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Rigidity and vanishing theorems for almost quaternionic manifolds

Haydeé Herrera · Rafael Herrera

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Abstract We prove the rigidity under (compatible) circle actions of several twisted Dirac operators on almost quaternionic manifolds, and the vanishing of the indices of some of them as a consequence.

Keywords Almost quaternionic · Circle action · Rigidity · Twisted Dirac operators · Index theory

Mathematics Subject Classification (2000) 53C15 · 53C26 · 57S15 · 57R20 · 58G10

1 Introduction

The rigidity of elliptic operators under group actions has been widely discussed in the context of the elliptic genera on Spin manifolds [4,8,15,19,25–27], complex manifolds [10,14], quasi-symplectic manifolds [20], Spin^c manifolds [6,7,9] and non-spin π_2-finite manifolds [11–13]. Twisted signature and Dirac operators play a fundamental role in this subject, in which several suitable twists have been proved to be rigid under group actions. The twists consist of tensoring the (locally defined) spin bundle with appropriate vector bundles (endowed with connections) prescribed by certain infinite (K-theoretic) product [15,27].

In this paper, we consider twisted Dirac operators on almost quaternionic manifolds resembling those studied on quaternion-Kähler manifolds [18,24]. Here, we prove the subtle property of rigidity under group actions for the operators in a similar fashion to [1], and obtain the vanishing of some of the indices as a corollary. Notice that the rigidity of an operator does not imply, in general, the vanishing of its index as the signature operator shows. It is

H. Herrera
Department of Mathematical Sciences, Rutgers University, Camden, NJ 08102, USA
e-mail: haydeeh@camden.rutgers.edu

R. Herrera (✉)
Centro de Investigación en Matemáticas, A. P. 402,Guanajuato GTO, C.P. 36000, México
e-mail: rherrera@cimat.mx

worth recalling that vanishing theorems of this type have been useful in the classification of positive quaternion-Kähler manifolds in dimension 8 and 12 (cf. [22] and [11], respectively).

The note is organized as follows. In Sect. 2 we recall some preliminaries on almost quaternionic manifolds. In Sect. 3 we recall the definition of Dirac operators with coefficients in auxiliary bundles and state our main Theorem 3.1. In Sect. 4, we describe some properties of the fixed point sets of smooth circle actions preserving the almost quaternionic structure. In Sect. 5 we prove Theorem 3.1 and one more rigidity theorem (Theorem 5.1).

2 Preliminaries

2.1 Almost quaternionic manifolds

A $4n$-dimensional manifold M, $n > 1$, is called *almost quaternionic* if there is a rank 3 sub-bundle Q of the endomorphism bundle $\mathrm{End}(TM) = T^*M \otimes TM$ such that locally Q has an (admissible) basis $\{I, J, K\}$ satisfying the relations $I^2 = J^2 = -1$ and $K = IJ = -JI$. The existence of the sub-bundle Q implies the reduction of structure of the frame bundle of M to a sub-bundle P with structure group $GL_n(\mathbb{H}) \times_{\mathbb{Z}_2} Sp(1) \subset GL_{4n}(\mathbb{R})$. Thus, the complexified tangent bundle of M has the form

$$TM_c = E \otimes H,$$

where E and H correspond to the standard complex representations $\mathbb{C}^{2n}$ and $\mathbb{C}^2$ of $GL_n(\mathbb{H})$ and $Sp(1)$, respectively.

Given any Riemannian metric g_1 on an almost quaternionic manifold we can obtain a quaternion-Hermitian metric g_2 as follows

$$g_2(v, w) = g_1(v, w) + g_1(Iv, Iw) + g_1(Jv, Jw) + g_1(Kv, Kw),$$

where v, w are tangent vectors to M, so that

$$g_2(Iv, Iw) = g_2(v, w), \quad g_2(Jv, Jw) = g_2(v, w), \quad g_2(Kv, Kw) = g_2(v, w).$$

With this choice, M becomes an almost quaternion-Hermitian manifold. Thus, M has a further reduction of its structure group to $Sp(n)Sp(1)$. There are three local Kähler forms

$$\omega_I(v, w) = g_2(v, Iw), \quad \omega_J(v, w) = g_2(v, Jw), \quad \omega_K(v, w) = g_2(v, Kw),$$

and a globally defined, non-degenerate 4-form Ω, called the *fundamental form*, given by the local formula

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$

Since the stabilizer of Ω in $GL_{4n}(\mathbb{R})$ is $Sp(n)Sp(1)$, the fundamental form characterizes such a reduction of structure group [23].

Observe that now, E and H correspond to the standard representations of $\mathbb{C}^{2n}$ of $Sp(n)$ and $\mathbb{C}^2$ of $Sp(1)$, respectively. Note that

$$\begin{aligned}
\mathrm{End}(TM)_c &\cong (E \otimes H) \otimes (E \otimes H)\\
&= (E \otimes E) \otimes (H \otimes H)\\
&= (\textstyle\bigwedge_0^2 E \oplus 1 \oplus S^2 E) \otimes (1 \oplus S^2 H)\\
&= \textstyle\bigwedge_0^2 E \oplus 1 \oplus S^2 E \oplus \bigwedge_0^2 E \otimes S^2 H \oplus S^2 H \oplus S^2 E \otimes S^2 H
\end{aligned}$$

where $\bigwedge_0^p E$ denotes the primitive subspace in $\bigwedge^p E$ with respect to a symplectic form. The Lie algebra $\mathfrak{sp}(1)$ has the structure of the space of imaginary quaternions and, therefore, Q is the associated rank 3 real vector bundle underlying $S^2 H$.

2.2 Characteristic classes

In order to compute characteristic numbers of M we will use the splitting principle. Since $E \cong E^*$ and $H \cong H^*$, we can write formally

$$E = L_1 + L_1^{-1} + \cdots + L_n + L_n^{-1} \quad \text{and } H = L + L^{-1}, \tag{1}$$

so that

$$\begin{aligned} TM_c &= (L_1 + L_1^{-1} + \cdots + L_n + L_n^{-1}) \otimes (L + L^{-1}) \\ &= L_1 L + L_1^{-1} L + \cdots + L_n L + L_n^{-1} L \\ &\quad + L_1 L^{-1} + L_1^{-1} L^{-1} + \cdots + L_n L^{-1} + L_n^{-1} L^{-1} \end{aligned} \tag{2}$$

where juxtaposition means tensor product. The total Chern classes of E and H can be formally written as follows

$$c(E) = (1 + x_1)(1 - x_1) \cdots (1 + x_n)(1 - x_n), \quad c(H) = (1 + l)(1 - l),$$

where x_i and l are formal roots corresponding to first Chern classes of L_i and L, respectively, and, therefore,

$$c(TM_c) = (1 - (x_1 + l)^2)(1 - (x_1 - l)^2) \cdots (1 - (x_n + l)^2)(1 - (x_n - l)^2),$$

The Pontrjagin class of M is

$$p(TM) = (1 + (x_1 + l)^2)(1 + (x_1 - l)^2) \cdots (1 + (x_n + l)^2)(1 + (x_n - l)^2),$$

and the $\widehat{A}$-genus is given by

$$\begin{aligned} \widehat{A}(M) &= \prod_{j=1}^{n} \frac{(x_i + l)/2}{\sinh(x_i + l)/2} \cdot \frac{(x_i - l)/2}{\sinh(x_i - l)/2} \\ &= \prod_{j=1}^{n} \frac{x_i + l}{e^{(x_i+l)/2} - e^{-(x_i+l)/2}} \cdot \frac{x_i - l}{e^{(x_i-l)/2} - e^{-(x_i-l)/2}} \end{aligned}$$

Note that we could change $(x_i + l)$ for $-(x_i + l)$ or $(x_i - l)$ for $-(x_i - l)$.

The Chern characters of the relevant coefficient bundles are

$$\begin{aligned} \mathrm{ch}(E) &= e^{x_1} + e^{-x_1} + \cdots + e^{x_n} + e^{-x_n} \\ \mathrm{ch}(H) &= e^{l} + e^{-l}, \\ \mathrm{ch}(S^q H) &= \sum_{t=0}^{q} e^{(q-2t)l}, \end{aligned}$$

and the ones of the exterior powers $\bigwedge^p E$ of E are given by the coefficients of the powers of t in the following expression

$$\mathrm{ch}(\textstyle\bigwedge_t E) = \mathrm{ch}\left(\displaystyle\sum_{p=0}^{n} (\textstyle\bigwedge^p E) \cdot t^p \right) = \displaystyle\prod_{i=1}^{n} \left((1 + t e^{x_i}) \cdot (1 + t e^{-x_i}) \right),$$

where

$$\bigwedge_t E = \sum_{p=0}^{\infty} (\bigwedge^p E) \cdot t^p.$$

3 Rigidity and Dirac operators

3.1 Rigidity of elliptic operators

Definition 3.1 Let $D\colon \Gamma(E) \longrightarrow \Gamma(F)$ be an elliptic operator acting on sections of the vector bundles E and F over a compact manifold M. The index of D is the virtual vector space $\operatorname{ind}(D) = \ker(D) - \operatorname{coker}(D)$. If M admits a circle action preserving D, i.e. such that S^1 acts on E and F, and commutes with D, $\operatorname{ind}(D)$ admits a Fourier decomposition into complex one-dimensional irreducible representations of S^1 $\operatorname{ind}(D) = \sum a_m L^m$, where $a_m \in \mathbb{Z}$ and L^m is the representation of S^1 on $\mathbb{C}$ given by $\lambda \mapsto \lambda^m$. The elliptic operator D is called *rigid* if $a_m = 0$ for all $m \neq 0$, i.e. $\operatorname{ind}(D)$ consists only of the trivial representation with multiplicity a_0.

Let us recall three examples.

Example. The deRham complex

$$d + d^*\colon \Omega^{even} \longrightarrow \Omega^{odd}$$

from even-dimensional forms to odd-dimensional ones, where d^* denotes the adjoint of the exterior derivative d, is *rigid* for any circle action on M by isometries since by Hodge theory the kernel and the cokernel of this operator consist of harmonic forms, which by homotopy invariance stay fixed under the circle action.

Example. The signature operator on an oriented manifold

$$d_s\colon \Omega_c^+ \longrightarrow \Omega_c^-$$

from even to odd complex forms under the Hodge $*$ operator is *rigid* for any circle action on M by isometries since the kernel and cokernel of this operator consist of harmonic forms.

Example. The Dirac operator on a Spin manifold is *rigid* for any circle action by isometries [1]. This time, however, the rigidity is not due to homotopy invariance.

3.2 Twisted Dirac operators

In this subsection, let M be a $4n$-dimensional oriented Riemannian manifold. M is Spin if its orthonormal frame bundle $P_{SO(4n)}$ admits a double cover by a principal bundle $P_{Spin(4n)}$ with structure group $Spin(4n)$, the universal cover of $SO(4n)$. This gives rise to new bundles over M such as the spinor bundle Δ corresponding to the faithful complex representation of $Spin(4n)$ of dimension 2^{2n}. By using a real or quaternionic structure on Δ (depending on the parity of n) and an invariant Hermitian metric we get an equivariant isomorphism $\Delta \cong \Delta^*$. The spinor representation decomposes as

$$\Delta = \Delta_+ \oplus \Delta_-,$$

where Δ_+ and Δ_- are irreducible representations of equal dimension.

The representation of $Spin(4n)$ on $\mathrm{End}(\Delta) \cong \Delta \otimes \Delta$ has kernel $\{\pm 1\}$, so that it factors through $SO(4n)$ and

$$\Delta \otimes \Delta \cong \bigoplus_{k=0}^{4n} \bigwedge^k T,$$

where $T = TM_c$ denotes the complexified tangent bundle of M and $\bigwedge^k T$ denotes its kth exterior power. The inclusion of $T = \bigwedge^1 T$ in $\mathrm{End}(\Delta) \cong \Delta \otimes \Delta$ gives rise to a $Spin(4n)$-equivariant homomorphism $\mu \colon T^* \otimes \Delta \longrightarrow \Delta$ called Clifford multiplication with the property

$$\mu(T^* \otimes \Delta_\pm) = \Delta_\mp.$$

The Levi-Civita connection on $P_{SO(4n)}$ can be lifted to $P_{Spin(4n)}$ to define a covariant differentiation ∇ on Δ

$$\nabla \colon \Gamma(\Delta) \longrightarrow \Gamma(T^* \otimes \Delta),$$

and the Dirac operator is defined as the composition

$$\partial\!\!\!/ = \mu \circ \nabla : \Gamma(\Delta) \longrightarrow \Gamma(\Delta),$$

which is elliptic and self-adjoint. Since the spin representation decomposes, the Dirac operator can be split into two parts

$$\partial\!\!\!/ \colon \Gamma(\Delta_+) \longrightarrow \Gamma(\Delta_-), \quad \partial\!\!\!/^* \colon \Gamma(\Delta_-) \longrightarrow \Gamma(\Delta_+).$$

In terms of dual bases $\{e_i\}$ and $\{e^i\}$ of T and T^*, respectively,

$$\partial\!\!\!/(\psi) = \sum_{i=1}^{4n} \mu(e^i \otimes \nabla_{e_i} \psi)$$

for $\psi \in \Gamma(\Delta_+)$.

We are interested in Dirac operators with coefficients in auxiliary vector bundles F equipped with a covariant derivative $\nabla^F : \Gamma(F) \longrightarrow \Gamma(T^* \otimes F)$. The Dirac operator twisted by F (or with coefficients in F)

$$(\partial\!\!\!/ \otimes F) \colon \Gamma(\Delta_+ \otimes F) \longrightarrow \Gamma(\Delta_- \otimes F),$$

is defined by

$$(\partial\!\!\!/ \otimes F)(\psi \otimes f) = \partial\!\!\!/(\psi) \otimes f + \sum_{i=0}^{4n} \mu(e^i \otimes \psi) \otimes \nabla^F_{e_i} f,$$

where $\psi \in \Gamma(\Delta)$, $f \in \Gamma(F)$. Note that, even if the manifold is not Spin, the spinor bundle Δ can be defined locally. Furthermore, by taking the tensor product of Δ with another suitably chosen locally defined vector bundle F, we can still get a globally defined vector bundle $\Delta \otimes F$ and a Dirac operator $\partial\!\!\!/ \otimes F$ defined by the same formulae.

3.3 Dirac operators on almost quaternionic manifolds

In this subsection, let M be a compact, almost quaternion-Hermitian manifold and Δ denote the (locally defined) Spin bundle of M. Depending on whether M is Spin or not, one can consider tensor products of Δ with other bundles such as $F^{p,q} = \bigwedge^p E \otimes S^q H$ to get a globally defined vector bundle

$$\Delta \otimes F^{p,q} = \Delta \otimes \bigwedge^p E \otimes S^q H.$$

In general, for $\Delta \otimes F^{p,q}$ to be defined, $p+q$ must be even if M is Spin, and $p+q$ must be odd if M is not Spin. After choosing connections on $\bigwedge^p E$ and $S^q H$, there is a Dirac operator for sections of these bundles

$$\partial\!\!\!/ \otimes F^{p,q} = \partial\!\!\!/ \otimes (\bigwedge^p E \otimes S^q H),$$

and by the Atiyah–Singer Theorem, the index of this operator can be computed by

$$\mathrm{ind}(\partial\!\!\!/ \otimes F^{p,q}) = \left\langle \mathrm{ch}(\bigwedge^p E)\mathrm{ch}(S^q H)\widehat{A}(M), [M] \right\rangle.$$

We are now ready to state our main theorem.

Theorem 3.1 *Let M be a compact $4n$-dimensional almost quaternionic manifold admitting a smooth circle action preserving the almost quaternionic structure. Then there exists a compatible quaternion-Hermitian metric so that the circle action is isometric and the twisted Dirac operators $\partial\!\!\!/ \otimes F^{p,q}$ are rigid (under the action) if $p+q \leq n$ and $p+q \equiv n \pmod 2$.*

Furthermore,

$$\mathrm{ind}(\partial\!\!\!/ \otimes F^{p,q}) = \left\langle \mathrm{ch}(F^{p,q})\widehat{A}(M), [M] \right\rangle = 0$$

if $p+q < n$.

Remark 3.1 On any compact, oriented, smooth manifold, the signature operator can be seen as a Dirac operator twisted by the Spinor vector bundle

$$d_s = \partial\!\!\!/ \otimes \Delta.$$

With respect to $Sp(n)Sp(1)$, the spin representation splits as follows

$$\Delta = F^{n,0} \oplus F^{n-1,1} \oplus \cdots \oplus F^{0,n}.$$

Although the rigidity of the signature operator is known, it is interesting to notice that each one of the pieces in

$$d_s = \partial\!\!\!/ \otimes (F^{n,0} \oplus F^{n-1,1} \oplus \cdots \oplus F^{0,n})$$

is also rigid.

4 S^1 action and fixed points

In this section, let M be a compact, almost quaternionic manifold admitting a smooth circle action preserving the almost quaternionic structure.

4.1 Infinitesimal automorphism of the structure

First, choose a Riemannian metric g_0 on M and average it over the circle action to get a S^1-invariant metric g_1

$$g_1(v, w) = \int_{S^1} g_0(\lambda_* v, \lambda_* w) d\lambda,$$

where $v, w \in TM$. Now, given a local admissible basis $\{I, J, K\}$, consider the almost quaternion-Hermitian metric g_2 given, as before, by

$$g_2(v, w) = g_1(v, w) + g_1(Iv, Iw) + g_1(Jv, Jw) + g_1(Kv, Kw).$$

With these choices, the manifold M is now an almost quaternion-Hermitian with an isometric circle action which preserves the almost quaternionic structure. Let ∇ denote the Levi-Civita connection on M.

Let X denote the Killing vector field of the circle action. Now the circle action preserves the almost quaternion-Hermitian structure if and only if X is an infinitesimal automorphism of the structure, i.e.,

$$\begin{aligned}(\mathcal{L}_X g_2)(v, w) &= X(g_2(v, w)) - g_2(\mathcal{L}_X v, w) - g_2(v, \mathcal{L}_X w) = 0,\\ (\mathcal{L}_X \Omega)(v, w, x, y) &= X(\Omega(v, w, x, y)) - \Omega(\mathcal{L}_X v, w, x, y) - \Omega(v, \mathcal{L}_X w, x, y)\\ &\quad -\Omega(v, w, \mathcal{L}_X x, y) - \Omega(v, w, x, \mathcal{L}_X y) = 0.\end{aligned}$$

Since we have the Levi-Civita connection, the equations are equivalent to

$$\begin{aligned}0 &= X(g_2(v, w)) - g_2(\nabla_X v - \nabla_v X, w) - g_2(v, \nabla_X w - \nabla_w X),\\ 0 &= X(\Omega(v, w, x, y)) - \Omega(\nabla_X v - \nabla_v X, w, x, y) - \Omega(v, \nabla_X w - \nabla_w X, x, y)\\ &\quad -\Omega(v, w, \nabla_X x - \nabla_x X, y) - \Omega(v, w, x, \nabla_X y - \nabla_y X).\end{aligned}$$

At a fixed point p of the circle action, $X_p = 0$, $(X(g_2(v, w)))|_p = 0$, $(X(\Omega(v, w, x, y)))|_p = 0$, $(\nabla_X v)|_p = 0$, etc., so that

$$0 = (g_2(\nabla_v X, w) + g_2(v, \nabla_w X))|_p, \tag{3}$$

$$\begin{aligned}0 &= (\Omega(\nabla_v X, w, x, y) + \Omega(v, \nabla_w X, x, y) + \Omega(v, w, \nabla_x X, y)\\ &\quad +\Omega(v, w, x, \nabla_y X))|_p.\end{aligned} \tag{4}$$

Lemma 4.1 *Let V be a real vector space and $\Phi\colon V^{\times k} \longrightarrow \mathbb{R}$ a multilinear map. Let $A \in GL(V)$ and $a \in \mathfrak{gl}(V)$ such that $A = e^{ta} = 1 + ta + t^2a^2/2! + \cdots$. Then, A preserves Φ*

$$\Phi(Av_1, \ldots, Av_k) = \Phi(v_1, \ldots, v_k)$$

if and only if

$$\Phi(av_1, v_2, \ldots, v_k) + \cdots + \Phi(v_1, \ldots, v_{k-1}, av_k) = 0.$$

Proof If A preserves Φ

$$\begin{aligned}\Phi(v_1, \ldots, v_k) &= \Phi(Av_1, \ldots, Av_k)\\ &= \Phi(v_1 + tav_1 + \cdots, \cdots, v_k + tav_k + \cdots)\\ &= \Phi(v_1, \ldots, v_k) + t(\Phi(av_1, v_2, \ldots, v_k) + \cdots + \Phi(v_1, \ldots, v_{k-1}, av_k))\\ &\quad +O(t^2),\end{aligned}$$

which implies

$$\Phi(av_1, v_2, \ldots, v_k) + \cdots + \Phi(v_1, \ldots, v_{k-1}, av_k) = 0.$$

Conversely, assume

$$\Phi(av_1, v_2, \ldots, v_k) + \cdots + \Phi(v_1, \ldots, v_{k-1}, av_k) = 0.$$

Examine

$$\begin{aligned}\Phi(Av_1,\ldots,Av_k) &= \Phi(v_1+tav_1+\cdots,\cdots,v_k+tav_k+\cdots)\\ &= \Phi(v_1,\ldots,v_k)+t(\Phi(av_1,v_2,\ldots,v_k)+\cdots+\Phi(v_1,\ldots,v_{k-1},av_k)),\\ &\quad+\frac{1}{2!}t^2[\Phi(a^2v_1,v_2,\ldots,v_k)+\cdots+\Phi(v_1,\ldots,v_{k-1},a^2v_k)\\ &\quad+2\Phi(av_1,av_2,\ldots,v_k)+\cdots+2\Phi(v_1,\ldots,av_{k-1},av_k)]+O(t^3)\end{aligned}$$

Observe that the coefficient of t^2 (up to a factor) is

$$\begin{aligned}&\Phi(a^2v_1,v_2,\ldots,v_k)+\cdots+\Phi(v_1,\ldots,v_{k-1},a^2v_k)\\ &\quad+2\Phi(av_1,av_2,\ldots,v_k)+\cdots+2\Phi(v_1,\ldots,av_{k-1},av_k)\\ &= \Phi(a(av_1),v_2,\ldots,v_k)+\Phi(av_1,av_2,\ldots,v_k)+\cdots+\Phi(av_1,\ldots,v_{k-1},av_k)\\ &\quad+\Phi(v_1,a(av_2),\ldots,v_k)+\Phi(av_1,av_2,\ldots,v_k)+\cdots+\Phi(v_1,av_2,\ldots,av_k)\\ &\quad\vdots\\ &\quad+\Phi(v_1,v_2,\ldots,a(av_k))+\Phi(av_1,v_2,\ldots,av_k)+\cdots+\Phi(v_1,\ldots,av_{k-1},av_k)\\ &= 0,\end{aligned}$$

and similarly for the coefficients of the higher powers of t. Hence,

$$\Phi(Av_1,\ldots,Av_k)=\Phi(v_1,\ldots,v_k).$$

□

Lemma 4.1, (3), (4) and [23, Lemma 9.1] prove the following.

Lemma 4.2 *Let M be a compact $4n$-dimensional almost quaternionic manifold admitting a smooth circle action preserving the almost quaternionic structure. Then, there exists a compatible quaternion-Hermitian metric so that the circle action is isometric, and at a fixed point p, $(\nabla X)|_p$ belongs to the Lie algebra of the structure group, $\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)$.* □

Remark 4.1 Notice that we have chosen a particular metric so that the vector field X becomes a Killing vector field. In general, however, Killing vector fields may not preserve the $Sp(n)$ $Sp(1)$ structure since this structure is parallel only when the Riemannian manifold is quaternion-Kähler.

4.2 Exponents of the S^1 action and fixed point submanifolds

In this section, let M be a compact, almost quaternion-Hermitian manifold with an isometric circle action preserving the almost quaternionic structure.

Let $P\subset M^{S^1}$ be an S^1-fixed submanifold. The tangent space at a point $p\in P$ can be decomposed as follows

$$TM_p=\mathcal{L}^{m_1}+\cdots+\mathcal{L}^{m_{2n}}$$

where $m_i\in\mathbb{Z}$ are the exponents of the action so that $z\in S^1$ acts via multiplication by z^{m_i} on $\mathcal{L}^{m_i}$. The sign of these exponents can be changed in lots of two.

We can be more precise about these exponents. Since the S^1 action on M is isometric, it lifts to the tangent bundle and to the bundle of orthonormal frames while preserving the Levi-Civita connection. On a fixed submanifold P, S^1 acts on the fibers of the restricted bundles $S^2H|_P$, $\bigwedge^2E|_P$, etc., since by Lemma 4.2, $(\nabla X)|_p\in\mathfrak{sp}(n)\oplus\mathfrak{sp}(1)$. By composing with projections to both factors we see that S^1 lies inside maximal tori of $Sp(n)$ and $Sp(1)$, respectively. In this way, E_p, H_p, S^2H_p, $\bigwedge^2 E_p$, etc., can be decomposed formally into sums of complex line bundles under the S^1 action, and we can take the formal decompositions (1) to be the ones given by S^1 at $p \in P$. Thus, $z \in S^1$ acts on $L_j^2|_p$ and $L^2|_p$ by multiplication by z^{e_j} and z^h, respectively, for some integers $e_j = e_j(P) \in \mathbb{Z}$ and $h = h(P) \in \mathbb{Z}$. In this way, by (2), the m_i become either

$$\pm m_i = \frac{e_a + h}{2} \quad \text{or} \quad \pm m_i = \frac{e_a - h}{2}$$

for some $1 \leq a \leq n$, where n of the exponents must be of the first type and the remaining n exponents must be of the second type.

Now, let us suppose that P has positive dimension. That means that over P some m_i are equal to zero. For instance, if $m_j = e_j - h = 0$, there are two possibilities: either $e_j + h = 0$ or not.

Case $\boldsymbol{e_j + h = 0}$. In this case $e_j = h = 0$. Since not all of the exponents can be zero, we see that the real dimension of P is a multiple of 4. Furthermore, $h = 0$ implies that the action of S^1 on $S^2H|_p$ is trivial, so that no almost complex structure is being distinguished by the S^1 action, but we will still have a quaternionic structure over P (see below).

Case $\boldsymbol{e_j + h \neq 0}$. Here $h \neq 0$, which means that the real dimension of P increases by 2 for each $e_j \neq -h$, and there can only be up to n of them. Furthermore, since $h \neq 0$, the action on $S^2H|_p$ is non-trivial, so that there is a distinguished one-dimensional subspace of $S^2H|_p$ as a trivial summand giving us an almost complex structure on TP (see below).

Proposition 4.1 *Let M be a $4n$-dimensional almost quaternion-Hermitian manifold admitting a non-trivial isometric circle action preserving the almost quaternionic structure. If M^{S^1} is not empty, the connected components $P \subset M^{S^1}$ are either*

- *almost quaternionic submanifolds of real dimension a multiple of 4,*
- *or almost complex submanifolds of real dimension at most $2n$.*

Proof Let X denote the Killing vector field of the S^1 action. The tangent space to a component $P \subset M^{S^1}$ is given as

$$T_pP = \{Y \in T_pM | \nabla_Y X = 0\}.$$

At an S^1-fixed point $p \in P$, X is determined by $(\nabla X)|_p$ and, by Lemma 4.2, $(\nabla X)|_p$ belongs to $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$.

For the first case, the action of S^1 on the fibers of S^2H_p is trivial, i.e. $h = 0$, so that the projection on the $\mathfrak{sp}(1)$-factor is also zero and

$$(\nabla X)|_p = \sigma \in \Gamma(S^2E|_p).$$

Thus, for any almost complex structure $\eta \in \Gamma(Q|_P)$ and $Y \in T_pP$

$$\begin{aligned} \nabla_{\eta Y} X &= \sigma(\eta Y) \\ &= \eta\sigma Y \\ &= \eta \nabla_Y X \\ &= 0, \end{aligned}$$

so that $\eta Y \in T_pP$, since the action of $\mathfrak{sp}(1)$ commutes with the one of $\mathfrak{sp}(n)$. In other words, $Q|_P$ gives endomorphisms of TP, thus making P an almost quaternionic submanifold of M.

For the second case, since $h \neq 0$, the action of S^1 on the fibers of $S^2 H_p$ is non-trivial, i.e., the splitting

$$S^2 H_p = L^2 + 1 + L^{-2}$$

contains an honest trivial line bundle determined by the projection of ∇X to $\mathfrak{sp}(1)$, which is nonzero. Thus

$$(\nabla X)|_p = t\xi + \sigma \in \Gamma(S^2 H_p \oplus S^2 E_p),$$

for some ξ of constant unit length and $t \in \mathbb{R}$. Now, for $Y \in T_p P$

$$\begin{aligned} \nabla_{\xi Y} X &= (t\xi + \sigma)(\xi Y) \\ &= t\xi\xi Y + \sigma\xi Y \\ &= \xi(t\xi + \sigma)Y \\ &= \xi \nabla_Y X \\ &= 0, \end{aligned}$$

so that $\xi Y \in T_p P$, and ξ is an almost complex structure on TP. □

This proposition generalizes results of [5] and [3].

5 Rigidity of twisted Dirac operators

First, let us make the following remark.

Remark 5.1 Consider the function of $z \in \mathbb{C}$

$$F(z) = \frac{z^k}{z^{-m}e^x - z^m e^{-x}}$$

where x is an unknown, $k, m \in \mathbb{Q}$ and, for simplicity, let us assume $m > 0$. We wish to control the behaviour of the function $F(z)$ at 0 and ∞. Thus,

$$\lim_{z\to 0} F(z) = \lim_{z\to 0} \frac{z^{k+m}}{e^x - z^{2m}e^{-x}} = 0 \tag{5}$$

if $k > -m$, and

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{z^{k-m}}{z^{-2m}e^x - e^{-x}} = 0 \tag{6}$$

if $k < m$. Hence, F has zeroes at 0 and ∞ if $|k| < |m|$.

In fact, even if $k + m = 0$

$$\lim_{z\to 0} F(z) = \lim_{z\to 0} \frac{1}{e^x - z^{2m}e^{-x}} = \frac{1}{e^x} \tag{7}$$

and

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{z^{-2m}}{z^{-2m}e^x - e^{-x}} = 0, \tag{8}$$

and analogously if $k - m = 0$.

Proof of Theorem 3.1. Consider the bundle $F^{p,q}$ and assume the corresponding twisted Dirac operator is defined. By the Atiyah–Singer index theorem, the index of the elliptic operator $\partial\!\!\!/ \otimes F^{p,q}$ can be computed as follows

$$\begin{aligned}\operatorname{ind}(\partial\!\!\!/ \otimes F^{p,q}) &= \left\langle \operatorname{ch}\left(\bigwedge^p E\right)\cdot \operatorname{ch}(S^q H)\cdot \widehat{A}(M), [M]\right\rangle \\ &= \left\langle \left(\sum (e^{((-1)^{\varepsilon_1}x_{i_1}+\cdots+(-1)^{\varepsilon_r}x_{i_r})} + e^{-((-1)^{\varepsilon_1}x_{i_1}+\cdots+(-1)^{\varepsilon_r}x_{i_r})}\right)\right. \\ &\quad\times\left(\sum_{t=0}^{q} e^{(q-2t)l}\right) \\ &\quad\times\left.\left(\prod_{j=1}^{n}\frac{x_i+l}{e^{(x_i+l)/2}-e^{-(x_i+l)/2}}\cdot\frac{x_i-l}{e^{(x_i-l)/2}-e^{-(x_i-l)/2}}\right), [M]\right\rangle.\end{aligned}$$

where $1 \le i_1 < \ldots < i_r \le n$, $r \le p$, $r \equiv p \pmod 2$, and $\varepsilon_i = 0, 1$. By the Atiyah–Singer fixed point theorem [2], the equivariant version of the index can be written in terms of the local data of M^{S^1} (see [16, p. 67])

$$\operatorname{ind}(\partial\!\!\!/ \otimes F^{p,q})_z = \sum_{P\subset M^{S^1}} \mu^{p,q}(P, z)$$

where $z \in S^1$ be a generic element of S^1, $\mu^{p,q}(P, z)$ is the local contribution of the fixed point connected component (submanifold) $P \subset M^{S^1}$

$$\begin{aligned}\mu^{p,q}(P,z) &= \left\langle\left(\sum (z^{-((-1)^{\varepsilon_1}e_{i_1}+\cdots+(-1)^{\varepsilon_r}e_{i_r})/2}e^{((-1)^{\varepsilon_1}x_{i_1}+\cdots+(-1)^{\varepsilon_r}x_{i_r})}\right.\right. \\ &\quad\left.+z^{((-1)^{\varepsilon_1}e_{i_1}+\cdots+(-1)^{\varepsilon_r}e_{i_r})/2}e^{-((-1)^{\varepsilon_1}x_{i_1}+\cdots+(-1)^{\varepsilon_r}x_{i_r})})\right) \\ &\quad\times\left(\sum_{t=0}^{q} z^{(q-2t)h/2}e^{-(q-2t)l}\right)\times\prod_{e_i+h=0}(x_i+l)\prod_{e_j-h=0}(x_j-l) \\ &\quad\times\prod_{i=1}^{n}\frac{1}{z^{-(e_i+h)/4}e^{(x_i+l)/2}-z^{(e_i+h)/4}e^{-(x_i+l)/2}} \\ &\quad\times\left.\prod_{i=1}^{n}\frac{1}{z^{-(e_i-h)/4}e^{(x_i-l)/2}-z^{(e_i-h)/4}e^{-(x_i-l)/2}}, [P]\right\rangle.\end{aligned}$$

Here, we have substituted

$$\begin{array}{rcl} M & \text{by} & P, \\ e^{\pm x_i} & \text{by} & z^{\mp e_i/2}e^{\pm x_i}, \\ e^{\pm l} & \text{by} & z^{\mp h/2}e^{\pm l}. \end{array}$$

The exponents of $\bigwedge^p E \otimes S^q H$ are of the form

$$(-1)^{\varepsilon_1}e_{i_1}+\cdots+(-1)^{\varepsilon_p}e_{i_r}\pm sh$$

where $1 \le i_1 < \ldots < i_r \le n$, $r \le p$, $r \equiv p \pmod 2$, $s \le q$, $s \equiv q \pmod 2$, and $\varepsilon_i = 0, 1$.

From Remark 5.1 we see that $\partial\!\!\!/ \otimes F^{p,q}$ is rigid, i.e. does not depend on z, if

$$\left|\sum_{a=1}^{r}(-1)^{\varepsilon_a}e_{i_a} \pm sh\right| = \left|\sum_{a=1}^{r}(-1)^{\varepsilon_a}\left(\frac{e_{i_a}+h}{2}+\frac{e_{i_a}-h}{2}\right) \pm \sum_{b=1}^{s}\left(\frac{h+e_{j_b}}{2}+\frac{h-e_{j_b}}{2}\right)\right|$$
$$\leq \frac{1}{2}\left[\sum_{a=1}^{r}(|e_{i_a}+h|+|e_{i_a}-h|)+\sum_{b=1}^{s}(|h+e_{j_b}|+|h-e_{j_b}|)\right]$$
$$\leq \frac{1}{2}\sum_{i=1}^{n}(|e_i+h|+|e_i-h|),$$

which is fulfilled as long as there exists an s-tuple of indices $j_1 < \cdots < j_s$ such that $\{j_1,\ldots,j_s\} \subset \{1,\ldots,n\} - \{i_1,\ldots,i_r\}$. More precisely, the last inequality holds if $p+q \leq n$ so that the function $\mu^{p,q}(P,z)$ has finite values at 0 and ∞. Each equivariant index $\mathrm{ind}(\partial\!\!\!/ \otimes \bigwedge^p E \otimes S^q H)_z$, as well as being a rational function of the complex variable z, it also belongs to the representation ring $R(S^1)$ of S^1 which can be identified with the Laurent polynomial ring $\mathbb{Z}[z, z^{-1}]$. Notice that as an equivariant index, it can only have poles on the unit circle; as a Laurent series it can only have poles at 0 and ∞. Thus it has no poles at all and is constant in the variable z.

Furthermore, if $p+q < n$, and $p+q \equiv n \pmod 2$, $\mu^{p,q}(P,z)$ has zeroes at 0 and ∞, so that the equivariant index $\mathrm{ind}(\partial\!\!\!/ \otimes F^{p,q})_z$ vanishes at 0 and ∞. Hence,

$$\mathrm{ind}(\partial\!\!\!/ \otimes \textstyle\bigwedge^p E \otimes S^q H) = \mathrm{ind}(\partial\!\!\!/ \otimes \bigwedge^p E \otimes S^q H)_1 = 0. \qquad \square$$

Remark 5.2 We can see from the proof of Theorem 3.1 that the exponents of the S^1 action at the fixed points determine the rigidity (and vanishing) of the equivariant indices of several twisted Dirac operators. We cannot, however, apply the same arguments to obtain general rigidity theorems for other twisted Dirac operators, such as $\partial\!\!\!/ \otimes S^2 E$ since we would have the following inequalities

$$|e_i+e_j| < \frac{1}{2}\sum_{i=1}^{n}(|e_i+h|+|e_i-h|) \quad \text{for } i<j,$$
$$|e_i-e_j| < \frac{1}{2}\sum_{i=1}^{n}(|e_i+h|+|e_i-h|) \quad \text{for any } i, j,$$
$$2|e_i| < \frac{1}{2}\sum_{i=1}^{n}(|e_i+h|+|e_i-h|) \quad \text{for all } i.$$

The last set of inequalities may not be fulfilled in general.

Despite Remark 5.2 we can prove the following rigidity theorem.

Theorem 5.1 *Let M be an $8n$-dimensional almost quaternion-Hermitian manifold admitting an isometric circle action preserving the almost quaternionic structure. Then the operators*

$$\partial\!\!\!/ \otimes TM_c \quad \textit{and}\ \partial\!\!\!/ \otimes S^2E$$

are rigid.

Proof For $\partial\!\!\!/ \otimes TM_c$ (the Rarita-Schwinger operator) recall that

$$TM_c = E \otimes H = F^{1,1},$$

so that Theorem 3.1 gives another proof of its rigidity not requiring the Witten rigidity theorem for the elliptic genus [4]. Note that in this dimension, almost quaternionic manifolds are Spin [21].

For $\partial\!\!\!/ \otimes S^2E$, on the other hand, observe that

$$\bigwedge^2(TM_c) = S^2E \oplus \bigwedge^2 E \otimes S^2H \oplus S^2H.$$

Given that M is Spin [21], the Witten rigidity theorem [4] says that the operator $\partial\!\!\!/ \otimes \bigwedge^2(TM_c)$ is rigid. Since the operators $\partial\!\!\!/ \otimes F^{2,2}$ and $\partial\!\!\!/ \otimes F^{0,2}$ are also rigid, the operator $\partial\!\!\!/ \otimes S^2E$ must also be rigid. □

Remark 5.3 Theorems 3.1 and 5.1 are applicable to the compact homogeneous quaternionic manifolds given by Joyce [17].

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