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SOLUTIONS OF THE YAMABE EQUATION BY LYAPUNOV-SCHMIDT REDUCTION

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Jorge Dávila Ortiz

Contents

Abstract

Given any closed Riemannian manifold (M, g) we use the Lyapunov-Schmidt finite dimensional reduction method to prove multiplicity results for positive solutions of a subcritical Yamabe type equation on (M, g). If (N, h) is a closed Riemannian manifold of constant positive scalar curvature, our result gives multiplicity of solutions for the Yamabe equation on the Riemannian product $(M \times N, g + \varepsilon^2 h)$, for $\varepsilon > 0$ small.

$$-a_{m+n}\Delta_{g+\varepsilon^2 h}u + \left(S_g + \varepsilon^{-2}S_h\right)u = u^{p_{m+n}-1}. \tag{0.0.1}$$

We restrict our study of solutions to functions that only depend on M, $u : M \to \mathbb{R}$. Normalizing h, such that $S_h = a_{m+n}$, we have that u solves the Yamabe equation if and only if

$$-\varepsilon^2\Delta_g u + \left(\frac{S_g}{a_{m+n}}\varepsilon^2 + 1\right)u = u^{p_{m+n}-1}. \tag{0.0.2}$$

We define a functional J_ε on a space of Sobolev H_ε, such that its critical points, are positive solutions to our equation 0.0.2. The method of Lyapunov-Schmidt will allow us to reduce our poblem to one of finite dimension. Also in this process we will obtain a function function C^2, $W : M \to H_\varepsilon$. We conclude our problem, proving that the critical points of the function $F_\varepsilon : M \to \mathbb{R}$, defined by $F_\varepsilon = J_\varepsilon \circ W$, induces critical points to the functional J_ε, that is, solutions to the equation 0.0.2.

We have that the critical points of the map F_ε give solutions to 0.0.2. This allows to apply classical results of Lusternik-Schnirelmann category of M . We obtain the following result.

There is $\varepsilon_o > 0$ such that for $\varepsilon \in (0, \varepsilon_o)$ the Yamabe equation on the Riemannian product $(M \times N, g + \varepsilon^2 h)$ has at least $Cat(M)$ (Lusternick-Schnirelmann category) solutions which depend only on M.

Applying Morse theory, we have that if $\varepsilon \in (0, \varepsilon_o)$ all the critical points of the function $F_\varepsilon : M \to \mathbb{R}$ are non-degenerate, then the Yamabe equation on the Riemannian product $(M \times N, g + \varepsilon^2 h)$ has at least $b(M)$ solutions which depend only on M, where $b_i(M) \doteq \dim(H_i(M, \mathbb{R}))$ and $b(M) \doteq b_1(M) + \cdots + b_n(M)$.

1 Introduction

How long will it take me
solve my problem?

Not a minute more of what you
need to understanding this, said the
master.

In [37] H. Yamabe considered the following question: Let (M, g) be a closed Riemannian manifold of dimension $n \geq 3$. Is there a metric h which is conformal to g and has constant scalar curvature? If we express the conformal metric h as $h = u^{\frac{4}{n-2}} g$ for a positive function u, the scalar curvature S_h of h is related to the scalar curvature of g by

$$-a_n \Delta_g u + S_g u = S_h u^{p_n - 1},$$

where Δ_g is the Laplacian operator associated with the metric g, $a_n = \dfrac{4(n-1)}{(n-2)}$ and $p_n = \dfrac{2n}{n-2}$. It follows that the metric h has constant scalar curvature $\lambda \in \mathbb{R}$ if and only if u is a positive solution of the *Yamabe equation*:

$$-a_n \Delta_g u + S_g u = \lambda u^{p_n - 1}. \qquad \textbf{The Yamabe Equation} \tag{1.0.1}$$

It is easy to check that Eq. 1.0.1 is the Euler-Lagrange equation of the *Yamabe functional*, Y_g, defined by:

$$Y_g(u) = \frac{\int_M \left(a_n |\nabla u|^2 + S_g u^2\right) d\mu_g}{\left(\int_M u^{p_n}\, d\mu_g\right)^{\frac{n-2}{n}}} = \frac{\int_M \left(a_n |\nabla u|^2 + S_g u^2\right) d\mu_g}{\|u\|_{p_n}^2}. \tag{1.0.2}$$

If $\mathcal{E}$ denotes the normalized Hilbert-Einstein functional

$$\mathcal{E}(g) = \frac{\int\limits_M S_g d\mu_g}{Vol(M,g)^{\frac{n-2}{n}}},$$

it follows that $Y_g(u) = \mathcal{E}(u^{\frac{4}{n-2}} g)$.

The Yamabe constant of g is defined as the infimum of the Yamabe functional Y_g :

$$Y(M,[g]) = \inf_{u \in H^1(M)-\{0\}} Y_g(u). \qquad \textbf{Yamabe Constant} \qquad (1.0.3)$$

A minimizer for the Yamabe constant is therefore a solution of (1.0.1) and, moreover it follows elliptic theory that it must be strictly positive and smooth. H. Yamabe presented a proof that a minimizer always exists, but his argument contained an error which was pointed out (and fixed under certain conditions) by N. Trüdinger in [35]. Later T. Aubin [2] and R. Schoen [34] completed the proof that for any metric g the infimum of the Yamabe functional is achieved. Therefore there is always at least one (positive) solution to the Yamabe equation (1.0.1). If $Y(M,g) \leq 0$ the solution is unique (up to homothecies). In the case of $Y(M,g) > 0$ uniqueness in general fails. We consider the standard metric g_0^2 in the sphere S^2. We now define the Riemannian metric $g = ag_0^2 + bg_0^2$, with $a, b > 0$, over $\mathbb{S}^2 \times \mathbb{S}^2$. It has positive scalar curvature, moreover g is conformal to the metric $g_{ab} = \frac{a}{b} g_0^2 + g_0^2$ and $S_{g_{ab}} = \frac{2(a+b)}{a}$. Evaluating g_{ab} in the Yamabe functional $\mathcal{E}$ we see that

$$\begin{aligned}
\mathcal{E}(g_{ab}) &= \frac{\int_{\mathbb{S}^2 \times \mathbb{S}^2} \frac{2(a+b)}{a} d\mu_{g_{ab}}}{vol(\mathbb{S}^2 \times \mathbb{S}^2, g_{ab})^{1/2}} \\
&= \frac{2(a+b)}{a} Vol(\mathbb{S}^2 \times \mathbb{S}^2, g_{ab})^{1/2} \\
&= \frac{2(a+b)}{\sqrt{ab}} Vol(\mathbb{S}^2, g_0^2) \\
&= 2\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right) Vol(\mathbb{S}^2, g_0^2).
\end{aligned}$$

Note that $\mathcal{E}(g_{ab}) \to \infty$ if either $\frac{a}{b} \to \infty$ or $\frac{b}{a} \to 0$. On the other hand a fundamental result proved by Aubin [2] is that for any closed Riemanniana manifold (M, g) of dimension n, $Y(M,[g]) \leq Y(\mathbb{S}^n, [g_0^n]$. In particular $Y(\mathbb{S}^2 \times \mathbb{S}^2, [g_{ab}]) \leq Y(\mathbb{S}^4, g_0^4)$, if $\frac{a}{b}$ suffiently small or large.

Therefore, although g_{ab_0} has constant scalar curvature, this metric does not minimize $\mathcal{E}$. However, by we know that there is a minimizing metric, from which it follows that there must be at least two metrics with constant scalar curvature in the class $[g_{ab_0}]$.

Another very importan example of multiplicity of solutions is the sphere (S^n, g_o^n) with the canonical metric g_0^n. The case of the sphere is very special because it has a non-compact family of conformal transformations which induces a noncompact family of solutions to the Yamabe equation. By a result of Obata [27] each metric of constant scalar curvature which is conformal to the round metric on S^n is obtained as the pull-back of the round metric under a conformal diffeomorphism. Therefore, if g_o^n is the round metric over S^n, every solution to (1.0.1) is minimizing.

Another important example was considered by R. Schoen in [33] (and also by O. Kobayashi in [18]), the product metric on $S^{n-1} \times S^1(L)$ (the circle of radius L). R. Schoen prescribed out that all solutions to (1.0.1) are constant along the $(n-1)$-spheres and, therefore, the Yamabe equation reduces to an ordinary differential equation. By a careful analysis of this equation, R. Schoen proved that there are many non-mimizing solutions if L is large.

In general, for the positive case there will be non-minimizing solutions. For instance, D. Pollack proved in [30] that every conformal class with positive Yamabe constant can be C^0-approximated by a conformal class with an arbitrary number of (non-isometric) metrics of constant scalar curvature which are not minimizers. M. Berti and A. Malchiodi proved in [6], that if $k \geq 2$ and $n \geq 4k+3$, then exists a family of metrics g_ε of the clas C^k over $\mathbf{S}^n$, such that g_ε is close to g_0^n in the norm $C^k(\mathbf{S}^n)$, when ε is close to 0, and such that Yamabe equation on $(\mathbf{S}^n, g_\varepsilon)$ has a non-compact family of solutions.

Also, S. Brendle in [4] proved, using blow-up theory, that if $n \geq 52$, then there is a non-conformaly flat metric g in S^n and a non-compact family of class C^∞ solutions, of the Yamabe problem in (S^n, g). S. Brendle and F.C. Marques proved in [5], an extension of this result for the case where $25 \leq n \leq 51$. The previous results show that in general, the solution space is not compact if $n \geq 25$.

Similarly to the case of $S^{n-1} \times S^1(L)$, particular interest arises in the study of products of the form $(M \times N, g + \delta h)$, where the constant $\delta > 0$ goes to 0 (or ∞). The Yamabe constants of such Riemannian products were studied in [1].

In the case $(M, g) = (S^n, g_o^n)$, where g_{0^n} is the canonical metric on the sphere, radial solutions of the resulting subcritical equation have been obtained by Qinian Jin, YanYan Li and Haoyuan Xu in [32]. The authors proved that there is a sequence of positive numbers $\delta_i \to 0$ such that for $\delta < \delta_i$ the Yamabe equation corresponding to g_δ has at least i different solutions, which are radial functions on S^n . The authors obtain this result by showing that the δ_i's are bifurcation instants (local bifurcation) and then using the global bifurcation theory of Rabinowitz to prove that the branches of solutions appearing at these bifurcation instants persist to give solutions for every $\delta < \delta_i$.

J. Petean proved in [28] multiplicity of solutions on Riemannian products $(N \times \mathbb{S}^n, g + g_0^n)$, where the scalar curvature of g over N is constant and the solutions depend only on $\mathbb{S}^n$. In [16] G. Henry and J. Petean. obtained multiplicity of solutions over spheres product $(S^n \times S^m, g_0^n + \delta g_0^m)$ using isoparametric functions. The solutions found are non-radial and also depend only on one of the factors of the product. The authors use local bifurcation theory to proved that there is an increasing sequence of bifurcation points, where branches of non-trivial solutions emerge.

L.L De Lima, P. Piccione and M. Zedda [23], proved using bifurcation theory, multiplicity of solutions on arbitrary products $(M \times N, g + \lambda h)$, where M and N have constant scalar curvature and $\lambda > 0$. The solutions obtained are points of accumulation of solutions to the problem of Yamabe in the product.
All these multiplicity results for the Yamabe equation were obtained in using bifurcation theory and assuming that the scalar curvatures of g and h are constant.

In the present thesis we consider the case of Riemannian products where one of the scalar curvatures is not constant. Let (M^n, g) be any closed Riemannian manifold and (N^m, h) be a Riemannian manifold of constant positive scalar curvature. The function $u : M \to \mathbb{R}_{>0}$ is a solution of the Yamabe equation in $(W, g_\varepsilon) = (M \times N, g + \varepsilon^2 h)$ if it satisfies

$$-a_{n+m}\Delta_g u + \left(s_g + \varepsilon^{-2} S_h\right) u = u^{p_{m+n}-1}.$$

This is of course equivalent to finding solutions of the equation

$$-a_{n+m}\Delta_g u + \left(s_g + \varepsilon^{-2} S_h\right) u = \varepsilon^{-2} S_h u^{p_{m+n}-1}. \tag{1.0.4}$$

Moreover, we can normalize h and assume that $S_h = a_{m+n}$. Then Eq. (1.0.4) is equivalent to:

$$-\varepsilon^2 \Delta_g u + \left(\frac{s_g}{a_{m+n}}\varepsilon^2 + 1\right) u = u^{p_{m+n}-1}. \tag{1.0.5}$$

We will find solutions of (1.0.5) using the Lyapunov-Schmidt reduction technique, which was introduced in [8, 12, 20], for instance. The same technique was also used by Micheletti and Pistoia in [26] to study the sub-critical equation equation $-\varepsilon^2 \Delta_g u + u = u^{p-1}$ on a Riemannian manifold. Here we will use a similar approach. We now give a brief description of this method and state the results we have obtained.

Let $H_\varepsilon(M)$ be the Hilbert space $H_g^1(M)$ equipped with the inner product

$$\langle u, v\rangle_\varepsilon \doteq \frac{1}{\varepsilon^n}\left(\varepsilon^2\int_M \langle\nabla_g u, \nabla_g v\rangle \, d\mu_g + \int_M uv\, d\mu_g\right),$$

and the induced norm

$$\|u\|_\varepsilon^2 \doteq \frac{1}{\varepsilon^n}\left(\varepsilon^2\int_M |\nabla_g u|^2 d\mu_g + \int_M u^2 d\mu_g\right).$$

Consider the functional $J_\varepsilon : H_\varepsilon(M) \to \mathbb{R}$ given by

$$J_\varepsilon(u) = \varepsilon^{-n}\int_M \left(\frac{1}{2}\varepsilon^2\|\nabla u\|^2 + \frac{\mathbf{s}_g\varepsilon^2 + a_{m+n}}{2a_{m+n}}u^2 - \frac{1}{p_{m+n}}(u^+)^{p_{m+n}}\right) d\mu_g.$$

where $u^+ = \max\{u, 0\}$. The critical points of the functional J_ε are the positive solutions of Eq. (1.0.5). Let us consider the map

$$S_\varepsilon \doteq \nabla J_\varepsilon : H_\varepsilon \to H_\varepsilon.$$

The Yamabe equation (1.0.5) is then equivalent to $S_\varepsilon(u) = 0$.

Note that $p_{m+n} < p_n$. From now on we let $q \in (2, p_n)$. There exists a unique (up to translation) positive finite-energy solution U of the equation on $\mathbb{R}^n$

$$-\Delta U + U = U^{q-1}.$$

The function U is radial (around some fixed point). We also consider the linear equation

$$-\Delta\psi + \psi = (q-1)U^{q-2}\psi \quad \text{in } \mathbb{R}^n.$$

It is well known that all solutions of the above equation are the directional derivatives of U, i.e., the solutions are of the form

$$\psi^v(z) \doteq \frac{\partial U}{\partial v}(z),\ v \in \mathbb{R}^n.$$

The function $U_\varepsilon(x) = U((1/\varepsilon)x)$ is a solution of

$$-\varepsilon^2\Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{q-1}.$$

Similarly, we have that $\psi_\varepsilon^v(x) \doteq \psi^v((1/\varepsilon)x)$ solves

$$-\varepsilon^2\Delta\psi_\varepsilon + \psi_\varepsilon = (q-1)U_\varepsilon^{q-2}\psi_\varepsilon.$$

Using the exponential map $exp_x : B(0, r) \to B_g(x, r)$, we define

$$U_{\varepsilon,x}(y) \doteq \begin{cases} U_\varepsilon(\exp_x^{-1}(y))\chi_r(\exp_x^{-1}(y)) & \text{if } y \in B_g(x,r), \\ 0 & \text{otherwise.} \end{cases}$$

We regard $U_{\varepsilon,x}$ as an approximate solution of Eq. (1.0.5), and we will try to find an exact solution of the form $u \doteq U_{\varepsilon,x} + \phi$, where ϕ is a small perturbation. For that we consider the following subspace of $H_\varepsilon(M)$:

$$K_{\varepsilon,x} = \left\{ W_{\varepsilon,x}^v : v \in \mathbb{R}^n \right\},$$

where

$$W_{\varepsilon,x}^v(y) \doteq \begin{cases} \psi_\varepsilon^v(\exp_x^{-1}(y))\chi_r(\exp_x^{-1}(y)) & \text{if } y \in B_g(x,r), \\ 0 & \text{otherwise.} \end{cases}$$

$W_{\varepsilon,x}^v$ is an approximate solution of the linearized equation $S'_\varepsilon(U_{\varepsilon,x})(v) = 0$, and $K_{\varepsilon,x}$ an approximation to the kernel of $S'_\varepsilon(U_{\varepsilon,x})$.

We are going to solve our equation modulo $K_{\varepsilon,x}$ for ϕ in the orthogonal complement $K_{\varepsilon,x}^\perp$ of $K_{\varepsilon,x}$ in H_ε. In other words, for $\varepsilon > 0$ small and $x \in M$, we will find $\phi_{\varepsilon,x} \in K_{\varepsilon,x}^\perp$ such that

$$\Pi_{\varepsilon,x}^\perp \left\{ S_\varepsilon \left(U_{\varepsilon,x} + \phi_{\varepsilon,x} \right) \right\} = 0.$$

Hence, if for some $x_o \in M$ we have

$$\Pi_{\varepsilon,x_o} \left\{ S_\varepsilon \left(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o} \right) \right\} = 0,$$

then $U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}$ is a solution of Eq. (1.0.5). In this way, the problem is reduced to a problem in finite dimensions. This is called the Lyapunov-Schmidt finite-dimensional reduction.

The following theorem is the first main result.

Theorem 1.0.1. *There exists $\varepsilon_o > 0$ such that for $\varepsilon \in (0, \varepsilon_o)$ and for any $x \in M$ there exists a unique $\phi_{\varepsilon,x} \in K_{\varepsilon,x}^\perp$ such that*

$$\Pi_{\varepsilon,x}^\perp \left\{ S_\varepsilon \left(U_{\varepsilon,x} + \phi_{\varepsilon,x} \right) \right\} = 0,$$

and $\|\phi_{\varepsilon,x}\|_\varepsilon = O(\varepsilon^2)$. The map $x \in M \mapsto J_\varepsilon(U_{\varepsilon,x} + \phi_{\varepsilon,x})$ is C^2, and if x_o is a critical point of this map then $U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}$ is a positive solution of equation (1.0.5).

Therefore, critical points of a C^2 function on M give solutions to our equation. This allows to apply classical results about the number of critical points of functions on closed manifolds.

Recall that the Lusternik-Schnirelmann category of M, $Cat(M)$, is the minimal integer k such that M can be covered by k subsets, $M \subset M_1 \cup M_2 ... \cup M_k$, with M_i closed and contractible in M. The classical result of Lusternick-Schnirelmann theory says that any C^1 function on a closed manifold M has at least $Cat(M)$ critical points. Therefore, from Theorem 1.0.1 (and the discussion above) we obtain the following result.

Theorem 1.0.2. *Let (M,g) be any closed Riemannian manifold and (N, h) be a Riemannian manifold of constant positive scalar curvature. There exist $\varepsilon_o > 0$ such that for $0 < \varepsilon < \varepsilon_o$ the Yamabe equation on the Riemannian product $(M \times N, g + \varepsilon^2 h)$ has at least $Cat(M)$ solutions which depend only on M.*

In [29] J. Petean also considered the product $(M \times N, g + \delta h)$, where the scalar curvature S_h is constant and positive, on the scalar curvature of g, no condition is imposed. Using topological techniques J. Petean proved that existence of $Cat(M) + 1$ solutions for δ small, where $Cat(M)$ of this solutions have low energy and one of higher energy. The solutions obtained are functions of M.
The solutions provided in our theorem have low energy and they are close to the explicit approximate solutions. Rey and Ruiz [31] also applied the Lyapunov-Schmidt reduction technique to construct *multipeak* high-energy solutions under certain conditions on the scalar curvature of g. These seem to be the only known results when the scalar curvature of g is not a constant.

We can apply Morse theory as well. Let $b_i(M) \doteq \dim(H_i(M, \mathbb{R}))$ and $b(M) \doteq b_1(M) + \cdots + b_n(M)$. Then, if f is a Morse function on M (which means that all of its critical points are non-degenerate) then f has at least $b(M)$ critical points [25]. Then, we get the following result.

Theorem 1.0.3. *Let (M,g) be any closed Riemannian manifold and (N, h) be a Riemannian manifold of constant positive scalar curvature. There exist $\varepsilon_o > 0$ such that if for $0 < \varepsilon < \varepsilon_o$ all the critical points of the function $J_\varepsilon(U_{\varepsilon,x} + \phi_{\varepsilon,x}) : M \to \mathbb{R}$ are non-degenerate, then the Yamabe equation on the Riemannian product $(M \times N, g + \varepsilon^2 h)$ has at least $b(M)$ solutions which depend only on M.*

This thesis is organized as follows. In Chapter 2 we introduce a background over the Yamabe Problem. In Section 2.1 we introduce Yamabe problem and the Yamabe equation is given. In section 2.2 the Hilbert-Einstein is defined and proved that critical point of the Hilbert-Einstein restricted to the conformal class of the metric $[g]$, are solutions of the Yamabe Equation. Is proved the Yamabe functional is bounded from below and we define the Yamabe constant $Y(M, [g])$ as the infimum of the Yamabe functional over the conformal class. The section

2.3 The problem of Yamabe in the sphere (S^n, g_0^n) is analyzed and we describe the solutions of the Yamabe Problem. In Section 2.4 we work on the problem of uniqueness and multiplicity of solutions to the Yamabe Problem. We give conditions for uniqueness and we will see examples of multiplicity solutions of the Yamabe equation.

In chapter 3, we prove the main results of this thesis. Section 3.1 establishes the problem and theorem of this thesis. Sections 3.2-3.3 we describe the framework to solve the problem. Finally in sections 3.4-3.5 we use the Lyapunov-Schmidt-reduction technique comprehensively to prove the problem of this thesis.

Appendix A contains demonstrations of estimates and known results that are used throughout the present work.

2 The Yamabe Problem

2.1 Introduction

Let M be a smooth, connected, oriented closed two-dimensional manifold. We say that a metric h is conformal to the metric g, if $h = f \cdot g$ with $f \in C^{\infty}_{>0}(M)$. A Riemannian metric g in M determines a conformal class

$$[g] = \{f \cdot g : f \in C^{\infty}_{>0}(M)\}.$$

Let g be a Riemannian metric in M, with Gaussian curvature function K_g. If $f = e^{2u}$ with $u \in C^{\infty}(M)$ and $h = e^{2u} \cdot g$. Then the curvatures Gaussian, K_g and K_h are related by the so-called Gauss curvature equation

$$-\Delta_g u + K_g = K_h e^{2u}. \tag{2.1.1}$$

The classical uniformization Theorem establishes that every conformal class $[g]$ there is a metric $h \in [g]$ with constant Gaussian curvature, that is, existe $u \in C^{\infty}(M)$ solution to 2.1.1 with K_h constant.

In an article published in 1960 [37], H. Yamabe considered the case of dimension $n \geq 3$. Let (M, g) be a Riemannian manifold closed of dimension $n \geq 3$. Is there a metric h which is conformal to metric g and has constant scalar curvature $S_h \in \mathbb{R}$?

It is well known that the Yamabe problem is equivalent to solve the following semilinear elliptical PDE equation 2.1.3. If $h = u^{p_n - 2} g$ with $u > 0$, the scalar curvature S_h is related to the scalar curvature S_g of g by the equation (see [7] or [11] pag. 90. for details)

$$-a_n \Delta_g u + S_g u = S_h u^{p_n - 1} \tag{2.1.2}$$

Here, Δ_g is the Laplacian operator associated with the metric g, $a_n = \dfrac{4(n-1)}{(n-2)}$ and $p_n = \dfrac{2n}{n-2}$. The metric $h = u^{p_n-2}g$ has constant scalar curvature λ if and only if u is a positive solution of the equation

$$-a_n\Delta_g u + S_g u = \lambda u^{p_n-1}. \tag{2.1.3}$$

The above equation is known as **the Yamabe equation**. The Yamabe equation can be seen as a nonlinear eigenvalue problem

$$L_g(u) = \lambda u^{p_n-1},$$

where $L_g := -a_n\Delta_g + S_g$ is the conformal Laplacian operator with respect to the metric g.

2.2 Hilbert-Einstein & Yamabe Functional

The Yamabe problem admits a variational formulation. We define **the normalized Hilbert-Einstein functional** by

$$\mathcal{E}(g) := \frac{\displaystyle\int_M S_g d\mu_g}{Vol(M,g)^{\frac{n-2}{n}}}, \tag{2.2.1}$$

if $h = u^{p_n-2}g$, it follows from 2.1.2 and $dv_h = u^{p_n}d\mu_g$ that

$$\mathcal{E}(h) = \frac{\displaystyle\int_M u^{1-p_n}L(u)\,dv_h}{Vol(M,h)^{\frac{2}{p_n}}} = \frac{\displaystyle\int_M uL(u)\,d\mu_g}{\left(\displaystyle\int_M dv_h\right)^{\frac{2}{p_n}}} = \frac{\displaystyle\int_M \left(a_n|\nabla_g u|_g^2 + S_g u^2\right) d\mu_g}{\|u\|_{p_n}^2}.$$

We define the **Yamabe Functional** by

$$Y_g(u) := \frac{\displaystyle\int_M \left(a_n|\nabla_g u|_g^2 + S_g u^2\right) d\mu_g}{\left(\displaystyle\int_M u^{p_n} d\mu_g\right)^{\frac{2}{p_n}}}. \tag{2.2.2}$$

Remark 2.2.1. *Notice that* $\left|\nabla_g|u|\right|_g = \left|\nabla_g u\right|_g$ *for almost every point in* M*, we have that* $Y_g(|u|) = Y_g(Y)$*, so, the functional of Yamabe is defined in* $C^\infty(M) - \{0\}$*. It is possible to define* Y_g *in space of Sobolev* $H^1(M) - \{0\}$*.*

A function u satisfies 2.1.3 for some $\lambda \in \mathbb{R}$, if and only if u is a critical point of **the Yamabe functional**

Proposition 2.2.2. *A function $u \in C^\infty(M)$ is a critical point of Y_g if and only if u is a solution to the Yamabe Equation, with constant $\lambda = \dfrac{Y_g(u)}{\|u\|_{p_n}^{p_n-2}}$.*

Proof. By definition u is a critical point of Y_g if for all $\varphi \in C^\infty(M)$ we have that

$$\frac{\partial}{\partial t} Y_g(u+t\varphi)\Big|_{t=0} = 0.$$

Now

$$\frac{\partial}{\partial t} Y_g(u+t\varphi)\Big|_{t=0} = \frac{\partial}{\partial t}\left(\frac{\int_M L_g(u+t\varphi)(u+t\varphi)d\mu_g}{\|u+t\varphi\|_{p_n}^2} \right)\Big|_{t=0},$$

developing the Yamabe functional Y_g and using that $\int_M L_g(u)\varphi d\mu_g = \int_M L_g(\varphi) u d\mu_g$. (Using Green formula)

$$= \frac{\partial}{\partial t}\left(\frac{\int_M L_g(u)u + 2tL_g(u)\varphi + t^2 L_g(\varphi)\varphi d\mu_g}{\|u+t\varphi\|_{p_n}^2} \right)\Big|_{t=0}.$$

So,

$$\begin{aligned}
\frac{\partial}{\partial t} Y_g(u+t\varphi)\Big|_{t=0} &= \left(2\|u\|_{p_n}^2 \int_M L_g(u)\varphi d\mu_g - 2\|u\|_{p_n}^{\frac{2-p_n}{p_n}} \int_M u^{p_n-1}\varphi d\mu_g \int_M L_g(u)u d\mu_g\right)\|u\|_{p_n}^{-4} \\
&= 2\left(\|u\|_{p_n}^2 \int_M L_g(u)\varphi d\mu_g - Y_g(u)\|u\|_{p_n}^{\frac{2-p_n}{p_n}} \int_M u^{p_n-1}\varphi d\mu_g\right)\|u\|_{p_n}^{-4} \\
&= 2\left(\int_M L_g(u)\varphi d\mu_g - Y_g(u)\|u\|_{p_n}^{2-p_n} \int_M u^{p_n-1}\varphi d\mu_g\right)\|u\|_{p_n}^{-2} \\
&= 2\|u\|_{p_n}^{-2} \int_M \left(L_g(u) - Y_g(u)\|u\|_{p_n}^{2-p_n} u^{p_n-1}\right)\varphi d\mu_g.
\end{aligned}$$

Therefore, u is a critical point of Y_g, if and only if

$$L_g(u) = \frac{Y_g(u)}{\|u\|_{p_n}^{p_n-2}} u^{p_n-1}.$$

□

Proposition 2.2.3. *The Yamabe functional is bounded below.*

Proof. By the Hölder inequality, with $\frac{2}{p_n} + \frac{2}{n} = 1$ we have

$$\|u\|_2^2 \leq \|u\|_{p_n}^2 Vol(M,g)^{2/n}. \tag{2.2.3}$$

Moreover

$$Y_g = \frac{\displaystyle\int_M \left(a_n \left|\nabla_g u\right|_g^2 + S_g u^2\right) d\mu_g}{\|u\|_{p_n}^2} \geq \frac{\displaystyle\int_M S_g u^2}{\|u\|_{p_n}^2} \geq \left(\inf_M S_g\right) \frac{\|u\|_2^2}{\|u\|_{p_n}^2}. \tag{2.2.4}$$

Now if $S_g \geq 0$, then $Y_g(u) \geq 0$.
If $\inf_M S_g < 0$, by 2.2.3, we have that

$$Y_g \geq \frac{\displaystyle\int_M S_g u^2}{\|u\|_{p_n}^2} \geq \left(\inf_M S_g\right) \frac{\|u\|_2^2}{\|u\|_{p_n}^2} \geq \left(\inf_M S_g\right) Vol(M,g)^{2/n}. \tag{2.2.5}$$

Therefore Y_g is bounded below. □

Definition 2.2.4. ***The Yamabe constant*** *of* (M, g)*, is defined as:*

$$Y(M,[g]) = \inf_{h\in[g]} \mathcal{E}(h) = \inf_{u\in C^{\infty}_{>0}(M)} Y_g(u). \tag{2.2.6}$$

The metrics that realize the infimum are called **Yamabe metric**.
By remark 2.2.1 and since $C^{\infty}(M)$ is dense in $H^1(M)$, we can define

$$Y(M,[g]) = \inf_{u\in H^1(M)-0} Y_g(u).$$

Note that if $h \in [g]$ satisfies $E(h) = Y(M,[g])$, that is to say, realize the infimum of E, then h has constant scalar curvature. Now if a function u in $H^1(M)$ realize the infimum, then $|u|$ also realize and is a non-negative solution of the Yamabe equation. By elliptic regularity it follows that is must be strictly positive and C^{∞}. (See [21] and [14] for details).
If $f\colon (M,g) \to (N,h)$ is an isometry, i.e; $f^*(h) = g$ and $v \in C^2(M)$ then,

$$f^*(\Delta_h(v)) = \Delta_h(v) \circ f = \Delta_g(v \circ f) = \Delta_{f^*(h)}(f^*(v)). \tag{2.2.7}$$

(see [10] pag. 27 and [9] pag. 46). Now, we have that

$$Y_h(v) = Y_g(v \circ f) \tag{2.2.8}$$

and therefore

$$Y(M,[g]) = Y(N,[h]). \tag{2.2.9}$$

On the other hand, if $h \in [g]$ with $h = u^{p_n-2}g$, it follows (see [17])

$$Y_h(v) = Y_g(uv) \tag{2.2.10}$$

So, if $f\colon (M,g) \to (N,h)$ is a conformal diffeomorphism, with $f^*(h) = u^{p_n-2}g$, we have

$$Y_g(u(v\circ f)) = Y_h(v). \tag{2.2.11}$$

In this way, from the previous arguments, we obtain the following proposition

Proposition 2.2.5. *Let (M,g) and (N,h) be a closed Riemannian manifold. If $f : (M,g) \to (N,h)$ is a conformal diffeomorphism, then*

$$Y(M,[g]) = Y(N,[h]). \tag{2.2.12}$$

Moreover

Proposition 2.2.6. *If v is a solution to the Yamabe equation in (N,h) with constant λ, i.e. $L_h(v) = \lambda v^{p_n-1}$ and $f\colon (M,g) \to (N,h)$ is a conformal diffeomorphism. We have that*

$$u\cdot f^*(v) = u\cdot (v\circ f),$$

is a solution to the Yamabe equation over (M,g), with the same constant λ, i.e:

$$L_g(u\cdot f^*(v)) = \lambda\Big(u\cdot f^*(v)\Big)^{p_n-1}. \tag{2.2.13}$$

Proof. We have

$$L_{u^{p_n-2}g}(f^*(v)) = a_n\Delta_{u^{p_n-2}g}f^*(v) + S_{u^{p_n-2}g}f^*(v).$$

By 2.2.7, $\Delta_{u^{p_n-2}g}f^*(v) = \Delta_h(v)\circ f$ and $f^*(S_h) = S_{u^{p_n-2}g}$. So

$$\begin{aligned}
L_{u^{p_n-2}g}(f^*(v)) &= a_n\Delta_h(v)\circ f + f^*(v)f^*(S_h)\\
&= (a_n\Delta_h(v) + S_h v)\circ f\\
&= (\lambda v^{p_n-1})\circ f\\
&= f^*\left(\lambda v^{p_n-1}\right) = \lambda(f^*(v))^{p_n-1}.
\end{aligned}$$

Now, by conformal invariance of the Laplacian (see [17]),

$$L_{u^{p_n-1}g}(\varphi) = u^{1-p_n} L_g(u\varphi).$$

It follows

$$L_{u^{p_n-2}g}(f^*(v)) = u^{1-p_n} L_g(u \cdot f^*(v)) = \lambda (f^*(v))^{p_n-1}.$$

Therefore

$$L_g(uf^*(v)) = \lambda (uf^*(v))^{p_n-1}.$$

□

2.3 The Yamabe Problem in the Sphere (S^n, g_0^n)

Consider the family of functions

$$U_\lambda(x) := \left(\frac{\lambda}{\lambda^2 + |x|^2}\right)^{\frac{n-2}{2}} = \frac{1}{\lambda^{\frac{n-2}{2}}}\left(\frac{1}{1+|\lambda^{-1}x|^2}\right)^{\frac{n-2}{2}} \quad \forall x \in \mathbb{R}^n. \tag{2.3.1}$$

where λ is a positive number. Note that

$$\lambda^{\frac{n-2}{2}} U_\lambda(\lambda x) = U_1(x), \tag{2.3.2}$$

where $U_1 = \left(\frac{1}{1+\|x\|^2}\right)^{\frac{n-2}{2}}$.
A direct calculus show that

$$\Delta U_\lambda + n(n-2) U_\lambda^{p_n-1} = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{2.3.3}$$

and

$$\int_{\mathbb{R}^n} |\nabla U_\lambda|^2 dx = n(n-2) \int_{\mathbb{R}^n} U_\lambda^{p_n} dx. \tag{2.3.4}$$

Therefore, using $\lambda^{\frac{n-2}{2}} U_\lambda(\lambda x) = U_1(x)$ and $x = \lambda y$

$$\int_{\mathbb{R}^n} U_\lambda^{p_n}(x)dx = \int_{\mathbb{R}^n} U_\lambda(\lambda y)^{p_n}\lambda^n dy = \int_{\mathbb{R}^n}\left(\lambda^{\frac{n-2}{2}} U_\lambda(\lambda y)\right)^{p_n} dy = \int_{\mathbb{R}^n} U_1^{p_n}(y)dy. \tag{2.3.5}$$

We observe that the last term is independent on λ. On the other hand,

$$\lim_{\lambda\to 0^+} U_\lambda(x) = \Big(\lim_{\lambda\to 0^+} \frac{\lambda}{\lambda^2+|x|^2}\Big)^{\frac{n}{p_n}} = \begin{cases} 0 & \text{for} \quad x\neq 0 \\ \infty & \text{for} \quad x=0. \end{cases} \tag{2.3.6}$$

Note that

$$\int_{\mathbb{R}^n} U_\lambda^q dx = \lambda^{\frac{n-2}{2}(p_n-q)} \int_{\mathbb{R}^n} U_1^q(y)dy \to \begin{cases} 0 & \text{as } \lambda\to 0^+ & \text{for} & 0<q<p_n \\ \infty & \text{as } \lambda\to 0^+ & \text{for} & p_n<q<\infty. \end{cases} \tag{2.3.7}$$

Thus $\{U_\lambda\}$ is bounded in $H^1(\mathbb{R}^n)$ for $n\geq 3$.
It follows from 2.3.5, that $\{U_\lambda\}$ does not have a convergent subsequence in $L^{p_n}(\mathbb{R}^n)$.

Proposition 2.3.1. *For $\lambda>0$, the metric $4U_\lambda^{p_n-2}g_e^n$ on $\mathbb{R}^n$ is isometric to the standard spherical metric g_0^n on $\mathbb{S}^n-\{e_{n+1}\}$, where g_e^n is the Euclidean metric.*

Proof. Consider the stereographic projections

$$\pi : \mathbb{S}^n - \{e_{n+1}\} \to \mathbb{R}^n.$$

If $(x', x_{n+1})\in\mathbb{S}^n$, $x'=(x_1,\dots,x_n)$, the explicit expression of π, is given by

$$\pi(x',x_{n+1}) = y \quad \text{where} \quad y=\frac{x'}{1-x_{n+1}}.$$

while for the inverse map there holds

$$\pi^{-1}(y) = (x',x_{n+1}) = \Big(\frac{2y}{1+r^2},\frac{r^2-1}{r^2+1}\Big) \quad \text{where} \quad r=|y|.$$

The pull-back $(\pi^{-1})^*(g_0^n) = 4U_1^{p_n-2}g_e^n$. Taking the derivative of π^{-1} we have

$$\begin{aligned}
\pi_*^{-1}(e_1) &= \frac{2}{(1+r^2)^2}(r^2+1-2y_1^2,-2y_1y_2,\dots,-2y_1y_n,2y_1). \\
\pi_*^{-1}(e_2) &= \frac{2}{(1+r^2)^2}(-2y_1y_2,r^2+1-2y_2^2,-2y_2y_3,\dots,-2y_2y_n,2y_2). \\
\pi_*^{-1}(e_i) &= \frac{2}{(1+r^2)^2}(-2y_1y_i,-2y_2y_i,\dots,r^2+1-2y_i^2,-2y_iy_{i+1}\dots,-2y_iy_n,2y_i).
\end{aligned}$$

So

$$\big\langle \pi_*^{-1}(e_i),\pi_*^{-1}(e_j)\big\rangle = \delta_{ij}\frac{4}{(1+r^2)^2}.$$

Therefore,

$$(\pi^{-1})^*(g_0^n) = 4\left(\frac{1}{1+r^2}\right)^2 dy^2 = 4\left[\left(\frac{1}{1+r^2}\right)^{\frac{n-2}{2}}\right]^{\frac{4}{n-2}} g_e^n = 4U_1^{p_n-2}g_e^n.$$

Similarly, if we consider the function $\pi^{-1}\circ\Phi_c : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$, where $\Phi_c(x) = c\cdot x$ and $c > 0$. We have

$$(\pi^{-1}\circ\Phi_c)^*(g_0^n) = \Big((\Phi_c)^*\circ(\pi^{-1})^*\Big)(g_0^n) = \Phi_c^*\Big(4\Big(\frac{1}{1+r^2}\Big)^2 dy^2\Big)\Big) = 4\Big[\Big(\frac{c}{c^2|y|^2+1}\Big)^{\frac{n-2}{2}}\Big]^{\frac{4}{n-2}} g_e^n.$$

Now if $c = \frac{1}{\lambda}$,

$$= 4\Big[\Big(\frac{\lambda}{\lambda^2+|y|^2}\Big)^{\frac{n-2}{2}}\Big]^{\frac{4}{n-2}} g_e^n = 4U_\lambda^{p_n-2}g_e^n.$$

□

Let $f_c(y) = \left(\dfrac{2c}{c^2|y|^2+1}\right)^{\frac{n-2}{n}}$, by the above arguments, $(\pi^{-1}\circ\Phi_c)^*(g_0^n) = f_c^{p_n-2}g_e^n = \left(\dfrac{f_c}{f_1}\right)^{\frac{4}{n-2}} f_1^{\frac{4}{n-2}} g_e^n$. Since $f_1 = 2^{\frac{n-2}{2}}U_1$ and U_1 resolve the equation 2.3.3, to follow

$$-a_n\Delta f_1 = n(n-1)f_1^{p_n-1},$$

is solution of the Yamabe equation in $\mathbb{R}^n$. Moreover the metric $\left(\dfrac{f_c}{f_1}\right)^{\frac{4}{n-2}} g_0^n$ is isometric to g_0^n, so it has the same scalar curvature $n(n-1)$. Therefore the functions $\dfrac{f_c}{f_1}$ satisfy the Yamabe equation in (S^n, g_0^n)

$$-a_n\Delta_{g_o^n}\Big(\frac{f_c}{f_1}\Big) + n(n-1)\Big(\frac{f_c}{f_1}\Big) = n(n-1)\Big(\frac{f_c}{f_1}\Big)^{p_n-1},$$

and constitute a non-compact family of solutions, note that if $c = \frac{1}{\lambda}$ then $f_{\frac{1}{\lambda}} = 2^{\frac{n-2}{2}}U_\lambda$.

Let's now determine the Yamabe constant of the sphere. If (M, g) is a non-compact Riemannian manifold, we can define the Yamabe constant in the following form

$$Y(M,[g]) := \inf_{u\in C_0^\infty(M)-\{0\}} \frac{\displaystyle\int_M a_n|\nabla u|^2 + S_g u^2 d\mu_g}{\|u\|_{p_n}^2}$$

Given

$$\rho := \pi^{-1}\colon \big(\mathbb{R}^n, g_e^0\big) \to \big(\mathbb{S}^n\setminus\{e_{n+1}\}, g_0^n\big)$$

is a conformal diffeomorphism, by 2.2.8 and 2.2.10, we have that for every $v \in C_0^\infty\big(\mathbb{S}^n - \{e_{n+1}\}\big) - \{0\}$.

$$Y_{g_0^n}(v) = \frac{\displaystyle\int_{\mathbb{R}^n} a_n |\nabla(u \cdot \rho^*(v))|^2_{g_e^n} dv_{g_e^n}}{\|u \cdot \rho^*(v)\|^2_{p_n}}$$

where $u = \left(\frac{4}{1+\|x\|^2}\right)^{\frac{n-2}{2}}$. Therefore

$$\begin{aligned}
Y(\mathbb{S}^n, [g_0^n]) &= \inf_{v \in C^\infty(\mathbb{S}^n)-\{0\}} Y_{g_0^n}(v) \\
&= \inf_{\substack{v \in C^\infty(\mathbb{S}^n)-\{0\} \\ \text{supp}(v) \subset \mathbb{S}^n - \{e_{n+1}\}}} Y_{g_0^n}(v) \\
&= \inf_{\substack{v \in C^\infty(\mathbb{S}^n)-\{0\} \\ \text{supp}(v) \subset \mathbb{S}^n - \{e_{n+1}\}}} \frac{\displaystyle\int_{\mathbb{R}^n} a_n |\nabla(u \cdot \rho^*(v))|^2_{g_e^n} dv_{g_e^n}}{\|u \cdot \rho^*(v)\|^2_{p_n}} \\
&= \inf_{w \in C_0^\infty(\mathbb{R}^n)-\{0\}} \frac{\int_{\mathbb{R}^n} a_n |\nabla(w)|^2_{g_e^n} dv_{g_e^n}}{\|w\|^2_{p_n}} \\
&= \inf_{w \in C_0^\infty(\mathbb{R}^n)-\{0\}} \frac{a_n \|\nabla(w)\|^2_{L_2(\mathbb{R}^n)}}{\|w\|^2_{p_n}}.
\end{aligned}$$

By Sobolev inequality in $\mathbb{R}^n$, there is a constant σ such that

$$\|w\|^2_{p_n} \leq \sigma \|\nabla w\|^2_2 \quad \forall w \in C_0^\infty(\mathbb{R}^n)$$

the smallest of these constants σ_n, is called the best Sobolev constant in $\mathbb{R}^n$. From the variational point of view it is characterized by

$$\frac{1}{\sigma_n} = \inf_{w \in C_0^\infty(\mathbb{R}^n)-\{0\}} \frac{\|\nabla w\|^2_2}{\|w\|^2_{p_n}}. \tag{2.3.8}$$

So, we have

$$\sigma_n = \frac{a_n}{Y(\mathbb{S}^n, [g_0^n])}.$$

T. Aubin and G. Talenti proved that (see [3], [38])

$$\sigma_n = \frac{4}{n(n-2)Vol(\mathbb{S}^n, g_0^n)^{2/n}}.$$

and the infimum of 2.3.8 is reached in the functions $U_\lambda(x) = \lambda^{\frac{2-n}{2}} U_1(\frac{x}{\lambda})$, i.e:

$$\frac{1}{\sigma_n} = \frac{\|\nabla U_\lambda\|^2_2}{\|U_\lambda\|^2_{p_n}} \quad \forall \lambda > 0$$

Therefore the invariant of Yamabe of the sphere $(\mathbb{S}^n, g_0^n)$ is

$$Y(\mathbb{S}^n, [g_0^n]) = n(n-1)Vol(\mathbb{S}^n)^{2/n}. \tag{2.3.9}$$

By to finalize, we describe the results that solve the problem of Yamabe

T. Aubin proves in [2] that if (M, h) is a closed Riemannian manifold with $dim(M) \geq 3$. Then $Y(M, [g]) \leq Y_n$, where $Y_n := Y(\mathbb{S}^n, [g_0^n])$ is the Yamabe constant of the sphere. Also T. Aubin, shows that if the inequality is strict, i.e, $Y(M, [g]) < Y_n$. Then the constant of Yamabe $Y(M, [g])$ is always reached. In [2], T. Aubin proves that if (M, h) it is a Riemannian closed with $dim(M) \geq 6$ and M is non-conformally flat then $Y(M, [g]) < Y_n$.

Finally, R. Schoen [34] proves that given a (M, h) closed Riemannian manifold, with dimension $3, 4$ or 5, or M is locally conformally flat, then $Y(M, [g]) < Y_n$ or (M, g) is conformal to $(\mathbb{S}^n, g_o^n)$. Consult [21, 15] to see details of the results mentioned above.

We can summarize the previous results. Theorem, let (M^n, h) be a Riemannian closed with $n \geq 3$. The infimum of the Yamabe functional is reached, that is to say, exists $h \in [g]$ such that $Y(M, [g]) = \mathcal{E}(h)$. Therefore there is at least one solution to the problem of Yamabe.

2.4 Yamabe Constant Sign and Multiplicity of Solutions

We started the section with the following proposition

Proposition 2.4.1. *Two conformal metric, h and g in a manifold closed they can not have scalar curvatures of different sign.*

Proof. Let $h = u^{p_n-2}g$ with $u \in C^\infty_{>0}(M)$ such that

$$-a_n\Delta_g u + S_g u = S_h u^{p_n-1}.$$

Integrating over M

$$\int_M a_n\Delta_g u d\mu_g + \int_M S_g u d\mu_g = \int S_h u^{p_n-1} d\mu_g. \tag{2.4.1}$$

Now by the divergence theorem $\int_M \Delta_g u d\mu_g = 0$, therefore

$$\int_M S_g u d\mu_g = \int_M S_h u^{p_n-1} d\mu_g.$$

So, if $S_g > 0$ then $S_h > 0$ or if $S_g < 0$ then $S_h < 0$. Now if $S_g = 0$ then

$$\int_M S_h u^{p_n - 1} d\mu_g = 0.$$

Therefore $S_h = 0$ or S_h has changes of sign. □

In the following theorem we will see that the sign of the Yamabe constant determines the sign of the scalar curvature functions of the conformal class.

Theorem 2.4.2. *Let $(M.g)$ be a Riemannian closed with dimension $n \geq 3$ then:*

$$1) \quad Y(M,[g]) > 0 \iff \exists h \in [g] \text{ such that } S_h > 0.$$

$$2) \quad Y(M,[g]) = 0 \iff \exists h \in [g] \text{ such that } S_h = 0.$$

$$3) \quad Y(M,[g]) < 0 \iff \exists h \in [g] \text{ such that } S_h < 0.$$

Proof. Case $1)$. Suppose that $Y(M,[g]) > 0$. Let $h = u^{p_n-2}g$ a metric such that realize $Y(M,[g])$, i.e., $Y_g(u) = Y(M,[g])$. We know that $S_h = \dfrac{Y_g(u)}{\|u\|_{p_n}^{p_n-2}}$, therefore $S_h > 0$.

Conversely, suppose that there $h \in [g]$ such that $S_h > 0$. We know that there a metric $\bar{h} \in [g]$ with $S_{\bar{h}}$ constant such that realize $Y(M,[g])$, that is to say,

$$Y(M,[g]) = \mathcal{E}(\bar{h}) = \frac{\int_M S_{\bar{h}} d\mu_g}{Vol(M,\bar{h})^{\frac{n-2}{n}}} = S_{\bar{h}} Vol(M,\bar{h})^{2/n}.$$

Now by the previous proposition we have that $S_{\bar{h}} > 0$, so, $Y(M,[g]) > 0$.

The same arguments are used to prove $2)$ and $3)$. □

When $Y(M,g) \leq 0$ there is only one metric of constant scalar curvature up to homothecia.

Proposition 2.4.3. *Let (M,g) be Riemannian manifold with dimension $n \geq 3$ and $Y(M,g) \leq 0$. If h_1 and $h_2 \in [g]$ are metrics with constant scalar curvature then exist $c \in \mathbb{R}^+$ such that*

$$h_1 = ch_2.$$

Proof. if $Y(M,g) = 0$, we know by theorem 2.4.2 that $S_{h_1} = S_{h_2} = 0$. So if $h_1 = u^{p_n-2}h_2$ we have

$$0 = \int_M \Delta_g u d\mu_g = \int_M |\nabla_g u|^2 d\mu_g.$$

Therefore u is a constant function.

If $Y(M,g) < 0$, we have by 2.4.2 that $S_{h_1} < 0$ and $S_{h_2} < 0$. We know that

$$a_n \Delta_{h_1} u + S_{h_1} u = S_{h_2} u^{p_n-1}.$$

Multiplying h_2 by a constant $c > 0$ so that $S_{ch_2} = S_{h_1}$, (and $c \cdot h_2 = u^{p_n-2}h_1$) it follows that

$$a_n \Delta_{h_1} u = S_{h_1}(u^{p_n-1} - u). \tag{2.4.2}$$

If x_0 is a global maximum of u (or global minimum), then

$$\Delta_{h_1} u(x_0) \geq 0, (\Delta_{h_1} u(x_0) \leq 0)$$

Because $S_{h_1} < 0$ then $u^{p_n-1}(x_0) - u(x_0) \leq 0$. Therefore

$$u(x) \leq u(x_0) \leq 1.$$

Analogously with case of global minimum

$$u(x) \geq 1.$$

Therefore $u = 1$ and $c \cdot h_2 = h_1$. □

In the case of $Y(M,g) > 0$ in general uniqueness fails. There are many known examples of multiplicity of metric of constant scalar curvature in this case, as was mentioned in the introduction.

3 The Problem by Lyapunov-Schmidt Reduction

We shall consider Riemannian products of closed manifolds (M^n, g) and (N^m, h), such that (M^n, g) is any closed Riemannian manifold and (N^m, h) is a Riemannian manifold of constant positive scalar curvature. The Yamabe equation in $(W, g_\varepsilon) = (M \times N, g + \varepsilon^2 h)$ is

$$-a_{n+m}\Delta_{g_\varepsilon} u + \left(S_g + \varepsilon^{-2} S_h\right) u = u^{p_{m+n}-1},$$

where $\varepsilon > 0$ is small enough so that $S_g + \varepsilon^{-2} S_h$ is positive. This is of course equivalent to finding solutions of the equation

$$- a_{n+m}\Delta_{g_\varepsilon} u + \left(S_g + \varepsilon^{-2} S_h\right) u = \varepsilon^{-2} S_h u^{p_{m+n}-1}. \tag{3.0.1}$$

Moreover, we can normalize h and assume that $S_h = a_{m+n}$. Then the previous equation is equivalent to:

$$- \varepsilon^2 \Delta_{g_\varepsilon} u + \left(\frac{S_g}{a_{m+n}}\varepsilon^2 + 1\right) u = u^{p-1}. \tag{3.0.2}$$

where $p \doteq p_{m+n}$.
In the present work , we consider solutions to the above equation that only depend on the first component, that is, they will be functions

$$u \colon M \to \mathbb{R}_{>0}.$$

Such function is solution of equation 3.0.2 if and only if

$$- \varepsilon^2 \Delta_g u + \left(\frac{S_g}{a_{m+n}}\varepsilon^2 + 1\right) u = u^{p-1}. \tag{3.0.3}$$

Note that the above equation is subcritical since $p_{n+m} < p_n$.

3.1 The Limiting Equation and its Solution on $\mathbb{R}^n$

In order to resolve the equation 3.0.3, we use Lyapunov-Schmidt reduction techniques. We consider the equation in $\mathbb{R}^n$

$$-\Delta U(y) + U(y) = U^{q-1}(y). \tag{3.1.1}$$

where $2 < q < 2^*$, if $n > 2$. It is well known that there exists a unique (up to translation) positive finite-energy solution U of the equation on $\mathbb{R}^n$. Moreover, the function U is radial around some chosen point, and it is exponentially decreasing at infinity. (see [13]):

$$U(x) \leq Ce^{-c\|x\|} \quad \text{and} \quad \|\nabla U(x)\| \leq Ce^{-c\|x\|}.$$

If we consider the change of variable $D_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$D_\varepsilon(x) = \frac{x}{\varepsilon} = y \quad y_i = x_i/\varepsilon,$$

we have that the function $(U_\varepsilon(x) = U(y)$ $(U_\varepsilon = D^*_\varepsilon(U)$ pullback of U) satisfies the equation

$$-\varepsilon^2 \Delta U_\varepsilon(x) + U_\varepsilon(x) = U_\varepsilon^{q-1}(x),. \tag{3.1.2}$$

This is true because

$$\frac{\partial U_\varepsilon}{\partial x_i} = \frac{1}{\varepsilon}\frac{\partial U}{\partial y_i},$$

therefore,

$$\varepsilon^2 \frac{\partial^2 U_\varepsilon}{\partial^2 x_i} = \frac{\partial^2 U}{\partial^2 y_i}$$

From the previous equality we have that

$$\varepsilon^2 \Delta U_\varepsilon(x) = \Delta U(y), \quad \text{and} \quad dy = \frac{1}{\varepsilon^n}dx. \tag{3.1.3}$$

Now consider the space $H^1(\mathbb{R}^n)$ with norm

$$\|U\|_{H^1} := \int_{\mathbb{R}^n} \left(|\nabla U|^2 + U^2\right) dy.$$

It follows from 3.1.3 that

$$\|U\|_{H^1} := \int_{\mathbb{R}^n} \left(|\nabla U|^2 + U^2\right) dy = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \left(\varepsilon^2|\nabla U_\varepsilon|^2 + U_\varepsilon^2\right) dx.$$

Definition 3.1.1. *We define the space $H_\varepsilon(\mathbb{R}^n)$ of functions $v : \mathbb{R}^n \to \mathbb{R}$ with norm*

$$\|v\|_\varepsilon := \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \left(\varepsilon^2 |\nabla v|^2 + v^2\right) dx$$

Remark 3.1.2. *By construction, the spaces H_ε and H^1 are isometrics with D_ε as isometry, and by definition $\|U\|_{H^1} = \|U_\varepsilon\|_\varepsilon$.*

Consider the functional $E : H^1(\mathbb{R}^n) \to \mathbb{R}$,

$$E(f) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla f|^2 + \frac{1}{2} f^2 - \frac{1}{q}(f^+)^q\right) \, dx,$$

where $f^+(x) := \max\{f(x), 0\}$. The positive solutions to 3.1.1 are the critical points of the functional E restricted to the corresponding Nehari manifold :

$$N(E) := \left\{u \in H^1(\mathbb{R}^n) - \{0\} : \int_{\mathbb{R}^n} \left(|\nabla u|^2 + u^2\right) \, dx = \int_{\mathbb{R}^n} \left(u^+\right)^q \, dx\right\}.$$

The function U is actually the minimizer of the functional E restricted to $N(E)$. The minimum is then

$$\mathbf{m}(E) = \min\{E(u) \ : \ u \in N(E)\} = \frac{q-2}{2q} \|U\|_q^q. \tag{3.1.4}$$

For any $\varepsilon > 0$, let

$$E_\varepsilon(f) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \left(\frac{\varepsilon^2}{2} |\nabla f|^2 + \frac{1}{2} f^2 - \frac{1}{q} \left(f^+\right)^q\right) \, dx$$

and

$$N(E_\varepsilon) := \left\{u \in H^1\left(\mathbb{R}^n\right) - \{0\} : \int_{\mathbb{R}^n} \left(\varepsilon^2 |\nabla u|^2 + u^2\right) \, dx = \int_{\mathbb{R}^n} (u^+)^q \, dx\right\}.$$

Let $U_\varepsilon(x) = U\left((\frac{1}{\varepsilon})x\right)$. Then $U_\varepsilon \in N(E_\varepsilon)$, and it is a solution of

$$-\varepsilon^2 \Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{q-1}.$$

U_ε is a minimizer of E_ε restricted to $N(E_\varepsilon)$. By the previous discussion (see Remark 3.1.2), we have that the minimum is equal to $\mathbf{m}(E)$.

Now, if

$$S_0 : H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$$

is

$$S_0 := \nabla E$$

let us consider the linearized equation of S_0 of U. We have that $S_0'(U)(\psi) = 0$ is equivalent to

$$-\Delta\psi + \psi = (p-1)U^{p-2}\psi \quad \text{in } \mathbb{R}^n. \tag{3.1.5}$$

Is well known that all solutions of the above equation are the directional derivatives of U, i.e., the solutions are of the form

$$\psi^v(z) \doteq \frac{\partial U}{\partial v}(z), \; v \in \mathbb{R}^n.$$

We have that the set $\{\psi^1, \ldots, \psi^n\}$ is orthogonal in $H^1(\mathbb{R}^n)$, where $\psi^i(y) = \dfrac{\partial U}{\partial y_i}$, i.e.

$$\int_{\mathbb{R}^n} \Big\{\nabla\psi^i\nabla\psi^j + \psi^i(z)\psi^j(z)\Big\}dz = 0, \quad \text{for } i \neq j. \tag{3.1.6}$$

Proof.

$$\begin{aligned}\int_{\mathbb{R}^n} \Big\{\nabla\psi^i\nabla\psi^j + \psi^i(z)\psi^j(z)\Big\}dz &= \int_{\mathbb{R}^n} \big(-\Delta\psi^i(z) + \psi^i(z)\big)\psi^j(z)dz \\ &= (p-1)\int_{\mathbb{R}^n} U^{P-2}(z)\psi^i(z)\psi^j(z)dz \\ &= (p-1)\int_{\mathbb{R}^n} U^{P-2}(z)\frac{\partial U}{\partial z_i}(z)\frac{\partial U}{\partial z_j}(z)dz.\end{aligned}$$

Since U is radial

$$\int\limits_{S^{n-1}} d\theta \int_0^r U^{P-2}(\rho)(U')^2(\rho)\frac{z_i}{\rho}\frac{z_j}{\rho}\rho^{n-1}d\rho = \int\limits_{S^{n-1}} z_i(\rho,\theta)z_j(\rho,\theta)d\theta \int_0^r U^{P-2}(\rho)(U')^2(\rho)\rho^{n-1}d\rho$$

But a straightforward calculation show that

$$\int\limits_{S^{n-1}} z_i(\rho,\theta)z_j(\rho,\theta)d\theta = 0.$$

□

In a similar way, for the functional

$$E_\varepsilon(f) = \varepsilon^{-n}\int_{\mathbb{R}^n} (\varepsilon^2/2)|\nabla f|^2 + (1/2)f^2 - (1/q)(f^+)^q \, dx,$$

the function U_ε is a non-degenerate solution of the equation

$$-\varepsilon^2 \Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{q-1}. \tag{3.1.7}$$

i.e, the functions $\psi_\varepsilon^i(x) = \dfrac{\partial U_\varepsilon}{\partial x_i}$ span the kernel of $(\nabla E_\varepsilon)'(U_\varepsilon)$.
For a detailed study of this equation see [36], appendix pag. 299.

3.2 The setting on a Riemannian manifold

Let H_ε be the Hilbert space $H_g^1(M)$ equipped with the inner product

$$\langle u, v\rangle_\varepsilon \doteq \frac{1}{\varepsilon^n}\left(\varepsilon^2 \int_M \nabla_g u \nabla_g v d\mu_g + \int_M uv d\mu_g\right), \tag{3.2.1}$$

and the induced norm.

$$\|u\|_\varepsilon^2 \doteq \frac{1}{\varepsilon^n}\left(\varepsilon^2 \int_M |\nabla_g u|^2 d\mu_g + \int_M u^2 d\mu_g\right). \tag{3.2.2}$$

Let L_ε^q be the Banach spaces $L_g^q(M)$ with the norm

$$|u|_{q,\varepsilon} \doteq \left(\frac{1}{\varepsilon^n}\int_M |u|^q d\mu_g\right)^{1/q}. \tag{3.2.3}$$

Remark 3.2.1. *For $u \in H^1(\mathbb{R}^n)$ we let $u_\varepsilon(x) = u(\varepsilon^{-1}x)$. For any $\varepsilon > 0$ we have*

$$\|u_\varepsilon\|_\varepsilon = \|u\|_{H^1} \tag{3.2.4}$$

and

$$|u_\varepsilon|_{q,\varepsilon} = |u|_{L^q}. \tag{3.2.5}$$

Remark 3.2.2. *By Sobolev's embedding theorem we have that for $q \in [1, 2^*)$ if $n \geq 3$ or $q \geq 2$ if $n = 2$, the embedding $i_\varepsilon : H_\varepsilon \hookrightarrow L_\varepsilon^q$ is a continuous map. Moreover, one can easily check that*

$$|i_\varepsilon(u)|_{q,\varepsilon} \leq c\|u\|_\varepsilon, \quad \textit{for any } u \in H_\varepsilon, \tag{3.2.6}$$

where the constant c is independent of ε.

Let $p' = \frac{p}{p-1}$. Notice that for $v \in L_\varepsilon^{p'}$, the map

$$\varphi \to \langle v, i_\varepsilon(\varphi)\rangle \doteq \frac{1}{\varepsilon^n}\int_M v \cdot i_\varepsilon(\varphi), \quad \varphi \in H_\varepsilon,$$

is a continuous functional by the compact embedding $i_\varepsilon : H_\varepsilon \hookrightarrow L^p_\varepsilon$. Hence, if $v \in L^{p'}_\varepsilon$, then a function u is a solution of

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = \ v \quad \text{in } M \\ u \ \in \ H^1_g(M) \end{cases} \tag{3.2.7}$$

if and only if $u \in H^1_g(M)$ and it satisfies

$$\frac{1}{\varepsilon^n}\left(\varepsilon^2 \int_M \nabla_g u \nabla_g \varphi d\mu_g + \int_M u\varphi d\mu_g\right) = \frac{1}{\varepsilon^n}\int_M v \cdot i_\varepsilon(\varphi), \quad \forall\, \varphi \in H_\varepsilon.$$

Recall that the adjoint operator $i^*_\varepsilon : L^{p'}_\varepsilon \to H_\varepsilon$ is a continuous map such that

$$\langle i^*_\varepsilon(v), \varphi\rangle_\varepsilon = \langle v, i_\varepsilon(\varphi)\rangle, \quad \forall\, v \in L^{p'}_\varepsilon \text{ and } \forall\, \varphi \in H_\varepsilon.$$

Observe that

$$\|i^*_\varepsilon(v)\|_\varepsilon \le c|v|_{p',\varepsilon}, \quad \text{for any } v \in L^{p'}_\varepsilon, \tag{3.2.8}$$

where the constant $c > 0$ does not depend on $\varepsilon > 0$ (the same constant as in Remark 2.1 works).

If we define $u \doteq i^*_\varepsilon(v)$, with $v \in L^{p'}_\varepsilon$, then u is a solution of (3.2.7). So if $v \in C^k(M)$ then $u \in C^{k+2}(M)$.

Now, let $u \in H_\varepsilon$, then

$$|f(u)|^{p'}_{p',\varepsilon} = \frac{1}{\varepsilon^n}\int_M |(u^+)^{p-1}|^{p'} d\mu_g \le \frac{1}{\varepsilon^n}\int_M |u|^p d\mu_g \le c^p \|u\|^p_\varepsilon.$$

Moreover,

$$\begin{aligned}
\left|\frac{s_g(x)}{a_{m+n}}\varepsilon^2 u\right|_{p',\varepsilon} &= \left[\int_M \frac{1}{\varepsilon^n}\left(\frac{\varepsilon^2 S_g u}{a_{m+n}}\right)^{\frac{p}{p-1}} d\mu_g\right]^{\frac{p-1}{p}} \\
&= \left[\left(\int_M \frac{1}{\varepsilon^n}\left(\frac{\varepsilon^2 S_g u}{a_{m+n}}\right)^{\frac{p}{p-1}} d\mu_g\right)^{p-1}\right]^{\frac{1}{p}} \\
\text{by Jensen's inequality} &\le \left[\int_M \left(\frac{1}{\varepsilon^n}\left(\frac{\varepsilon^2 S_g u}{a_{m+n}}\right)^{\frac{p}{p-1}} d\mu_g\right)^{p-1}\right]^{\frac{1}{p}} \\
&= \left[\int_M \frac{1}{\varepsilon^{n(p-1)}}\left(\frac{\varepsilon^2 S_g u}{a_{m+n}}\right)^{p} d\mu_g\right]^{\frac{1}{p}} \\
\text{since } \ p-1 = \frac{p}{p'} \text{ so} &= \left[\int_M \frac{1}{\varepsilon^{n\frac{p}{p'}}}\left(\frac{\varepsilon^2 S_g u}{a_{m+n}}\right)^{p} d\mu_g\right]^{\frac{1}{p}} \\
&= \frac{1}{\varepsilon^{\frac{n}{p'}}}\left[\int_M \left(\frac{\varepsilon^2 S_g u}{a_{m+n}}\right)^{p} d\mu_g\right]^{\frac{1}{p}} \\
&\le c\varepsilon^{2+\frac{n}{p}-\frac{n}{p'}}|u|_{p,\varepsilon}.
\end{aligned}$$

where $c > 0$ depends only on M. It is easy to see that

$$2 + \frac{n}{p} - \frac{n}{p'} > 0, \quad \text{since} \quad 2 < p < \frac{2n}{n-2} = 2^*.$$

[1]

Now, we set $p \doteq p_{m+n}$. It follows that if $u \in H_\varepsilon$, then

$$F(u) \doteq (u^+(x))^{p_{m+n}-1} - \frac{s_g(x)}{a_{m+n}}\varepsilon^2 u(x) \in L_\varepsilon^{p'_{m+n}}.$$

Consider the functional $J_\varepsilon : H_\varepsilon(M) \to \mathbb{R}$ given by

$$J_\varepsilon(u) = \frac{1}{\varepsilon^n}\int_M \left(\frac{\varepsilon^2\|\nabla u\|^2}{2} + \frac{u^2}{2} - F(u)\right) d\mu_g, \tag{3.2.9}$$

It is well known that critical points of J_ε are positive solutions of Eq. (3.0.2).
Let $\nabla J_\varepsilon : H_\varepsilon \to L(H_\varepsilon, \mathbb{R})$, the derivative of J_ε, where

$$\begin{aligned}\nabla J_\varepsilon(u)(v) &= \frac{1}{\varepsilon^n}\int_M \left(\varepsilon^2 \nabla u \nabla v + \left(1 + \frac{S_g\varepsilon^2}{a}\right) uv - u^{p-1}v\right) d\mu_g \\ &= < u, v >_\varepsilon - \frac{1}{\varepsilon^n}\int_M F'(u) v d\mu_g.\end{aligned}$$

We define the operator $S_\varepsilon : H_\varepsilon \to H_\varepsilon$ by (the Riesz representation theorem).

$$\nabla J_\varepsilon(u)(v) = \langle S_\varepsilon(u), v\rangle_\varepsilon \quad \forall v \in H_\varepsilon. \tag{3.2.10}$$

Now, we have that

$$S_\varepsilon(u) = u - i_\varepsilon^*\left(F(u)\right). \tag{3.2.11}$$

The second derivative $\nabla^2 J_\varepsilon : H_\varepsilon \to \text{Bil}(H_\varepsilon, \mathbb{R})$ of J_ε, is given by

$$\begin{aligned}\nabla^2 J_\varepsilon(u)(v, w) &= \frac{1}{\varepsilon^n}\int_M \left(\varepsilon^2 \nabla v \nabla w + \left(1 + \frac{S_g\varepsilon^2}{a}\right) vw - f'(u) vw\right) d\mu_g \\ &= \langle v, w\rangle_\varepsilon - \frac{1}{\varepsilon^n}\int_M F''(u) vw d\mu_g.\end{aligned}$$

[1]**Note:**

$$2 + \frac{n}{p} - \frac{n}{p'} = 2 + n\Big(\frac{1}{p} - \frac{1}{p'}\Big) = 2 + n\Big(\frac{1}{p} + \frac{1}{p} - 1\Big) = 2 + n\Big(\frac{2}{p} - 1\Big).$$

Now, $2 < p < \frac{2n}{n-2}$, then $\frac{n-2}{2n} < \frac{1}{p} < \frac{1}{2}$. So $0 < 2 + n\Big(\frac{2}{p} - 1\Big) < 2$.

We define the operator $S'_\varepsilon(u)(v)$, for all $u, v \in H_\varepsilon$, by

$$\langle S'_\varepsilon(u)(v), w\rangle_\varepsilon := \nabla^2 J_\varepsilon(u)(v,w) \quad \text{for all} \;\; w \in H_\varepsilon$$

.

Moreover:

$$S'_\varepsilon(u)(\psi) = 0 \text{ if } -\varepsilon^2\Delta\psi + \psi = F'(u)\psi.$$

Note also that

$$S'_\varepsilon(u)\varphi = \varphi - \; i^*_\varepsilon\left((p_{m+n}-1)(u^+)^{p_{m+n}-2}\varphi - \frac{s_g(x)}{a_{m+n}}\varepsilon^2\varphi\right), \quad \varphi \;\in\; H_\varepsilon(M). \tag{3.2.12}$$

In particular, $S_\varepsilon(u) = 0$ if and only if u is a critical point of the functional J_ε.

Therefore, we can rewrite problem (3.0.2) in the equivalent way

$$u = i^*_\varepsilon\left(F(u)\right), \quad u \in H_\varepsilon, \tag{3.2.13}$$

3.3 Approximate Solutions

Let U be the solution of Eq. (3.1.1) with $p \doteq p_{m+n}$ and define as in the introduction

$$U_{\varepsilon,x}(y) \doteq \begin{cases} U_\varepsilon(\exp_x^{-1}(y))\chi_r(\exp_x^{-1}(y)) & \text{if } y \in B_g(x,r), \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.1}$$

Since U_ε solves (3.1.7), we consider $U_{\varepsilon,x}$ as an approximate solution of equation (3.0.2). In this section we will prove some estimates related to $U_{\varepsilon,x}$. Similar estimates have been obtained before, see for instance in [26]. We give the proofs of the estimates for completeness and to point out the necessary adjustments to handle the extra term $\frac{S_g\varepsilon^2}{a_{m+n}}$ in Eq. (3.0.2). Some details will be given in the appendix A.

The function $U_{\varepsilon,x}$ is an approximate solution in the following sense.

Lemma 3.3.1. *There exists an $\varepsilon_0 > 0$ and $C > 0$ such that for any $x \in M$ and any $\varepsilon \in (0,\varepsilon_0)$ we have*

$$\|S_\varepsilon(U_{\varepsilon,x})\|_\varepsilon \leq C\varepsilon^2.$$

Proof. Observe (By Riesz representation theorem)

$$\|S_\varepsilon(U_{\varepsilon,x})\|_\varepsilon = \sup_{\|v\|_\varepsilon=1} \langle S_\varepsilon(U_{\varepsilon,x}), v\rangle_\varepsilon.$$

Now

$$\langle S_\varepsilon(U_{\varepsilon,x}), v\rangle_\varepsilon = \nabla J_\varepsilon(U_{\varepsilon,x})(v) = \frac{1}{\varepsilon^n}\int_M \left[\varepsilon^2\langle\nabla U_{\varepsilon,x},\nabla v\rangle + \left(1+\frac{S_g\varepsilon^2}{a_{m+n}}\right)U_{\varepsilon,x}v - U_{\varepsilon,x}^{p-1}v\right]d\mu_g$$

$$= \frac{1}{\varepsilon^n}\int_M \Big(-\varepsilon^2\Delta U_{\varepsilon,x} + U_{\varepsilon,x} - U_{\varepsilon,x}^{p-1}\Big)v\, d\mu_g + \frac{1}{\varepsilon^n}\int_M \frac{S_g\varepsilon^2}{a_{m+n}}U_{\varepsilon,x}v\, d\mu_g.$$

On one hand

$$\begin{aligned}\left|\frac{1}{\varepsilon^n}\int_M \frac{S_g\varepsilon^2}{a}U_{\varepsilon,x}v d\mu_g\right| &\leq C_1\frac{\varepsilon^2}{\varepsilon^n}\int_M |U_{\varepsilon,x}v|d\mu_g \\ &= C_1\varepsilon^2\frac{1}{\varepsilon^{n/p'}}\|U_{\varepsilon,x}\|_{L^{p'}}\frac{1}{\varepsilon^{n/p}}\|v\|_{L^p} \\ &= C_1\varepsilon^2\frac{1}{\varepsilon^{n/p'}}\|U_{\varepsilon,x}\|_{p',\varepsilon}\|v\|_{p,\varepsilon} \\ &\leq C_1\varepsilon^2\|U_{\varepsilon,x}\|_{p',\varepsilon}\|v\|_\varepsilon.\end{aligned}$$

Such that $n = \dfrac{n}{p} + \dfrac{n}{p'}$.

Using Hölder inequality and Remark 3.2.2, because $\|v\|_\varepsilon = 1$, $v \in L^p$ and $U_{\varepsilon,x} \in L^{p'}$. Now, by Lemma A.1.4 (in appendix A), we know that

$$\lim_{\varepsilon\to 0}\|U_{\varepsilon,x}\|_{p',\varepsilon}^{p'} = \int_{\mathbb{R}^n} U_0^{p'}dx = \|U_0\|_{p'}^{p'} < \infty.$$

Therefore there exists $C > 0$ such that

$$\frac{\varepsilon^2}{\varepsilon^n}\left|\int_M \frac{S_g}{a_{m+n}}U_{\varepsilon,x}v d\mu_g\right| \leq C\varepsilon^2$$

On the other hand we have

$$\begin{aligned}
\left|\frac{1}{\varepsilon^n}\int_M\Big(-\varepsilon^2\Delta U_{\varepsilon,x}+U_{\varepsilon,x}-U_{\varepsilon,x}^{p-1}\Big)vd\mu_g\right| &\le \frac{1}{\varepsilon^n}\Big\|-\varepsilon^2\Delta U_{\varepsilon,x}+U_{\varepsilon,x}-U_{\varepsilon,x}^{p-1}\Big\|_{L^{p'}}\|v\|_{L^p}\\
&= \frac{\varepsilon^{n/p}}{\varepsilon^n}\Big\|-\varepsilon^2\Delta U_{\varepsilon,x}+U_{\varepsilon,x}-U_{\varepsilon,x}^{p-1}\Big\|_{L^{p'}}\frac{1}{\varepsilon^{n/p}}\|v\|_{L^p}\\
&\le \frac{\varepsilon^{n/p}}{\varepsilon^n}\Big\|-\varepsilon^2\Delta U_{\varepsilon,x}+U_{\varepsilon,x}-U_{\varepsilon,x}^{p-1}\Big\|_{L^{p'}}\|v\|_{\varepsilon}\\
&= \frac{\varepsilon^{n/p}}{\varepsilon^n}\Big\|-\varepsilon^2\Delta U_{\varepsilon,x}+U_{\varepsilon,x}-U_{\varepsilon,x}^{p-1}\Big\|_{L^{p'}}\\
&= \frac{\varepsilon^{n/p}\varepsilon^{n/p'}}{\varepsilon^n}\Big\|-\varepsilon^2\Delta U_{\varepsilon,x}+U_{\varepsilon,x}-U_{\varepsilon,x}^{p-1}\Big\|_{p',\varepsilon}\\
&= \Big\|-\varepsilon^2\Delta U_{\varepsilon,x}+U_{\varepsilon,x}-U_{\varepsilon,x}^{p-1}\Big\|_{p',\varepsilon}.
\end{aligned}$$

Moreover, by lemma A.1.2 we have that there is positive constant C such that

$$\|-\varepsilon^2\Delta U_{\varepsilon,x}+U_{\varepsilon,x}-U_{\varepsilon,x}^{p-1}\|_{p',\varepsilon}\le C\varepsilon^2. \tag{3.3.2}$$

This completes the proof of the lemma.

□

We consider now the kernel of the linearized equation at the approximate solution, $\{v\in H^1(M): S'_\varepsilon(U_{\varepsilon,x})(v)=0\}$. In order to have information about the kernel we consider $\varepsilon>0$, $x\in M$, and pick an orthonormal basis of T_xM to identified it with $\mathbb{R}^n$. Using normal coordinates we define the following subspace of $H^1(M)$:

$$K_{\varepsilon,x}=\Big\{W_{\varepsilon,x}^v : v\in\mathbb{R}^n\Big\},$$

where

$$W_{\varepsilon,x}^v(y)\doteq\begin{cases}\psi_\varepsilon^v(\exp_x^{-1}(y))\chi_r(\exp_x^{-1}(y)) & \text{if } y\in B_g(x,r),\\ 0 & \text{otherwise,}\end{cases} \tag{3.3.3}$$

with $\psi_\varepsilon^v(z)=\psi^v(\frac{z}{\varepsilon})$ (as in the introduction). Note that $W_{\varepsilon,x}^v$ depends on the election of the orthonormal basis but the space itself $K_{\varepsilon,x}$ does not. We will also denote by $W_{\varepsilon,x}^i=W_{\varepsilon,x}^{e_i}$.

It is easy to see from (3.1.6) and Remark 2.1 that

$$\lim_{\varepsilon\to0}\langle W_{\varepsilon,x}^i,W_{\varepsilon,x}^i\rangle_\varepsilon\to C,\quad \langle W_{\varepsilon,x}^i,W_{\varepsilon,x}^j\rangle_\varepsilon\to0\quad \text{if } i\neq j,\quad \text{as } \varepsilon\to0, \tag{3.3.4}$$

where the constant $C=\int_{\mathbb{R}^n}(\langle\nabla\psi^i,\nabla\psi^i\rangle+\psi^i\psi^i)dx>0$, $i=1,\dots,n$, is independent of i and $x\in M$. In efect, we have that

$$\langle W_{\varepsilon,x}^i,W_{\varepsilon,x}^j\rangle_\varepsilon=\frac{1}{\varepsilon^n}\int_M\Big(\varepsilon^2\nabla W_{\varepsilon,x}^i\cdot\nabla W_{\varepsilon,x}^j+W_{\varepsilon,x}^iW_{\varepsilon,x}^j\Big)d\mu_g=$$

$$\int_{B(o,r/\varepsilon)}\Big(\sum_{k,l}g^{kj}(\varepsilon x)\frac{\partial}{\partial x_k}(\psi^i(x)\chi_r(\varepsilon x)\,))\frac{\partial}{\partial x_l}(\psi^j(x)\chi_r(\varepsilon x))+(\psi^i(x)\chi_r(\varepsilon x)\,)(\psi^j(x)\chi_r(\varepsilon x)\,)\Big)|g(\varepsilon x)|^{1/2}dx$$

$$=\int_{\mathbb{R}^n}(\nabla\psi^i\nabla\psi^j+\psi^i\psi^j)dx+o(1)=C+o(1).$$

Therefore by remark 3.1.6,the result is followed.

One can also show the following

Proposition 3.3.2.

$$\lim_{\varepsilon\to 0}\ \varepsilon^2\ \|\frac{\partial}{\partial v}W^v_{\varepsilon,x_o}\|_\varepsilon\ =0. \tag{3.3.5}$$

and

$$\lim_{\varepsilon\to 0}\varepsilon\langle\frac{\partial}{\partial v}(U_{\varepsilon,x}),W^v_{\varepsilon,x_o}\rangle_\varepsilon=\langle\psi^v,\psi^v\rangle_{H^1}>0. \tag{3.3.6}$$

Proof. See [26, Lemmas 6.1 and 6.2].

□

The function $W^v_{\varepsilon,x}$ is an approximate solution of the linearized equation in the following sense.

Lemma 3.3.3. *For any $v\in\mathbb{R}^n$ there exists an $\varepsilon_o>0$ and $C>0$ such that for every $x\in M$ and all $\varepsilon\in(0,\varepsilon_o)$ we have*

$$\|S'_\varepsilon(U_{\varepsilon,x})(W^v_{\varepsilon,x})\|_\varepsilon\le C\varepsilon^2\|v\|.$$

Proof. It is enough to consider the case $v=e_i$. We have

$$\|S'_\varepsilon(U_{\varepsilon,x})(W^i_{\varepsilon,x})\|_\varepsilon=\sup_{\|v\|_\varepsilon=1}\langle S'_\varepsilon(U_{\varepsilon,x})(W^i_{\varepsilon,x}),v\rangle_\varepsilon.$$

Now, we have that

$$\langle S'_\varepsilon(U_{\varepsilon,x})(W^i_{\varepsilon,x}),v\rangle_\varepsilon=\nabla^2J_\varepsilon(U_{\varepsilon,x})(W^i_{\varepsilon,x})=$$
$$\frac{1}{\varepsilon^n}\int_M\Big[\varepsilon^2\langle\nabla W^i_{\varepsilon,x},\nabla v\rangle+\Big(1+\frac{s_g\varepsilon^2}{a_{m+n}}\Big)W^i_{\varepsilon,x}v-(p_{m+n}-1)(U_{\varepsilon,x})^{p_{m+n}-2}W^i_{\varepsilon,x}v\Big]d\mu_g\ =$$
$$=\frac{1}{\varepsilon^n}\int_M\Big(-\varepsilon^2\Delta W^i_{\varepsilon,x}+W^i_{\varepsilon,x}-(p_{m+n}-1)(U_{\varepsilon,x})^{p-2}W^i_{\varepsilon,x}\Big)vd\mu_g$$
$$+\frac{1}{\varepsilon^n}\int_M\frac{s_g\varepsilon^2}{a_{m+n}}W^i_{\varepsilon,x}vd\mu_g.$$

Observe that

$$\begin{aligned}\frac{\varepsilon^2}{\varepsilon^n}\left|\int_M \frac{s_g}{a_{m+n}} W^i_{\varepsilon,x} v d\mu_g\right| &\leq C\frac{\varepsilon^2}{\varepsilon^n}\int_M |W^i_{\varepsilon,x} v| d\mu_g \\ &\leq C\frac{\varepsilon^2}{\varepsilon^n}\|W^i_{\varepsilon,x}\|_{L^{p'_{m+n}}}\|v\|_{L^{p_{m+n}}} \leq C\varepsilon^2\|W^i_{\varepsilon,x}\|_{p',\varepsilon},\end{aligned}$$

by a similar argument as in (3.3.2).

It follows form the exponential decay of ψ^i and change of variables that $\lim_{\varepsilon\to 0}\|W^i_{\varepsilon,x}\|_{\varepsilon,p'} = \|\psi^i\|_{L^{p'}}$. We conclude that

$$\frac{\varepsilon^2}{\varepsilon^n}\left|\int_M \frac{s_g}{a_{m+n}} W^i_{\varepsilon,x} v\, d\mu_g\right| \leq \overline{C}\varepsilon^2. \tag{3.3.7}$$

Moreover,

$$\begin{aligned}&\left|\frac{1}{\varepsilon^n}\int_M \Big(-\varepsilon^2\Delta W^i_{\varepsilon,x} + W^i_{\varepsilon,x} - (p_{m+n}-1)(U_{\varepsilon,x})^{p-2}W^i_{\varepsilon,x}\Big) v d\mu_g\right| \\ \leq\ & \frac{1}{\varepsilon^n}\int_M |-\varepsilon^2\Delta W^i_{\varepsilon,x} + W^i_{\varepsilon,x} - (p_{m+n}-1)(U_{\varepsilon,x})^{p-2}W^i_{\varepsilon,x}||v| d\mu_g \\ \leq\ & \frac{1}{\varepsilon^n}\|-\varepsilon^2\Delta W^i_{\varepsilon,x} + W^i_{\varepsilon,x} - (p_{m+n}-1)(U_{\varepsilon,x})^{p_{m+n}-2}W^i_{\varepsilon,x}\|_{L^{p'}}\|v\|_{L^p} \\ =\ & \frac{1}{\varepsilon^{n/p'}}\|-\varepsilon^2\Delta W^i_{\varepsilon,x} + W^i_{\varepsilon,x} - (p_{m+n}-1)(U_{\varepsilon,x})^{p_{m+n}-2}W^i_{\varepsilon,x}\|_{L^{p'}}\frac{1}{\varepsilon^{n/p}}\|v\|_{L^p} \\ \leq\ & \frac{1}{\varepsilon^{n/p'}}\|-\varepsilon^2\Delta W^i_{\varepsilon,x} + W^i_{\varepsilon,x} - (p_{m+n}-1)(U_{\varepsilon,x})^{p_{m+n}-2}W^i_{\varepsilon,x}\|_{L^{p'}}\|v\|_{\varepsilon} \\ =\ & \|-\varepsilon^2\Delta W^i_{\varepsilon,x} + W^i_{\varepsilon,x} - (p_{m+n}-1)(U_{\varepsilon,x})^{p_{m+n}-2}W^i_{\varepsilon,x}\|_{p',\varepsilon}.\end{aligned}$$

It is shown in Lemma A.1.3 that

$$\|-\varepsilon^2\Delta W^i_{\varepsilon,x} + W^i_{\varepsilon,x} - (p_{m+n}-1)(U_{\varepsilon,x})^{p_{m+n}-2}W^i_{\varepsilon,x}\|_{p',\varepsilon} \leq C\varepsilon^2, \tag{3.3.8}$$

Estimate (3.3.8) together with (3.3.7) finishes the proof of the lemma.

□

We now solve $S_\varepsilon(u) = 0$ modulo $K_{\varepsilon,x}$. We consider the orthogonal complement $K^\perp_{\varepsilon,x}$ of $K_{\varepsilon,x}$ in H_ε and we find $\phi_{\varepsilon,x} \in K^\perp_{\varepsilon,x}$ such that

$$\Pi^\perp_{\varepsilon,x}\Big\{S_\varepsilon\left(U_{\varepsilon,x} + \phi_{\varepsilon,x}\right)\Big\} = 0, \ \textbf{Auxiliar Equation} \tag{3.3.9}$$

where $\Pi^\perp_{\varepsilon,x} : H_\varepsilon \to K^\perp_{\varepsilon,x}$ is the orthogonal projection. In the next section we will show that there exists $\varepsilon_o = \varepsilon_o(M) > 0$, such that for every $x \in M$ and $\varepsilon \in (0, \varepsilon_o)$, there is a unique $\phi_{\varepsilon,x} \in K^\perp_{\varepsilon,x}$ that solves Eq. (3.3.9). It will remain then to find points $x \in M$ for which

$$\Pi_{\varepsilon,x}\Big\{S_\varepsilon\left(U_{\varepsilon,x} + \phi_{\varepsilon,x}\right)\Big\} = 0, \ \textbf{Bifurcation Equation} \tag{3.3.10}$$

where $\Pi_{\varepsilon,x} : H_\varepsilon \to K_{\varepsilon,x}$ is the orthogonal projection.

3.4 The Finite-Dimensional Reduction

This section is devoted to solve Eq. (3.3.9). For $x \in M$, $\varepsilon > 0$ we consider the linear operator $L_{\varepsilon,x} : K^{\perp}_{\varepsilon,x} \to K^{\perp}_{\varepsilon,x}$ defined by

$$L_{\varepsilon,x}(\phi) \doteq \Pi^{\perp}_{\varepsilon,x}\Big\{S'(U_{\varepsilon,x})\phi\Big\},$$

where by (3.2.12)

$$S'(U_{\varepsilon,x})\phi = \phi - i^*_{\varepsilon}\Big[(p-1)(U_{\varepsilon,x})^{p_{m+n}-2}\phi - \varepsilon^2 \frac{S_g}{a_{m+n}}\phi\Big]$$

In the following proposition we show that the bounded operator $L_{\varepsilon,x}$ satisfies a coercivity estimate for $\varepsilon > 0$ small enough, uniformly on M. From this result it follows the invertibility of $L_{\varepsilon,x}$.

Proposition 3.4.1. *There exists $\varepsilon_0 > 0$ and $c > 0$ such that for any point $x \in M$ and for any $\varepsilon \in (0, \varepsilon_0)$*

$$\|L_{\varepsilon,x}(\phi)\|_{\varepsilon} \geq c\|\phi\|_{\varepsilon} \quad \textit{for all } \phi \in K^{\perp}_{\varepsilon,x}.$$

Proof. Assume the proposition is not true. Then there exists a sequence of positive numbers ε_i, with $\lim_{i\to\infty} \varepsilon_i = 0$, and sequences $\{x_i\} \subset M$, $\{\phi_i\} \subset K^{\perp}_{\varepsilon_i,x_i}$ with $\|\phi_i\|_{\varepsilon_i} = 1$, such that $\|L_{\varepsilon_i,x_i}(\phi_i)\|_{\varepsilon_i} \to 0$. Moreover, since M is compact we can assume that there exists $x \in M$ such that $x_i \to x$.

Claim 3.4.1.1. *Let $\omega_i \doteq L_{\varepsilon_i,x_i}(\phi_i)$ and set*

$$\xi_i \doteq S'_{\varepsilon_i}(U_{\varepsilon_i,x_i})\phi_i - \omega_i \in K_{\varepsilon_i,x_i}. \tag{3.4.1}$$

Then,

$$\|\xi_i\|_{\varepsilon_i} \to 0, \quad \textit{as } i \to \infty.$$

Proof of Claim 3.4.1.1. To prove the claim note that for any $v \in \mathbb{R}^n$,

$$\langle \xi_i, W^v_{\varepsilon_i,x_i}\rangle_{\varepsilon_i} = \langle S'_{\varepsilon_i}(U_{\varepsilon_i,x_i})\phi_i, W^v_{\varepsilon_i,x_i}\rangle_{\varepsilon_i} = \langle \phi_i, S'_{\varepsilon_i}(U_{\varepsilon_i,x_i})(W^v_{\varepsilon_i,x_i})\rangle_{\varepsilon_i}.$$

The claim then follows from Lemma 3.3.3. □

Now, we have

$$u_i \doteq \phi_i - \omega_i - \xi_i = \phi_i - S'_{\varepsilon_i}(U_{\varepsilon_i,x_i})\phi_i = i^*_{\varepsilon_i}\left((p_{m+n}-1)(U_{\varepsilon_i,x_i})^{p_{m+n}-2}\phi_i - \frac{S_g(x)}{a_{m+n}}\varepsilon_i^2\phi_i\right), \tag{3.4.2}$$

by (3.2.12). It follows from Claim 3.4.1.1 that

$$\|u_i\|_{\varepsilon_i} \to 1. \tag{3.4.3}$$

From Remark 2.2 and Eq. (3.4.2), u_i solves

$$-\varepsilon_i^2 \Delta_g u_i + u_i = (p_{m+n}-1)(U_{\varepsilon_i,x_i})^{p_{m+n}-2}\phi_i - \frac{S_g(x)}{a_{m+n}}\varepsilon_i^2\phi_i. \tag{3.4.4}$$

Let

$$v_i \doteq i^*_{\varepsilon_i}\left((p_{m+n}-1)(U_{\varepsilon_i,x_i})^{p_{m+n}-2}\phi_i\right) = u_i + i^*_{\varepsilon_i}\left(\frac{S_g(x)}{a_{m+n}}\varepsilon_i^2\phi_i\right).$$

Then v_i is supported in $B(x_i,r)$ and

$$\|v_i\|_{\varepsilon_i} \to 1 \quad , \quad \|v_i - \phi_i\|_{\varepsilon_i} \to 0. \tag{3.4.5}$$

Moreover, it solves

$$-\varepsilon_i^2 \Delta_g v_i + v_i = (p_{m+n}-1)(U_{\varepsilon_i,x_i})^{p_{m+n}-2}\phi_i. \tag{3.4.6}$$

Claim 3.4.1.1. *Let*

$$\widetilde{v}_i(y) \doteq v_i\left(\exp_{x_i}(\varepsilon_i y)\right), \quad y \in B\left(0, r/\varepsilon_i\right) \subset \mathbb{R}^n.$$

Then,

$$\widetilde{v}_i \to 0 \quad \textit{weakly in } H^1(\mathbb{R}^n) \textit{ and strongly in } L^q_{loc}(\mathbb{R}^n), \tag{3.4.7}$$

for any $q \in (2, p_n)$ if $n \geq 3$ or $q > 2$ if n=2.

Proof of Claim 3.4.1.1. Let $\widetilde{v}_{i_{\varepsilon_i}}(y) = \widetilde{v}_i(\varepsilon_i^{-1}y) = v_i\left(\exp_{x_i}(y)\right)$. Observe that

$$\|\widetilde{v}_i\|_{H^1(\mathbb{R}^n)} = \|\widetilde{v}_{i_{\varepsilon_i}}\|_{H_{\varepsilon_i}(\mathbb{R}^n)} \leq C\|v_i\|_{\varepsilon_i} \leq C, \quad \text{for all } i \in \mathbb{N}. \tag{3.4.8}$$

Therefore, by taking a subsequence we can assume that there exists $\widetilde{v} \in H^1(\mathbb{R}^n)$ such that $\widetilde{v}_i \to \widetilde{v}$ weakly in $H^1(\mathbb{R}^n)$, and strongly in $L^q_{loc}(\mathbb{R}^n)$ for any $q \in (2, p_n)$ if $n \geq 3$ or $q > 2$ if $n = 2$.

Now, observe that by Claim 3.4.1.1 for $j = 1, \ldots, n$,

$$\langle W^j_{\varepsilon_i,x_i}, v_i\rangle_{\varepsilon_i} = \langle W^j_{\varepsilon_i,x_i}, u_i\rangle_{\varepsilon_i} + o(\varepsilon_i) = -\langle W^j_{\varepsilon_i,x_i}, \xi_i\rangle_{\varepsilon_i} + o(\varepsilon_i) \to 0, \quad \text{as } i \to \infty, \tag{3.4.9}$$

and (by change of variables and the exponential decay of ψ^j)

$$\langle W^j_{\varepsilon_i,x_i}, v_i\rangle_{\varepsilon_i} \to \int_{\mathbb{R}^n}\left(\nabla\psi^j\nabla\widetilde{v} + \psi^j\widetilde{v}\right)dy, \quad \text{as } i \to \infty. \tag{3.4.10}$$

We have from (3.4.5) and (3.4.6) that $\widetilde{v}$ solves

$$-\Delta\widetilde{v}+\widetilde{v}=(p_{m+n}-1)(U)^{p_{m+n}-2}\widetilde{v}\quad \text{in } \mathbb{R}^n. \tag{3.4.11}$$

Therefore, $\widetilde{v}\in \text{span}\{\psi^1,\dots,\psi^n\}$. From Eq.'s (3.4.9) and (3.4.10), we have that $\widetilde{v}$ is orthogonal to $\{\psi^1,\dots,\psi^n\}$, hence $\widetilde{v}\equiv 0$. □

Multiplying Eq. (3.4.6) by $v_i\in H_\varepsilon$, we obtain from (3.4.5)

$$\|v_i\|^2_{\varepsilon_i} \;=\; \frac{1}{\varepsilon_i^n}\int_M \Big\{(p_{m+n}-1)(U_{\varepsilon_i,x_i})^{p_{m+n}-2}\Big\}v_i\,\phi_i\to 1 \tag{3.4.12}$$

But, by Claim 3.4.1.1 we have

$$\frac{1}{\varepsilon_i^n}\int_M \Big\{(p_{m+n}-1)(U_{\varepsilon_i,x_i})^{p_{m+n}-2}\Big\}v_i\,\phi_i\to \int_{\mathbb{R}^n}(p_{m+n}-1)(U)^{p_{m+n}-2}\widetilde{v}^2=0. \tag{3.4.13}$$

This is a contradiction, thus proving the proposition. □

Now, we write for $\phi\in K^{\perp}_{\varepsilon,x}$,

$$S_\varepsilon(U_{\varepsilon,x}+\phi)=S_\varepsilon(U_{\varepsilon,x})+S'_\varepsilon(U_{\varepsilon,x})\phi+\widetilde{N}_{\varepsilon,x}(\phi), \tag{3.4.14}$$

where

$$\begin{aligned}\widetilde{N}_{\varepsilon,x}(\phi) &= S_\varepsilon(U_{\varepsilon,x}+\phi)-S_\varepsilon(U_{\varepsilon,x})-S'_\varepsilon(U_{\varepsilon,x})\phi\\ &= -i^*_\varepsilon\left(((U_{\varepsilon,x}+\phi)^+)^{p_{m+n}-1}-(U_{\varepsilon,x})^{p_{m+n}-1}-(p_{m+n}-1)(U_{\varepsilon,x})^{p_{m+n}-2}\phi\right).\end{aligned}$$

Applying $\Pi^{\perp}_{\varepsilon,x}$ to 3.4.14 we see that (3.3.9) is equivalent to

$$L_{\varepsilon,x}(\phi)=N_{\varepsilon,x}(\phi)-\Pi^{\perp}_{\varepsilon,x}(S_\varepsilon(U_{\varepsilon,x})), \tag{3.4.15}$$

where

$$N_{\varepsilon,x}(\phi)\doteq-\Pi^{\perp}_{\varepsilon,x}(\widetilde{N}_{\varepsilon,x}(\phi))=\Pi^{\perp}_{\varepsilon,x}\Big\{i^*_\varepsilon\Big[((U_{\varepsilon,x}+\phi)^+)^{p_{m+n}-1}-(U_{\varepsilon,x})^{p_{m+n}-1}-(p_{m+n}-1)(U_{\varepsilon,x})^{p_{m+n}-2}\phi\Big]\Big\}.$$

We are now ready to prove the main result of this section.

Proposition 3.4.2. *There exists an $\varepsilon_o>0$ and $A>0$ such that for any $x\in M$ and for any $\varepsilon\in(0,\varepsilon_o)$ there exists a unique $\phi_{\varepsilon,x}=\phi(\varepsilon,x)\in K^{\perp}_{\varepsilon,x}$ that solves Eq. (3.3.9) with $\|\phi_{\varepsilon,x}\|_\varepsilon\le A$. Moreover, there exists a constant $c_o>0$ independent of ε such that*

$$\|\phi_{\varepsilon,x}\|_\varepsilon\le c_o\varepsilon^2,$$

and $x\to\phi_{\varepsilon,x}$ is a C^2 map.

Proof. In order to solve Eq. (3.3.9), or equivalently Eq. (3.4.15), we have to find a fixed point of the operator $T_{\varepsilon,x} : K_{\varepsilon,x}^{\perp} \to K_{\varepsilon,x}^{\perp}$ given by

$$T_{\varepsilon,x}(\phi) \doteq L_{\varepsilon,x}^{-1}\left(N_{\varepsilon,x}(\phi) - \Pi_{\varepsilon,x}^{\perp}(S_\varepsilon(U_{\varepsilon,x}))\right).$$

Now, from Proposition 3.4.1 we have that there is a constant $C > 0$ such that

$$\|T_{\varepsilon,x}(\phi)\|_\varepsilon \leq C\left(\|N_{\varepsilon,x}(\phi)\|_\varepsilon + \|\Pi_{\varepsilon,x}^{\perp}(S_\varepsilon(U_{\varepsilon,x}))\|_\varepsilon\right), \quad \forall \phi \in K_{\varepsilon,x}^{\perp}. \tag{3.4.16}$$

Claim 3.4.2.1. *For any $b \in (0,1)$ there exist constants $a, \varepsilon_o > 0$ such that for any $\varepsilon \in (0, \varepsilon_o)$, if $\phi_1, \phi_2 \in K_{\varepsilon,x}^{\perp}$, $\|\phi_1\|_\varepsilon$, with $\|\phi_2\|_\varepsilon < a$, then $\|N_{\varepsilon,x}(\phi_1) - N_{\varepsilon,x}(\phi_2)\|_\varepsilon \leq b\|\phi_1 - \phi_2\|_\varepsilon$.*

Proof of Claim 3.4.2.1.

$$N_{\varepsilon,x}(\phi_1) - N_{\varepsilon,x}(\phi_2) = \Pi^{\perp}\{S_\varepsilon(U_{\varepsilon,x} + \phi_2) - S_\varepsilon(U_{\varepsilon,x} + \phi_1) - S_\varepsilon^{'}(U_{\varepsilon,x})(\phi_2 - \phi_1)\}$$

Therefore,

$$\|N_{\varepsilon,x}(\phi_1) - N_{\varepsilon,x}(\phi_2)\|_\varepsilon \leq \|S_\varepsilon(U_{\varepsilon,x} + \phi_2) - S_\varepsilon(U_{\varepsilon,x} + \phi_1) - S_\varepsilon^{'}(U_{\varepsilon,x})(\phi_2 - \phi_1)\|_\varepsilon$$

$$= \|i_\varepsilon^*\left(((U_{\varepsilon,x} + \phi_1)^+)^{p_{m+n}-1} - ((U_{\varepsilon,x} + \phi_2)^+)^{p_{m+n}-1} + (p_{m+n} - 1)U_{\varepsilon,x}^{p_{m+n}-2}(\phi_2 - \phi_1)\right)\|_\varepsilon$$

$$\leq c|((U_{\varepsilon,x} + \phi_1)^+)^{p_{m+n}-1} - ((U_{\varepsilon,x} + \phi_2)^+)^{p_{m+n}-1} - (p_{m+n} - 1)(U_{\varepsilon,x})^{p_{m+n}-2}(\phi_1 - \phi_2)|_{p',\varepsilon}$$

By the Intermediate Value Theorem, there is a $\lambda \in [0,1]$ such that

$$((U_{\varepsilon,x}+\phi_1)^+)^{p_{m+n}-1} - ((U_{\varepsilon,x}+\phi_2)^+)^{p_{m+n}-1} = (p_{m+n}-1)(U_{\varepsilon,x}+\phi_1+\lambda(\phi_2-\phi_1))^{p_{m+n}-2}(\phi_2-\phi_1).$$

Then, we have that

$$\begin{aligned}
& |((U_{\varepsilon,x} + \phi_1)^+)^{p_{m+n}-1} - ((U_{\varepsilon,x} + \phi_2)^+)^{p_{m+n}-1} - (p_{m+n} - 1)(U_{\varepsilon,x})^{p_{m+n}-2}(\phi_1 - \phi_2)|_{p',\varepsilon} \\
= \ & |[(p_{m+n} - 1)(U_{\varepsilon,x} + \phi_1 + \lambda(\phi_2 - \phi_1))^{p_{m+n}-2} - (p_{m+n} - 1)(U_{\varepsilon,x})^{p_{m+n}-2}](\phi_1 - \phi_2)|_{p',\varepsilon} \\
\leq \ & c|(U_{\varepsilon,x} + \phi_1 + \lambda(\phi_2 - \phi_1))^{p_{m+n}-2} - (U_{\varepsilon,x})^{p_{m+n}-2}|_{\frac{p}{p-2},\varepsilon}|(\phi_2 - \phi_1)|_{p,\varepsilon} \\
\leq \ & c|(U_{\varepsilon,x} + \phi_1\lambda(\phi_2 - \phi_1))^{p_{m+n}-2} - (U_{\varepsilon,x})^{p_{m+n}-2}|_{\frac{p}{p-2},\varepsilon}\|(\phi_2 - \phi_1)\|_\varepsilon.
\end{aligned}$$

In order to complete the estimate we need the following elementary observation which appeared in [20, Lemma 2.1]. Let $a > 0$ and $b \in \mathbb{R}$, then

$$||a + b|^\beta - a^\beta| \leq \begin{cases} C(\beta)\min\{|b|^\beta, a^{\beta-1}|b|\} & \text{if } 0 < \beta < 1. \\ C(\beta)(|a|^{\beta-1}|b| + |b|^\beta) & if \beta \geq 1. \end{cases} \tag{3.4.17}$$

See lemma A.1.1, for details.

Using 3.4.17 and setting $p := p_{m+n}$, we see that for all $v \in H_\varepsilon$

$$|(U_{\varepsilon,x}+v)^{p-2}-(U_{\varepsilon,x})^{p-2}| \leq \begin{cases} C(p)|v|^{p-2} & \text{if} \quad 2<p<3. \\ C(p)\Big(|U_{\varepsilon,x}|^{p-3}|v|+|v|^{p-2}\Big) & \text{if} \quad p \geq 3. \end{cases} \tag{3.4.18}$$

Now

$$\begin{aligned}
\Big\|(U_{\varepsilon,x}+v)^{p-2}-(U_{\varepsilon,x})^{p-2}\Big\|_{\frac{p}{p-2},\varepsilon} &\leq C(p)\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M\Big(U_{\varepsilon,x}^{p-3}|v|+|v|^{p-2}\Big)^{\frac{p}{p-2}}\right)^{\dfrac{p-2}{p}} \\
&\leq C(p)\left\{\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M\Big(U_{\varepsilon,x}^{p-3}|v|\Big)^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}+\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M\Big(|v|^{p-2}\Big)^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}\right\} \\
&= C(p)\left\{\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M\Big(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}|v|^{\frac{p}{p-2}}\Big)\right)^{\frac{p-2}{p}}+\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\|v\|_p^{p-2}\right\} \\
&= C(p)\left\{\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M\Big(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}|v|^{\frac{p}{p-2}}\Big)\right)^{\frac{p-2}{p}}+\|v\|_{p,\varepsilon}^{p-2}\right\}
\end{aligned}$$

otherwise if $q=p-2$ and $q'=\frac{p-2}{p-3}$, by the Hölder's inequality

$$\begin{aligned}
\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M\Big(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}|v|^{\frac{p}{p-2}}\Big)\right)^{\frac{p-2}{p}} &= \left(\frac{1}{\varepsilon^n}\int_M\Big(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}|v|^{\frac{p}{p-2}}\Big)\right)^{\frac{p-2}{p}} \\
&\leq \left(\frac{1}{\varepsilon^{n(1/q+1/q')}}\left[\int_M\Big(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}\Big)^{\frac{p-2}{p-3}}\right]^{\frac{p-3}{p-2}}\left[\int_M\Big(|v|^{\frac{p}{p-2}}\Big)^{p-2}\right]^{\frac{1}{p-2}}\right)^{\frac{p-2}{p}} \\
&= \left(\left[\frac{1}{\varepsilon^{nq'}}\int_M\Big(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}\Big)^{\frac{p-2}{p-3}}\right]^{\frac{p-3}{p-2}}\left[\frac{1}{\varepsilon^{nq}}\int_M\Big(|v|^{\frac{p}{p-2}}\Big)^{p-2}\right]^{\frac{1}{p-2}}\right)^{\frac{p-2}{p}} \\
&= \left(\left[\frac{1}{\varepsilon^n}\int_M|U_{\varepsilon,x}|^p\right]^{p-3}\left[\frac{1}{\varepsilon^n}\int_M|v|^p\right]\right)^{\frac{1}{p}} \\
&= \Big\|U_{\varepsilon,x}\Big\|_{p,\varepsilon}^{p-3}\|v\|_{p,\varepsilon}.
\end{aligned}$$

Therefore

$$\Big\|(U_{\varepsilon,x}+v)^{p-2}-(U_{\varepsilon,x})^{p-2}\Big\|_{\frac{p}{p-2},\varepsilon} \leq \begin{cases} C(p)\|v\|_{p,\varepsilon}^{p-2} & \text{if } 2<p<3. \\ C(p)\left(\Big\|U_{\varepsilon,x}\Big\|_{p,\varepsilon}^{p-3}\|v\|_{p,\varepsilon}+\|v\|_{p,\varepsilon}^{p-2}\right) & \text{if } p \geq 3. \end{cases} \tag{3.4.19}$$

Now by 3.4.17, we have

$$\|(U_{\varepsilon,x}+\phi_2+\lambda(\phi_1-\phi_2))^{p-2}+(U_{\varepsilon,x})^{p-2}\|_{\frac{p}{p-2},\varepsilon}\leq C\Big(\|\phi_2\|^{p-2}_{p,\varepsilon}+\|\phi_1-\phi_2\|^{p-2}_{p,\varepsilon}\Big)\quad \text{if } 2<p<3.$$

Now if

$$C\Big(\|\phi_2\|^{p-2}_{p,\varepsilon}+\|\phi_1-\phi_2\|^{p-2}_{p,\varepsilon}\Big)\leq b,$$

with $\|\phi_1\|_\varepsilon$ and $\|\phi_2\|_\varepsilon<a$ and a sufficiently small such that

$$C\Big(\|\phi_2\|^{p-2}_{p,\varepsilon}+\|\phi_1-\phi_2\|^{p-2}_{p,\varepsilon}\Big)\leq 3Ca^{p-2}\leq b,\quad a\leq\Big(\frac{b}{3C}\Big)^{1/p-2}$$

and if $p\geq 3$, then

$$\|(U_{\varepsilon,x}+\phi_2+\lambda(\phi_1-\phi_2))^{p-2}+(U_{\varepsilon,x})^{p-2}\|_{\frac{p}{p-2},\varepsilon}\leq C\Big(\|U_{\varepsilon,x}\|^{p-3}_{p,\varepsilon}(\|\phi_2\|_{p,\varepsilon}+\|\phi_1-\phi_2\|_{p,\varepsilon})+\|\phi_2\|^{p-2}_{p,\varepsilon}+\|\phi_1-\phi_2\|^{p-2}_{p,\varepsilon}\Big).$$

In the same way

$$\begin{aligned}
&\leq C\Big(\|U_{\varepsilon,x}\|^{p-3}_{p,\varepsilon}(\|\phi_2\|_{p,\varepsilon}+\|\phi_1-\phi_2\|_{p,\varepsilon})+\|\phi_2\|^{p-2}_{p,\varepsilon}+\|\phi_1-\phi_2\|^{p-2}_{p,\varepsilon}\Big)\\
&\leq C\Big(C_1 3a+3a^{p-2}\Big)\\
&\leq C\Big(C_1 3a^{p-2}+3a^{p-2}\Big)\\
&\leq C_2a^{p-2}\\
&\leq b
\end{aligned}$$

We can see that if a is small enough then

$$\|(U_{\varepsilon,x}+\phi_2+\lambda(\phi_2-\phi_1))^{p_{m+n}-2}-(U_{\varepsilon,x})^{p_{m+n}-2}\|_{\frac{p}{p-2},\varepsilon}<b,$$

proving the claim.

□

In similar fashion we can prove the following claim.

Claim 3.4.2.1. *For any $b\in(0,1)$ there exist constants $a>0$ and $\varepsilon_o>0$ such that for any $\varepsilon\in(0,\varepsilon_o)$, if $\|\phi\|_\varepsilon<a$ then $\|N_{\varepsilon,x}(\phi)\|_\varepsilon\leq b\|\phi\|_\varepsilon$.*

Proof of Claim 3.4.2.1.

$$\begin{aligned}
\|N_{\varepsilon,x}(\phi)\|_\varepsilon &= \Big\| N_{\varepsilon,x}(\phi)\|_\varepsilon = \|\Pi^\perp\{S_\varepsilon(U_{\varepsilon,x}+\phi) - S_\varepsilon(U_{\varepsilon,x}) - S_\varepsilon^{'}(U_{\varepsilon,x})(\phi)\}\Big\|_\varepsilon \\
&= \Big\| i_\varepsilon^*(F(U_{\varepsilon,x}) - F(U_{\varepsilon,x}+\phi) + F^{'}(U_{\varepsilon,x})\phi)\Big\|_\varepsilon \\
&\leq c\Big| F(U_{\varepsilon,x}) - F(U_{\varepsilon,x}+\phi) + F^{'}(U_{\varepsilon,x})\phi\Big|_{p',\varepsilon}
\end{aligned}$$

Apply Remark 4.2.2 and $F(U) = f(U) - \dfrac{S_g\varepsilon^2}{a}U$ defined as in the introduction. Now

$$\begin{aligned}
&F(U_{\varepsilon,x}) - F(U_{\varepsilon,x}+\phi) + F^{'}(U_{\varepsilon,x})\phi \\
&\quad = f(U_{\varepsilon,x}) - \frac{S_g\varepsilon^2}{a}U_{\varepsilon,x} - f(U_{\varepsilon,x}+\phi) + \frac{S_g\varepsilon^2}{a}(U_{\varepsilon,x}+\phi) + f^{'}(U_{\varepsilon,x})\phi - \frac{S_g\varepsilon^2}{a}\phi \\
&\quad = f(U_{\varepsilon,x}) - f(U_{\varepsilon,x}+\phi) + f^{'}(U_{\varepsilon,x})\phi.
\end{aligned}$$

Therefore

$$\|N_{\varepsilon,x}(\phi)\|_\varepsilon \leq C|f(U_{\varepsilon,x}) - f(U_{\varepsilon,x}+\phi) + f^{'}(U_{\varepsilon,x})\phi)|_{p',\varepsilon},$$

by mean value theorem, there is some $\lambda \in [0,1]$ such that

$$\begin{aligned}
&\leq C|[f'(U_{\varepsilon,x}+\lambda\phi) + f^{'}(U_{\varepsilon,x})]\phi|_{p',\varepsilon} \\
&\leq C|[f'(U_{\varepsilon,x}+\lambda\phi) + f^{'}(U_{\varepsilon,x})]|_{\frac{p}{p-2},\varepsilon}|\phi|_{p,\varepsilon} \\
&\leq C|[f'(U_{\varepsilon,x}+\lambda\phi) + f^{'}(U_{\varepsilon,x})]|_{\frac{p}{p-2},\varepsilon}\|\phi\|_\varepsilon
\end{aligned}$$

Then, by 3.4.19 we have

$$\|f'(U_{\varepsilon,x}+\lambda\phi) - f'(U_{\varepsilon,x})\|_{\frac{p}{p-2},\varepsilon} \leq \begin{cases} C(p)\|\phi\|_{p,\varepsilon}^{p-2} & \text{if } 2<p<3. \\ C(p)\Big(\|U_{\varepsilon,x}\|_{p,\varepsilon}^{p-3}\|\phi\|_{p,\varepsilon} + \|\phi\|_{p,\varepsilon}^{p-2}\Big) & \text{if } p\geq 3. \end{cases} \tag{3.4.20}$$

In this form

$$\|f'(U_{\varepsilon,x}+\lambda\phi) - f'(U_{\varepsilon,x})\|_{\frac{p}{p-2},\varepsilon} \leq \begin{cases} C(p)\|\phi\|_\varepsilon^{p-2} & \text{if } 2<p<3. \\ C(p)\Big(\|\phi\|_\varepsilon + \|\phi\|_\varepsilon^{p-2}\Big) & \text{if } p\geq 3. \end{cases} \tag{3.4.21}$$

So,

$$\|N_{\varepsilon,x}(\phi)\|_\varepsilon \leq C\Big(\|\phi\|_\varepsilon^{p-1} + \|\phi\|_\varepsilon^2\Big).$$

Now for a small enough such that, $\|\phi\|_\varepsilon < a$ and $\beta := max\{p-1, 2\}$

$$\|N_{\varepsilon,x}(\phi)\|_\varepsilon \leq C(a^{p-1} + a^2) \leq 2Ca^\beta \leq b\|\phi\|_\varepsilon, \quad 2Ca^\beta \leq b.$$

□

Now we prove the first statements of the proposition using the claims. Let C be the constant in (3.4.16) and take $b = \frac{1}{2C}$. Let a be the constant given by Claim 3.4.2.1 and Claim 3.4.2.1 (the minimum of the two, to be precise). From Lemma 3.3.1 and Claim 3.4.2.1 there exists $\varepsilon_o > 0$ such that if $\varepsilon \in (0, \varepsilon_o)$ then $T_{\varepsilon,x}$ sends the ball of radius a in $K_{\varepsilon,x}^\perp$ to itself.

If $\|\phi_1\|_\varepsilon$, $\|\phi_2\|_\varepsilon < a$, we have that

$$\|T_{\varepsilon,x}(\phi_1) - T_{\varepsilon,x}(\phi_2)\|_\varepsilon \leq C\|N_{\varepsilon,x}(\phi_1) - N_{\varepsilon,x}(\phi_2)\|_\varepsilon \leq \frac{1}{2}\|\phi_1 - \phi_2\|_\varepsilon.$$

We see then that $T_{\varepsilon,x}$ is a contraction in the ball of radius a. It follows that it has a unique fixed point there. The fixed point is obtained for instance as the limit ot the sequence $a_k = T_{\varepsilon,x}^k(0)$. Note that $\|a_1\|_\varepsilon \leq C\varepsilon^2$ by Lemma 3.3.1 and then from Claim 3.4.2.1 we have that for all k, $\|a_k\|_\varepsilon \leq 2C\varepsilon^2$.

It remains to prove that the map $x \to \phi_{\varepsilon,x}$ is C^2. In order to show this, we apply the Implicit Function Theorem to the C^2-function $G : M \times H_\varepsilon \to H_\varepsilon$ defined by

$$G(x, u) = \Pi_{\varepsilon,x}^\perp\Big\{S_\varepsilon(U_{\varepsilon,x} + \Pi_{\varepsilon,x}^\perp u)\Big\} + \Pi_{\varepsilon,x}u.$$

Observe that $G(x, \phi_{\varepsilon,x}) = 0$, and that the derivative $\frac{\partial G}{\partial u}(x, \phi_{\varepsilon,x}) : H_\varepsilon \to H_\varepsilon$ is given by

$$\frac{\partial G}{\partial u}(x, \phi_{\varepsilon,x})(u) = \Pi_{\varepsilon,x}^\perp\Big\{S_\varepsilon'(U_{\varepsilon,x} + \phi_{\varepsilon,x})\Pi_{\varepsilon,x}^\perp u\Big\} + \Pi_{\varepsilon,x}u$$

The proof would be complete if we show the next claim.

Claim 3.4.2.1. *For $\varepsilon > 0$ small enough, there is $C > 0$ such that*

$$\Big\|\frac{\partial G}{\partial u}(x, \phi_{\varepsilon,x})(u)\Big\|_\varepsilon \geq C\|u\|_\varepsilon,$$

for every $x \in M$.

Proof of Claim 3.4.2.1. We have that for $c = \frac{1}{\sqrt{2}}$ that

$$\left\|\frac{\partial G}{\partial u}(x, \phi_{\varepsilon,x})(u)\right\|_{\varepsilon} \geq c\left\|\Pi_{\varepsilon,x}^{\perp}\left\{S_{\varepsilon}'(U_{\varepsilon,x} + \phi_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u)\right\}\right\|_{\varepsilon} + c\left\|\Pi_{\varepsilon,x}(u)\right\|_{\varepsilon}$$

$$= c\left\|\Pi_{\varepsilon,x}^{\perp}\left\{S_{\varepsilon}'(U_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u) + S_{\varepsilon}'(U_{\varepsilon,x} + \phi_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u) - S_{\varepsilon}'(U_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u)\right\}\right\|_{\varepsilon} + c\left\|\Pi_{\varepsilon,x}(u)\right\|_{\varepsilon}$$

$$\geq c\left\|\Pi_{\varepsilon,x}(u)\right\|_{\varepsilon} + c\left\|L_{\varepsilon,x}(\Pi_{\varepsilon,x}^{\perp}(u))\right\|_{\varepsilon} - c\left\|\Pi_{\varepsilon,x}^{\perp}\left\{S_{\varepsilon}'(U_{\varepsilon,x} + \phi_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u) - S_{\varepsilon}'(U_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u)\right\}\right\|_{\varepsilon}$$

It follows from Proposition 3.4.1 that, for another constant $c > 0$, $\left\|L_{\varepsilon,x}(\Pi_{\varepsilon,x}^{\perp}(u))\right\|_{\varepsilon} \geq c\left\|\Pi_{\varepsilon,x}^{\perp}(u)\right\|_{\varepsilon}$. Then we have that for some constant $C > 0$,

$$c\left\|\Pi_{\varepsilon,x}(u)\right\|_{\varepsilon} + c\left\|L_{\varepsilon,x}(\Pi_{\varepsilon,x}^{\perp}(u))\right\|_{\varepsilon} \geq C\|u\|_{\varepsilon}.$$

Therefore, it only remains to prove that

$$\lim_{\varepsilon\to 0}\left\|\Pi_{\varepsilon,x}^{\perp}\left\{S_{\varepsilon}'(U_{\varepsilon,x} + \phi_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u) - S_{\varepsilon}'(U_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u)\right\}\right\|_{\varepsilon} = 0.$$

But,

$$S_{\varepsilon}'(U_{\varepsilon,x}+\phi_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u) - S_{\varepsilon}'(U_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u) = (p_{m+n}-1)i_{\varepsilon}^{*}((U_{\varepsilon,x}+\phi_{\varepsilon,x})^{p_{m+n}-2} - (U_{\varepsilon,x})^{p_{m+n}-2}\Pi_{\varepsilon,x}^{\perp}(u)).$$

Hence, as in the proof of Claim 3.4.2.1,

$$\left\|S_{\varepsilon}'(U_{\varepsilon,x}+\phi_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u) - S_{\varepsilon}'(U_{\varepsilon,x})\Pi_{\varepsilon,x}^{\perp}(u)\right\|_{\varepsilon} \leq c|((U_{\varepsilon,x}+\phi_{\varepsilon,x})^{p_{m+n}-2} - (U_{\varepsilon,x})^{p_{m+n}-2})\Pi_{\varepsilon,x}^{\perp}(u)|_{p',\varepsilon}$$

$$\leq c|((U_{\varepsilon,x} + \phi_{\varepsilon,x})^{p_{m+n}-2} - (U_{\varepsilon,x})^{p_{m+n}-2})|_{\frac{p}{p-2},\varepsilon}|\Pi_{\varepsilon,x}^{\perp}(u)|_{p,\varepsilon}$$

$$\leq c|((U_{\varepsilon,x} + \phi_{\varepsilon,x})^{p_{m+n}-2} - (U_{\varepsilon,x})^{p_{m+n}-2})|_{\frac{p}{p-2},\varepsilon}|\|u\|_{\varepsilon}.$$

Arguing as in the end of the proof of Claim 3.4.2.1 we can see that

$$\lim_{\varepsilon\to 0}|((U_{\varepsilon,x} + \phi_{\varepsilon,x})^{p_{m+n}-2} - (U_{\varepsilon,x})^{p_{m+n}-2})|_{\frac{p}{p-2},\varepsilon} = 0,$$

thus completing the proof of the claim.

□

This finishes the proof of the proposition.

□

3.5 Proof of Theorem 1.0.1

Recall that the critical points of the functional $J_\varepsilon : H^1(M) \to \mathbb{R}$ given by

$$J_\varepsilon(u) = \varepsilon^{-n} \int_M \left(\frac{1}{2}\varepsilon^2 \|\nabla u\|^2 + \frac{\mathbf{s}_g \varepsilon^2 + a_{m+n}}{2a_{m+n}} u^2 - \frac{1}{p_{m+n}} (u^+)^{p_{m+n}} \right) d\mu_g,$$

are the positive solutions of Eq. (3.0.2).

Proposition 4.2 tells us that there exists $\varepsilon_o > 0$ such that for $\varepsilon \in (0, \varepsilon_o)$ and $x \in M$ there exists a uniquely defined $\phi_{\varepsilon,x} \in K^\perp_{\varepsilon,x}$ such that $U_{\varepsilon,x} + \phi_{\varepsilon,x}$ solves Eq. (3.3.9). In order to finish the proof of Theorem 1.0.1 we have to establish the following result.

Proposition 3.5.1. *There exists $\varepsilon_o > 0$ such that if $\varepsilon \in (0, \varepsilon_o)$ and $x_o \in M$ is a critical point of $F_\varepsilon : M \to \mathbb{R}$, where*

$$F_\varepsilon(x) \doteq J_\varepsilon(U_{\varepsilon,x} + \phi_{\varepsilon,x}), \tag{3.5.1}$$

then $U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}$ is a positive solution of Eq. (3.0.2).

Proof. Let $x_o \in M$ be a critical point of F_ε where $\varepsilon > 0$. We need to show that for each $\varphi \in H_\varepsilon(M)$ one has that

$$\langle S_\varepsilon(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}), \varphi\rangle_\varepsilon = 0.$$

If $\varphi \in K^\perp_{\varepsilon,x_o}$ then

$$\langle S_\varepsilon\left(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}\right), \varphi\rangle_\varepsilon = \langle \Pi^\perp_{\varepsilon,x}(S_\varepsilon(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o})), \varphi\rangle_\varepsilon = 0,$$

since $U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}$ solves Eq. (3.3.9).

Then it is enough to show that $\langle S_\varepsilon(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}), \varphi\rangle_\varepsilon = 0$ if $\varphi \in K_{\varepsilon,x_o}$. On the other hand we know that $\langle S_\varepsilon(U_{\varepsilon,x_o} + \phi_{\varepsilon,x_o}), \varphi\rangle_\varepsilon = 0$ if φ is tangent to the map $x \mapsto V(x) = U_{\varepsilon,x} + \phi_{\varepsilon,x}$ at x_o. And since M and K_{ε,x_o} have the same dimension it is enough to see that the projection $\Pi_{\varepsilon,x_o} \circ D_{x_o}V : T_{x_o}M \to K_{\varepsilon,x_o}$ is injective.

Then to finish the proof it is enough to show that, fixing geodesic coordinates centered at x_o, for any $v \in \mathbb{R}^n$

$$\langle \frac{\partial}{\partial v}(U_{\varepsilon,x} + \phi_{\varepsilon,x})(x_o), W^v_{\varepsilon,x_o}\rangle_\varepsilon \neq 0. \tag{3.5.2}$$

Note that $\langle \phi_{\varepsilon,x}, W^v_{\varepsilon,x}\rangle_\varepsilon = 0$. Then

$$\langle \frac{\partial}{\partial v}(\phi_{\varepsilon,x}), W^v_{\varepsilon,x_o}\rangle_\varepsilon = -\langle \phi_{\varepsilon,x}, \frac{\partial}{\partial v} W^v_{\varepsilon,x_o}\rangle_\varepsilon.$$

As we pointed out in (3.3.5), we have

$$\lim_{\varepsilon\to 0}\ \varepsilon^2\ \|\frac{\partial}{\partial v}W^v_{\varepsilon,x_o}\|_\varepsilon = 0.$$

Then, it follows from Cauchy-Schwarz inequality and Proposition 3.4.2 that

$$\lim_{\varepsilon\to 0}\langle\frac{\partial}{\partial v}(\phi_{\varepsilon,x}), W^v_{\varepsilon,x_o}\rangle_\varepsilon = 0.$$

From (3.3.6),

$$\lim_{\varepsilon\to 0}\varepsilon\langle\frac{\partial}{\partial v}(U_{\varepsilon,x}), W^v_{\varepsilon,x_o}\rangle_\varepsilon = \langle\psi^v,\psi^v\rangle > 0.$$

Then, for $\varepsilon > 0$ small enough (3.5.2) holds, and the proposition is proved.

□

A Basic Lemmas

A.1 known results and details

We consider a function $\chi : \mathbb{R}^n \to [0,1]$ defined by

$$\chi(x) \doteq \begin{cases} 1 & \text{if } |x| \leq 1/2, \\ 0 & \text{if } |x| \geq 1, \end{cases} \tag{A.1.1}$$

Also χ satifies that $|\nabla\chi(x)| \leq 2$ and $|\Delta\chi(x)| \leq 2$ if $1/2 \leq |x| \leq 1$.
Is mentioned in the introduction the nonlinear elliptic equation

$$-\Delta U + U = U^{p-1} \quad \text{on } \mathbb{R}^n, \tag{A.1.2}$$

has a positive radial smooth solution $U \in C^\infty(\mathbb{R}^n)$ which vanishes at infinity. It is known that U is unique up to translation. Since U is radial, we think it as a function $U \in C^\infty([0,\infty),\mathbb{R})$. $U(r)$ is a monotone decreasing as $r \to \infty$ and satisfies that for some $c > 0$

$$\begin{aligned} U(r)r^{n-1}e^r &\to c \quad \text{as } r \to \infty \\ U'(r)r^{n-1}e^r &\to -c \quad \text{as } r \to \infty \end{aligned} \tag{A.1.3}$$

By these properties of U, we can find $\varepsilon_0 < 1$ such that

$$U(r) \leq e^{-r} \quad \text{and} \quad |U'(r)| \leq e^{-r} \text{ for all } r \geq \frac{1}{2\varepsilon_o}. \tag{A.1.4}$$

Note that $U(r)$ satisfies the equation

$$-U'' - \frac{n-1}{r}U' + U = U^{p-1} \text{ on } [0,\infty).$$

Now, if $\varepsilon > 0$ we define

$$U_\varepsilon(x) = U(\frac{x}{\varepsilon})$$

on $\mathbb{R}^n$. We have that U_ε is a positive radial solution of the equation

$$-\varepsilon^2 \Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{p-1} \text{ on } \mathbb{R}^n.$$

Then we have that $U_\varepsilon(r)$ satisfies

$$-\varepsilon^2 U_\varepsilon'' - \frac{\varepsilon^2(n-1)}{r} U_\varepsilon' + U_\varepsilon = U_\varepsilon^{p-1} \text{ on } [0,\infty).$$

We first give details of the proof of the following elementary estimate

Lemma A.1.1. *For any* $a > 0$ *and* $b \in \mathbb{R}$

$$||a+b|^\beta - a^\beta| \leq \begin{cases} C(\beta)\min\{|b|^\beta, a^{\beta-1}|b|\} & \textit{if } 0 < \beta < 1. \\ C(\beta)(|a|^{\beta-1}|b| + |b|^\beta) & if \beta \geq 1. \end{cases} \tag{A.1.5}$$

Proof. **Case** $\beta \geq 1$:
if $b = 0$ the inequality is satisfied for any $C > 0$. Now consider the function

$$g(t) = a^\beta \Big| |1+t|^\beta - 1 \Big| = a^\beta \Big| (1+t)^\beta - 1 \Big| = a^\beta \Big((1+t)^\beta - 1 \Big) \text{ on the interval } [0, |b/a|].$$

Applying the mean value theorem, there is $c \in (0, |b/a|)$ such that

$$\begin{aligned} g\Big(\Big|\frac{b}{a}\Big|\Big) - g(0) &= g'(c)\Big|\frac{b}{a}\Big| \\ &= a^\beta \beta (c+1)^{\beta-1}\Big|\frac{b}{a}\Big| \\ &\leq a^\beta \beta (1+|b/a|)^{\beta-1}\Big|\frac{b}{a}\Big| \end{aligned}$$

If $\beta = 1$ then

$$g\Big(\Big|\frac{b}{a}\Big|\Big) \leq a\Big|\frac{b}{a}\Big| = |b|, \text{ let } C(\beta) = 2.$$

If $\beta - 1 > 0$, then

$$\begin{aligned}\left(1+\left|\frac{b}{a}\right|\right)^{\beta-1} &\leq \left(2\max\left\{1,\frac{|b|}{a}\right\}\right)^{\beta-1}\\ &= 2^{\beta-1}\max\left\{1,\frac{|b|^{\beta-1}}{a^{\beta-1}}\right\}\\ &= 2^{\beta-1}\left(1+\frac{|b|^{\beta-1}}{a^{\beta-1}}\right).\end{aligned}$$

Therefore

$$a^\beta\beta\left(1+\left|\frac{b}{a}\right|\right)^{\beta-1}\left|\frac{b}{a}\right| \leq 2^{\beta-1}\beta a^{\beta-1}|b|\left(1+\left|\frac{b}{a}\right|^{\beta-1}\right) = 2^{\beta-1}\beta\left(|a|^{\beta-1}|b|+|b|^\beta\right).$$

Case $0<\beta<1$: Similarly we have for this case that

$$g(t) = a^\beta\Big||1+t|^\beta-1\Big| = a^\beta\Big|(1+t)^\beta-1\Big| = a^\beta\Big((1+t)^\beta-1\Big) \text{ on the interval } [0,|b/a|].$$

and

$$\begin{aligned}g\Big(\left|\frac{b}{a}\right|\Big)-g(0) &= g^{'}(c)\left|\frac{b}{a}\right|\\ &= a^\beta\beta(c+1)^{\beta-1}\left|\frac{b}{a}\right|\\ &= a^\beta\beta\frac{(1+c)^\beta}{1+c}\left|\frac{b}{a}\right|\\ &\leq \beta a^{\beta-1}|b|\left(1+\left|\frac{b}{a}\right|\right)^\beta.\end{aligned}$$

Therefore, by previous arguments we have

$$\beta a^{\beta-1}|b|\left(1+\left|\frac{b}{a}\right|\right)^\beta \leq 2^\beta\beta a^{\beta-1}|b|\left(1+\left|\frac{b}{a}\right|^\beta\right).$$

Now if $\dfrac{|b|}{a}\leq 1$, then

$$2^\beta\beta a^{\beta-1}|b|\left(1+\left|\frac{b}{a}\right|^\beta\right) \leq 2^{\beta+1}\beta a^{\beta-1}|b| \leq 2^{\beta+1}a^{\beta-1}|b|.$$

On the other hand

$$g\left(\frac{|b|}{a}\right) = a^\beta\left(\left(1+\frac{|b|}{a}\right)^\beta - 1\right) \leq a^\beta\left(1+\frac{|b|}{a}\right)^\beta.$$

If $1 < \frac{|b|}{a}$, then

$$1+\frac{|b|}{a} < 2\frac{|b|}{a}.$$

Therefore

$$\left(1+\frac{|b|}{a}\right)^\beta < \left(2\frac{|b|}{a}\right)^\beta = 2^\beta\frac{|b|^\beta}{a^\beta},$$

so

$$\begin{aligned} g\left(\frac{|b|}{a}\right) &\leq a^\beta(1+|b|/a)^\beta \\ &\leq a^\beta 2^\beta\frac{|b|^\beta}{a^\beta} \\ &= 2^\beta|b|^\beta \\ &\leq 2^{\beta+1}|b|^\beta. \end{aligned}$$

Let $C(\beta) := 2^{\beta+1}$, then

$$g\left(\frac{|b|}{a}\right) \leq C(\beta)\min\left\{a^{\beta-1}|b|, |b|^\beta\right\}.$$

□

Using the previous lemma we get that for all $v \in H_\varepsilon$. (Where $f(u) = u^{p-1}$)

$$|f'(U_{\varepsilon,x}+v) - f'(U_{\varepsilon,x})| \leq \begin{cases} C(p)|v|^{p-2} & \text{if } \ 2 < p < 3. \\ C(p)\left(|U_{\varepsilon,x}|^{p-3}|v| + |v|^{p-2}\right) & \text{if } \ p \geq 3. \end{cases} \quad \text{(A.1.6)}$$

Now

$$\Big\|f'(U_{\varepsilon,x}+v)-f'(U_{\varepsilon,x})\Big\|_{\frac{p}{p-2},\varepsilon} \le C(p)\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M \left(U_{\varepsilon,x}^{p-3}|v|+|v|^{p-2}\right)^{\frac{p}{p-2}}\right)^{\dfrac{p-2}{p}}$$

$$\le\ C(p)\left\{\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M \left(U_{\varepsilon,x}^{p-3}|v|\right)^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}+\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M \left(|v|^{p-2}\right)^{\frac{p}{p-2}}\right)^{\frac{p-2}{p}}\right\}$$

$$=\ C(p)\left\{\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M \left(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}|v|^{\frac{p}{p-2}}\right)\right)^{\frac{p-2}{p}}+\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\|v\|_p^{p-2}\right\}$$

$$=\ C(p)\left\{\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M \left(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}|v|^{\frac{p}{p-2}}\right)\right)^{\frac{p-2}{p}}+\|v\|_{p,\varepsilon}^{p-2}\right\}$$

otherwise if $q=p-2$ and $q'=\frac{p-2}{p-3}$, by the Hölder's inequality

$$\frac{1}{\varepsilon^{n\frac{p-2}{p}}}\left(\int_M \left(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}|v|^{\frac{p}{p-2}}\right)\right)^{\frac{p-2}{p}} = \left(\frac{1}{\varepsilon^{n}}\int_M \left(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}|v|^{\frac{p}{p-2}}\right)\right)^{\frac{p-2}{p}}$$

$$\le\ \left(\frac{1}{\varepsilon^{n(1/q+1/q')}}\left[\int_M \left(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}\right)^{\frac{p-2}{p-3}}\right]^{\frac{p-3}{p-2}}\left[\int_M \left(|v|^{\frac{p}{p-2}}\right)^{p-2}\right]^{\frac{1}{p-2}}\right)^{\frac{p-2}{p}}$$

$$=\ \left(\left[\frac{1}{\varepsilon^{nq'}}\int_M \left(|U_{\varepsilon,x}|^{\frac{p(p-3)}{p-2}}\right)^{\frac{p-2}{p-3}}\right]^{\frac{p-3}{p-2}}\left[\frac{1}{\varepsilon^{nq}}\int_M \left(|v|^{\frac{p}{p-2}}\right)^{p-2}\right]^{\frac{1}{p-2}}\right)^{\frac{p-2}{p}}$$

$$=\ \left(\left[\frac{1}{\varepsilon^{n}}\int_M |U_{\varepsilon,x}|^{p}\right]^{p-3}\left[\frac{1}{\varepsilon^{n}}\int_M |v|^{p}\right]\right)^{\frac{1}{p}}$$

$$=\Big\|U_{\varepsilon,x}\Big\|_{p,\varepsilon}^{p-3}\|v\|_{p,\varepsilon}.$$

Therefore

$$\Big\|f'(U_{\varepsilon,x}+v)-f'(U_{\varepsilon,x})\Big\|_{\frac{p}{p-2},\varepsilon} \le \begin{cases} C(p)\|v\|_{p,\varepsilon}^{p-2} & \text{if } 2<p<3. \\ C(p)\left(\Big\|U_{\varepsilon,x}\Big\|_{p,\varepsilon}^{p-3}\|v\|_{p,\varepsilon}+\|v\|_{p,\varepsilon}^{p-2}\right) & \text{if } p\ge 3. \end{cases} \tag{A.1.7}$$

Lemma A.1.2.

$$\left|\frac{1}{\varepsilon^n}\int_M \left(-\varepsilon^2\Delta U_{x,\varepsilon}+U_{x,\varepsilon}-U_{x,\varepsilon}^{p-1}\right)^{p'}d\mu_g\right|\le \varepsilon^2 C. \tag{A.1.8}$$

Proof.

$$
\begin{aligned}
&\left|\frac{1}{\varepsilon^n}\int_M \Big(-\varepsilon^2\Delta U_{x,\varepsilon}+U_{x,\varepsilon}-U_{x,\varepsilon}^{p-1}\Big)^{p'} d\mu_g\right| \le \frac{1}{\varepsilon^n}\int_M \left|-\varepsilon^2\Delta U_{x,\varepsilon}+U_{x,\varepsilon}-U_{x,\varepsilon}^{p-1}\right|^{p'} d\mu_g\\
&\quad = \frac{1}{\varepsilon^n}\int\limits_{B(0,1)} |-\varepsilon^2\Delta_g(U_\varepsilon\chi)+U_\varepsilon\chi-U_\varepsilon^{p-1}\chi^{p-1}|^{p'}\sqrt{|g_x|}dx\\
&\quad \le \frac{C}{\varepsilon^n}\int\limits_{B(0,1)} |-\varepsilon^2\Delta_g(U_\varepsilon\chi)+U_\varepsilon\chi-U_\varepsilon^{p-1}\chi^{p-1}|^{p'}dx\\
&\quad = \frac{C}{\varepsilon^n}\int\limits_{S^{n-1}} d\theta\int\limits_0^1 |-\varepsilon^2\Big(\Delta_0(U_\varepsilon\chi)+O(r)\partial_r(U_\varepsilon\chi)\Big)+U_\varepsilon\chi-U_\varepsilon^{p-1}\chi^{p-1}|^{p'}r^{n-1}dr\\
&\quad = \frac{C}{\varepsilon^n}\int\limits_{B(0,1)} |-\varepsilon^2\Big(\chi\Delta_0U_\varepsilon+U_\varepsilon\Delta_0\chi-2\nabla U_\varepsilon\cdot\nabla\chi+O(r)\partial_r(U_\varepsilon\chi)\Big)+U_\varepsilon\chi-U_\varepsilon^{p-1}\chi^{p-1}|^{p'}dx\\
&\quad = \frac{C}{\varepsilon^n}\int\limits_{B(0,1)} |\chi(-\varepsilon^2\Delta_0U_\varepsilon+U_\varepsilon)-\varepsilon^2U_\varepsilon\Delta_0\chi+2\varepsilon^2\nabla U_\varepsilon\cdot\nabla\chi-\varepsilon^2\partial_r(U_\varepsilon\chi)O(r)-U_\varepsilon^{p-1}\chi^{p-1}|^{p'}dx\\
&\quad = \frac{C}{\varepsilon^n}\int\limits_{B(0,1)} |U_\varepsilon^{p-1}(\chi-\chi^{p-1})-\varepsilon^2U_\varepsilon\Delta_0\chi+2\varepsilon^2\nabla U_\varepsilon\cdot\nabla\chi-\varepsilon^2\partial_r(U_\varepsilon\chi)O(r)|^{p'}dx\\
&\quad \le \frac{C}{\varepsilon^n}\int\limits_{B(0,1)} |U_\varepsilon^{p-1}(\chi-\chi^{p-1})|^{p'}dx \quad \textbf{(A)}\\
&\quad + \frac{C\varepsilon^{2p'}}{\varepsilon^n}\int\limits_{B(0,1)} U_\varepsilon^{p'}|\Delta_0\chi|^{p'}dx \quad \textbf{(B)}\\
&\quad + \frac{C\varepsilon^{2p'}}{\varepsilon^n}\int\limits_{B(0,1)} |\nabla U_\varepsilon\cdot\nabla\chi|^{p'}dx \quad \textbf{(C)}\\
&\quad = \frac{C\varepsilon^{2p'}}{\varepsilon^n}\int\limits_{B(0,1)} |\partial_r(U_\varepsilon\chi)O(r)|^{p'}dx \quad \textbf{(D)}
\end{aligned}
$$

Part A: with $r=\varepsilon\rho$. Since $\frac{1}{2\varepsilon}<\rho<\frac{1}{\varepsilon}$ we have that $e^{-\rho p'}<e^{-\frac{p'}{2\varepsilon}}$.

$$
\begin{aligned}
\frac{C}{\varepsilon^n}\int\limits_{B(0,1)} |U_\varepsilon^{p-1}(\chi-\chi^{p-1})|^{p'}dx &\leq \frac{C}{\varepsilon^n}\int\limits_{B(0,1)\setminus B(0,1/2)} U_\varepsilon^p(x)dx \\
&= \frac{C}{\varepsilon^n}\int\limits_{S^{n-1}} d\theta \int_{1/2}^{1} U_\varepsilon^p(r)r^{n-1}dr \\
&= C\int_{1/2\varepsilon}^{1/\varepsilon} U_o^p(\rho)\rho^{n-1}d\rho \\
\text{Using the exponential decay of } U_0 &\leq C\int\limits_{S^{n-1}} d\theta\int_{1/2\varepsilon}^{1/\varepsilon} e^{-\rho p}\rho^{n-1}d\rho \\
&\leq Ce^{-\frac{p}{2\varepsilon}}\int\limits_{S^{n-1}} d\theta\int_{1/2\varepsilon}^{1/\varepsilon} \rho^{n-1}d\rho \\
&= C\frac{e^{-\frac{p}{2\varepsilon}}}{\varepsilon^n} \\
&\leq Ce^{-\frac{p}{2\varepsilon}} \\
&= o(\varepsilon^{2p'}).
\end{aligned}
$$

Part B:

$$
\begin{aligned}
\frac{C\varepsilon^{2p'}}{\varepsilon^n}\int\limits_{B(0,1)} U_\varepsilon^{p'}(x)|\Delta_0\chi|^{p'}dx &\leq \frac{C\varepsilon^{2p'}}{\varepsilon^n}\int\limits_{S^{n-1}} d\theta\int_{1/2}^{1} U_\varepsilon^{p'}(r)r^{n-1}dr \\
&\leq C\varepsilon^{2p'}\int_{1/2\varepsilon}^{1/\varepsilon} U_o^{p'}(\rho)\rho^{n-1}d\rho \\
&\leq C\varepsilon^{2p'}\frac{e^{-\frac{p'}{2\varepsilon}}}{\varepsilon^n} \\
&\leq C\varepsilon^{2p'}e^{-\frac{p'}{2\varepsilon}} \\
&= o(\varepsilon^{2p'}).
\end{aligned}
$$

Part C:

$$
\frac{C\varepsilon^{2p'}}{\varepsilon^n}\int\limits_{B(0,1)} |\nabla U_\varepsilon\cdot\nabla\chi|^{p'}dx = \frac{C\varepsilon^{2p'}}{\varepsilon^n}\int\limits_{S^{n-1}} d\theta\int\limits_0^1 |U_\varepsilon'(r)\chi'(r)|^{p'}r^{n-1}dr
$$

now $|\chi'(r)| = |\chi'(\rho)/\varepsilon| \leq 2$, so $|\chi'(\rho)| \leq 2\varepsilon$.

$$
\begin{aligned}
&\leq C\varepsilon^{2p'}\int_{S^{n-1}}d\theta\int_{\frac{1}{2\varepsilon}}^{1/\varepsilon}|U_o^{'}(\rho)|^{p'}\rho^{n-1}d\rho\\
&= C\varepsilon^{2p'}Vol(S^{n-1})\int_{\frac{1}{2\varepsilon}}^{1/\varepsilon}|U_o^{'}(\rho)|^{p'}\rho^{n-1}d\rho\\
&\leq C\varepsilon^{2p'}\int_{\frac{1}{2\varepsilon}}^{1/\varepsilon}e^{-\rho p'}\rho^{n-1}d\rho\\
&\leq C\varepsilon^{2p'}e^{-\frac{p'}{2\varepsilon}}\int_{\frac{1}{2\varepsilon}}^{1/\varepsilon}\rho^{n-1}d\rho\\
&\leq C\varepsilon^{2p'}\frac{e^{-\frac{p'}{2\varepsilon}}}{\varepsilon^n}\\
&\leq C\varepsilon^{2p'}e^{-\frac{p'}{2\varepsilon}}\\
&= o(\varepsilon^{2p'}).
\end{aligned}
$$

At last **Part D:**

$$
\begin{aligned}
&\frac{C\varepsilon^{2p'}}{\varepsilon^n}\int_{B(0,1)}|\partial_r(U_\varepsilon\chi)O(r)|^{p'}dx\\
&\leq \frac{C\varepsilon^{2p'}}{\varepsilon^n}\int_{S^{n-1}}d\theta\int_{\frac{1}{2}}^{1}|U_\varepsilon(r)\chi'(r)r|^{p'}r^{n-1}dr+\frac{C\varepsilon^{2p'}}{\varepsilon^n}\int_{S^{n-1}}d\theta\int_0^1|U_\varepsilon^{'}(r)\chi(r)r|^{p'}r^{n-1}dr
\end{aligned}
$$

now $|\chi^{'}(r)| = |\chi^{'}(\rho)/\varepsilon| \leq 2$, so $|\chi^{'}(\rho)| \leq 2\varepsilon$.

$$
\leq C\varepsilon^{2p'}\int_{\frac{1}{2\varepsilon}}^{1/\varepsilon}U_o^{p'}(\rho)\varepsilon^{p'}\rho^{p'}\rho^{n-1}d\rho+C\varepsilon^{2p'}\int_0^{1/\varepsilon}|U_o^{'}(\rho)\frac{1}{\varepsilon}\varepsilon\rho|^{p'}\rho^{n-1}d\rho
$$

Now by exponential decay of U_o we get that

$$
C\varepsilon^{2p'}\varepsilon^{p'}\int_{\frac{1}{2\varepsilon}}^{1/\varepsilon}U_o^{p'}(\rho)\rho^{p'}\rho^{n-1}d\rho\leq C\varepsilon^{2p'}\varepsilon^{p'}\frac{e^{-\frac{p'}{2\varepsilon}}}{\varepsilon^{n+p'}}\leq C\varepsilon^{2p'}e^{-\frac{p'}{2\varepsilon}}=o(\varepsilon^{2p'}).
$$

and

$$
\begin{aligned}
&C\varepsilon^{2p'}\int_0^{1/\varepsilon}|U_o^{'}(\rho)\frac{1}{\varepsilon}\varepsilon\rho|^{p'}\rho^{n-1}d\rho\\
&= C\varepsilon^{2p'}\int_0^{1/2\varepsilon_0}|U_o^{'}(\rho)|^{p'}\rho^{p'}\rho^{n-1}d\rho+C\varepsilon^{2p'}\int_{1/2\varepsilon_0}^{1/\varepsilon}|U_o^{'}(\rho)|^{p'}\rho^{p'}\rho^{n-1}d\rho
\end{aligned}
$$

Similarly by exponential decay of $U_o^{'}$.

$$
= C\varepsilon^{2p'}+C\varepsilon^{2p'}\int_{1/2\varepsilon_0}^{1/\varepsilon}|U_o^{'}(\rho)|^{p'}\rho^{p'}\rho^{n-1}d\rho\leq C\varepsilon^{2p'}+C\varepsilon^{p'}\int_{1/2\varepsilon_0}^{1/\varepsilon}e^{-\rho p'}\rho^{p'+n-1}d\rho=o(\varepsilon^{2p'})
$$

□

Lemma A.1.3.

$$\frac{1}{\varepsilon^n}\int\limits_M |-\varepsilon^2\Delta W^i_{\varepsilon,x} + W^i_{\varepsilon,x} - f'(U_{x,\varepsilon})W^i_{\varepsilon,x}|^{p'}d\mu_g = O(\varepsilon^2)$$

.

Proof. On a normal coordinate system we have that $v = u \circ \exp_x$, so

$$(*)\ \ \Delta v = \Delta_g u + (g^{ij} - \delta_{ij})\partial_{ij}u - g^{ij}\Gamma^k_{ij}\partial_k u.$$

$$(**)\ \ |g^{ij}(x) - \delta_{ij}| = O(|x|).$$

$$(***)\ \ \Gamma^k_{ij}(\varepsilon y) = \Gamma^k_{ij}(0) + O(\varepsilon|y|).$$

Then

$$\begin{aligned}
&\int_M \left|-\varepsilon^2\Delta_g W^i_{\varepsilon,x} + W^i_{\varepsilon,x} - f'(U_{x,\varepsilon})W^i_{\varepsilon,x}\right|^{p'} d\mu_g \\
&= \int_{B(0,1)} \left|-\varepsilon^2\Delta_g(\psi^i_\varepsilon\chi) + \psi^i_\varepsilon\chi - f'(U_\varepsilon\chi)\psi^i_\varepsilon\chi\right|^{p'} \sqrt{|g_x|}dx \\
&\le C\int_{B(0,1)} \left|-\varepsilon^2\Delta_g(\psi^i_\varepsilon\chi) + \psi^i_\varepsilon\chi - f'(U_\varepsilon\chi)\psi^i_\varepsilon\chi\right|^{p'} dx \\
&= C\int_{B(0,1)} \Big|-\varepsilon^2\Delta(\psi^i_\varepsilon\chi) - \varepsilon^2(g^{ij} - \delta_{ij})\partial_{ij}(\psi^i_\varepsilon\chi) \\
&\quad + \varepsilon^2 g^{ij}\Gamma^k_{ij}\partial_k(\psi^i_\varepsilon\chi) + \psi^i_\varepsilon\chi - f'(U_\varepsilon\chi)\psi^i_\varepsilon\chi\Big|^{p'} dx \quad \text{by } (*) \\
&= C\int_{B(0,1)} \Big|-\varepsilon^2\chi\Delta\psi^i_\varepsilon - \varepsilon^2\psi^i_\varepsilon\Delta\chi - 2\varepsilon^2\nabla\psi^i_\varepsilon\nabla\chi - \varepsilon^2(g^{ij} - \delta_{ij})\partial_{ij}(\psi^i_\varepsilon\chi) \\
&\quad + \varepsilon^2 g^{ij}\Gamma^k_{ij}\partial_k(\psi^i_\varepsilon\chi) + \psi^i_\varepsilon\chi - f'(U_\varepsilon\chi)\psi^i_\varepsilon\chi\Big|^{p'} dx
\end{aligned}$$

Since $-\varepsilon^2\Delta\psi^i_\varepsilon + \psi^i_\varepsilon = f'(U_\varepsilon)\psi^i_\varepsilon$ then, the last expression is equal to

$$C\int_{B(0,1)}\Big|(f'(U_\varepsilon)-f'(U_\varepsilon\chi))\psi_\varepsilon^i\chi-\varepsilon^2\psi_\varepsilon^i\Delta\chi-2\varepsilon^2\nabla\psi_\varepsilon^i\nabla\chi-\varepsilon^2(g^{ij}-\delta_{ij})\partial_{ij}(\psi_\varepsilon^i\chi)+\varepsilon^2 g^{ij}\Gamma_{ij}^k\partial_k(\psi_\varepsilon^i\chi)\Big|^{p'}dx.$$

$$\leq \frac{C}{\varepsilon^n}\int_{B(0,1)}\Big|\psi_\varepsilon^i U_\varepsilon^{p-2}(x)\Big(\chi(x)-\chi^{p-2}(x)\Big)\Big|^{p'}dx \quad \textbf{(A)}$$

$$+\frac{C\varepsilon^{2p'}}{\varepsilon^n}\int_{B(0,1)}|\psi_\varepsilon^i\Delta\chi|^{p'}dx \quad \textbf{(B)}$$

$$+\frac{C\varepsilon^{2p'}}{\varepsilon^n}\int_{B(0,1)}|\nabla\chi\cdot\nabla\psi_\varepsilon^i|^{p'}dx \quad \textbf{(C)}$$

$$+\frac{C\varepsilon^{2p'}}{\varepsilon^n}\int_{B(0,1)}|(g^{ij}-\delta_{ij})\partial_{ij}(\psi_\varepsilon^i\chi)|^{p'}dx \quad \textbf{(D)}$$

$$+\frac{C\varepsilon^{2p'}}{\varepsilon^n}\int_{B(0,1)}|g^{ij}\Gamma_{ij}^k\partial_k(\psi_\varepsilon^i\chi)|^{p'}dx \quad \textbf{(E)}$$

Parte D: If $x=\varepsilon y$ and $r=\varepsilon\rho$ with $r=|x|$, $\rho=|y|$, then

$$\begin{aligned}
\frac{C\varepsilon^{2p'}}{\varepsilon^n}\int_{B(0,1)}|(g^{ij}(x)-\delta_{ij})\partial_{ij}(\psi_\varepsilon^i\chi)|^{p'}dx &= \frac{C\varepsilon^{2p'}}{\varepsilon^n}\int_{S^{n-1}}d\theta\int_{1/2\varepsilon}^{1/\varepsilon}\Big|(g^{ij}(\varepsilon\rho)-\delta_{ij}\Big|^{p'}\Big|\Big(\frac{1}{\varepsilon}U_0'(\rho)\chi(\rho)\Big)''\Big|^{p'}\varepsilon^n\rho^{n-1}d\rho \\
&= C\varepsilon^{2p'}\int_{S^{n-1}}d\theta\int_{1/2\varepsilon}^{1/\varepsilon}O((\varepsilon\rho)^{p'})\Big|\Big(\frac{1}{\varepsilon}U_0'(\rho)\chi(\rho)\Big)''\Big|^{p'}\rho^{n-1}d\rho \\
&\leq C\varepsilon^{2p'}\int_{S^{n-1}}d\theta\int_{1/2\varepsilon}^{1/\varepsilon}\Big|\Big(U_0'(\rho)\chi(\rho)\Big)''\Big|^{p'}\rho^{n-1}d\rho \\
&= O(e^{-1/2\varepsilon})
\end{aligned}$$

□

Lemma A.1.4. *For $1\leq p<\infty$ we have*

$$\lim_{\varepsilon\to 0}\|U_{\varepsilon,x}\|_{\varepsilon,p}^p=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\int_M|U_{\varepsilon,x}|^p d\mu_g=\int_{\mathbb{R}^n}U_0^p dx=\|U_0\|_p^p.$$

Proof. We have

$$\frac{1}{\varepsilon^n}\int_M|U_{\varepsilon,x}|^p d\mu_g=\frac{1}{\varepsilon^n}\int_{B(0,1)}|\widetilde{U}_\varepsilon|^p\sqrt{|g_x|}dx=\frac{1}{\varepsilon^n}\int_{S^{n-1}}d\theta\int_0^1|U_\varepsilon(r)\chi(r)|^p\eta(r,\theta)r^{n-1}dr$$

if $r = \varepsilon\rho$, and $\eta = 1 + O(r)$. over normal coordinate system.

$$= \int\limits_{S^{n-1}} d\theta \int_0^{1/\varepsilon} |U_0(\rho)\chi(\rho)|^p \eta(\varepsilon\rho, \theta)\rho^{n-1} d\rho$$

.

$$= \int\limits_{S^{n-1}} d\theta \int_0^{1/\varepsilon} |U_0(\rho)\chi(\rho)|^p \Big(1 + O(\varepsilon\rho)\Big)\rho^{n-1} d\rho$$

$$= \int\limits_{S^{n-1}} d\theta \int_0^{1/\varepsilon} |U_0(\rho)\chi(\rho)|^p \rho^{n-1} d\rho \qquad \textbf{(I)}$$

$$+ \int\limits_{S^{n-1}} d\theta \int_0^{1/\varepsilon} |U_0(\rho)\chi(\rho)|^p \rho^{n-1} O(\varepsilon\rho) d\rho \qquad \textbf{(II)}$$

Since that $\chi(\rho) = 1$ in $[0, 1/2\varepsilon)$ we have that

$$\textbf{(I)} \quad \lim_{\varepsilon\to 0} \int\limits_{S^{n-1}} d\theta \int_0^{1/\varepsilon} |U_0(\rho)\chi(\rho)|^p \rho^{n-1} d\rho = \lim_{\varepsilon\to 0} \int\limits_{S^{n-1}} d\theta \int_0^{1/2\varepsilon} |U_0(\rho)|^p \rho^{n-1} d\rho = \int\limits_{\mathbb{R}^n} |U_0(y)|^p dy.$$

On the other hand, by exponential decay of U_0

$$\textbf{(II)} \quad \int\limits_{S^{n-1}} d\theta \int_0^{1/\varepsilon} |U_0(\rho)\chi(\rho)|^p \rho^{n-1} O(\varepsilon\rho) d\rho$$

$$\leq C\varepsilon \int_0^{1/\varepsilon} U_0^p(\rho)\rho^n d\rho \leq C\varepsilon \int_0^{\infty} U_0^p(\rho)\rho^n d\rho = o(\varepsilon).$$

□

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