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BLOWUP AND LIFE SPAN BOUNDS FOR A REACTION-DIFFUSION EQUATION WITH A TIME-DEPENDENT GENERATOR

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Abstract. We consider the nonlinear equation

$$\frac{\partial}{\partial t}u(t) = k(t)\Delta_\alpha u(t) + u^{1+\beta}(t), \quad u(0,x) = \lambda\varphi(x),\ x \in \mathbb{R}^d,$$

where $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ denotes the fractional power of the Laplacian; $0 < \alpha \le 2$, λ, $\beta > 0$ are constants; φ is bounded, continuous, nonnegative function that does not vanish identically; and k is a locally integrable function. We prove that any combination of positive parameters d, α, ρ, β, obeying $0 < d\rho\beta/\alpha < 1$, yields finite time blow up of any nontrivial positive solution. Also we obtain upper and lower bounds for the life span of the solution, and show that the life span satisfies $T_{\lambda\varphi} \sim \lambda^{-\alpha\beta/(\alpha - d\rho\beta)}$ near $\lambda = 0$.

1. Introduction

We study positive solutions for the semilinear non-autonomous Cauchy problem

$$\begin{gathered}\frac{\partial u(t,x)}{\partial t} = k(t)\Delta_\alpha u(t,x) + u^{1+\beta}(t,x), \\ u(0,x) = \varphi(x) \ge 0, \quad x \in \mathbb{R}^d,\end{gathered} \tag{1.1}$$

where $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ denotes the fractional power of the Laplacian, $0 < \alpha \le 2$ and $\beta \in (0,\infty)$ are constants, the initial value φ is bounded, continuous and not identically zero, and $k : [0,\infty) \to [0,\infty)$ is a locally integrable function satisfying

$$\varepsilon_1 t^\rho \le \int_0^t k(r)dr \le \varepsilon_2 t^\rho \tag{1.2}$$

for all t large enough, where ε_1, ε_2 and ρ are given positive constants. Solutions are understood in the mild sense, so that (1.1) is meaningful for any bounded measurable initial value.

Recall (see e.g. [16], Chapter 6) that there exists a number $T_\varphi \in (0,\infty]$ such that (1.1) has a unique continuous solution u on $[0,T_\varphi) \times \mathbb{R}^d$, which is given by

$$u(t,x) = U(t,0)\varphi(x) + \int_0^t (U(t,s)u^{1+\beta}(s,\cdot))(x)\,ds, \tag{1.3}$$

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and is bounded on $[0,T]\times\mathbb{R}^d$ for any $0<T<T_\varphi$. Moreover, if $T_\varphi<\infty$, then $\|u(t,\cdot)\|_\infty\to\infty$ as $t\uparrow T_\varphi$. Here $\{U(t,s), 0\le s\le t\}$ denotes the evolution system corresponding to the family of generators $\{k(t)\Delta_\alpha,\ t\ge 0\}$. When $T_\varphi=\infty$ we say that u is a global solution, and when $T_\varphi<\infty$ we say that u blows up in finite time or that u is nonglobal. The extended real number T_φ is termed life span of (1.1).

Equations of the form (1.1) arise in fields like statistical physics, hydrodynamics and molecular biology [19]. Generators of the form $k(t)\Delta_\alpha$, $\alpha\in(0,2]$, allow for non-local integro-differential or pseudodifferential terms that have been used in models of anomalous growth of certain fractal interfaces [14] and in hydrodynamic models with modified diffusivity [2].

This work can be considered as a continuation of our previous article [12], to which we refer for more background and additional references. In [12] we proved that $d>\frac{\alpha}{\rho\beta}$ implies existence of non-trivial global solutions of (1.1) for all sufficiently small initial values, and that, under the additional assumption $\beta\in\{1,2,\dots,\}$, the condition $d<\frac{\alpha}{\rho\beta}$ yields finite time blowup of any positive solution. Moreover, the case $\rho=0$, which under condition (1.2) corresponds to an integrable k, yields finite time blow up of (1.1) for any non-trivial initial value, regardless of the spatial dimension and the stability exponent α. Here we consider the case $d<\frac{\alpha}{\rho\beta}$ with $\beta\in(0,\infty)$, and focus on the asymptotic behavior of the life span of (1.1) when the initial value is of the form $\lambda\varphi$, where $\lambda>0$ is a parameter.

The life span asymptotics of semilinear parabolic Cauchy problems give insight on how the "size" of the initial value affects the blowup time of their positive solutions; see [7, 8, 10, 11, 15, 17] and the references therein. Given two functions $f,g:[0,\infty)\to[0,\infty)$, let us say that $f\sim g$ near $c\in\{0,\infty\}$ if there exist two positive constants C_1, C_2 such that $C_1f(r)\ge g(r)\ge C_2f(r)$ for all r which are sufficiently close to c. In [11] it was proved, initially for $k(t)\equiv 1$ and $\alpha=2$, that $T_{\lambda\varphi}\sim\lambda^{-\beta}$ near ∞ provided $\varphi\ge 0$ is bounded, continuous and does not vanish identically. Later on, Gui and Wang [7] showed that, in fact, $\lim_{\lambda\to\infty}T_{\lambda\varphi}\cdot\lambda^\beta=\beta^{-1}\|\varphi\|^{-\beta}_{L^\infty(\mathbb{R}^d)}$. The behavior of $T_{\lambda\varphi}$ as λ approaches 0 was also investigated by Lee and Ni in [11]. One of their results addresses the case of initial values $\varphi\ge 0$ obeying growth conditions of the form $0<\liminf_{|x|\to\infty}|x|^a\varphi(x)$ and $\limsup_{|x|\to\infty}|x|^a\varphi(x)<\infty$, where $a>0$ is a given constant different from d. They proved that, in this case,

$$T_{\lambda\varphi}\sim(1/\lambda)^{(1/\beta-\frac{1}{2}\min\{a,d\})^{-1}}\quad\text{as }\lambda\to 0. \tag{1.4}$$

In the present paper we obtain upper and lower bounds for the life span $T_{\lambda\varphi}$ of (1.1), and provide in this way a description of the behavior of $T_{\lambda\varphi}$ as $\lambda\to\infty$ and $\lambda\to 0$. Here is a brief outline.

First we prove that any combination of positive parameters d,α,ρ,β, obeying $0<d\rho\beta/\alpha<1$, yields finite time blow up of any nontrivial positive solution of (1.1). This is carried out by bounding from below the mild solution of (1.1) by a subsolution which locally grows to ∞. Finite-time blowup of (1.1) is then inferred from a classical comparison procedure that dates back to [9] (see also [3], Sect. 3). The construction of our subsolution uses the Feynman-Kac representation of (1.1), and requires to control the decay of certain conditional probabilities of $W\equiv\{W(t),\ t\ge 0\}$, where W is the $\mathbb{R}^d$-valued Markov process corresponding to the evolution system $\{U(t,s),\ t\ge s\ge 0\}$; see [3], [4] and [13] for the time-homogeneous case.

A further consequence of the Feynman-Kac representation of (1.1) is the inequality $T_{\lambda\varphi} \le \text{Const.}\lambda^{-\frac{\alpha\beta}{\alpha-d\rho\beta}}$, which holds for small positive λ when $0 < d\rho\beta/\alpha < 1$. This, together with the lower bound of $T_{\lambda\varphi}$ given in Section 6, implies (again under the condition $0 < d\rho\beta/\alpha < 1$) that

$$T_{\lambda\varphi} \sim \lambda^{-\frac{\alpha\beta}{\alpha-d\rho\beta}} \tag{1.5}$$

near 0. Note that (1.5) yields (1.4) when $d < a$, $\alpha = 2$ and $\rho = 1$. We also provide an upper bound for $T_{\lambda\varphi}$ which is valid for all $\lambda > 0$, namely

$$T_{\lambda\varphi} \le \Big(C\lambda^{-\beta} + [(10\varepsilon_2/\varepsilon_1)^{1/\rho}\theta]^{\frac{\alpha-d\rho\beta}{\alpha}}\Big)^{\frac{\alpha}{\alpha-d\rho\beta}} + \eta, \tag{1.6}$$

where C, θ and η are suitable positive constants.

We remark that many of our arguments rely on the assumption $d\rho\beta/\alpha < 1$. Thus, the blowup behavior and life span asymptotics of (1.1) in the "critical" case $d\rho\beta/\alpha = 1$ remain to be investigated.

As this paper is partly aimed at the multidisciplinary reader, in the next section we recall some basic facts regarding the Feynman-Kac formula. In Section 3 we obtain semigroup and bridge estimates that we shall need in the sequel. Section 4 is devoted to prove that (1.1) does not admit nontrivial global solutions if $d < \frac{\alpha}{\rho\beta}$. In the remaining sections 5 and 6 we prove our bounds for the life span of (1.1).

2. The Feynman-Kac representation and subsolutions

For any $T > 0$ let us consider the initial-value problem

$$\begin{gathered}\frac{\partial\varrho(t,x)}{\partial t} = k(t)\Delta_\alpha\varrho(t,x) + \zeta(t,x)\varrho(t,x), \quad 0 < t \le T,\\ \varrho(0,x) = \varphi(x), \quad x \in \mathbb{R}^d,\end{gathered} \tag{2.1}$$

where $k : [0,\infty) \to [0,\infty)$ is integrable on any bounded interval, and ζ and φ are nonnegative bounded continuous functions on $[0,T] \times \mathbb{R}^d$ and $\mathbb{R}^d$, respectively. It is well known that in the classical setting $k \equiv 1$, $\alpha = 2$, $\zeta(t,x) \equiv \zeta(x)$, the solution of (2.1) can be expressed via the Feynman-Kac formula, see e.g. [5]. Theorem 2.1 below gives the Feynman-Kac representation corresponding to (2.1).

Let $W \equiv \{W(t)\}_{t\ge0}$ be the (time-inhomogeneous) càdlàg Feller process corresponding to the family of generators $\{k(t)\Delta_\alpha\}_{t\ge0}$. Note that W can be constructed by performing a deterministic time change of the symmetric α-stable process $Z \equiv \{Z(t)\}_{t\ge0}$. We designate P_x the distribution of $\{W(t)\}_{t\ge0}$ such that $P_x[W(0) = x] = 1$, and write E_x for the expectation with respect to P_x, $x \in \mathbb{R}^d$.

Theorem 2.1. *Let k, ζ and φ be as above. Then, the solution of* (2.1) *admits the Feynman-Kac representation*

$$\varrho(t,x) = E_x\Big[\varphi(W(t))\exp\{\int_0^t \zeta(t-s,W(s))\,ds\}\Big], \quad (t,x) \in [0,T] \times \mathbb{R}^d. \tag{2.2}$$

Proof. The proof is a simple adaptation to our time-inhomogeneous context of the approach used in [1]. We shall assume that $0 < \alpha < 2$; the case $\alpha = 2$ is simpler and can be handled in a similar fashion.

Recall [18] that there exists a Poisson random measure $N(dt,dx)$ on $[0,\infty)\times\mathbb{R}^d\backslash\{0\}$ having expectation $EN(dt,dx)=dt\,\nu(dx)$, with

$$\nu(dx)=\frac{\alpha 2^{\alpha-1}\Gamma((\alpha+d)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)\|x\|^{\alpha+d}}\,dx,$$

and such that the paths of Z admit the Lévy-Itô decomposition

$$Z(t)=\int_{|x|<1}x\widetilde{N}(t,dx)+\int_{|x|\geq 1}xN(t,dx),\quad t\geq 0, \tag{2.3}$$

where $N(t,dx):=\int_0^t N(dt,dx)$, and $\widetilde{N}(t,dx)$ is the compensated Poisson random measure

$$\widetilde{N}(t,B)=N(t,B)-t\nu(B),\quad t\geq 0,\ B\in\mathcal{B}(\mathbb{R}^d);$$

here $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-algebra in $\mathbb{R}^d$. The process W also admits a Lévy-Itô decomposition, with corresponding Poisson random measure $k(t)N(dt,dx)$.

Let us write $W(p^-)$ for the limit of W from the left of p. From the integration by parts formula we obtain

$$\begin{aligned}
&d\big[\varrho(t-s,W(s))\exp\big\{\int_0^s\zeta(t-r,W(r))dr\big\}\big]\\
&=\varrho(t-s,W(s^-))\zeta(t-s,W(s^-))\exp\big\{\int_0^s\zeta(t-r,W(r^-))dr\big\}\,ds\\
&\quad+\exp\big\{\int_0^s\zeta(t-r,W(r^-))\,dr\big\}\,d\varrho(t-s,W(s)).
\end{aligned}$$

Using Itô's formula [1, Section 4.4], to calculate $d\varrho(t-s,W(s))$, we obtain

$$\begin{aligned}
&d\big[\varrho(t-s,W(s))\exp\big\{\int_0^s\zeta(t-r,W(r))dr\big\}\big]\\
&=\exp\big\{\int_0^s\zeta(t-r,W(r^-))\,dr\big\}\\
&\quad\times\Big\{\varrho(t-s,W(s^-))\zeta(t-s,W(s^-))\,ds-\frac{d}{ds}\varrho(t-s,W(s^-))\\
&\quad+k(s)\int_{|x|<1}\big[\varrho(t-s,W(s^-)+x)-\varrho(t-s,W(s^-))\big]\widetilde{N}(ds,dx)\\
&\quad+k(s)\int_{|x|\geq 1}\big[\varrho(t-s,W(s^-)+x)-\varrho(t-s,W(s^-))\big]N(ds,dx)\\
&\quad+k(s)\int_{|x|<1}\Big[\varrho(t-s,W(s^-)+x)-\varrho(t-s,W(s^-))\\
&\quad-\sum_i x_i\frac{d}{dx_i}\varrho(t-s,W(s^-))\Big]\nu(dx)ds\Big\}.
\end{aligned}$$

Integrating from 0 to t, and taking expectation with respect to P_x, yields

$$
\begin{aligned}
&E_x\Big[\varphi(W(t))\exp\Big\{\int_0^t \zeta(t-s,W(s))\,ds\Big\}\Big]-\varrho(t,x)\\
&=E_x\int_0^t \exp\Big\{\int_0^s \zeta(t-r,W(r^-))\,dr\Big\}\\
&\quad\times\Big\{\varrho(t-s,W(s^-))\zeta(t-s,W(s^-))-\frac{d}{ds}\varrho(t-s,W(s^-))\\
&\quad+k(s)\int_{|x|<1}\Big[\varrho(t-s,W(s^-)+x)-\varrho(t-s,W(s^-))\\
&\quad-\sum_i x_i\frac{d}{dx_i}\varrho(t-s,W(s^-))\Big]\nu(dx)\\
&\quad+k(s)\int_{|x|\ge 1}\big[\varrho(t-s,W(s^-)+x)-\varrho(t-s,W(s^-))\big]\,\nu(dx)\Big\}\,ds\\
&=0,
\end{aligned}
$$

where in the first equality we used the identity $\widetilde{N}(ds,dx)=N(ds,dx)-ds\,\nu(dx)$, and the fact that the stochastic integrals with respect to $\widetilde{N}(ds,dx)$ are martingales, and therefore have expectation 0. □

The Feynman-Kac representation is suitable for constructing subsolutions of reaction-diffusion equations of the type

$$\frac{\partial w(t,y)}{\partial t}=k(t)\Delta_\alpha w(t,y)+w^{1+\beta}(t,y),\qquad w(0,y)=\varphi(y),\quad y\in\mathbb{R}^d, \tag{2.4}$$

where $\beta>0$ is a constant, and k, φ are as in (2.1). From Theorem 2.1 we know that

$$w(t,y)=E_y\Big[\varphi(W(t))\exp\Big(\int_0^t w^\beta(t-s,W(s))\,ds\Big)\Big],\quad (t,y)\in[0,T]\times\mathbb{R}^d,$$

for every positive $T<T_\varphi$. Hence, for every $y\in\mathbb{R}^d$,

$$w(t,y)\;\ge\;E_y[\varphi(W(t))]=:v_0(t,y),\quad t\ge 0,$$

so that v_0 is a subsolution of (2.4); i.e., $w(0,\cdot)=v_0(0,\cdot)$ and $w(t,\cdot)\ge v_0(t,\cdot)$ for every $t>0$. The next lemma, which we will need in the following section, is a direct consequence of the Feynman-Kac representation.

Lemma 2.2. *Let k, φ be as in* (2.1)*, and let $\zeta(\cdot,\cdot)$ be a nonnegative, bounded and continuous subsolution of* (2.4)*. Then, any solution of*

$$\frac{\partial\varrho(t,y)}{\partial t}=k(t)\Delta_\alpha\varrho(t,y)+\zeta^\beta(t,y)\varrho(t,y),\qquad \varrho(0,\cdot)=\varphi,$$

remains a subsolution of (2.4)*.*

3. Bridge and semigroup bounds

Let us denote by $p(t,x)$, $t\ge 0$, $x\in\mathbb{R}^d$, the transition densities of the d-dimensional symmetric α-stable process $\{Z(t)\}_{t\ge 0}$. Recall that $p(t,\cdot)$, $t>0$, are strictly positive, radially symmetric continuous functions that satisfy the following properties.

Lemma 3.1. *For any $s,t>0$, and $x,y\in\mathbb{R}^d$, $p(t,x)$ satisfies*

(i) $p(ts,x)=t^{-\frac{d}{\alpha}}p(s,t^{-\frac{1}{\alpha}}x)$,
(ii) $p(t,x)\le p(t,y)$ *when* $|x|\ge|y|$,
(iii) $p(t,x)\ge(\frac{s}{t})^{d/\alpha}p(s,x)$ *for* $t\ge s$,
(iv) $p(t,\frac{1}{\tau}(x-y))\ge p(t,x)p(t,y)$ *if* $p(t,0)\le 1$ *and* $\tau\ge 2$.

For a proof of the above lemma, see [6, page 493] or [20, pages 46 and 47].

Let $\varphi:\mathbb{R}^d\to[0,\infty)$ be bounded and measurable, and let $k:[0,\infty)\to[0,\infty)$ be locally integrable. Notice that the transition probabilities of the Markov process $\{W(t),\, t\ge 0\}$ are given by

$$P(W(t)\in dy|W(s)=x)=p(\textstyle\int_s^t k(r)\,dr,y-x)\,dy,\quad 0\le s\le t,\ x\in\mathbb{R}^d. \tag{3.1}$$

We define the function

$$v_0(t,x)=E_x\left[\varphi(W(t))\right]=E_x\left[\varphi(Z(K(t,0)))\right]=\int p(K(t,0),y-x)\varphi(y)\,dy, \tag{3.2}$$

where $t\ge 0$, $x\in\mathbb{R}^d$, $K(t,s):=\int_s^t k(r)dr$, $0\le s\le t$, and write $B(r)\equiv B_r\subset\mathbb{R}^d$ for the ball of radius r, centered at the origin.

Lemma 3.2. *There exists a constant $c_0>0$ satisfying*

$$v_0(t,x)\ge c_0K^{-d/\alpha}(t,0)1_{B_1}\big(K^{-1/\alpha}(t,0)x\big) \tag{3.3}$$

for all $x\in\mathbb{R}^d$, and all $t>0$ such that $K(t,0)\ge 1$.

Proof. From Lemma 3.1 (i), (ii) and the radial symmetry of $p(t,\cdot)$ we have, for $K^{1/\alpha}(t,0)\ge 1$, $x\in B_{K^{1/\alpha}(t,0)}$ and $z\in\partial B_2$, that

$$\begin{aligned}
v_0(t,x)&=E_0[\varphi(Z(K(t,0))+x)]\\
&=E_0\big[\varphi(K^{1/\alpha}(t,0)(Z(1)+K^{-1/\alpha}(t,0)x))\big]\\
&\ge\int_{B_1}\varphi(K^{1/\alpha}(t,0)y)P\Big[Z(1)\in dy-K^{-1/\alpha}(t,0)x\Big]\\
&=\int_{B_1}\varphi(K^{1/\alpha}(t,0)y)p(1,y-K^{-1/\alpha}(t,0)x)dy\\
&\ge p(1,z)\int_{B_1}\varphi(K^{1/\alpha}(t,0)y)\,dy\\
&=p(1,z)K^{-d/\alpha}(t,0)\int_{B_{K^{1/\alpha}(t,0)}}\varphi(y)dy\\
&\ge p(1,z)K^{-d/\alpha}(t,0)1_{B_1}(K^{-1/\alpha}(t,0)x)\int_{B_1}\varphi(y)dy.
\end{aligned}$$

Letting $c_0=p(1,z)\int_{B_1}\varphi(y)dy$ yields (3.3). $\square$

Fix $\theta>0$ such that (1.2) holds for all $t\ge\theta$ and such that $K(\theta,0)\ge 1$. Define $\delta_0=\min\{(\frac{\varepsilon_1}{2\varepsilon_2})^{1/\rho},1-(\frac{\varepsilon_1}{2\varepsilon_2})^{1/\rho}\}$.

Lemma 3.3. *There exists $c>0$ such that for all x, $y\in B_1$ and $t\ge\theta/\delta_0$,*

$$P_x\big[W(s)\in B_{K^{1/\alpha}(t-s,0)}:W(t)=y\big]\ge c$$

for $s\in[\theta,\delta_0 t]$.

Proof. Using (1.2) and Lemma 3.1 (i), we obtain

$$\begin{aligned} &P_x\big[W(s) \in B_{K^{1/\alpha}(t-s,0)} : W(t) = y\big] \\ &= \int_{B_{K^{1/\alpha}(t-s,0)}} \frac{p(K(s,0), x-z)p(K(t,s), z-y)}{p(K(t,0), x-y)}\,dz \\ &\geq \int_{B_{rs^{\rho/\alpha}}} \frac{K^{-d/\alpha}(s,0)}{K^{-d/\alpha}(t,0)} \\ &\quad\times \frac{p(1, K^{-1/\alpha}(s,0)(x-z))K^{-d/\alpha}(t,s)p(1, K^{-1/\alpha}(t,s)(z-y))}{p(1, K^{-1/\alpha}(t,0)(x-y))}\,dz, \end{aligned} \tag{3.4}$$

with $r = \varepsilon_1^{1/\alpha}(\frac{1}{\delta_0} - 1)^{\rho/\alpha}$. It is straightforward to verify that $K^{-1/\alpha}(s,0)(x-z) \in B_{r_1}$, where $r_1 = 2(\frac{\varepsilon_2}{\varepsilon_1})^{1/\alpha}\delta^{\frac{\rho}{\alpha}}$ with $\delta = \max\{1, \frac{1}{\delta_0} - 1\}$, hence Lemma 3.1 (ii) and radial symmetry of $p(t,\cdot)$ imply

$$p(1, K^{-1/\alpha}(s,0)(x-z)) \geq p(1, \varsigma) \equiv c_1$$

for any $\varsigma \in \partial B_{r_1}$.

Thus, the term in the right-hand side of (3.4) is bounded from below by

$$\int_{B_{rs^{\rho/\alpha}}} \frac{c_1 K^{-d/\alpha}(s,0)K^{-d/\alpha}(t,s)p(1, K^{-\frac{1}{\alpha}}(t,s)(z-y))}{K^{-d/\alpha}(t,0)p(1, K^{-1/\alpha}(t,0)(x-y))}\,dz.$$

Using (1.2), and the facts that $K(t,0) \geq K(t,s)$ and $p(t,x) \leq p(t,0)$ for all $t > 0$ and $x \in \mathbb{R}^d$, it follows that

$$P_x\big[W(s) \in B_{K^{1/\alpha}(t-s,0)} : W(t) = y\big] \geq \int_{B_{rs^{\rho/\alpha}}} c_2 s^{-d\rho/\alpha} p(1, K^{-1/\alpha}(t,s)(z-y))dz, \tag{3.5}$$

where $c_2 = \frac{c_1\varepsilon_2^{-d/\alpha}}{p(1,0)}$. Since $\theta \leq s \leq \delta_0 t$, we have from (1.2) and the definition of δ_0 that

$$K^{-1/\alpha}(t,s) = [K(t,0) - K(s,0)]^{-1/\alpha} \leq (\varepsilon_1 t^\rho - \varepsilon_2\delta_0^\rho t^\rho)^{-1/\alpha} \leq c_3 t^{-\rho/\alpha},$$

where $c_3 = (\frac{2}{\varepsilon_1})^{1/\alpha}$. Since $y \in B_1$, $z \in B_{rs^{\rho/\alpha}}$ and $\theta \leq s \leq \delta_0 t$, we deduce that, for $t \geq 1$, $y \in B_{t^{\rho/\alpha}}$ and $z \in B_{r\delta_0^{\rho/\alpha}t^{\rho/\alpha}}$. Letting $\gamma = \max\{1,\ r\delta_0^{\rho/\alpha}\}$, it follows that $z - y \in B_{2\gamma t^{\rho/\alpha}}$, and thus $K^{-1/\alpha}(t,s)(z-y) \in B_{2\gamma c_3}$. Therefore,

$$p(1, K^{-1/\alpha}(t,s)(z-y)) \geq p(1,\varsigma) \equiv c_4$$

for any $\varsigma \in \partial B_{2\gamma c_3}$. From (3.5) we conclude that

$$P_x\big[W(s) \in B_{K^{1/\alpha}(t-s,0)}\ : W(t) = y\big] \geq \int_{B_{rs^{\rho/\alpha}}} c_5 s^{-d\rho/\alpha}dz \equiv c.$$

□

4. Nonexistence of positive global solutions

In this section we shall use the Feynman-Kac representation to construct a subsolution of (1.1) which grows to infinity uniformly on the unit ball. As we are going to prove afterward, this guarantees nonexistence of nontrivial positive solutions of (1.1).

Let v solve the semilinear nonautonomous equation

$$\begin{gathered}\frac{\partial v(t,x)}{\partial t} = k(t)\Delta_\alpha v(t,x) + v_0^\beta(t,x)v(t,x),\\ v(0,x)=\varphi(x), \quad x\in\mathbb{R}^d,\end{gathered} \tag{4.1}$$

where k and φ are as in (1.1), and v_0 is defined in (3.2). Since $v_0 \le u$, where u is the solution of (1.1), it follows from Lemma 2.2 that $v \le u$ as well. Without loss of generality we shall assume that φ does not a.e. vanish on the unit ball.

Proposition 4.1. *There exist c', $c'' > 0$ such that, for all $x \in B_1$ and all $t > 0$ large enough,*

$$v(t,x) \ge c' t^{-d\rho/\alpha} \exp(c'' t^{1-\frac{d\rho\beta}{\alpha}}).$$

Proof. In the sequel, $c_0, c_1, \dots, c_8$ denote suitable positive constants, $c_0, \dots, c_5$ being defined in Lemma 3.3. From Theorem 2.1 we know that

$$v(t,x) = \int_{\mathbb{R}^d} \varphi(y)p(K(t,0),x-y)E_x\Big[\exp\int_0^t v_0^\beta(t-s,W(s))ds \,\Big|\, W(t)=y\Big]dy.$$

Let θ and δ_0 be as in Lemma 3.3. For any $\theta \le s \le \delta_0 t$, we have $t-s \ge t-\delta_0 t = (1-\delta_0)t \ge \delta_0 t \ge \theta$, and therefore $K^{\frac{1}{\alpha}}(t-s,0) \ge 1$. From here, using (3.3) and Jensen's inequality, we get

$$\begin{aligned} v(t,x) &\ge \int_{\mathbb{R}^d} \varphi(y)p(K(t,0),x-y)\\ &\quad\times E_x\Big[\exp\int_\theta^{\delta_0 t} c_0^\beta K^{-d\beta/\alpha}(t-s,0)1_{B_{K^{\frac{1}{\alpha}}(t-s,0)}}(W(s))\,ds \,\Big|\, W(t)=y\Big]dy\\ &\ge \int_{B_1}\varphi(y)p(K(t,0),x-y)\\ &\quad\times \exp\Big\{\int_\theta^{\delta_0 t} c_0^\beta K^{-\frac{d\beta}{\alpha}}(t-s,0)P_x\Big[W(s)\in B_{K^{\frac{1}{\alpha}}(t-s,0)} \,\Big|\, W(t)=y\Big]\,ds\Big\}\,dy. \end{aligned}$$

It follows from Lemma 3.1 and Lemma 3.3 that

$$\begin{aligned} v(t,x) &\ge \int_{B_1}\varphi(y)p(K(t,0),x-y)\exp\int_\theta^{\delta_0 t} c_6K^{-d\beta/\alpha}(t-s,0)\,ds\,dy\\ &= \int_{B_1}\varphi(y)K^{-d/\alpha}(t,0)p(1,K^{-1/\alpha}(t,0)(x-y))dy\\ &\quad\times\exp\int_\theta^{\delta_0 t} c_6K^{-d\beta/\alpha}(t-s,0)\,ds. \end{aligned}$$

Let x, $y \in B_1$. Then $K^{-1/\alpha}(t,0)(x-y)\in B_2$. Radial symmetry of $p(t,\cdot)$ implies

$$p\big(1,K^{-1/\alpha}(t,0)(x-y)\big) \ge p(1,\varsigma) \equiv c_7$$

for any $\varsigma\in\partial B_2$. Therefore,

$$v(t,x) \ge \int_{B_1} c_7\varphi(y)K^{-d/\alpha}(t,0)dy\exp\int_\theta^{\delta_0 t} c_6K^{-d\beta/\alpha}(t-s,0)ds. \tag{4.2}$$

Let $c_8 = c_7 \int_{B_1} \varphi(y)dy$. Using (1.2) and the fact that $K(t,0) \geq K(t-s,0)$, the term in the right of (4.2) is bounded below by

$$\begin{aligned} &c_8 K^{-d/\alpha}(t,0)\exp\Big(c_6\int_\theta^{\delta_0 t} K^{-\frac{d\beta}{\alpha}}(t,0)ds\Big)\\ &\geq c_8\varepsilon_2^{-d/\alpha} t^{-d\rho/\alpha}\exp\big[c_6\varepsilon_2^{-d\beta/\alpha}(\delta_0 t^{1-\frac{d\beta\rho}{\alpha}} - \theta t^{-\frac{d\beta\rho}{\alpha}})\big]\end{aligned}$$

if $t > 0$ is large. It follows that

$$v(t,x) \geq c' t^{-\frac{d\rho}{\alpha}} \exp(c'' t^{1-\frac{d\beta\rho}{\alpha}}) \tag{4.3}$$

for sufficiently large t, where $c' = c_8\varepsilon_2^{-d/\alpha}\exp\big(-c_6\theta\varepsilon_2^{-d\beta/\alpha}\big)$ and $c'' = c_6\delta_0\varepsilon_2^{-d\beta/\alpha}$. □

As a consequence of Proposition 4.1, if $0 < \frac{d\beta\rho}{\alpha} < 1$, then $\inf_{x\in B_1} v(t,x) \to \infty$ when $t \to \infty$. As v is subsolution of Equation (1.1), this implies that

$$C(t) := \inf_{x\in B_1} u(t,x) \to \infty \quad \text{as } t \to \infty. \tag{4.4}$$

We need the following lemma.

Lemma 4.2. *Let*

$$\xi_1 := \min_{x\in B_1}\min_{0\leq r\leq 1} P_x[Z(r)\in B_1],$$

where $\{Z(t),\, t\geq 0\}$ denotes the symmetric α-stable process. Then

(i) $\xi_1 > 0$
(ii) *For any $0 \leq s \leq t$,*

$$\xi := \int_{B_1} p(K(t_0+t,t_0), y-x)\,dy \geq \int_{B_1} p(K(t+t_0,s+t_0),y-x)\,dv \geq \xi_1^{\lfloor K(t+t_0,s+t_0)\rfloor},$$

where $\lfloor x\rfloor$ denotes the least integer no smaller than $x\in[0,\infty)$.

Proof. Let $\epsilon > 0$, and let $f_\epsilon : \mathbb{R}^d \to \mathbb{R}_+$ be a continuous function bounded by 1 such that $f_\epsilon = 1$ on B_1, and $f_\epsilon = 0$ on the complement of the ball $B_{1+\epsilon}$. Because of strong continuity of the semigroup $\{S(t)\}_{t\geq 0}$ corresponding to $\{Z(t),\, t\geq 0\}$, $\lim_{t\to 0} S(t)f_\epsilon = f_\epsilon$ uniformly on $\mathbb{R}^d$. Moreover, uniformly on B_1,

$$1_{B_1} = \lim_{\epsilon\downarrow 0} f_\epsilon = \lim_{\epsilon\downarrow 0}\lim_{t\to 0} S(t)f_\epsilon = \lim_{t\to 0}\lim_{\epsilon\downarrow 0} S(t)f_\epsilon = \lim_{t\to 0} S(t)1_{B_1},$$

where the last equality follows from the bounded convergence theorem. Hence, there exists $\epsilon \equiv \epsilon_{1/2} > 0$ such that $\sup_{x\in B_1}|S(t)1_{B_1}(x) - 1_{B_1}(x)| < \frac{1}{2}$ for all $t < \epsilon$, which implies

$$P_x[Z(r)\in B_1] = S(r)1_{B_1}(x) \geq \frac{1}{2}$$

for all $r < \epsilon$ and $x \in B_1$. Therefore, in order to prove (i) it suffices to show that

$$\inf_{\epsilon\leq r\leq 1} P_x[Z(r)\in B_1] > \mathrm{A} \tag{4.5}$$

for all $x \in B_1$, where the constant A > 0 does not depend on x. Using Lemma 3.1 (i) we get

$$\begin{aligned}
P_x[Z(r) \in B_1] &= \int_{B_1} p(r, z-x)\, dz \\
&= r^{-d/\alpha} \int_{B_1} p(1, r^{-1/\alpha}(z-x))\, dz \\
&\geq \inf_{y \in B_{2\epsilon^{-1/\alpha}}} p(1,y) \int_{B_1} dz,
\end{aligned}$$

which yields (4.5).

The assertion in part (ii) is deduced from the Chapman-Kolmogorov equation as follows. If $l := K(t_0+t, t_0+s) \leq 1$, the statement follows directly from part (i). If $l > 1$, then

$$\begin{aligned}
P_x[Z(l) \in B_1] &= \int_{B_1} \int_{\mathbb{R}^d} p(l-1, y-x) p(1, z-y)\, dy\, dz \\
&\geq \int_{B_1} \int_{B_1} p(l-1, y-x) p(1, z-y)\, dy\, dz \\
&= S(l-1)((S(1)1_{B_1})1_{B_1}) \\
&\geq \xi_1 P_x[Z(l-1) \in B_1].
\end{aligned}$$

Thus, applying the above procedure $\lfloor l-1 \rfloor$ times we obtain the assertion. $\square$

Now we are ready to prove that (4.4) is sufficient to guarantee finite-time blow up of (1.1).

Theorem 4.3. *If $0 < \frac{d\rho\beta}{\alpha} < 1$, then all nontrivial positive solutions of* (1.1) *are nonglobal.*

Proof. Let u be the solution of (1.1), and let $t_0 > 0$ be such that $\|u(t_0, \cdot)\|_\infty < \infty$. Then

$$\begin{aligned}
u(t+t_0, x) &= \int_{\mathbb{R}^d} p(K(t+t_0, t_0), y-x) u(t_0, y) dy \\
&\quad + \int_0^t \int_{\mathbb{R}^d} p(K(t+t_0, s+t_0), y-x) u^{1+\beta}(s+t_0, y)\, dy\, ds \\
&\geq \int_{B_1} p(K(t+t_0, t_0), y-x) u(t_0, y) dy \\
&\quad + \int_0^t \int_{B_1} p(K(t+t_0, s+t_0), y-x) u^{1+\beta}(s+t_0, y)\, dy\, ds.
\end{aligned}$$

Therefore, $w(t, \cdot) := u(t_0+t, \cdot)$ satisfies

$$\begin{aligned}
w(t,x) &\geq C(t_0) \int_{B_1} p(K(t+t_0, t_0), y-x)\, dy \\
&\quad + \int_0^t \int_{B_1} p(K(t+t_0, s+t_0), y-x) \big(\min_{z \in B_1} w(s,z)\big)^{1+\beta}\, dy\, ds.
\end{aligned}$$

It follows from Lemma 4.2 that for all $t \in [0,1]$,

$$\min_{x \in B_1} w(t,x) \geq \xi C(t_0) + \xi \int_0^t \big(\min_{z \in B_1} w(s,z)\big)^{1+\beta} ds.$$

We put $\mathrm{w}(t) \equiv \min_{z\in B_1} w(t,z)$, $t \geq 0$, and consider the integral equation

$$v(t) = \xi C(t_0) + \xi \int_0^t v^{1+\beta}(s)\,ds,$$

whose solution satisfies

$$v^\beta(t) = \frac{[\xi C(t_0)]^\beta}{1 - \beta \xi^{1+\beta} C^\beta(t_0) t}. \tag{4.6}$$

Choosing t_0 large enough that the blow up time of v is smaller than one, renders

$$\mathrm{w}(1) = \min_{x\in B_1} w(1,x) \geq v(1) = \infty,$$

which proves blow up of u at time $t_0 + 1$. □

5. Upper estimates for the life span

In this section we obtain two upper bounds for the life span of the solution to (1.1) with initial value $u(0,\cdot) = \lambda\varphi(\cdot)$, where λ is a positive parameter. We first consider the case of small and positive λ.

Proposition 5.1. *If $0 < d\rho\beta/\alpha \leq n/(n+1)$ with $n \in \mathbb{N}$, then there exists a constant $C_n > 0$ such that for all sufficiently small $\lambda > 0$,*

$$T_{\lambda\varphi} \leq C_n \lambda^{-\frac{\alpha\beta}{\alpha - d\beta\rho}}.$$

Proof. From (4.3) and (4.4) it follows that

$$C(t) \geq \lambda c' t^{-d\rho/\alpha} \exp\big(c'' t^{1-\frac{d\beta\rho}{\alpha}}\big)$$

for all $t \geq \frac{\theta}{\delta_0}$; here we require $t \geq \theta/\delta_0$ in order to have an interval $[\theta, \delta_0 t]$ so that $K^{-1/\alpha}(t,s) \leq \text{Const.} t^{-\rho/\alpha}$ for $s \in [\theta, \delta_0 t]$, and then to use (3.5).

Recall from (4.6) that $v(1) = \infty$ provided $\beta\xi^{1+\beta} C^\beta(t_0) = 1$. Note that $t_0 \leq t_1$, where t_1 is such that $t_1 \geq \frac{\theta}{\delta_0}$ and

$$\beta \xi^{1+\beta} \lambda^\beta (c')^\beta t_1^{-d\beta\rho/\alpha} \exp\big(\beta c'' t_1^{1-\frac{d\beta\rho}{\alpha}}\big) = 1.$$

Choosing $\theta \geq \delta_0$, from the inequality $e^x \geq \frac{x^{n+1}}{(n+1)!}$ and the fact that the condition $0 < \frac{d\rho\beta}{\alpha} \leq \frac{n}{n+1}$ implies $\frac{d\rho\beta}{n\alpha} \leq 1 - \frac{d\rho\beta}{\alpha}$, we have that $t_1 \leq t_2$, where t_2 is such that

$$t_2 \geq \frac{\theta}{\delta_0} \quad \text{and} \quad \frac{1}{(n+1)!} \beta^{n+2} \xi^{1+\beta} (c')^\beta (c'')^{n+1} \lambda^\beta t_2^{1-\frac{d\beta\rho}{\alpha}} = 1,$$

which is the same as

$$t_2 = \Big[\frac{(n+1)!}{\beta^{n+2}\xi^{1+\beta}(c')^\beta(c'')^{n+1}}\Big]^{\frac{\alpha}{\alpha - d\beta\rho}} \lambda^{-\frac{\alpha\beta}{\alpha - d\beta\rho}}.$$

Choosing

$$C_n = \Big[\frac{(n+1)!}{\beta^{n+2}\xi^{1+\beta}(c')^\beta(c'')^{n+1}}\Big]^{\frac{\alpha}{\alpha - d\beta\rho}}$$

renders $t_0 \leq t_1 \leq t_2 = C_n \lambda^{-\frac{\alpha\beta}{\alpha - d\beta\rho}}$. Hence $T_{\lambda\varphi} \leq C_n \lambda^{-\frac{\alpha\beta}{\alpha - d\beta\rho}}$ for all $\lambda > 0$ such that $C_n \lambda^{-\frac{\alpha\beta}{\alpha - d\beta\rho}} \geq \frac{\theta}{\delta_0}$. □

Let us define
$$\upsilon(t)=\int_{\mathbb{R}^d}p(K(t,0),x)u(t,x)dx,$$
where u is the solution of (1.1), and let $\theta>0$ be such that (1.2) holds for all $t\ge\theta$.

Lemma 5.2. *If there exist $\tau_0\ge\theta$ such that $\upsilon(t)=\infty$ for $t\ge\tau_0$, then the solution to* (1.1) *blows up in finite time.*

Proof. Since $p(K(t,0),0)=K(t,0)^{-d/\alpha}p(1,0)$ and $K(t,0)\ge\varepsilon_1t^\rho$ for all $t\ge\theta$, we can assume, by taking τ_0 bigger if necessary, that $p(K(t,0),0)\le 1$ for all $t\ge\tau_0$.

If $\tau_0\le\varepsilon_1^{1/\rho}t$ and $\varepsilon_1^{1/\rho}t\le r\le(2\varepsilon_1)^{1/\rho}t$, we have, from the conditions on $k(t)$, that
$$\begin{aligned}
\tau&\equiv\Big[\frac{K((10\varepsilon_2)^{1/\rho}t,r)}{K(r,0)}\Big]^{1/\alpha}=\Big[\frac{K((10\varepsilon_2)^{1/\rho}t,0)-K(r,0)}{K(r,0)}\Big]^{1/\alpha}\\
&\ge\Big[\frac{K((10\varepsilon_2)^{1/\rho}t,0)}{K((2\varepsilon_1)^{1/\rho}t,0)}-1\Big]^{1/\alpha}\\
&\ge\Big[\frac{\varepsilon_1(10\varepsilon_2)t^\rho}{\varepsilon_2(2\varepsilon_1)t^\rho}-1\Big]^{1/\alpha}=4^{1/\alpha}\ge 2.
\end{aligned}$$
Using properties (i) and (iv) in Lemma 3.1, with $\tau=\big[\frac{K((10\varepsilon_2)^{1/\rho}t,r)}{K(r,0)}\big]^{1/\alpha}$, yields
$$\begin{aligned}
&p\Big(K\big((10\varepsilon_2)^{1/\rho}t,r\big),x-y\Big)\\
&=p\Big(K(r,0)\Big[\frac{K((10\varepsilon_2)^{1/\rho}t,r)}{K(r,0)}\Big],x-y\Big)\\
&=\Big[\frac{K(r,0)}{K((10\varepsilon_2)^{1/\rho}t,r)}\Big]^{d/\alpha}p(K(r,0),\Big[\frac{K(r,0)}{K((10\varepsilon_2)^{1/\rho}t,r)}\Big]^{1/\alpha}(x-y))\\
&\ge\Big[\frac{K(r,0)}{K((10\varepsilon_2)^{1/\rho}t,r)}\Big]^{d/\alpha}p(K(r,0),x)p(K(r,0),y).
\end{aligned}$$
Since $\upsilon(t)=\infty$ for all $t\ge\tau_0$, it follows that
$$\begin{aligned}
&\int_{\mathbb{R}^d}p\big(K((10\varepsilon_2)^{1/\rho}t,r),x-y\big)u(r,y)dy\\
&\ge\Big[\frac{K(r,0)}{K((10\varepsilon_2)^{1/\rho}t,r)}\Big]^{d/\alpha}p(K(r,0),x)\upsilon(r)=\infty.
\end{aligned}$$
The solution $u(t,x)$ of (1.1) satisfies
$$\begin{aligned}
u(t,x)&=\lambda\int_{\mathbb{R}^d}p(K(t,0),x-y)\varphi(y)dy+\int_0^t\Big(\int_{\mathbb{R}^d}p(K(t,r),x-y)u^{1+\beta}(r,y)dy\Big)dr\\
&\ge\int_0^t\Big(\int_{\mathbb{R}^d}p(K(t,r),x-y)u^{1+\beta}(r,y)dy\Big)dr.
\end{aligned}$$
Thus,
$$u\big((10\varepsilon_2)^{1/\rho}t,x\big)\ge\int_0^{(10\varepsilon_2)^{1/\rho}t}\Big(\int_{\mathbb{R}^d}p(K((10\varepsilon_2)^{1/\rho}t,r),x-y)u^{1+\beta}(r,y)dy\Big)\,dr.$$
Jensen's inequality renders
$$u\big((10\varepsilon_2)^{1/\rho}t,x\big)\ge\int_{\varepsilon_1^{1/\rho}t}^{(2\varepsilon_1)^{1/\rho}t}\Big(\int_{\mathbb{R}^d}p(K((10\varepsilon_2)^{1/\rho}t,r),x-y)u(r,y)dy\Big)^{1+\beta}dr=\infty,$$

so that $u(t,x)=\infty$ for any $t\geq(10\frac{\varepsilon_2}{\varepsilon_1})^{1/\rho}\tau_0$ and $x\in\mathbb{R}^d$. $\square$

Proposition 5.3. *Let $0<\frac{d\rho\beta}{\alpha}<1$. There exists a constant $C>0$ depending on α, β, d, ε_1, ε_2, θ, ρ and φ, such that*

$$T_{\lambda\varphi}\leq\big\{C\lambda^{-\beta}+[(10\frac{\varepsilon_2}{\varepsilon_1})^{1/\rho}\theta]^{\frac{\alpha-d\rho\beta}{\alpha}}\big\}^{\frac{\alpha}{\alpha-d\rho\beta}}+\eta,\quad \lambda>0, \tag{5.1}$$

where η is any positive real number satisfying $p(K(\eta,0),0)\leq 1$.

Proof. From Lemma 3.1 we obtain

$$\begin{aligned}
p(K(\eta,0),x-y)&=p\big(K(\eta,0),\frac{1}{2}(2x-2y)\big)\\
&\geq p\big(K(\eta,0),2x\big)p(K(\eta,0),2y)\\
&=2^{-d}p(2^{-\alpha}K(\eta,0),x)p(K(\eta,0),2y).
\end{aligned}$$

Therefore

$$\begin{aligned}
u(\eta,x)&\geq\lambda\int_{\mathbb{R}^d}p(K(\eta,0),x-y)\varphi(y)dy\\
&\geq 2^{-d}\lambda p(2^{-\alpha}K(\eta,0),x)\int_{\mathbb{R}^d}p(K(\eta,0),2y)\varphi(y)dy\\
&=\lambda N_0p(2^{-\alpha}K(\eta,0),x),
\end{aligned}$$

where $N_0=2^{-d}\int_{\mathbb{R}^d}p(K(\eta,0),2y)\varphi(y)dy$. Thus, for any $\lambda>0$, $t\geq 0$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}
u(t+\eta,x)&=\int_{\mathbb{R}^d}p(K(t+\eta,\eta),x-y)u(\eta,y)\,dy\\
&\quad+\int_0^t\Big(\int_{\mathbb{R}^d}p(K(t+\eta,r+\eta),x-y)u^{1+\beta}(r+\eta,y)\,dy\Big)\,dr\\
&\geq\lambda N_0\int_{\mathbb{R}^d}p(K(t+\eta,\eta),x-y)p(2^{-\alpha}K(\eta,0),y)dy\\
&\quad+\int_0^t\Big(\int_{\mathbb{R}^d}p(K(t+\eta,r+\eta),x-y)u^{1+\beta}(r+\eta,y)\,dy\Big)\,dr\\
&\geq\lambda N_0p(K(t+\eta,\eta)+2^{-\alpha}K(\eta,0),x)\\
&\quad+\int_0^t\Big(\int_{\mathbb{R}^d}p(K(t+\eta,r+\eta),x-y)u^{1+\beta}(r+\eta,y)\,dy\Big)\,dr\\
&\geq w(t,x),
\end{aligned}$$

where w solves the equation

$$\begin{aligned}
w(t,x)&=\lambda N_0p(K(t+\eta,\eta)+2^{-\alpha}K(\eta,0),x)\\
&\quad+\int_0^t\Big(\int_{\mathbb{R}^d}p(K(t+\eta,r+\eta),x-y)w^{1+\beta}(r,y)dy\Big)dr,\quad t\geq 0,\ x\in\mathbb{R}^d.
\end{aligned}\tag{5.2}$$

Hence, it is sufficient to prove that w is non-global, and, because of Lemma 5.2, it suffices to show finite time blowup of

$$\upsilon(t)=\int_{\mathbb{R}^d}p(K(t,0),x)w(t,x)\,dx,\quad t\geq 0.$$

Multiplying both sides of (5.2) by $p(K(t,0),x)$ and integrating, we obtain

$$\begin{aligned}
\upsilon(t) &= \int_{\mathbb{R}^d} p(K(t,0),x)w(t,x)\,dx \\
&= \lambda N_0 \int_{\mathbb{R}^d} p(K(t+\eta,\eta)+2^{-\alpha}K(\eta,0),x)p(K(t,0),x)\,dx \\
&\quad + \int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} p(K(t+\eta,r+\eta),x-y)p(K(t,0),x)w^{1+\beta}(r,y)\,dy\,dr\,dx \\
&= \lambda N_0 p(K(t,0)+K(t+\eta,\eta)+2^{-\alpha}K(\eta,0),0) \\
&\quad + \int_0^t\int_{\mathbb{R}^d} p(K(t+\eta,r+\eta)+K(t,0),y)w^{1+\beta}(r,y)\,dy\,dr, \quad t\ge 0.
\end{aligned}$$

From Lemma 3.1 (i), we have $p(t,0)\le p(s,0)$ for all $0<s\le t$. Hence

$$\begin{aligned}
\upsilon(t) &\ge \lambda N_0 p(2K(t+\eta,0)+2^{-\alpha}K(\eta,0),0) \\
&\quad + \int_0^t\int_{\mathbb{R}^d} p(K(t+\eta,r+\eta)+K(t,0),y)w^{1+\beta}(r,y)\,dy\,dr.
\end{aligned}$$

Using now Lemma 3.1 (iii) we obtain,

$$\begin{aligned}
\upsilon(t) &\ge \lambda N_0 p(2K(t+\eta,0)+2^{-\alpha}K(\eta,0),0) \\
&\quad + \int_0^t \Big(\frac{K(r,0)}{K(t+\eta,r+\eta)+K(t,0)}\Big)^{d/\alpha}\int_{\mathbb{R}^d} p(K(r,0),y)w^{1+\beta}(r,y)\,dy\,dr.
\end{aligned}$$

Jensen's inequality together with Lemma 3.1 (i) gives

$$\begin{aligned}
\upsilon(t) &\ge \lambda N_0 p(2K(t+\eta,0)+2^{-\alpha}K(\eta,0),0) + \int_0^t \big(\frac{K(r,0)}{2K(t+\eta,0)}\big)^{d/\alpha}\upsilon^{1+\beta}(r)\,dr \\
&= \lambda N_0[2K(t+\eta,0)+2^{-\alpha}K(\eta,0)]^{-d/\alpha}p(1,0) \\
&\quad + \int_0^t \big(\frac{K(r,0)}{2K(t+\eta,0)}\big)^{d/\alpha}\upsilon^{1+\beta}(r)\,dr.
\end{aligned}$$

Let $f_1(t)=K^{d/\alpha}(t+\eta,0)\upsilon(t)$ and $t\ge\theta$. We have

$$\begin{aligned}
f_1(t) &\ge \lambda p(1,0)N_0\Big[\frac{K(\theta+\eta,0)}{2K(\theta+\eta,0)+2^{-\alpha}K(\eta,0)}\Big]^{d/\alpha} \\
&\quad + 2^{-d/\alpha}\int_\theta^t K^{-d\beta/\alpha}(r,0)f_1^{1+\beta}(r)\,dr,
\end{aligned}$$

and if $N:=p(1,0)N_0[\frac{K(\theta+\eta,0)}{2K(\theta+\eta,0)+2^{-\alpha}K(\eta,0)}]^{d/\alpha}$, then

$$f_1(t)\ge \lambda N + 2^{-d/\alpha}\int_\theta^t K^{-d\beta/\alpha}(r,0)f_1^{1+\beta}(r)dr, \quad t\ge\theta.$$

Let f_2 be the solution of the integral equation

$$f_2(t)= \lambda N + 2^{-d/\alpha}\int_\theta^t K^{-d\beta/\alpha}(r,0)f_2^{1+\beta}(r)dr, \quad t\ge\theta,$$

which satisfies

$$f_2^\beta(t)=\frac{(\lambda N)^\beta}{1-\beta(\lambda N)^\beta(\frac12)^{d/\alpha}H(t)} \tag{5.3}$$

with $H(t) \equiv \int_\theta^t K^{-d\beta/\alpha}(r,0)dr$. From (1.2) and the assumption $0 < \frac{d\rho\beta}{\alpha} < 1$, we get

$$H(t) \geq \varepsilon_2^{-d\beta/\alpha} \int_\theta^t r^{-\frac{d\rho\beta}{\alpha}} dr = \frac{\alpha}{\alpha - d\rho\beta} \varepsilon_2^{-d\beta/\alpha} [t^{\frac{\alpha-d\rho\beta}{\alpha}} - \theta^{\frac{\alpha-d\rho\beta}{\alpha}}] \to \infty \quad \text{as} \quad t \to \infty.$$

Hence, there exists $\tau_0 \geq \theta$ such that $\beta(\frac{1}{2})^{d/\alpha}(\lambda N)^\beta H(\tau_0) = 1$, and therefore,

$$\int_\theta^{\tau_0} K^{-d\beta/\alpha}(r,0)dr = \frac{2^{d/\alpha}}{\beta} N^{-\beta}\lambda^{-\beta},$$

which together with (1.2) gives $\int_\theta^{\tau_0} (\varepsilon_2 r^\rho)^{-\frac{d\beta}{\alpha}} dr \leq \frac{2^{d/\alpha}}{\beta} N^{-\beta}\lambda^{-\beta}$. Hence

$$\tau_0^{\frac{\alpha-d\rho\beta}{\alpha}} \leq \frac{2^{d/\alpha}[\alpha - d\rho\beta]}{\alpha\beta} N^{-\beta} \varepsilon_2^{\frac{d\beta}{\alpha}} \lambda^{-\beta} + \theta^{\frac{\alpha-d\rho\beta}{\alpha}},$$

or, equivalently,

$$\tau_0 \leq \Big\{ \frac{2^{d/\alpha}\,[\alpha - d\rho\beta]}{\alpha\beta} N^{-\beta} \varepsilon_2^{\frac{d\beta}{\alpha}} \lambda^{-\beta} + \theta^{\frac{\alpha-d\rho\beta}{\alpha}} \Big\}^{\frac{\alpha}{\alpha-d\rho\beta}}. \tag{5.4}$$

From (5.3), we deduce that $f_2(\tau_0) = \infty$. It follows that

$$K^{d/\alpha}(\tau_0 + \eta, 0)v(\tau_0) = f_1(\tau_0) \geq f_2(\tau_0) = \infty,$$

which implies (as in the proof of Lemma 5.2) that $w(t,x) = \infty$ if $t \geq (10\frac{\varepsilon_2}{\varepsilon_1})^{1/\rho}\tau_0$, and thus, $u(t,x) = \infty$ provided $t \geq (10\frac{\varepsilon_2}{\varepsilon_1})^{1/\rho}\tau_0 + \eta$. Therefore

$$T_{\lambda\varphi} \leq \big(10\frac{\varepsilon_2}{\varepsilon_1}\big)^{1/\rho}\tau_0 + \eta.$$

From (5.4) we conclude that there is a positive constant $C = C(\alpha, \beta, d, \varepsilon_1, \varepsilon_2, \theta, \rho, \varphi)$ satisfying (5.1). $\square$

6. A lower estimate for the life span

To bound from below the life span $T_{\lambda\varphi}$ of the initial-value problem (1.1), we need to assume that (1.2) holds for any $t \geq 0$, and that φ is integrable.

Let $\{U(t,s)\}_{t\geq s\geq 0}$ be the evolution family on $C_b(\mathbb{R}^d)$ generated by the family of operators $\{k(t)\Delta_\alpha\}_{t\geq 0}$, which is given by

$$U(t,s)\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)p(K(t,s), x-y)dy = S(K(t,s))\varphi(x),$$

where $\{S(t)\}_{t\geq 0}$ is the semigroup with the infinitesimal generator Δ_α.

Proposition 6.1. *Let $0 < \frac{d\rho\beta}{\alpha} < 1$. There exists a constant $c > 0$, depending on α, β, d, ε_1, ρ and φ, such that*

$$T_{\lambda\varphi} \geq c\lambda^{-\frac{\alpha\beta}{\alpha-d\rho\beta}}, \quad \lambda > 0. \tag{6.1}$$

Proof. The function

$$\overline{u}(t,x) := \Big[\lambda^{-\beta} - \beta\int_0^t \|U(r,0)\varphi\|_\infty^\beta dr\Big]^{-1/\beta} U(t,0)\varphi(x), \qquad t \geq 0, \quad x \in \mathbb{R}^d.$$

is a supersolution of (1.1). Indeed, $\overline{u}(0,\cdot) = \lambda\varphi(\cdot)$ and

$$\begin{aligned}\frac{\partial \overline{u}(t,x)}{\partial t} &= -\frac{1}{\beta}\Big[\lambda^{-\beta} - \beta\int_0^t \|U(r,0)\varphi\|_\infty^\beta dr\Big]^{-\frac{1}{\beta}-1}\Big[-\beta\|U(t,0)\varphi\|_\infty^\beta\Big]U(t,0)\varphi(x)\\ &\quad + \Big[\lambda^{-\beta} - \beta\int_0^t \|U(r,0)\varphi\|_\infty^\beta dr\Big]^{-1/\beta} k(t)\Delta_\alpha U(t,0)\varphi(x).\end{aligned}$$

Since $-\frac{1}{\beta} - 1 = -\frac{\beta+1}{\beta}$, we get

$$\begin{aligned}\frac{\partial \overline{u}(t,x)}{\partial t} &= \Big\{\Big[\lambda^{-\beta} - \beta\int_0^t \|U(r,0)\varphi\|_\infty^\beta dr\Big]^{-1/\beta}\Big\}^{\beta+1}\|U(t,0)\varphi\|_\infty^\beta U(t,0)\varphi(x)\\ &\quad + k(t)\Delta_\alpha\Big[\lambda^{-\beta} - \beta\int_0^t \|U(r,0)\varphi\|_\infty^\beta dr\Big]^{-1/\beta} U(t,0)\varphi(x).\end{aligned}$$

Using the inequality

$$\|U(t,0)\varphi\|_\infty^\beta U(t,0)\varphi(x) \ge [U(t,0)\varphi(x)]^{1+\beta}$$

it follows that

$$\frac{\partial \overline{u}(t,x)}{\partial t} \ge k(t)\Delta_\alpha \overline{u}(t,x) + \overline{u}^{1+\beta}(t,x),$$

showing that $\overline{u}$ is a supersolution of (1.1). Writing $\overline{L}(\lambda)$ for the life span of $\overline{u}$, it follows that

$$\overline{L}(\lambda) \le T_{\lambda\varphi}, \quad \lambda \ge 0.$$

Now,

$$\overline{u}(t,x) = \Big[\lambda^{-\beta} - \beta\int_0^t \|U(r,0)\varphi\|_\infty^\beta dr\Big]^{-1/\beta} U(t,0)\varphi(x) = \infty$$

when $\lambda^{-\beta} = \beta\int_0^t \|U(r,0)\varphi\|_\infty^\beta \, dr$. By definition of $\overline{L}(\lambda)$,

$$\beta^{-1}\lambda^{-\beta} = \int_0^{\overline{L}(\lambda)} \|U(r,0)\varphi\|_\infty^\beta dr. \tag{6.2}$$

Note that, by Lemma 3.1 (i), (ii),

$$\begin{aligned}U(t,0)\varphi(x) &= S(K(t,0))\varphi(x)\\ &= \int_{\mathbb{R}^d} \varphi(y)p(K(t,0), x-y)dy\\ &\le p(1,0)K^{-d/\alpha}(t,0)\|\varphi\|_1, \quad t>0,\ x\in\mathbb{R}^d.\end{aligned}$$

Since, by assumption, (1.2) holds for any $t \ge 0$, we obtain

$$\|U(t,0)\varphi\|_\infty \le p(1,0)(\varepsilon_1 t^\rho)^{-d/\alpha}\|\varphi\|_1.$$

Inserting this inequality in (6.2) and using that $0 < \frac{d\rho\beta}{\alpha} < 1$, we get

$$\begin{aligned}\beta^{-1}\lambda^{-\beta} &\le (p(1,0)\|\varphi\|_1)^\beta \varepsilon_1^{-d\beta/\alpha}\int_0^{\overline{L}(\lambda)} r^{-d\rho\beta/\alpha}\, dr\\ &= \frac{\alpha}{\alpha - d\rho\beta}(p(1,0)\|\varphi\|_1)^\beta \varepsilon_1^{-d\beta/\alpha}\overline{L}(\lambda)^{\frac{\alpha-d\rho\beta}{\alpha}},\end{aligned}$$

which gives

$$\overline{L}(\lambda)^{\frac{\alpha-d\rho\beta}{\alpha}} \ge \frac{\alpha - d\rho\beta}{\alpha\beta}(p(1,0)\|\varphi\|_1)^{-\beta}\varepsilon_1^{\frac{d\beta}{\alpha}}\lambda^{-\beta}.$$

In this way we obtain the inequality

$$T_{\lambda\varphi} \geq \Big[\frac{\alpha - d\rho\beta}{\alpha\beta}\Big]^{\frac{\alpha}{\alpha-d\rho\beta}} (p(1,0)\|\varphi\|_1)^{-\frac{\alpha\beta}{\alpha-d\rho\beta}} \varepsilon_1^{\frac{d\beta}{\alpha-d\rho\beta}} \lambda^{-\frac{\alpha\beta}{\alpha-d\rho\beta}},$$

which proves the existence of a constant $c \equiv c(\alpha, \beta, d, \varepsilon_1, \rho, \varphi) > 0$ that satisfies (6.1). □

Summarizing both, upper and lower bounds for the life span of (1.1), we get the following statement.

Theorem 6.2. *Let $0 < \frac{d\rho\beta}{\alpha} < 1$, and let $T_{\lambda\varphi}$ be the life span of the nonautonomous semilinear equation*

$$\begin{gathered} \frac{\partial u(t,x)}{\partial t} = k(t)\Delta_\alpha u(t,x) + u^{1+\beta}(t,x) \\ u(0,x) = \lambda\varphi(x) \geq 0, \quad x \in \mathbb{R}^d, \end{gathered}$$

where $\lambda > 0$. Then

$$\lim_{\lambda\to 0} T_{\lambda\varphi} = \infty, \quad \lim_{\lambda\to\infty} T_{\lambda\varphi} \in \big[0, (10\frac{\varepsilon_2}{\varepsilon_1})^{1/\rho}\theta + \eta\big], \tag{6.3}$$

where θ and η are any positive numbers such that $\varepsilon_1\theta^\rho \leq K(\theta, 0) \leq \varepsilon_2\theta^\rho$ and $p(K(\eta, 0), 0) \leq 1$, respectively.

Proof. Due to (5.1) and (6.1),

$$c\lambda^{-\frac{\alpha\beta}{\alpha-d\rho\beta}} \leq T_{\lambda\varphi} \leq \big\{C\lambda^{-\beta} + \big[(10\frac{\varepsilon_2}{\varepsilon_1})^{1/\rho}\theta\big]^{\frac{\alpha-d\rho\beta}{\alpha}}\big\}^{\frac{\alpha}{\alpha-d\rho\beta}} + \eta,$$

from which (6.3) follows directly using the fact that $0 < \frac{d\rho\beta}{\alpha} < 1$. □

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