a complex with i-th graded pieces $\bigoplus_{k+l=i}(C_k \otimes C'_l)$ and an endomorphism*

$$C_k \otimes C'_l \longrightarrow (C_{k-1} \otimes C'_l) \oplus (C_k \otimes C'_{l-1})$$

defined by

$$x \otimes y \mapsto (d_k(x) \otimes y) \oplus ((-1)^k x \otimes d'_l(y)).$$

Definition 4.2. Let $x \in R$. The **Koszul complex** of x is

$$K_\bullet(x; R) : \quad 0 \longrightarrow R_1 \xrightarrow{x} R_0 \longrightarrow 0,$$

where $R_0 = R_1 = R$ and the map labeled x is the multiplication by x. For $x_1, \dots, x_l \in R$, the **Koszul complex** of $x_1, \dots, x_l$ is defined inductively as

$$K_\bullet(x_1, \dots, x_l; R) \coloneqq K_\bullet(x_1, \dots, x_{l-1}; R) \otimes_R K_\bullet(x_l; R)$$

where $K_i = \wedge^i(R^l)$ and the map $d_i : K_i \longrightarrow K_{i-1}$ is defined by

$$d_i(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{k=1}^{i} (-1)^{k+1} x_k (e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_k} \wedge \cdots \wedge e_{j_i}).$$

Let M be an R-module. The Koszul complex of M with respect $x_1, \dots, x_l$ is defined by

$$K_\bullet(x_1, \dots, x_l; M) \coloneqq K_\bullet(x_1, \dots, x_l; R) \otimes_R M.$$

Let $(C_\bullet, d_\bullet)$ be a complex over R and let $K_\bullet = K_\bullet(x; R)$ be the Koszul complex of $x \in R$, by Remark 4.1 we have

$$(C_\bullet \otimes K_\bullet)_i = (C_i \otimes R_0) \oplus (C_{i-1} \otimes R_1) \cong C_i \oplus C_{i-1}.$$

Therefore, we have the short exact sequence of complexes

$$0 \longrightarrow C_\bullet \xrightarrow{\alpha} C_\bullet \otimes K_\bullet \xrightarrow{\beta} C_\bullet(-1) \longrightarrow 0$$

defined by $\alpha(a) = (a, 0)$, $\beta(a, b) = b$, $(C_\bullet(-1))_i = C_{i-1}$ and the differential δ on $C_\bullet \otimes K_\bullet$ is $\delta_i(a, b) = (d_i(a) + (-1)^{i-1} x b, d_{i-1}(b))$.

Observe that $H_i(C_\bullet) \cong H_{i+1}(C_\bullet(-1))$. By Theorem 1.9, we have the following long exact sequence of homology

$$\cdots \xrightarrow{x} H_{i+1}(C_\bullet) \longrightarrow H_{i+1}(C_\bullet \otimes K_\bullet) \longrightarrow H_i(C_\bullet) \xrightarrow{x} H_i(C_\bullet) \longrightarrow H_i(C_\bullet \otimes K_\bullet) \longrightarrow H_{i-1}(C_\bullet) \xrightarrow{x} \cdots$$

This breaks up into short exact sequences

$$0 \longrightarrow \frac{H_i(C_\bullet)}{x H_i(C_\bullet)} \longrightarrow H_i(C_\bullet \otimes K_\bullet) \longrightarrow \operatorname{Ann}_{H_{i-1}(C_\bullet)}(x) \longrightarrow 0 \tag{4.2.1}$$

for all i.

Theorem 4.3. Let M be an R-module and $x_1, \ldots, x_l \in R$. Then, $H_0(K_\bullet(x_1, \ldots, x_l; M))$ is isomorphic to $M/(x_1, \ldots, x_l)M$. Moreover, if $x_1, \ldots, x_l$ is an M-sequence, then $H_i(K_\bullet(x_1, \ldots, x_l; M)) = 0$ for all $i \geq 1$.

Proof. We proceed by induction on l. If $l = 1$, then $K_\bullet(x_1; M)$ is the complex

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow 0.$$

Thus, $H_0(K_\bullet(x_1; M)) \cong M/x_1 M$.

For $l > 1$, let $C_\bullet = K_\bullet(x_1, \ldots, x_{l-1}; M)$. Then, by (4.2.1) with $i = 0$ and $x = x_l$, we have

$$0 \longrightarrow \frac{H_0(C_\bullet)}{x_l H_0(C_\bullet)} \longrightarrow H_0(C_\bullet \otimes K_\bullet(x_l; R)) \longrightarrow 0 \longrightarrow 0.$$

By induction hypothesis we have

$$\begin{aligned} H_0(K_\bullet(x_1, \ldots, x_l; M)) &\cong \frac{H_0(C_\bullet)}{x_l H_0(C_\bullet)} \cong (M/(x_1, \ldots, x_{l-1})M)/(x_l(M/(x_1, \ldots, x_{l-1})M)) \\ &\cong M/(x_1, \ldots, x_l)M. \end{aligned}$$

For the second claim we proceed again by induction on l. If $l = 1$, then x_1 is an M-sequence if and only if the complex

$$0 \longrightarrow M \xrightarrow{x_1} R \longrightarrow 0$$

is exact. This happens if and only if $H_1(K_\bullet(x_1; M)) = 0$. For $l > 1$, let $C_\bullet = K_\bullet(x_1, \ldots, x_{l-1}; M)$. Then by induction hypothesis, $H_i(C_\bullet) = 0$ for all $i \geq 1$. Thus $H_i(C_\bullet \otimes K_\bullet(x_l; R)) = 0$ for all $i \geq 2$, by (4.2.1).

Since x_l is a non-zero divisor on $M/(x_1, \ldots, x_{l-1}) \cong H_0(C_\bullet)$, we have $\operatorname{Ann}_{H_0(C_\bullet)}(x_l) = 0$. Then by induction hypothesis and (4.2.1) with $i = 1$, we have

$$0 \longrightarrow 0 \longrightarrow H_i(C_\bullet \otimes K_\bullet(x_l; R)) \longrightarrow 0 \longrightarrow 0$$

Therefore, $H_i(C_\bullet \otimes K_\bullet(x_l; R)) \cong H_i(K_\bullet(x_1, \ldots, x_l; M)) = 0$. ∎

Proposition 4.4. The Koszul complex $K_\bullet = K_\bullet(x_1, \ldots, x_n; S)$ is a minimal free resolution of $\mathbb{K} = S/\mathfrak{m}$.

Proof. Since $x_1, \ldots, x_n$ is an S-sequence. By Theorem 4.3, we conclude that $K_\bullet$ is a free resolution of $\mathbb{K}$. By definition of d_i on $K_\bullet$, $d_i(K_i) \subseteq \mathrm{m}K_{i-1}$, i.e., $K_\bullet$ is a minimal free resolution. ∎

Remark 4.5. *The above complex looks as*

$$0 \longrightarrow S^{\binom{n}{n}} \longrightarrow S^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow S^{\binom{n}{1}} \longrightarrow S \longrightarrow S/\mathfrak{m} \longrightarrow 0.$$

Let V the subspace of degree one forms of S. This implies that $\wedge^d V \cong \mathbb{K}^{\binom{n}{d}} \otimes_{\mathbb{K}} S = S^{\binom{n}{d}}$. Then we write $K_\bullet$ as

$$0 \longrightarrow \wedge^n V \longrightarrow \cdots \longrightarrow \wedge^1 V \longrightarrow \wedge^0 V \longrightarrow 0,$$

4.2 Simplicial homology

Let Δ be a simplicial complex on X. Let $F_i(\Delta)$ denote the set of i-faces and let $\mathbb{K}^{F_i(\Delta)}$ be the free $\mathbb{K}$-vector space on $F_i(\Delta)$.

Definition 4.6. The (**augmented** or **reduced**) **chain complex** of Δ over $\mathbb{K}$ is the complex $\widetilde{C_\bullet}(\Delta; \mathbb{K})$:

$$0 \longleftarrow \mathbb{K}^{F_{-1}(\Delta)} \xleftarrow{\delta_0} \cdots \longleftarrow \mathbb{K}^{F_{i-1}(\Delta)} \xleftarrow{\delta_i} \mathbb{K}^{F_i(\Delta)} \longleftarrow \cdots \xleftarrow{\delta_{n-1}} \mathbb{K}^{F_{n-1}(\Delta)} \longleftarrow 0$$

where the **boundary maps** δ_i are defined by setting $\operatorname{sign}(j, \sigma) = (-1)^{r-1}$ if x_j is the r-th element of the set $\sigma \subseteq X$, written in increasing order, and

$$\delta_i(e_\sigma) = \sum_{x_j \in \sigma} \operatorname{sign}(j, \sigma) e_{\sigma - \{x_j\}}.$$

For all $i \in \mathbb{Z}$, the i-**th (reduced) homology** of Δ over $\mathbb{K}$ is the $\mathbb{K}$-vector space

$$\widetilde{H}_i(\Delta; \mathbb{K}) = \ker \delta_i / \operatorname{im} \delta_{i+1}.$$

We write $(-)^*$ for vector space duality $\operatorname{Hom}_{\mathbb{K}}(-, \mathbb{K})$.

Definition 4.7. The (**reduced**) **cochain complex** of Δ over $\mathbb{K}$ is the vector space dual $\widetilde{C^\bullet}(\Delta; \mathbb{K}) = (\widetilde{C_\bullet}(\Delta; \mathbb{K}))^*$ of the chain complex, with **coboundary maps** $\delta^i = \delta_i^*$. For all $i \in \mathbb{Z}$, the i-**th (reduced) cohomology** of Δ over $\mathbb{K}$ is the $\mathbb{K}$-vector space

$$\widetilde{H}^i(\Delta; \mathbb{K}) = \ker \delta^{i+1} / \operatorname{im} \delta^i.$$

Although we do not study cohomology, it is in general useful in order to have other way to study homology, as the following two results show.

Theorem 4.8 (cf. [Mun84, Theorem 53.5])**.** Let Δ be a simplicial complex and let $\mathbb{K}$ be a field. Then $\dim_{\mathbb{K}} \widetilde{H}_i(\Delta; \mathbb{K}) = \dim_{\mathbb{K}} \widetilde{H}^i(\Delta; \mathbb{K})$.

Theorem 4.9 (Alexander duality (cf. [Mun84, Theorem 71.1]))**.** Let Δ be a simplicial complex on l vertices. Then $\widetilde{H}_i(\Delta^\vee; \mathbb{K}) \cong \widetilde{H}^{l-i-3}(\Delta; \mathbb{K})$.

Notation 4.10. Now, we establish a bijection between the elements of $\mathbb{N}^n$ and the subsets of X. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. Then its correspondent subset on X is defined by $F_{\mathbf{a}} \coloneqq \{x_i \in X : a_i \neq 0\}$. Analogously, if $F \subseteq X$. Then its correspondent n-tuple is defined by $V_F \coloneqq (f_1, \ldots, f_n) \in \mathbb{N}^n$, in which $f_i = 1$ if $x_i \in F$ or $f_i = 0$ if $x_i \notin F$. For the rest of the text, we abuse notation by writing $\mathbf{a}$ instead of $F_{\mathbf{a}}$ and F instead of V_F.

Definition 4.11. The **upper Koszul simplicial complex** of a monomial ideal I in multi-degree $\mathbf{b} \in \mathbb{N}^n$ is defined by
$$K^{\mathbf{b}}(I) = \{F \subseteq X : \mathbf{x}^{\mathbf{b}-F} \in I\}.$$

Theorem 4.12. Let $\mathbf{b} \in \mathbb{N}^n$ and let I be a monomial ideal. Then
$$\beta_{i,\mathbf{b}}(I) = \beta_{i+1,\mathbf{b}}(S/I) = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(K^{\mathbf{b}}(I);\mathbb{K}).$$

Proof. For the first equality let
$$F_\bullet : \qquad 0 \longrightarrow F_l \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$
be a minimal free resolution of I. Then, we generate a minimal free resolution for S/I from $F_\bullet$:
$$0 \longrightarrow F_l \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$
Hence, $\beta_{i,\mathbf{b}}(I) = \beta_{i+1,\mathbf{b}}(S/I)$ for all $i \geq 0$.

Now, we prove the second equality. Let $K_\bullet = K_\bullet(x_1,\dots,x_n;S)$ be the Koszul complex of S. Then by Proposition 1.17 and Theorem 1.14
$$\beta_{i,\mathbf{b}}(I) = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(I,\mathbb{K})_{\mathbf{b}} = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(\mathbb{K},I)_{\mathbf{b}} \tag{4.12.1}$$

By Remark 4.5 we have
$$I \otimes K_\bullet : \quad 0 \longrightarrow I \otimes \wedge^n V \longrightarrow \cdots \longrightarrow I \otimes \wedge^1 V \longrightarrow I \otimes \wedge^0 V \longrightarrow 0.$$
Then, $I \otimes \wedge^i V = \operatorname{span}\{m \otimes (x_{j_1} \wedge \cdots \wedge x_{j_i}) : m \in \operatorname{gens}(I)\}$. If $x \in V$, then
$$x(m \otimes (x_{j_1} \wedge \cdots \wedge x_{j_i})) = xm \otimes (x_{j_1} \wedge \cdots \wedge x_{j_i}) = m \otimes (x \wedge x_{j_1} \wedge \cdots \wedge x_{j_i}).$$

Let $\mathbf{b} \in \mathbb{N}^n$ and observe that
$$\mathbf{x}^{\mathbf{b}} \cong \mathbf{x}^{\mathbf{b}-F} \otimes (x_{j_1} \wedge \cdots \wedge x_{j_i}),$$
where $F = \{x_{j_1},\dots,x_{j_i}\}$.

Now from the previous observations, $(I \otimes \wedge^i V)_{\mathbf{b}}$ has a basis consisting of all expressions of the form
$$\mathbf{x}^{\mathbf{b}-F} \otimes (x_{j_1} \wedge \cdots \wedge x_{j_i})$$
if and only if $\mathbf{x}^{\mathbf{b}-F} \in I$, where $F = \{x_{j_1},\dots,x_{j_i}\}$. These expressions are on bijection with the $(i-1)$-faces F of $K^{\mathbf{b}}(I)$. This, one recognizes $(I \otimes K_\bullet)_{\mathbf{b}}$ as the augmented chain complex used to compute $\widetilde{H}_{i-1}(K^{\mathbf{b}}(I);\mathbb{K})$. Therefore by (4.12.1), $\beta_{i,\mathbf{b}}(I) = \dim_{\mathbb{K}} \operatorname{Tor}_i^S(I,\mathbb{K})_{\mathbf{b}} = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(K^{\mathbf{b}}(I);\mathbb{K})$. ∎

Furthermore, by Theorem 4.12 we have the following equality
$$\beta_{i,j}(I) = \sum_{|\mathbf{a}|=j} \beta_{i,\mathbf{a}}(I) = \sum_{|\mathbf{a}|=j} \beta_{i+1,\mathbf{a}}(S/I) = \beta_{i+1,j}(S/I). \tag{4.12.2}$$

Definition 4.13. The **link** of F inside the simplicial complex Δ is defined by
$$\operatorname{link}_\Delta(F) = \{G \in \Delta : G \cup F \in \Delta \text{ and } G \cap F = \varnothing\}.$$

Example 4.14. Let Δ be the simplicial complex from Example 2.3. Then,
$$\operatorname{link}_\Delta(\{x_2,x_4\}) = \{\{x_1\},\{x_3\}\} \quad \text{and} \quad \operatorname{link}_\Delta(\{x_5\}) = \{\{x_1\},\{x_2,x_3\}\}. \qquad \diamondsuit$$

Proposition 4.15. Let Δ be a simplicial complex and $Y \subseteq X$. Then,

(a) $\Delta^\vee = K^{\mathbf{1}}(I_\Delta)$;

(b) $K^Y(I_\Delta) = \operatorname{link}_{K^{\mathbf{1}}(I_\Delta)}(Y^c)$;

(c) $K^Y(I_\Delta) = (\Delta[Y])^\vee$.

Proof.

(a) By definition,

$$F \in K^{\mathbf{1}}(I_\Delta) \iff \mathbf{x}^{\mathbf{1}-F} \in I_\Delta \iff \mathbf{1} - F = F^c \notin \Delta \iff F \in \Delta^\vee.$$

(b) We use part (a) to prove the equality. By definition,

$$\begin{aligned} F \in K^Y(I_\Delta) &\iff \mathbf{x}^{Y-F} \in I_\Delta \\ &\iff Y - F \notin \Delta \text{ and } F \subsetneq Y \\ &\iff Y \cap F^c = Y - F \notin \Delta \text{ and } F \subsetneq Y \\ &\iff Y^c \cup F \in \Delta^\vee \text{ and } F \cap Y^c = \varnothing \\ &\iff F \in \operatorname{link}_{\Delta^\vee}(Y^c). \end{aligned}$$

(c) By definition,

$$F \in K^Y(I_\Delta) \iff \mathbf{x}^{Y-F} \in I_\Delta \iff F \subsetneq Y \text{ and } Y - F \notin \Delta \iff F \in (\Delta[Y])^\vee. \qquad \blacksquare$$

Therefore if $\mathbf{b} \in \{0,1\}^n$ by Proposition 4.15, we conclude that $\operatorname{link}_{\Delta^\vee}(\mathbf{b}^c) = K^{\mathbf{b}}(I_\Delta) = (\Delta[\mathbf{b}])^\vee$.

The next result is called the "dual version" of Hochster's formula because it gives Betti numbers of I_Δ by working with the Alexander dual complex $\Delta^\vee$, and because it is dual to Hochster's original formulation (Theorem 4.17).

Theorem 4.16 (Hochster's formula, dual form)**.**Let Δ be a simplicial complex and let $\mathbf{b} \in \{0,1\}^n$. Then,

$$\beta_{i,\mathbf{b}}(I_\Delta) = \beta_{i+1,\mathbf{b}}(S/I_\Delta) = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(\operatorname{link}_{\Delta^\vee}(\mathbf{b}^c);\mathbb{K}).$$

Proof. By Theorem 4.12 and Proposition 4.15 we have

$$\beta_{i,\mathbf{b}}(I_\Delta) = \beta_{i+1,\mathbf{b}}(S/I_\Delta) = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(K^{\mathbf{b}}(I_\Delta);\mathbb{K}) = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(\operatorname{link}_{\Delta^\vee}(\mathbf{b}^c);\mathbb{K}). \qquad \blacksquare$$

An immediate consequence of dual form of Hochster's formula is the Hochster's original formulation.

Theorem 4.17 (Hochster's formula)**.** Let Δ be a simplicial complex and let $\mathbf{b} \in \{0,1\}^n$. Then,

$$\beta_{i-1,\mathbf{b}}(I_\Delta) = \beta_{i,\mathbf{b}}(S/I_\Delta) = \dim_{\mathbb{K}} \widetilde{H}^{|\mathbf{b}|-i-1}(\Delta[\mathbf{b}];\mathbb{K}).$$

Proof. By the dual form of Hochster's formula, by Theorem 4.9 and Proposition 4.15

$$\begin{aligned} \beta_{i-1,\mathbf{b}}(I_\Delta) = \beta_{i,\mathbf{b}}(S/I_\Delta) &= \dim_{\mathbb{K}} \widetilde{H}_{i-2}(\operatorname{link}_{\Delta^\vee}(\mathbf{b}^c);\mathbb{K}) \\ &= \dim_{\mathbb{K}} \widetilde{H}_{i-2}((\Delta[\mathbf{b}])^\vee;\mathbb{K}) \\ &= \dim_{\mathbb{K}} \widetilde{H}^{|\mathbf{b}|-i+2-3}(\Delta[\mathbf{b}];\mathbb{K}) \\ &= \dim_{\mathbb{K}} \widetilde{H}^{|\mathbf{b}|-i-1}(\Delta[\mathbf{b}];\mathbb{K}). \end{aligned} \qquad \blacksquare$$

Chapter 5

Terai's Theorem

The goal of this chapter is prove the Terai's Theorem (Theorem 5.8) which relate the regularity of I_Δ with the projective dimension of $S/I_{\Delta^\vee}$. For this work it is necessary to study more extensively the Betti numbers. We also need to relate them with the simplicial homology and cohomology.

5.1 The Betti polynomial

Before we prove Terai's Theorem is necessary study more the Betti numbers and its properties.

Theorem 5.1 (Hochster's formula on the Betti numbers)**.** Let Δ be a simplicial complex. Then,

$$\beta_{i,j}(S/I_\Delta) = \sum_{\substack{F \subset X, \\ |F|=j}} \dim_{\mathbb{K}} \widetilde{H}_{j-i-1}(\Delta[F]; \mathbb{K}).$$

Proof. Fix j and let $F \subseteq X$ such that $|F| = j$. By the Hochster's formula

$$\beta_{i,F}(S/I_\Delta) = \dim_{\mathbb{K}} \widetilde{H}^{|F|-i-1}(\Delta[F]; \mathbb{K}) = \dim_{\mathbb{K}} \widetilde{H}_{|F|-i-1}(\Delta[F]; \mathbb{K}).$$

Then by definition,

$$\beta_{i,j}(S/I_\Delta) = \sum_{\substack{F \subset X, \\ |F|=j}} \beta_{i,F}(S/I_\Delta) = \sum_{\substack{F \subset X, \\ |F|=j}} \dim_{\mathbb{K}} \widetilde{H}_{j-i-1}(\Delta[F]; \mathbb{K}).$$

■

Definition 5.2. Let Δ be a simplicial complex. The **Betti polynomial** of S/I_Δ is defined by

$$T_i(S/I_\Delta, t) := \sum_{\mathbf{a} \in \mathbb{N}^n} \dim_{\mathbb{K}} \operatorname{Tor}_i^S(S/I_\Delta; \mathbb{K})_{\mathbf{a}} \mathbf{t}^{\mathbf{a}}.$$

Hochster gave the following formula for these Betti polynomials.

Theorem 5.3. Let Δ be a simplicial complex. Then

$$T_i(S/I_\Delta, t) = \sum_{F \subseteq X} \dim_{\mathbb{K}} \widetilde{H}_{|F|-i-1}(\Delta[F]; \mathbb{K}) \mathbf{t}^F.$$

Proof. By Proposition 1.17 and the proof of Hochster's formula

$$T_i(S/I_\Delta, t) = \sum_{F \subseteq X} \beta_{i,F}(S/I_\Delta) \mathbf{t}^F = \sum_{F \subseteq X} \dim_{\mathbb{K}} \widetilde{H}_{|F|-i-1}(\Delta[F]; \mathbb{K}) \mathbf{t}^F.$$

■

Proposition 5.4. Let Δ be a simplicial complex and $i \geq 1$. Then,

$$T_i(S/I_\Delta, t) = \sum_{F \in \Delta^\vee} \dim_{\mathbb{K}} \widetilde{H}_{i-2}(\mathrm{link}_{\Delta^\vee}(F); \mathbb{K}) \mathbf{t}^{X-F}.$$

Proof. If $G \in \Delta$ then $\Delta[G] = \{G\}$ and hence has no reduced homology. Therefore we only need to consider $G \subsetneq X$ such that $G \notin \Delta$. Let $F = G^c \in \Delta^\vee$. By the proof of Hochster's formula we know that $\dim_{\mathbb{K}} \widetilde{H}_{i-2}(\mathrm{link}_{\Delta^\vee}(F); \mathbb{K}) = \dim_{\mathbb{K}} \widetilde{H}_{|G|-i-1}(\Delta[G]; \mathbb{K})$. Then by Theorem 5.3,

$$T_i(S/I_\Delta, t) = \sum_{F \in \Delta^\vee} \dim_{\mathbb{K}} \widetilde{H}_{i-2}(\mathrm{link}_{\Delta^\vee}(F); \mathbb{K}) \mathbf{t}^{X-F}.$$

∎

5.2 Regularity

Theorem 5.5 (Hochster's formula on the local cohomology modules (cf. [Sta96, Theorem 4.1])). Let Δ be a simplicial complex, then

$$\mathrm{HS}(H^i_{\mathfrak{m}}(S/I_\Delta), t) = \sum_{F \in \Delta} \dim_{\mathbb{K}} \widetilde{H}_{i-|F|-1}(\mathrm{link}_\Delta(F); \mathbb{K}) \left(\frac{t^{-1}}{1-t^{-1}} \right)^{|F|},$$

where $H^i_{\mathfrak{m}}(S/I_\Delta)$ denote the i-th local cohomology module of S/I_Δ with respect to the homogeneous maximal ideal $\mathfrak{m}$.

We need to study some basic properties about regularity and initial degree, which are used to prove Theorem 5.7.

Lemma 5.6. Let I be a square-free monomial ideal. Then, we have the following properties:

(a) $\mathrm{indeg}(I) = \min\{\deg(m) : m \in \mathrm{gens}(I)\}$;

(b) $\mathrm{reg}(I) \geq \deg(m)$ for all $m \in \mathrm{gens}(I)$;

(c) $\mathrm{reg}(I) \geq \mathrm{indeg}(I)$;

(d) $\mathrm{reg}(I) \leq n$;

(e) $\mathrm{reg}(S) = 0$;

(f) If m is a square-free monomial of degree d, then $\mathrm{reg}(\langle m \rangle) = d$;

(g) $\mathrm{reg}(S/I) = \mathrm{reg}(I) - 1$.

Proof.

(a) Set $l_1 = \min\{\deg(m) : m \in \mathrm{gens}(I)\}$ and $l_2 = \max\{\deg(m) : m \in \mathrm{gens}(I)\}$. Then, we have the minimal free resolution

$$\cdots \longrightarrow S(-l_1)^{\beta_{0,l_1}} \oplus \cdots \oplus S(-l_2)^{\beta_{0,l_2}} \longrightarrow I \longrightarrow 0,$$

this implies that $\mathrm{indeg}(I) = l_1$.

(b) We have that $\beta_{0,\deg(m)} \neq 0$ for all $m \in \mathrm{gens}(I)$. Then by definition $\mathrm{reg}(I) \geq \deg(m)$.

(c) Since $\mathrm{reg}(I) \geq \deg(m)$ for all $m \in \mathrm{gens}(I)$. Then, $\mathrm{reg}(I) \geq \mathrm{indeg}(I)$.

(d) Since $\beta_{i,j}(I) = 0$ for all $i > n$ and $\beta_{i,j}(I) = 0$ for all $j > n$, Remark 1.20. Then, we only need to consider the Betti numbers $\beta_{i,j}(I)$ with $i, j \in \{0, \dots, n\}$. Hence, $\operatorname{reg}(I) \le n$.

(e) Observe that

$$0 \longrightarrow S \longrightarrow S \longrightarrow 0,$$

is a minimal free resolution of S. Then, $\operatorname{reg}(S) = 0$.

(f) Since

$$0 \longrightarrow S(-d) \xrightarrow{m} \langle m \rangle \longrightarrow 0,$$

is a minimal free resolution of $\langle m \rangle$. Hence, $\operatorname{reg}(\langle m \rangle) = d$.

(g) By (4.12.2), $\beta_{i,j}(I) = \beta_{i+1,j}(S/I)$ for all $i \ge 0$. Then by definition, $\operatorname{reg}(I) = \operatorname{reg}(S/I) + 1$. ■

In the rest of this chapter, we always assume $\dim(S/I_\Delta) = d$ and $\dim(S/I_{\Delta^\vee}) = d^*$.

Theorem 5.7 (Terai)**.** Let Δ be a $(d-1)$-dimensional simplicial complex on the vertex set X. If $d \le n-2$, then

$$\operatorname{reg}(I_\Delta) - \operatorname{indeg}(I_\Delta) = \dim(S/I_{\Delta^\vee}) - \operatorname{depth}_{\mathfrak{m}}(S/I_{\Delta^\vee}).$$

Proof. Since $d \le n-2$, then $\dim \Delta \le n-3$. Thus by Lemma 2.15, $\Delta^\vee$ is a simplicial complex on the vertex set X. This implies that I_Δ and $I_{\Delta^\vee}$ are ideals on S.

Let $\operatorname{depth}_{\mathfrak{m}}(S/I_{\Delta^\vee}) = \delta^*$. By Hochster's formula on the local cohomology modules, we have

$$\operatorname{HS}(H^i_{\mathfrak{m}}(S/I_{\Delta^\vee}), t) = \sum_{F \in \Delta^\vee} \dim_{\mathbb{K}} \widetilde{H}_{i-|F|-1}(\operatorname{link}_{\Delta^\vee}(F); \mathbb{K}) \left(\frac{t^{-1}}{1-t^{-1}} \right)^{|F|}.$$

If $l < \delta^*$, then $H^l_{\mathfrak{m}}(S/I_{\Delta^\vee}) = 0$. This implies that $\widetilde{H}_{l-|F|-1}(\operatorname{link}_{\Delta^\vee}(F); \mathbb{K}) = 0$ for all $F \in \Delta^\vee$.

Let $F \subseteq X$. If $F \in \Delta$, then $\Delta[F] = \{F\}$. Hence, there is no reduced homology. Suppose that, $F \notin \Delta$, then $F^c \in \Delta^\vee$. Hence,

$$\widetilde{H}_{n-l-2}(\Delta[F]; \mathbb{K}) \cong \widetilde{H}_{l-|F^c|-1}(\operatorname{link}_{\Delta^\vee}(F^c); \mathbb{K}) = 0.$$

Therefore, $\widetilde{H}_{n-l-2}(\Delta[F]; \mathbb{K}) = 0$ for all $F \subseteq X$. Then, by Hochster's formula on the Betti numbers

$$\beta_{i,i+n-l-1}(S/I_\Delta) = \sum_{|F|=i+n-l-1} \dim_{\mathbb{K}} \widetilde{H}_{n-l-2}(\Delta[F]; \mathbb{K}) = 0$$

for all $i \ge 1$ and $0 \le l \le \delta^* - 1$. Thus, by (4.12.2) we have

$$\beta_{i,i+n}(I_\Delta) = \beta_{i,i+n-1}(I_\Delta) = \cdots = \beta_{i,i+n-\delta^*+1}(I_\Delta) = 0 \tag{5.7.1}$$

for all $i \ge 0$. Similarly, since $H^{\delta^*}_{\mathfrak{m}}(S/I_{\Delta^\vee}) \ne 0$ we deduce that

$$\widetilde{H}_{n-\delta^*-2}(\Delta[F^c]; \mathbb{K}) \cong \widetilde{H}_{\delta^*-|F|-1}(\operatorname{link}_{\Delta^\vee}(F); \mathbb{K}) \ne 0$$

for some $F \in \Delta$. This implies,

$$\beta_{i,i+n-\delta^*}(I_\Delta) = \beta_{i+1,i+n-\delta^*}(S/I_\Delta) = \sum_{\substack{G \subseteq X, \\ |G| = i+n-\delta^*}} \dim_{\mathbb{K}} \widetilde{H}_{n-\delta^*-2}(\Delta[G]; \mathbb{K}) \ne 0 \tag{5.7.2}$$

for some $i \ge 0$. Hence, by Remark 1.20, (5.7.1) and (5.7.2) we conclude that $\operatorname{reg}(I_\Delta) = n - \delta^*$.

Set $l_1 = \min\{\deg(m) : m \in \mathrm{gens}(I_\Delta)\}$. Since $I_{\Delta^\vee} = \bigcap_{m\in\mathrm{gens}(I_\Delta)} P_m$, we obtain that

$$\begin{aligned}\dim(S/I_{\Delta^\vee}) &= \max\{\dim(S/P_m) : m \in \mathrm{gens}(I_\Delta)\}\\ &= \max\{n - \deg(m) : m \in \mathrm{gens}(I_\Delta)\}\\ &= n - l_1.\end{aligned}$$

Hence, $\mathrm{indeg}(I_\Delta) = n - d^* = l_1$. Therefore,

$$\mathrm{reg}(I_\Delta) - \mathrm{indeg}(I_\Delta) = d^* - \delta^*.$$

■

Theorem 5.8 (Terai's Theorem)**.** Let Δ be a $(d-1)$-dimensional simplicial complex on the vertex set X. Suppose $d \leq n-2$. Then,

$$\mathrm{reg}(I_\Delta) = \mathrm{pd}(S/I_{\Delta^\vee}).$$

Proof. By the Auslander-Buchsbaum Formula

$$\mathrm{pd}(S/I_{\Delta^\vee}) = \mathrm{depth}_{\mathfrak{m}}(S) - \mathrm{depth}_{\mathfrak{m}}(S/I_{\Delta^\vee}).$$

Furthermore $\mathrm{depth}_{\mathfrak{m}}(S) = \dim(S)$, because S is Cohen-Macaulay. By Theorem 5.7 we have that

$$\begin{aligned}\mathrm{reg}(I_\Delta) &= \dim(S/I_{\Delta^\vee}) - \mathrm{depth}_{\mathfrak{m}}(S/I_{\Delta^\vee}) + \mathrm{indeg}(I_\Delta)\\ &= \dim(S) - \mathrm{depth}_{\mathfrak{m}}(S/I_{\Delta^\vee}).\end{aligned}$$

Therefore, $\mathrm{reg}(I_\Delta) = \mathrm{pd}(S/I_{\Delta^\vee})$. ■

Definition 5.9. Let M be a finitely generated graded S-module. We say that M has a q-linear resolution, if M is generated by homogeneous elements of degree q and $\mathrm{reg}(M) = q$.

Proposition 5.10. Let I be a square-free monomial ideal generated by square-free monomials of degree q. Then, I has a q-linear resolution if and only if $\beta_{i,j}(I) = 0$ for all $j \neq i + q$ with $i \geq 0$.

Proof. Suppose that $\mathrm{reg}(I) = q$. Let

$$0 \longrightarrow F_l \xrightarrow{d_l} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} I \longrightarrow 0,$$

be a minimal free resolution of I. Since I is generated by square-free monomials of degree q we have that $F_0 = S(-q)^{\beta_{0,q}}$. We proceed by induction on l to prove that $\beta_{l,j}(I) = 0$ if $j < l + q$. Let $l = 1$ and suppose that $F_1 = S(-b_1) \oplus \cdots \oplus S(-b_r)$ and $F_0 = S(-c_1) \oplus \cdots \oplus S(-c_s)$ with $c_j = q$. Since d_1 is a graded homomorphism of degree zero, then d_1 is a matrix of size $s \times r$ where the non-zero uv-entry $a_{u,v} \in \mathfrak{m}$ is homogeneous of degree $b_v - c_u$, i.e., $\deg a_{u,v} = b_v - c_u \geq 1$. This implies that $b_v \geq c_u + 1 = q + 1$.

Suppose that it holds for $l > 1$ and consider the following graded homomorphism of degree zero $d_l : F_l \longrightarrow F_{l-1}$. By the induction hypothesis we have $F_{l-1} = \oplus_{j\in\mathbb{Z}_{\geq q+l-1}} S(-j)^{\beta_{l-1,j}}$. If $F_l = S(-b_1) \oplus \cdots \oplus S(-b_r)$ and $F_{l-1} = S(-c_1) \oplus \cdots \oplus S(-c_s)$. Then by the previous arguments, $b_v \geq c_u + 1 = q + l$. This implies that $F_l = \oplus_{j\in\mathbb{Z}_{\geq q+l}} S(-j)^{\beta_{l,j}}$. Now, suppose exists $\beta_{i,j}(I) \neq 0$ for some $j \neq i + q$. If $j < q + i$ then $\beta_{i,j}(I) = 0$ by the previous observations. If $j > q + i$, then $\mathrm{reg}(I) \geq j - i > q$ which is a contradiction. Therefore, $\beta_{i,j}(I) = 0$ for all $j \neq q + i$.

Conversely, suppose that $\beta_{i,j}(I) = 0$ for all $j \neq q + i$. Then I has the following minimal resolution

$$0 \longrightarrow S(-q-l)^{\beta_{l,q+l}} \xrightarrow{d_l} \cdots \longrightarrow S(-q-1)^{\beta_{1,q+1}} \xrightarrow{d_1} S(-q)^{\beta_{0,q}} \xrightarrow{d_0} I \longrightarrow 0.$$

Hence by definition, $\mathrm{reg}(I) = q$. ■

Corollary 5.11. Let Δ be a $(d-1)$-dimensional simplicial complex on the vertex set X. Suppose $d \leq n-2$. Then, I_Δ has a q-linear resolution if and only if $S/I_{\Delta^\vee}$ is Cohen-Macaulay of dimension $n-q$.

Proof. Suppose that I_Δ has a q-linear resolution. This implies that $\operatorname{reg}(I_\Delta) = \operatorname{indeg}(I_\Delta) = q$. By Auslander-Buchsbaum Formula and Terai's Theorem we have that

$$\begin{aligned}
\operatorname{depth}_{\mathfrak{m}}(S/I_{\Delta^\vee}) &= \operatorname{depth}_{\mathfrak{m}}(S) - \operatorname{pd}(S/I_{\Delta^\vee}) \\
&= n - \operatorname{pd}(S/I_{\Delta^\vee}) \\
&= n - \operatorname{reg}(I_\Delta) \\
&= n - q \\
&= \dim(S/I_{\Delta^\vee}).
\end{aligned}$$

Hence, $S/I_{\Delta^\vee}$ is Cohen-Macaulay of dimension $n-q$.

Conversaly, suppose that $S/I_{\Delta^\vee}$ is Cohen-Macaulay. Then by Auslander-Buchsbaum Formula and Terai's Theorem we have that

$$\operatorname{reg}(I_\Delta) = \operatorname{pd}(S/I_{\Delta^\vee}) = n - \operatorname{depth}_m(S/I_{\Delta^\vee}) = n - d^* = \operatorname{indeg}(I_\Delta).$$

Thus I_Δ is generated by square-free monomials of degree $n-d^*$, because $\operatorname{indeg}(I_\Delta) \leq \deg(m) \leq \operatorname{reg}(I_\Delta)$ for all $m \in \operatorname{gens}(I_\Delta)$. Therefore, I_Δ has a $(n-d^*)$-linear resolution. ∎

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