Int J Game Theory (2008) 36:369–392
DOI 10.1007/s00182-007-0079-0

ORIGINAL PAPER

Welfare and stability in senior matching markets

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Accepted: 16 March 2007 / Published online: 14 April 2007

Abstract We consider matching markets at a senior level, where workers are assigned to firms at an unstable matching—the status-quo—which might not be Pareto efficient. It might also be that none of the matchings Pareto superior to the status-quo are Core stable. We propose two weakenings of Core stability: status-quo stability and weakened stability, and the respective mechanisms which lead any status-quo to matchings meeting the stability requirements above mentioned. The first one is inspired by the Top trading cycle and Deferred Acceptance procedures, the other one belongs to the family of Branch and Bound algorithms. The last procedure finds a core stable matching in many-to-one markets whenever it exists, dispensing with the assumption of substitutability.

Keywords Matching · Core consistency · Status-quo stability · Weakened stability · Branch and bound algorithm

The authors thank Salvador Barberà, Stephanie Dunbar, participants to seminars at Universidad de Vigo, El Banco de Mexico, Concordia University, participants to the Wallis Conference on Resource Allocation and Game Theory held at Rochester, the 8th International meeting of the Society for Social Choice and Welfare, the 17th International Conference on Game Theory at Stony Brook and SED 2006. Support from CONACYT is gratefully acknowledged.

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1 Introduction

1.1 Motivation

The report by Roth (2002) leads to a non-ambiguous conclusion: matching institutions should provide core stable outcomes.[1] While in theoretical settings the normative appeal of the core yields from its characterization, the argument, here, is factual. Specifically, clearinghouses that produce core stable outcomes survive, others do not. In our view, the relevance of core stability for clearinghouses is partly tautological: a core stable outcome is robust to attempts of self-reassignment by coalitions of agents. Otherwise, groups of agents would have good reasons to oppose the outcome proposed by the central institution for their freedom to engage in economic activities. Thus, clearinghouses which design core stable outcomes make them easier to enforce.

Nevertheless, inefficiencies might prevent decentralized labor markets from reaching core stability. Among others, the agenda of offers and acceptances may bias the assignment of agents; a worker might accept an offer by a firm and, once committed, receive the offer of a preferred firm she cannot accept anymore. One might also think about changes in the preferences of agents. The adoption of centralized mechanisms in matching markets at a junior level allows to tackle these inefficiencies.

These are not the only difficulties experienced by decentralized markets at a senior level. Namely, markets where agents are matched to one another, and these matchings are disrupted by changes in the population of agents. The analysis is pioneered by Blum et al. (1997). Cantala (2004) observes two features that explain why instability might be persistent in those markets, as well as Pareto inefficiency: (1) in the case where workers do not have tenure, the market reaches stability again *only if* the disruption is the opening of positions and retirement of workers *and* firms make offers; (2) dynamics of offers might not find existing stable matchings.

We consider matching markets at a senior level, where workers and firms are matched to one another at a status-quo matching. The status-quo might be neither stable nor efficient. We elaborate two approaches—solution concepts and the respective mechanisms—that lead the market to a Pareto improvement.

1.2 Features of the problem

Our analysis is restricted in the sense that we guarantee to *all* agents in the market a match at least as preferred as the status-quo. The requirement is natural for senior workers who might have a protective status. We think of senior professors holding a tenure. This right to stay permanently in a job also ensures that any job switch will be for a preferred position. Assuming, however, that both sides of the market can enforce the status-quo unilaterally would be too strong an assumption. Our approach, instead, is motivated by the centralized nature of the problem: guaranteeing the status-quo is an incentive device for agents to take part in the mechanisms, comparable to the participation constraint in the contract theory literature.

[1] See Roth and Sotomayor (1990) for a complete introduction to matching markets.

Suppose that the set of matchings Pareto superior to the status-quo is non empty, is one of those matchings core stable? The answer is negative, as shown in Example 1.

Example 1 Consider the following market with three firms, f_1, f_2, and f_3, and three workers, w_1, w_2, and w_3, where preferences are given by the following profile

$$\begin{array}{ccc} \succ_{f_1} & \succ_{f_2} & \succ_{f_3} \\ \hline w_3 & w_1 & w_1 \\ w_1 & w_2 & w_2 \\ & & w_3 \end{array} \text{ and } \begin{array}{ccc} \succ_{w_1} & \succ_{w_2} & \succ_{w_3} \\ \hline f_3 & f_3 & f_1 \\ f_2 & f_2 & f_3 \\ f_1 & & \end{array}.$$

Suppose that the status-quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

The only matching Pareto superior to μ^0 is

$$\mu = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3 & w_1 & w_2 \end{pmatrix},$$

which is blocked by (f_3, w_1).

Hence, we are restricted to looking for core consistent procedures, namely those which select a core stable matching whenever it exists.

1.3 On manipulability

Alcalde and Barberà (1994) extend a result by Roth (1982) and show that there is no matching rule that is strategy-proof, efficient and individually rational. The negative result applies to our approach since they deal with the particular case where the status-quo is the empty matching, namely it is such that all agents are unmatched. We believe, however, that clearinghouses should not worry so much about the negative result. Dubins and Freedman (1981) and Roth (1982, 1984) consider markets where preferences are strict and show that mechanisms which select the optimal stable matching for one side of the market is strategy-proof for this side of the market. Demange et al. (1986) establish a general result, when preferences might be not-strict and, thus, the optimal stable matching defined above may not exist. Strategic questions for the other side of the market are analyzed in Roth (1982, 1984) and Gale and Sotomayor (1985). More recently and specifically about the DA algorithm, Ehlers (2004) considers that workers assign a probability to be matched to desirable firms. In this setting, manipulating seems to be a very sophisticated behavior.

Our issue is also related to the literature on a one-sided assignment when agents own property rights, which is comparable to holding a tenure. While in these markets there is conflict between equal treatment of equals, Pareto optimality and strategy-proofness (Zhou 1990), there is literature, following Shapley and Scarf (1974) and their Top trading cycle procedure, where the authors show that it is possible to combine core stability and group-strategy proofness (Roth 1982; Ma 1994; Svenson 1999; Bird 1984; Moulin 1995; Abdulkadiroğly and Sönmez 1998; Abdulkadiroğly and Sönmez 1999; Papaï 2000[2] and others). From Alcalde and Barberà (1994), in contrast, we know that there is no core consistent procedure that is strategy-proof. Furthermore, we exhibit in Example 2 that new ways to manipulate a mechanism arise from core consistency.

1.4 Two core consistent solutions

We propose two weakenings of the core. Both intend to capture the idea developed earlier: the "less" agents oppose a matching, as formalized by blocking coalitions, the easier it is to enforce. First, *status-quo stability*. We guarantee to all agents an outcome at least as preferred as the status-quo. Thus, a blocking coalition that is not compatible with a re-assignment of all agents to matches at least as preferred as their status-quo is not a valid objection. Hence, a matching where all blocking coalitions are not valid, faces no legitimate opposition. In this sense it is stable as a status-quo, or status-quo stable.

Notice that there is no conflict between status-quo stability and Pareto efficiency, moreover the solution concept itself is not core consistent. We define a two-step procedure, the Status-Quo Stable (SQS) procedure: first Pareto efficiency is reached by a graph representation of the problem inspired by the Top trading cycle; second, we adapt the Deferred Acceptance (DA) algorithm, from Gale and Shapley (1962), to our setting to reach core consistency. We show that our status-quo stable procedure finds a status-quo stable matching. In particular, whenever a core stable matching exists, the procedure picks the core stable matching unanimously preferred by workers among all status-quo matchings Pareto superior to the status-quo and status-quo stable. However, it does not single out an outcome. Moreover, the procedure only applies to one-to-one markets.

Second, *weakened stability*. Consider again academic markets. Suppose that a centralizer has to choose between two matchings, both Pareto superior to the status-quo and neither Pareto dominates the other. The first matching is such that a university with a micro position hires a micro specialist and blocks the matching with a micro professor. The second matching is such that a university with a macro position hires a micro specialist and blocks the matching with a macro professor. We argue that the first blocking coalition is a weaker opposition than the second one. It is so because it is desirable, from an educational point of view, that a position be held by the adequate specialist. Formally it means that blocking coalitions are comparable and that this comparison follows from a social objective: the more a blocking pair impacts the social objective, the stronger objection it is. Matching models do not capture this feature of the market. To do so, we endow the economy with a societal welfare function.[3] Among all matchings Pareto superior to the status-quo, we choose the one with the

[2] Papaï (2000) does not assume property rights.

[3] In our leading example, the function only takes into account the utility of universities.

weakest opposition. Specifically, for all such matchings, we sum all utility improvements for firms from all blocking coalitions, and pick the matching which entails the smallest such summation [See (1)].

How would DA algorithms perform in our setting? First, the procedures, adapted in Blum et al. (1997) and Cantala (2004) to senior markets, do not take into account the welfare restrictions above mentioned, except individual rationality. Second, it might cycle. One type of cycling is harmless: even if we consider a case where a status-quo matching exists, one can easily design an example where a DA algorithm would cycle. To solve the difficulty one might adopt the solution proposed by Roth and Vande Vate John (1990), namely introduce loop detectors in the algorithm that detect them and launch a new sequence of offers until finding the one that leads to a stable matching. The solution provides no clue if such a matching does not exist. Finally, these procedures require firms to have substitutable preferences.

We make use of a much more versatile family of procedures: Branch and Bound Algorithms. Four of their properties motivate the choice: (a) they do not require any restriction on the preferences of firms, (b) by construction they do not cycle, (c) they can compute all the possible solutions of the problem—which means, in the case of junior markets, that they might compute all the stable matchings, (d) whenever the problem to solve has no solution, they specify it.

We establish that the outcome matching of our Weakened Stability (WS) Algorithm is the solution to our problem and it is status-quo stable. Moreover, when the input matching is the empty one, it is core stable whenever a core stable matching exists, even if the preferences of firms are not substitutable.[4]

Section 2 introduces notations, Sect. 3 deals with manipulability, status-quo stability is presented in Sect. 4 and Weakened Stability in Sect. 5. The Appendix contains the proof of theorems, the details of the WS Algorithm and conditions that guarantee the existence of a core stable matching Pareto superior to any status-quo.

2 Preliminaries

2.1 The market

A many-to-one matching market is a quadruple $(\mathcal{F}, \mathcal{W}, q, \succ)$ where $\mathcal{F}$ and $\mathcal{W}$ are two disjoint finite sets of agents. $\mathcal{F} = \{f_1, \ldots, f_m\}$ is the set of firms and $\mathcal{W} = \{w_1, \ldots, w_n\}$ is the set of workers; generic firms and workers will be denoted by f and w, respectively. Subsets of $\mathcal{F}$ and $\mathcal{W}$ are denoted by F and W. The vector of quotas associated with each firm is $q = \left(q_f\right)_{f \in \mathcal{F}}$, where q_f is the maximum number of workers that can be assigned to firm f. Preference relations are not symmetrically defined between firms and workers since a firm can be assigned to many workers whereas a worker can be assigned to at most one firm. Each firm f has a strict, transitive and complete preference relation $\succ_f$ over the family of subsets of workers $2^{\mathcal{W}}$. We interpret the empty set as firm f not being assigned to any worker. When a firm

4 Echenique and Oviedo (2004), among others, also dispense on the substitutability assumption using fixed point techniques.

ranks the empty set better than a subset, it means that it prefers remaining unmatched to being assigned to this subset. Each worker w has a strict, transitive and complete preference relation $\succ_w$ over the set $\mathcal{F} \cup \{\emptyset\}$. We interpret the empty set in $\succ_w$ as w being unemployed. Preference profiles are $(m+n)$-tuples of preference relations and they are represented by $\succ = \left(\succ_{f_1}, \ldots, \succ_{f_m}, \succ_{w_1}, \ldots, \succ_{w_n}\right)$.

For any firm f we define the *acceptable set of* f *under* q *and* $\succ$ to be the subsets of workers with cardinality smaller or equal to q_f, strictly preferred to the empty set; namely $A_f(q, \succ) \equiv \left\{S \subseteq \mathcal{W} \mid S \succ_f \emptyset \text{ and } |S| \leq q_f\right\}$. Subsets in $A_f(q, \succ)$ are called acceptable. Since only acceptable subsets will matter, we represent the preferences of the firm as a list of acceptable subsets. Likewise, for any worker w we define the *acceptable set of* w *under* $\succ$ to be the set of firms strictly preferred to $\emptyset$. We denote it by $A_w(\succ)$. Firms in $A_w(\succ)$ are called acceptable. We will represent the preferences of firms and workers by ordered lists of acceptable partners. A pair (w, f) is acceptable under q and $\succ$ if both agents are mutually acceptable.

Definition 1 A *matching* μ is a mapping from the set $\mathcal{F} \cup \mathcal{W}$ into the set of all subsets of $\mathcal{F} \cup \mathcal{W}$ such that for all $f \in \mathcal{F}$ and $w \in \mathcal{W}$:

(1) $\mu(f) \in 2^{\mathcal{W}}$ and $|\mu(f)| \leq q_f$,
(2) either $|\mu(w)| = 1$ and $\mu(w) \in \mathcal{F}$, or $\mu(w) = \emptyset$,
(3) $\mu(w) = f$ if and only if $w \in \mu(f)$.

Let $\mathcal{M}$ denote the space of all possible matchings.

2.2 Stability concepts

Let $\succ$ be a preference profile. Given a set $W \subseteq \mathcal{W}$, let the *Choice* of firm f, denoted as $Ch\left(W, q_f, \succ_f\right)$, be f's *most preferred* subset of W with cardinality at most q_f according to its preference ordering $\succ_f$.

A matching μ is *blocked by a worker* w if she prefers remaining alone than being matched to $\mu(w)$; i.e., $\emptyset \succ_w \mu(w)$. Similarly, μ is *blocked by a firm* f if $\mu(f) \neq Ch\left(\mu(f), q_f, \succ_f\right)$. We say that a matching is *individually rational* if it is not blocked by any individual agent. A matching is *blocked by a worker-firm pair* (w, f) if worker w prefers being matched to f than to $\mu(w)$ and f would like to hire w; i.e., $f \succ_w \mu(w)$ and $w \in Ch\left(\mu(f) \cup \{w\}, q_f, \succ_f\right)$.

Definition 2 A matching μ is *pair-wise stable* if it is not blocked by any individual agent or any worker-firm pair.

Let W be a subset of $\mathcal{W}$. A matching μ is *blocked by a workers-firm coalition* (W, f) if all workers w in W prefer being matched to f than to $\mu(w)$ and f would like to hire W; formally if for all $w \in W$, $f \succ_w \mu(w)$ and $W \subseteq Ch\left(\mu(f) \cup W, q_f, \succ_f\right)$. We say that (W, f) forms a blocking coalition of μ. Let $\mathcal{W}_{f,\mu}$ be the set of workers who prefer f to their match under μ and, thus, they are potential members of blocking coalitions of μ. Formally, $\mathcal{W}_{f,\mu} = \{w \in \mathcal{W} \mid f \succ_w \mu(w)\}$.

Definition 3 A matching μ is *group-stable* if it is not blocked by any individual agent or by any workers-firm coalition.

A group-stable matching is also pair-wise stable; moreover core stability defined by weak dominance and group stability coincide in such markets.[5]

3 Strategy proofness

We aim to design a core consistent procedure which assigns to all agents in the market a match at least as preferred as their status-quo, and Pareto undominated. Unfortunately, none of them is strategy-proof.

Definition 4 A mechanism is strategy-proof if it is a dominant strategy, for all agents, to report their true preferences.

We now state the negative result.

Theorem 1 *In senior matching markets, there is no core consistent and strategy-proof mechanism that chooses a Pareto undominated matching and which guarantees to all agents a match at least as preferred as the status-quo.*

With respect to Alcalde and Barberà (1994), Example 2 exhibits new manipulations due to core consistency.

Example 2 Consider the market $(\mathcal{F}, \mathcal{W}, q, P)$ where $\mathcal{F} = \{f_1, f_2, f_3\}$, $q_{f_1} = q_{f_2} = q_{f_3} = 1$, $\mathcal{W} = \{w_1, w_2, w_3\}$ and true preferences are

$\succ_{f_1}$	$\succ_{f_2}$	$\succ_{f_3}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
w_3	w_3	w_1	f_3	f_1	f_2
w_2	w_2	w_3	f_1	f_2	f_1
w_1					f_3

Suppose that the status-quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

There are two matchings which are Pareto superior to the status-quo:

$$\mu^1 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_2 & w_3 & w_1 \end{pmatrix} \text{ and } \mu^2 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_3 & w_2 & w_1 \end{pmatrix}.$$

Notice that μ^1 is stable while μ^2 is blocked by (f_2,w_3), thus a core consistent procedure should pick μ^1. Nevertheless, if f_1 reports $\succ'_{f_1}$ where w_3 is preferred to w_1 and w_2 is not acceptable, the only matching Pareto superior to the status-quo is μ^2, which has to be selected, even if it is not core stable. Thus, in this market, firm 1 would gain by misrepresenting its preferences through $\succ'_{f_1}$.

5 See Roth (1984).

4 Status-quo stability

The status-quo is guaranteed to all agents. Thus, to be considered as a valid objection to a matching, blocking coalitions have to be compatible with a reassignment that makes all agents at least as well off as in the status-quo. In this sense, in Example 2, if μ^2 becomes the new status-quo, the blocking pair (f_2,w_3) to μ^2 is not valid since, if f_2 and w_3 are matched, f_1 cannot be reassigned to a firm preferred to her status-quo, w_3.

Definition 5 Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$, a matching μ is *status-quo stable* if for all blocking coalitions $(f, W) \subseteq \mathcal{F} \times 2^{\mathcal{W}}$ to μ, there is no matching where f and W are assigned to each other, possibly with other workers, which it is Pareto superior to μ.

As for status-quo stability, there is no conflict between blocking coalitions and Pareto optimality. Thus, given a status-quo μ_o, looking for matchings that are status-quo stable and Pareto superior to μ_o is equivalent to looking for the set of matchings Pareto superior to μ_o which are not Pareto dominated by another matching. Denote the set $\mathrm{SQS}(\mu_o)$; by transitivity of preferences it is not empty whenever there is at least one matching Pareto superior to the status-quo. Example 1 also shows that picking a matching randomly in $\mathrm{SQS}(\mu_o)$ is not a core consistent procedure since both μ^1 and μ^2 are status-quo stable.

Our aim is not only to reach a matching in $\mathrm{SQS}(\mu_o)$ but to select a core stable matching, whenever it exists. The status-quo stability procedure performs the task for one-to-one markets.

4.1 The status-quo stability (SQS) procedure

The SQS procedure begins by a graph representation of our problem.

1. Each node represents a match as defined by the status-quo μ_o; if $\mu_o(f) = w$, (f, w) is assigned a node, if $\mu_o(f) = \emptyset$, f is assigned a node and if $\mu_o(w) = \emptyset$, w is assigned a node.
2. From each node with a worker w, draw all arrows towards[6] nodes with firms f such that both w and f prefer each other to their respective status-quo.
3. Identify all *cycles* and *paths* defined as follows.
   A *cycle* is an ordered set S of pairs (f, w) which appear only once in S, where, in the graph constructed as mentioned in 1 and 2
   (a) from each node (f, w) in S an arrow points to another node in S,
   (b) (f, w) is pointed by an arrow from another node in S, moreover
   (c) (f', w') follows (f, w) in S only if (f, w) points to (f', w') in the graph, finally the first pair in S is said to follow the last one.
   A *path* is an ordered set S with one and only one single worker w , one and only one firm f and possibly pairs (f', w'), they all appear only once in S and, in the graph constructed as mentioned in 1 and 2

[6] Thus, it is a directed graph.

(a) the node with the single worker w points to another node in S and is the first element in the set,
(b) for each node (f', w') in S there is one arrow that points to another node in S and (f', w') is pointed to by an arrow from another node in S,
(c) the node with the single firm f is pointed to by another node in S and is the last element in the set,
(d) $[(f', w')$ or $f']$ follows $[(f, w)$ or $w]$ in S only if $[(f, w)$ or $w]$ points to $[(f', w')$ or $f']$ in the graph.

Let $\mathcal{P}$ be the set of all paths and cycles and denote as p an element in $\mathcal{P}$. We are now ready to construct all possible Pareto improvements that may lead the market to status-quo stability, and select one of them.

4. A *composition* c is a subset of $\mathcal{P}$ such that:
   (a) for all $p, p' \in c$, $p \cap p' = \emptyset$ and
   (b) for all $p'' \in \mathcal{P}$ which do not belong to c, there is at least one $p \in c$ and $p'' \cap p \neq \emptyset$.

   Let $\mathcal{C}$ be the set of all compositions. We say that a worker w prefers composition c to composition c' if she prefers the firms which follow her in c to the one in c'.
5. Given a status-quo μ_o and a composition c in $\mathcal{C}$, the induced matching $\mu(\mu_o, c)$ is such that
   (a) if a firm f' belongs to a path or a cycle p in the composition c, it is assigned the worker w of the previous element (single worker or couple) in p;
   (b) otherwise it is assigned the same match as in μ_o.

   Let $I(\mu_o, \mathcal{C})$ be the set of induced matchings by all compositions in $\mathcal{C}$.
   5.1 If $I(\mu_o, \mathcal{C}) = \{\emptyset\}$ then $SQ - S(\mu_o) := \mu_o$, otherwise let $i := 1$,
   5.2 If $I(\mu_o, \mathcal{C}) = \{\emptyset\}$, go to 5.5.
   Otherwise, pick a worker and let her choose within $I(\mu_o, \mathcal{C})$ her favorite matching in $I(\mu_o, \mathcal{C})$; if she is indifferent between different matchings, pick a second worker to break the tie and so on and so forth until a single matching μ^i is selected. If $i := 1$, let $\overline{\mu} \equiv \mu^i$.
   5.3 Let all firms f make simultaneous offers to all groups of workers which are preferred to $\mu^i(f)$ for all f.
   5.4 If no offer is accepted by all workers in a group, $SQ - S(\mu_o) := \mu^i$, otherwise $I(\mu_o, \mathcal{C}) := I(\mu_o, \mathcal{C}) \setminus \mu^i$, $i := i + 1$; go to 5.2.
   5.5 $SQ - S(\mu_o) := \overline{\mu}$.

We now perform the SQS procedure in a simple market.

Example 3 Consider the market $(\mathcal{F}, \mathcal{W}, q, P)$ where $\mathcal{F} = \{f_1, f_2, f_3, f_4\}$, $q_{f_1} = q_{f_2} = q_{f_3} = q_{f_4} = 1$, $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$ and preferences are

$\succ_{f_1}$	$\succ_{f_2}$	$\succ_{f_3}$	$\succ_{f_4}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$	$\succ_{w_4}$
w_4	w_3	w_1	w_2	f_4	f_1	f_4	f_2
w_1	w_1	w_4	w_3	f_2	f_4	f_2	f_3
w_2	w_2	w_3	w_4	f_1	f_2	f_3	f_4

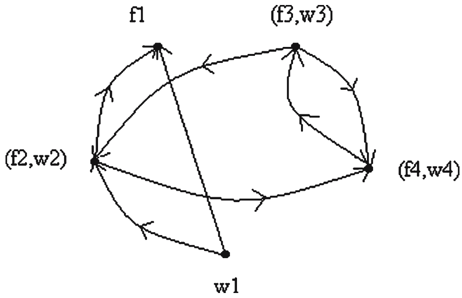

Fig. 1 Stages 1–2: the graph representation

Suppose that the status-quo is

$$\mu^0 = \begin{pmatrix} f_1 & \varnothing & f_2 & f_3 & f_4 \\ \varnothing & w_1 & w_2 & w_3 & w_4 \end{pmatrix}.$$

The SQS works as follows:
1–2. We represent the status-quo by the graph in Fig. 1.
3. There are two cycles:

$$\begin{aligned} p_1 &= [(f_2, w_2), (f_4, w_4), (f_3, w_3)] \quad \text{and} \\ p_2 &= [(f_3, w_3), (f_4, w_4)], \end{aligned}$$

and two paths:

$$\begin{aligned} p_3 &= [(w_1), (f_2, w_2), (f_1)] \quad \text{and} \\ p_4 &= [(w_1), (f_1)]. \end{aligned}$$

4. The set of compositions is $\mathcal{C} = \{c_1, c_2, c_3\}$ where

$$\begin{aligned} c_1 &= ([(w_1), (f_1)], [(f_2, w_2), (f_4, w_4), (f_3, w_3)]), \\ c_2 &= ([(w_1), (f_1)], [(f_3, w_3), (f_4, w_4)]), \quad \text{and} \\ c_3 &= ([(w_1), ((f_2, w_2)), (f_1)], [(f_3, w_3), (f_4, w_4)]). \end{aligned}$$

5. Then $I(\mu_o, \mathcal{C}) = \left\{\mu^1, \mu^2, \mu^3\right\}$ where

$$\mu^1 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1 & w_3 & w_4 & w_2 \end{pmatrix},$$

$$\mu^2 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_1 & w_2 & w_4 & w_3 \end{pmatrix} \text{ and}$$

$$\mu^3 = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ w_2 & w_1 & w_4 & w_3 \end{pmatrix}.$$

Now, if w_3 is selected to pick a matching in $I(\mu_o, \mathcal{C})$, she only discards μ^1, and if w_2 is selected, μ^3 is singled out. Offers are emitted by w_1 and w_4, respectively, to f_4 and f_2. Since none is accepted, μ^3 is stable, $SQ - S(\mu_o) = \mu^3$.

Theorem 2 states that our SQS procedure finds a status-quo stable matching and it is a core consistent procedure.

Theorem 2 *Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$, $q_f = 1$ for all $f \in \mathcal{F}$ and a status quo μ_o then*

1 *$SQ - S(\mu_o)$ is status-quo stable, and Pareto superior to μ_o or is μ_o,*
2 *whenever the set of core stable matchings Pareto superior to μ_o is non-empty, $SQ - S(\mu_o)$ is the core stable matching unanimously preferred by workers (and worst for firms).*

4.2 Comments

If none of the status-quo stable matchings is core stable, workers might not agree on a ranking of matching in SQS(μ_o), thus the order in which they are picked in the procedure might affect the output matching.

The procedure is not monotonic in the status-quo in the sense that a preferred status-quo does not ensure a preferred outcome of the SQS procedure. In particular the advantage of being guaranteed the status-quo might well be balanced by the fact that switching to a better position is conditioned by the simultaneous improvement of the match. Indeed, if a worker/firm is the best alternative for her/its match, she/it cannot switch to another position.

This simultaneous improvement requires a central intervention since, unlike in the Top trading cycle procedures, agents might belong to two different paths or cycles, hence compatible reassignments are not likely to occur without coordination. Moreover, Stage 5 is necessary for the SQS procedure to be core consistent.

Finally, the status-quo stability procedure is not adaptable to many-to-one markets when firms have preferences which are not responsive. The Weakened Stability (WS) algorithm, in contrast, does not require any assumption about the preferences of firms over subsets of workers.

5 Branch and bound algorithms and weakened stability

5.1 The optimization problem

We assume that preferences of firms are represented by utility functions, generically denoted u_f for firm f. We choose the reversed order representation: the lower the utility, the better; and the best subset of workers is assigned utility 0.[7] In this approach blocking coalitions are comparable: the more a blocking coalition impacts the utility of the firm, the stronger it is.

Formally, let (S, f) be a blocking coalition of μ, then $u_f(\mu(f)) - u_f(S)$ measures the welfare improvement of f if the blocking is completed. Our stability index is the sum of utility improvements for firms from all blocking coalitions.

[7] This is convenient for the algorithm, although not necessary.

Definition 6 Consider a market $(\mathcal{F}, \mathcal{W}, q, \succ)$; for a matching μ, let

$$i \equiv \sum_{\text{All blocking coalition } (S,f) \text{ of } \mu.} u_f(\mu(f)) - u_f(S),$$

then μ is said to be *weakened stable of order* i.

Notice that a matching weakened stable of order 0 is core stable. Denote by WS_i the set of matchings that are weakened stable of order i. We now define the utilitarian societal welfare function $W(\mu) = \sum_{f \in \mathcal{F}} u_f(\mu(f))$ that we aim to minimize, choosing a matching within the set of weakened stable matchings of the lowest order.

Formally, given a status-quo μ_o, our problem is

$$\begin{array}{lll} \min\limits_{\mu \text{ is Pareto superior to } \mu_0} & W(\mu) & \\ \text{s.t.} & \mu \in WS_i & \text{and} \\ & WS_j = \emptyset & \text{if } j < i. \end{array} \tag{1}$$

Hence, a matching μ is selected instead of another matching μ' in the following cases: (a) whenever the order of stability of μ is lower than the one of μ', (b) whenever the order of stability of μ or μ' are the same but $W(\mu) < W(\mu')$; otherwise μ and μ' are indifferent. Notice that the status-quo is the solution to the program when it is not Pareto dominated. The following algorithm finds this (these) optimal matching(s).

5.2 The weakened stability algorithm

For all firms $f \in \mathcal{F}$, let $B_f(\mu_o) = \{W \subseteq 2^{\mathcal{W}} | W \succeq_f \mu_o(f)\}$ be the set of subsets of workers f prefers to its status-quo, and for all workers $w \in \mathcal{W}$, let $B_w(\mu_o) = \{f \in \mathcal{W} | f \succeq_w \mu_o(w)\}$ be the set of firms w prefers to her status-quo. Let $A = \times_{f \in \mathcal{F}}(B_f(\mu_o) \cup \{\emptyset\})$, where for all elements in A, the profile of workers where the f^{th} entry is interpreted as being assigned to firm f. Notice that A contains all matchings Pareto superior to μ_o, that is why we will restrict our attention to assignments in A. We also observe that some of the matchings in A may not be Pareto superior to μ_o since preferences of workers are not taken into account in A. Finally, some assignments in A may not be matchings since, for instance, a worker might be assigned to many firms.

The WS algorithm belongs to the family of Branch and Bound (BB) algorithms. This technique is one of the most commonly used in optimization problems[8] when all or some of the decision variables are discrete (integer or mixed programing) and no characterization of optima exists; namely unlike first and second order conditions in differential calculus environments. As a consequence, the set of decision variables, A in our case, has to be scrutinized.

In our problem, there are as many decision variables as firms in the market, hence, the number of solutions can be very large: we call any matching a solution, and a

[8] Branch and Bound algorithms are used to solve, for instence, the classical assignment problem in operation research.

matching that solves (1) is an *optimal* solution. The efficiency of BB algorithms relies on the fact that, instead of analyzing a particular solution at a time, they discard sets of solutions. We denote $R \equiv (W_1, \ldots, W_n, \emptyset, \ldots, \emptyset)$, $R \subseteq A$, the set of solutions where the subset of workers W_f is assigned firm f for $f = 1, \ldots, n$, and there is no specific subset assigned to firms $f = n + 1, \ldots, F$.

For all $R = (W_1, \ldots, W_n, \emptyset, \ldots, \emptyset)$ we define $Z_L(R)$, the upper bound of the objective function of problem (1)[9] reached by solutions in R. Formally,

$$Z_L(R) = \sum_{f=1}^{n} u_f(W_f) + \sum_{f=n+1}^{F} \min\left\{u_f(W_f) | W_f \in B_f(\mu_o), W_f \subseteq \mathcal{W} \setminus \cup_{f=1}^{n} W_f\right\}.$$

Thus, $Z_L(R)$ is the minimal value reached by the objective function when all firms $n + 1, \ldots, F$ are assigned their favorite subset of workers among those not assigned at R. We call $\overline{R}$ the assignment in R for which the value of the objective function is $Z_L(R)$. It might be that $\overline{R}$ is neither a matching nor stable, in any case if this lower bound does not improve upon the tentative optimal solution when the last one is stable of order 0, no matching in R will be optimal, therefore solutions in R are discarded.[10]

We keep a record of the following information: in $WSP(\mu_0)$ the best current solution in the process, in i_t its order of weakened stability and in Z_U the value of its objective function.

The stack, S, is the set of solutions that the algorithm still has to scrutinize. At each iteration, the algorithm picks a set of solutions $R \equiv (W_1, \ldots, W_n, \emptyset, \ldots, \emptyset)$ in S, deletes it from the stack ($S := S \setminus R$), and performs the following tests:

(a) When the tentative optimal solution[11] is core stable, is the value of the objective function of the tentative solution smaller than the upper bound of R?
(b) Can one assign to each of the unassigned workers in R a firm preferred to the status-quo?
(c) Can one assign to each of the unassigned firms in R a group of workers preferred to the status-quo?

If the answer to at least one question is positive, the optimal solution cannot belong to R, another set of solutions in the stack is considered. Otherwise, one cannot discard solutions in $R = (W_1, \ldots, W_n, \emptyset, \ldots, \emptyset)$, we break off R in subfamilies of the form $R' = (W_1, \ldots, W_n, W_{n+1}, \emptyset, \ldots, \emptyset)$. There are as many subfamilies as subsets of workers unassigned in R preferred to the status-quo by firm f_{n+1}. Thus, for each W_{n+1} in B_{n+1} and $\mathcal{W} \setminus \cup_{f=1}^{n} W_f$ Pareto superior to the status-quo, a subfamily of

9 $Z_L(R) \geq \min_{\mu \in R} W(\mu)$

10 The use of the tentative optimal objective values motivates the term Bound in the expression “Branch and Bound Algorithm”.

11 The tentative optimal solution is the solution which is optimal within the set of solutions already scrutinized.

solutions $R' = (W_1, \ldots, W_{n+1}, \emptyset, \ldots, \emptyset)$ has to be inspected. These solutions are included in the stack; i.e., $S := S \cup \{R'\}$ for all such R'.

If a solution is such that all firms are assigned a subset of workers, its order of stability is computed as well as the value of the societal welfare function. If its order of stability of μ is lower than the one of the tentative solution, or the same but the value of the societal welfare function is lower, it becomes the new tentative solution. Otherwise, it is discarded.

The algorithm stops when the stack is empty. Then the last tentative solution becomes the outcome of the algorithm. The detail of the algorithm is in the Appendix.[12] Example 4 shows how our procedure works.

Example 4 Consider the market $(\mathcal{F}, \mathcal{W}, q, P)$ where $\mathcal{F} = \{f_1, f_2, f_3\}$, $q_{f_1} = q_{f_2} = q_{f_3} = 1$, $\mathcal{W} = \{w_1, w_2, w_3\}$ and preferences are

$\succ_{f_1}$	$\succ_{f_2}$	$\succ_{f_3}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
w_2	w_1	w_1	f_3	f_1	f_1
w_3	w_2	w_2	f_2	f_3	f_3
w_1	w_3	w_3	f_1	f_2	f_2

Moreover,

$$\begin{aligned} u_{f_1}(w_2) = 0 = u_{f_2}(w_1) = u_{f3}(w_1) \\ u_{f_1}(w_3) = 1 = u_{f_2}(w_2) = u_{f3}(w_2) \\ u_{f_1}(w_1) = 2 = u_{f_2}(w_3) = u_{f3}(w_3) \end{aligned}$$

The status-quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

Thus, at $t = 1$:

$$\begin{aligned} WSP(\mu_0) &= \mu_0, i = u_{f_1}(w_1) + u_{f_2}(w_2) + u_{f3}(w_3) = 2 + 1 + 2 \quad \text{and} \\ Z_U &= [u_{f_1}(w_3) - u_{f_1}(w_1)] + [u_{f_1}(w_2) - u_{f_1}(w_1)] + [u_{f_2}(w_1) - u_{f_2}(w_2)] \\ &\quad +[u_{f_3}(w_2) - u_{f_3}(w_3)] + [u_{f_3}(w_1) - u_{f_3}(w_3)]. \\ &= [2 - 1] + [2 - 0] + [1 - 0] + [2 - 1] + [2 - 0] \\ &= 1 + 2 + 1 + 1 + 2 = 7 \\ S &= (\varnothing, \varnothing, \varnothing). \end{aligned}$$

Iteration 1: pick $R = \{(\varnothing, \varnothing, \varnothing)\}$.

(a) When the tentative optimal solution is core stable, is the value of the objective function of the tentative solution smaller than the upper bound of R?
The tentative solution is not stable.

[12] Notation in the Example differs slightly from the notation used in Appendix.

(b) Can one assign to each of the unassigned workers in R a firm preferred to the status-quo?
Yes.
(c) Can one assign to each of the unassigned firms in R a group of workers preferred to the status-quo?
Yes.

Then $S = \{(w_2, \varnothing, \varnothing), (w_3, \varnothing, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 2: pick $R = (w_2, \varnothing, \varnothing)$.

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_2, w_1, \varnothing), (w_3, \varnothing, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 3: pick $R = (w_2, w_1, \varnothing)$.

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_2, w_1, w_3), (w_3, \varnothing, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 4: pick $R = (w_2, w_1, w_3)$.

$$i((w_2, w_1, w_3)) = [u_{f_3}(w_1) - u_{f_3}(w_3)] = 2 < 7$$
$$Z_L((w_2, w_1, w_3)) = 2$$

Thus

$$WSP(\mu_0) := (w_2, w_1, w_3)$$
$$i := 2$$
$$Z_U := 2.$$

and $S = \{(w_3, \varnothing, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 5: pick $R = (w_3, \varnothing, \varnothing)$.

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_3, w_2, \varnothing), (w_3, w_1, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 6: pick $R = (w_3, w_2, \varnothing)$.

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_3, w_2, w_1), (w_3, w_1, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 7: pick $R = (w_3, w_2, w_1)$,

$$i((w_3, w_2, w_1)) = [u_{f_1}(w_2) - u_{f_1}(w_3)] = 1 = i$$
$$Z_L((w_3, w_2, w_1)) = 2$$

Thus

$$
\begin{aligned}
WSP(\mu_0) &:= (w_2, w_1, w_3) \\
i &:= 1 \\
Z_U &:= 2.
\end{aligned}
$$

and $S = \{(w_3, w_1, \varnothing), (w_1, \varnothing, \varnothing)\}$.
Iteration 8: pick $R = (w_3, w_1, \varnothing)$.

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_3, w_1, w_2), (w_1, \varnothing, \varnothing)\}$.
Iteration 9: pick $R = (w_3, w_1, w_2)$

$$
\begin{aligned}
i((w_3, w_1, w_2)) &= [u_{f_3}(w_2) - u_{f_3}(w_1)] = 1 = i \\
Z_L((w_3, w_1, w_2)) &= 2 = Z_U
\end{aligned}
$$

Thus

$$
\begin{aligned}
WSP(\mu_0) &:= \{(w_2, w_1, w_3), (w_3, w_1, w_2)\} \\
i &:= 1 \\
Z_U &:= 2.
\end{aligned}
$$

and $S = \{(w_1, \varnothing, \varnothing)\}$.
Iteration 10: pick $R = (w_1, \varnothing, \varnothing)$

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_1, w_2, \varnothing)\}$.
Iteration 11: pick $R = (w_1, w_2, \varnothing)$

(a) The tentative solution is not stable.
(b) Yes.
(c) Yes.

Then $S = \{(w_1, w_2, w_3)\}$.
Iteration 12: pick $R = (w_1, w_2, w_3)$

$$
\begin{aligned}
i((w_1, w_2, w3)) &= 7 > i \\
Z_L((w_3, w_1, w_2)) &= 5 > Z_U
\end{aligned}
$$

It is discarded and $S = \{\varnothing\}$ and $WSP(\mu_0) := \{(w_2, w_1, w_3), (w_3, w_1, w_2)\}$. This concludes Example 4.

We are now ready to state our main result.

Theorem 3 *Consider a market* $(\mathcal{F}, \mathcal{W}, q, \succ)$ *and a status quo* μ_o *then* $WSP(\mu_0)$ *is a solution to* (1).

In particular, if the status quo is the empty matching, our algorithm finds a core stable matching whenever such matching exists, dispensing with the condition of q-substitutability.

Corollary 1 *Consider a market* $(\mathcal{F}, \mathcal{W}, q, \succ)$ *and let the status-quo* μ_o *be the empty matching. Then, when a core stable matching exists, the output of the WS algorithm is core stable.*

Moreover a solution to (1) cannot be Pareto dominated: let μ and μ' be two matchings such that μ dominates μ'. All blocking coalitions to μ are also blocking coalitions to μ' by transitivity of the preferences. The reverse is not true. Thus, the order of stability of μ is smaller than the order of stability of μ'.

Corollary 2 $WSP(\mu_0)$ *is status-quo stable.*

6 Concluding remarks

Since the WS algorithm selects the firm-best core stable matching whenever it exists, it recovers the same incentive properties of the mechanisms studied in Dubins and Freedman (1981) and Roth (1982, 1984) on the respective settings. Nevertheless, the simplicity of Example 1 and Proposition 1 suggest that the lack of existence of a core stable solution is not pathological. Moreover, manipulating core consistent procedures does not seem to require a sophisticated behavior, as suggests Example 2.

Our motivation to develop Weakened Stability is not primarily applicability. Rather, we are interested in capturing features of matching markets usually ignored by the modeler. Nevertheless, some properties of the solution concept look attractive. First, it is core consistent and core stability has shown to be a remarkable property of enforceability. Second, there is no conflict between Weakened stability and Pareto efficiency: if a matching dominates another in Pareto terms, its order of stability is lower. Third, in some countries, workers of the public sector are associated to an index which takes into account their seniority, professional performance or family situation. This index makes them comparable; in particular it is used to define priority orders. Thus, building up a societal welfare function might be seen as the formalization of a real life practice. Moreover, these functions depend on observable variables, coping partially with the problem of manipulability. Finally, Branch and Bound algorithms are versatile tools able to solve a large scope of variations from problem (1). In particular, we believe it is relevant to compare the performances of BB algorithms and the procedure used by the National Resident Matching Program.

Appendix

We investigate now the sufficient conditions which guarantee the existence of a group stable matching Pareto superior to a status-quo. We recall the following definitions.

Definition 7 A matching μ is *worker quasi-stable* if it is individually rational and for any blocking coalition (S, f), $\mu(w) = \emptyset$, for all $w \in S$.

Definition 8 A matching μ is *firm quasi-stable* if it is individually rational and for any firm f, worker $w \in \mu(f)$ and subset of workers $S \subseteq \mathcal{W}_{f,\mu}$, $w \in Ch(\mu(f) \cup S, q_f, \succ_f)$.

Definition 9 A matching μ is *quasi-stable* if it is individually rational and for every blocking coalition (S, f), for every $w \in \mu(f)$, $w \in Ch(\mu(f) \cup S, q_f, \succ_f)$ and $\mu(w) = \emptyset$, for all $w \in S$.

Proposition 1 *Consider a market* $(\mathcal{F}, \mathcal{W}, q, \succ)$ *and a matching* μ_0. *Assume that the set of matchings Pareto superior to* μ_o *is non-empty, we know that one of them is core stable when firms have q-substitutable preferences and the input matching is quasi-stable.*

Proof The argument is constructive: if the matching of departure is quasi-stable, in particular it is firm quasi-stable. Since firms have q-substitutable preferences, Proposition 1 in Cantala (2004) shows that applying his modified version of the DA algorithm leads to a core stable matching and that all along the sequence of tentative matchings, workers are never dismissed and all assignments are firm quasi-stable. by quasi-stability, original blocking pairs only involve unmatched workers, therefore resolving them makes no firm worse off and no new blocking coalition appears along the process. Thus, all agents get better assignments, no new blocking pair appears, all tentative matchings are quasi-stable and the resulting matching, say μ, is Pareto superior to the status-quo matching. Finally since μ is stable, it is Pareto efficient.

One cannot dispense with q-substitutability since, then, it might happen that no stable matching exists. The next example shows that quasi-stability is also necessary for Proposition 1 to hold.

Example 5 Consider the market $(\mathcal{F}, \mathcal{W}, q, P)$ where $\mathcal{F} = \{f_1, f_2, f_3\}$, $q_{f_1} = q_{f_2} = q_{f_3} = 1$, $\mathcal{W} = \{w_1, w_2, w_3\}$ and $\succ$ is given by the following profile

$$\begin{array}{cccccc} \succ_{f_1} & \succ_{f_2} & \succ_{f_3} & \succ_{w_1} & \succ_{w_2} & \succ_{w_3} \\ \hline \{w_1\} & \{w_3\} & \{w_2\} & \{f_1\} & \{f_1\} & \{f_3\} \\ \{w_2\} & \{w_1\} & \{w_3\} & \{f_2\} & \{f_3\} & \{f_2\} \end{array}.$$

Assume that the worker quasi-stable status-quo is

$$\mu^0 = \begin{pmatrix} f_1 & f_2 & f_3 & \emptyset \\ w_2 & \emptyset & w_3 & w_1 \end{pmatrix}.$$

The only matching Pareto superior to μ^0 is

$$\mu^1 = \begin{pmatrix} f_1 & f_2 & f_3 \\ w_2 & w_1 & w_3 \end{pmatrix},$$

which is blocked by (f_1, w_1).

Proof of Theorem 2 We observe that only arrows representing blocking pairs are drawn on the graph (step 2). Moreover, blocking pairs lead to a Pareto improvement only if dropped mates (if any) are also assigned a blocking mate (by definition preferred to the status-quo). Thus, one needs to identify all the ordered sets of blocking pairs (with the interpretation that $[(f', w')$ or $f']$ follows $[(f, w)$ or $w]$ if (w, f') is the blocking pair to be completed[13]) such that:

(a) if completed simultaneously, the market experiences a Pareto improvement and
(b) if one or some of them is withdrawn from the set, there is no such Pareto improvement.

Obviously Cycles and Paths are such sets; we show that they are the only ones. It is also clear that no blocking pair can appear twice in the sets since agents cannot complete two blocking pairs simultaneously. We adopt the convention that a set begins by a node with a worker (and possibly a firm) pointing towards another node (if there is no "pointing" in the set, neither are there blocking pairs). Since blocking is simultaneous, the order only matters in keeping track of who blocks with whom. Thus, if there is an unmatched worker in the set, there is no loss of generality in shifting all elements, ranking this unmatched worker first and following the original ordering; that is why, if there is an unmatched worker in the set, we put her first in the set.

Case 1 The set starts with an unmatched worker w.

If this worker blocks with an unmatched firm f, $\{w, f\}$ is the Pareto improving set as defined above, it is a path.

If this worker blocks with a matched firm f, the mate of f, w', will have to be assigned a firm f' in the set that is preferred to the status-quo. If this firm is unmatched, the set is $\{w, (f, w'), f'\}$, it is a path. Otherwise a pair (f', w'') has to follow (f, w') so as to assign w' a firm f' that is preferred to her status-quo. One can iterate the argument, until an unmatched firm appears in the sequence. If such unmatched firms did not exist, the blocking pairs specified by the ordered set would not be Pareto improving for the worker of the last pair, who would then remain unmatched. Therefore, the set is a path in any case. If there is more than one unmatched worker in the set, by the previous argument they would generate independent paths, since no pair can appear twice. Hence one of the paths might be withdrawn from the set without altering the Pareto improvement of agents in the other set.

Case 2 The set starts with a pair (f, w).

By our convention, there is no unmatched worker in the set. So as to compensate f for the fact that w blocks with another firm f', the last element of the set in the sequence has to be a couple (f^n, w^n) where w^n blocks with f. We observe that no unmatched firm can be included in the set, since the firm will not point to any other agent, in particular to couples, as required. Thus, the set is a cycle.

Hence, $\mathcal{P}$ contains all sets of blocking pairs such that, if they are completed simultaneously, all agents involved in the set will improve with respect to the status-quo. Of course, it might be that unmatched firms or workers, or matched worker-firm pairs are involved in many paths and cycles and, nevertheless one cannot complete

[13] Both w and f' might be involved in blocking pairs with other agents in the set.

simultaneously many blocking pairs. A composition of $\mathcal{P}$ (Step 4) is a set of compatible cycles and paths such that no other element in $\mathcal{P}$ is compatible with them.

We argue now that there is no matching Pareto superior to the one generated by a composition since the algorithm stops:

- either at step 5.4 when a matching is stable (in which case there is no matching Pareto superior to it, otherwise some agents would block);
- or at 5.5, when μ_1 is selected. Consider step 5.2 that leads to the selection of μ_1. If only one worker is necessary to select μ_1, it means that this worker strictly prefers μ_1 to any other matching in $I(\mu_o, \mathcal{C})$. If many workers are necessary to pick μ_1, notice that each time a matching in $I(\mu_o, \mathcal{C})$ is discarded by a worker, the discarded matching is strictly worse than μ_1 for this worker. Thus, μ_1 is not Pareto dominated by any matching in $I(\mu_o, \mathcal{C})$.

We prove now that the procedure picks the workers' optimal stable matching whenever it exists. We know from the lattice Lemma (Knuth 1976) that in one-to-one markets, if workers do not rank unanimously two stable matchings, by letting them choose their best mate between both matchings, not only does the picking function leads to a matching but leads to a stable one. Of course, if the two matchings are Pareto superior to the status-quo, so is the new matching. Thus, if there are stable matchings Pareto superior to the status-quo, one of them is unanimously preferred by workers. That is why we let workers choose their favorite matchings in $I(\mu_o, \mathcal{C})$ and check if those choices are stable. Specifically, if no offer emitted by firms is accepted by any worker, this is the outcome matching. Else another matching is chosen by new workers until a stable matching is found. If all sets of status-quo matching have been scrutinized and none of the matching is stable, the outcome matching is the first tentative matching. □

The WS algorithm

We keep a record of the following information: in $WSP(\mu_0)$ the best current solution in the process, in i_t its order of weakened stability and in Z_U the value of its objective function.

1. Initial round
   - For all $f \in \mathcal{F}$ define the function $B_f : \mathcal{M} \to 2^{2^{\mathcal{W}}}$ such that $B_f(\mu) = \{S \subseteq 2^{\mathcal{W}} |\ \#S \leq q_f \text{ and } S \succsim_f \mu(f)\}$. Let $A = \times_{f\in\mathcal{F}}(B_f(\mu) \cup \{\emptyset\})$.[Define the subsets of workers preferred by firms to a matching μ.]
   - For all $w \in \mathcal{W}$ define the function $B_w : \mathcal{M} \to 2^{\mathcal{F}\cup\{\emptyset\}}$ such that $B_w(\mu) = \{m \in \mathcal{F}\cup\{\emptyset\} | m \succsim_w \mu(w)\}$. [Define the set of firms preferred by workers to a matching μ.]
   - For all $f \in \mathcal{F}$ define the function $W_f : 2^{2^{\mathcal{W}}} \times 2^{\mathcal{F}\cup\{\emptyset\}} \to 2^{2^{\mathcal{W}}}$ such that $W_f(\mu) = \{W \in \mathcal{W} \mid W \subseteq B_f(\mu) \text{ and for all } w \in W, f \in B_w(\mu)\}$. [Define the set of subsets of workers who block μ with f.]
   - Define the function $i_0 : (2^{2^{\mathcal{W}}})^{\#\mathcal{F}} \to \Re$ such that $i(\mu) = \sum_{f\in F} \sum_{W\in W_f(\mu)} u_f(\mu(f)) - u_f(W)$ [$i(\mu)$ is the order of stability of matching μ.]

- For all $R \subset A$, define the function $Z_L : A \to \Re$ such that $Z_L(R) = \sum_{f=1}^{n} u_f(W_f) + \sum_{f=n+1}^{F} \min\{u_f(W_f) | W_f \in B_f(\mu_o), W_f \subseteq \mathcal{W} \setminus \cup_{f=1}^{n} W_f\}$.
- $WSP(\mu_0) = \mu_o$. [The initial tentative optimal solution is the status-quo.]
- $Z_U = Z_L(\mu_o)$. [The objective value of the initial tentative solution is the one of the status-quo.]
- $i_0 := i(\mu_o)$ [i is the order of stability of the status-quo.]
- $S = \{(\emptyset, \ldots, \emptyset)\}$. [At the beginning, we have to review all possible solutions.]
- $t \equiv 1$.

(a)Iteration

2. Selection within the stack S, of a solution.
   If $S = \emptyset$ then stop. [If the stack is empty, there are no more subsets to analyze and the tentative optimal solution is the solution to (1).]
   Otherwise, let R be such that $R = \arg\min_{R' \in S} Z_L(R')$, $S := S \setminus \{R\}$. [We select the family of solutions with minimal lower bound.][14]

3. Fathoms. One discards R or checks whether the optimal solution may belong to R.

   3.1 If $i_{t-1} = 0$ and $Z_U < Z_L(R)$ then go to 2. [If the tentative optimal solution is core stable and its objective function is smaller than the lower bound of R, solutions in R are discarded.]

   3.2 If $\{f_{n+1}, \ldots, f_F, \{\emptyset\}\} \cap B_w(\mu_o) = \emptyset$ for (at least) one $w \in \mathcal{W} \setminus \cup_{f=1}^{n} W_f$, then go to 2. [If all unassigned firms are worse than the status-quo, the solution cannot belong to R for (at least) one unassigned worker, R is discarded.]

   3.3 If for some firm $f \in \{f_{n+1}, \ldots, f_F\}$ no $W_f \in B_f(\mu_o)$ is such that $W_f \subseteq \{\mathcal{W} \cup \{\emptyset\}\} \setminus \cup_{f=1}^{n} W_f$, then go to 2. [If one cannot assign a group of workers preferred to the status-quo to each of the unassigned firms, R is discarded.]

   3.4 If $n + 1 < F$ go to 4 [If more than one firm is not assigned any subset of workers, the solution is partitioned in subsets of solutions ...].
   Else for $f = F$ define $W_F = \{W \subseteq \mathcal{W} \setminus \cup_{f=1}^{F-1} W_f$ such that [... else subsets in W_F are the only ones which complete R to form a matching Pareto superior to the status-quo ...]
   (a) $W \in B_F(\mu_o)$,
   (b) $F \in B_w(\mu_o)$ for all $w \in W$
   (c) if $\mu_o(w) \neq \emptyset$ for $w \in \mathcal{W} \setminus \cup_{f=1}^{F-1} W_f$, then $w \in W_F\}$. [... in particular matched workers at the status quo have to be included.]

   3.4.1 If $W_F = \emptyset$, go to 2.
   Else, let $N \equiv \# W_F$ and $l \equiv 1$.

   3.4.2.1 If $l \leq N$, select one $W \in W_F$, delete it from W_F and construct $R' = (W_1, \ldots, W)$.

14 In order to improve the efficiency of the algorithm, one would idealy choose the family of solution with lower bound of stability. Nevertheless this lower bound is not computable, that is why we use as lower bound the value of the objective function.

Else $t = t + 1$, go to 2. [One completes R assigning F to an acceptable subset of workers, including a fortiori those who are matched at the status-quo.]

3.4.2.2 If $i(R') < i_{t-1}$ or $(i(R') = i_{t-1}$ and $Z_L(R') \leq Z_U)$ then $i_t = i(R')$, $WSP(\mu_0) = R'$, $Z_U = Z_L(R')$. [A new tentative solution has been detected.]
In any case $l = l + 1$, go to 3.4.2.1.

4. Branching: in case we cannot discard R, we break it off in smaller subsets. Notice that only Pareto superior matchings are included in the stack.
$S := S \cup \{(W_1, \ldots, W_{n+1}, \emptyset, \ldots, \emptyset) \subseteq A$ such that
(a) $(W_1, \ldots, W_n, \emptyset, \ldots, \emptyset) = R$, [New solutions in S are subfamilies of $R \ldots$]
(b) $W_{n+1} \subseteq \mathcal{W} \setminus \cup_{f=1}^{n} W_f$, [... obtained by complementing R with subsets of available workers ...]
c $W_{n+1} \in B_{f_{n+1}}(\mu_o)$, $f_{n+1} \in B_w(\mu_o)$ for all $w \in W_{n+1}\}$. [... compatible with the Pareto criterion.]
Then go to 2.

Proof of Theorem 3 We observe that the algorithm is well behaved in the sense that it always ends. To see this, notice first that, when an iteration ends up by a branching, one does not add new solutions to the stack but keeps the subset of solutions selected within a partition of the solution (we consider only the solutions that might be Pareto superior to the status-quo). Since the number of firms is finite, so is the number of iterations which end up by a branching. Furthermore, solutions are deleted from the stack at iterations which do not end up by a branching, hence the stack will end up empty and the algorithm will end.

So as to prove that the output matching of the algorithm is the optimal solution to problem (1), we argue that none of the three following errors occurs.

Error 1. A solution has not been scrutinized that should have been scrutinized.

At the initial round, all possible solutions preferred to the status-quo by firms are included in the stack. Solutions are eliminated from the stack when it is analyzed. Then, either it is discarded, either it is selected as a new tentative solution or one proceeds to branching. In this case, only solutions which are Pareto superior to the status-quo are introduced in the stack (other solutions cannot be optimal for (1)) and, thus, will be analyzed later on.

Error 2. A solution has been discarded which should not have been discarded.

In a given iteration, assume that the tentative optimal solution, $WSP(\mu_0)$, is correct; i.e., it is optimal within the set of solutions already scrutinized. The solution R is discarded at the following steps:

3.1. When the tentative matching is core stable and the lower bound of R is greater than the objective value of the tentative solution, no solution in R can be optimal.

3.2. If for (at least) one worker unassigned at R none of the firms unassigned at R is at least as good as the status-quo, no solution in R can be incentive compatible with μ_o for this worker.

3.3. If for (at least) one firm unassigned at R none of the subsets of workers unassigned at R is at least as good as the status-quo, no solution in R can be incentive compatible with μ_o for this firm.

3.4. Solutions in R which are not Pareto Superior to the status-quo are discarded, they cannot be optimal solutions to (1).

3.4.2 and 3.4.3. All matchings in R which are Pareto superior to the status-quo are compared to the tentative solution and discarded if their order of stability is higher than the one of the tentative solution, or, in case of a tie, when their objective value is higher.

Hence, if the tentative solution is correct, so is the fact to discard families of solutions at 3.1, 3.2, 3.3, 3.4, 3.4.2 and 3.4.3.

Error 3. A solution has been selected as tentative optimal solution which should not have been selected.

In a given iteration, assume that the tentative optimal solution $WSP(\mu_0)$, is correct; i.e., it is optimal within the set of solutions which have been already scrutinized. The solution $\overline{R}$ is selected at 3.4.3. There, all solutions in R which are Pareto superior to the status-quo are compared sequentially to the tentative solution and selected as the new tentative solution if their indices of stability and their values are lower than those of the tentative solution.

Hence, if the tentative solution is correct, so is the fact to select a new tentative solution at 3.4.3. □

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