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An optimal investment strategy with maximal risk aversion and its ruin probability

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Abstract In this paper we study an optimal investment problem of an insurer when the company has the opportunity to invest in a risky asset using stochastic control techniques. A closed form solution is given when the risk preferences are exponential as well as an estimate of the ruin probability when the optimal strategy is used.

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Introduction

The ruin probability of the reserve of an insurance company, in finite and infinite horizon, when there is the possibility to invest in a risky asset, has recently received a lot of attention. It is well known that for the classical Cramér–Lundberg process (where there is no investment and the claims have exponential moments), the ruin probability decreases exponentially with respect to the initial wealth.

Hipp and Plum (2000), assuming that the price of stock is modeled by the Geometric Brownian Motion, determined the strategy of investment which minimizes the ruin probability using the Hamilton–Jacobi–Bellman equation. In Gaier et al. (2003), under the same hypotheses as Hipp and Plum, obtained an exponential bound with a rate that improves the classical Lundberg parameter by proposing a trading strategy that consists in investing in the stock a constant amount of money, independent of the current level of the reserve. Hipp and Schmidli (2004) showed that this strategy is asymptotically optimal as the initial capital tends to infinity.

In this paper we study the problem from a different point of view. We follow the approach introduced by Ferguson (1965) who conjectured that maximizing exponential utility from terminal wealth is intrinsically related to minimizing the probability of ruin. Ferguson studied this problem for a discrete time and discrete space investor. Browne (1995) verified the conjecture in a model without interest rate, where the stock follows a Geometric Brownian Motion, and the Risk Process is a Brownian Motion with drift, see Ferguson (1965), Browne (1995), and references therein. We consider the wealth process of the reserve of an insurance company,

$$X_t^\pi = x + ct - \sum_{j=0}^{N_t} Y_j + \int_0^t (a - \eta)\pi_r dr$$

$$+ \int_0^t \eta X_r^\pi dr + \int_0^t \sigma \pi_r dB_r,$$

where x is the initial surplus, c is the premium rate, the claims $(Y_j)_{j \geq 1}$ have exponential moments, $(\pi_t)_{t \geq 0}$ is the amount invested at time t in a risky asset that follows a Geometric Brownian Motion, and the rest of the wealth $(X_t^\pi - \pi_t)_{t \geq 0}$ is invested in a bond (see the complete formulation of the problem in Sect. 1). We first determine the optimal strategy that maximizes an exponential utility function $(-\exp^{-\gamma x})$ of the wealth process for a finite horizon of time (T). As a next step we examine the finite time ruin probability of the reserve process controlled by the optimal strategy. We obtain an exponential bound for the ruin probability that, when applied to the Classical Cramér Risk Process, improves the classical Lundberg parameter for some values of γ. If we take the particular value $\gamma = \widehat{r}$, with $\widehat{r}$ as in Gaier et al. (2003)—their modified Lundberg parameter—and with zero interest rate, our strategy is the same as the one obtained by Gaier, Grandits and Schachermayer, and Hipp and Schmidli. Hence, we can conclude that there is a deep relationship between maximizing the exponential utility function and minimizing the probability of ruin.

There are other works showing the relation between utility maximization and ruin probability minimization; when the claim process is a pure jump process (not necessarily compound Poisson) it has been studied by Wang (2007), while the case when the cash flow of the investor follows a jump diffusion process was analyzed in Yang and Zhang (2005). For the case of the classic Cramér–Lundberg model with different controls we refer to Guerra and Centeno (2008).

The organization of this paper is as follows. Section 1 is devoted to describe the problem. In Sect. 2 we prove the verification theorem for the optimization problem, Theorem 2.1. In Sect. 3 we give a closed form solution for an exponential utility function and find, explicitly, the optimal strategy in Theorem 3.1. In Sect. 4 we estimate a bound for the ruin probability in Proposition 4.1 and show that our results include those of other authors (see, in particular, Remark 4.3). Finally, in the last Section, we discuss numerical results.

1 Formulation of the problem

In this section we formulate an optimal investment problem for an insurance company which is allowed to invest in the securities market. Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space, where a Brownian motion $(B_t)_{t \geq 0}$, a Poisson process $(N_t)_{t \geq 0}$ with constant intensity λ, and a sequence of independent non-negative random variables $(Y_i)_{i \geq 1}$ with identical distribution ν are defined. It is assumed that $(B_t)_{t \geq 0}$, $(N_t)_{t \geq 0}$ and $(Y_i)_{i \geq 1}$ are independent, and for each $t > 0$ the filtration $(\mathcal{F}_t)_{t \geq 0}$ containing the information up to time t is defined by

$$\mathcal{F}_t = \sigma\{B_s, \ N_s, \ Y_j \mathbb{1}_{[j \leq N_s]}, \ s \leq t, \ j \geq 1\}.$$

The market where the insurer can invest is composed by a bank account S^0 and a risky asset S_t, whose dynamics satisfy

$$S_t^0 = S_0^0 e^{\eta t}, \quad S_0^0 = 1,$$
$$dS_t = S_t(a\,dt + \sigma\,dB_t), \quad S_0 = s_0,$$

where η, a, and σ are constants.

On the other hand the risk process is described by the classical Cramér–Lundberg model, using a compound Poisson process for the claims. Given the initial surplus x and the premium rate $c > 0$, the risk process

$$R_t = x + ct - \sum_{i=1}^{N_t} Y_i,$$

where $(Y_i)_{i \geq 1}$ model the claim amounts.

We are interested in a finite time horizon problem. Then, at each time $t \in [0, T]$, with $T > 0$ fixed, the insurer divides his wealth X_t between the risky and the riskless assets and if a claim is received at that time, it is paid immediately. Let π_t be the

amount of wealth invested in the risky asset at time t, which takes values in $\mathbb{R}$, while the rest of his wealth $X_t - \pi_t$ is invested in the bank account. Then, if at time $s < T$ the surplus of the company is x, the wealth process satisfies the dynamics

$$
\begin{aligned}
X_t^{s,x,\pi} = x + c(t-s) - \sum_{j=N_s+1}^{N_t} Y_j + \int_s^t (a-\eta)\pi_r dr \\
+ \int_s^t \eta X_r^{s,x,\pi} dr + \int_s^t \sigma \pi_r dB_r,
\end{aligned}
\tag{1.1}
$$

with the convention that $\sum_{j=1}^{0} = 0$. When $s = 0$, we write X_t^{π}.

Definition 1.1 We say that $\pi = (\pi_t)_{t \geq 0}$ is an admissible strategy if it is a $\mathcal{F}_t$-progressively measurable process such that

$$
\mathbf{P}[|\pi_t| \leq A_\pi, \ 0 \leq t \leq T] = 1.
$$

Note that the constant A_π may depend on the strategy, and Eq. (1.1) has a unique solution (see Theorem V.7 in Protter 2005). We denote the set of admissible strategies with $\mathcal{A}$.

A utility function $U : \mathbb{R} \to \mathbb{R}$ is defined as a twice continuously differentiable function, with the property that $U(\cdot)$ is strictly increasing and strictly concave. We consider the optimization problem consisting on maximizing the expected utility of the terminal wealth at time T, i.e. we are interested in the following value function:

$$
W(s, x) = \sup_{\pi \in \mathcal{A}} \mathbf{E}\left[U\left(X_T^{s,x,\pi}\right)\right].
\tag{1.2}
$$

We say that an admissible strategy π^* is optimal if $W(s, x) = \mathbf{E}[U(X_T^{s,x,\pi^*})]$.

The main results of this paper can be summarized as follows: when the risk preferences of the insurer are exponential, the optimal investment problem described above can be solved explicitly and, for the optimal investment strategy, it is possible to obtain an estimate of the associated ruin probability.

2 Verification theorem

In order to find a solution to the optimal investment problem formulated in (1.2) we will use dynamic programming techniques. The dynamic programming approach is suitable in view of the Markov structure of the stochastic processes involved in the model. See, for instance, Fleming and Soner (1993) for a general background in the theory of optimal stochastic control. The Hamilton–Jacobi–Bellman (HJB) equation

associated with the optimal stochastic control problem is given by

$$0 = \frac{\partial V}{\partial t}(t, x) + \max_{\tilde{\pi} \in \mathbb{R}} \left\{ \frac{\sigma^2}{2} \tilde{\pi}^2 \frac{\partial^2 V}{\partial x^2}(t, x) + (\tilde{\pi}(a - \eta) + \eta x) \frac{\partial V}{\partial x}(t, x) \right\}$$
$$+ c \frac{\partial V}{\partial x}(t, x) + \lambda \int_{\mathbb{R}} [V(t, x - y) - V(t, x)] \nu(dy), \tag{2.3}$$

with terminal condition $V(T, x) = U(x)$. Next we establish a verification theorem, which relates the solution of the HJB equation (when it exists) to the value function (1.2).

Theorem 2.1 *Assume that there exists a classical solution $V(t, x) \in C^{1,2}([0, T] \times \mathbb{R})$ to the HJB equation (2.3) with boundary condition $V(T, x) = U(x)$. Assume also that for each $\pi \in \mathcal{A}$*

$$\int_0^T \int_{\mathbb{R}} \mathbf{E} |V(s, X_{s-}^\pi - y) - V(s, X_{s-}^\pi)|^2 \nu(dy) ds < \infty \tag{2.4}$$

and

$$\int_0^T \mathbf{E} \left[\pi_{s-} \frac{\partial V}{\partial x}(s, X_{s-}^\pi) \right]^2 ds < \infty. \tag{2.5}$$

Then, for each $s \in [0, T]$, $x \in \mathbb{R}$,

$$V(s, x) \geq W(s, x).$$

If, in addition, there exists a bounded measurable function $\pi^* : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that

$$\pi^*(t, x) \in \text{argmax}_{\pi \in \mathbb{R}} \left\{ \frac{\sigma^2}{2} \pi^2 \frac{\partial^2 V}{\partial x^2}(t, x) + (\pi(a - \eta) + \eta x) \frac{\partial V}{\partial x}(t, x) \right\},$$

then $\pi_t^* = \pi^*(t, X_{t-}^{\pi^*})$ defines an optimal investment strategy in feedback form if (1.1) admits a unique solution $X_t^{\pi^*}$ and

$$V(s, x) = W(s, x) = \mathbf{E} U \left[X_T^{s, x, \pi^*} \right].$$

In order to prove this theorem we need the following lemma. Its proof can be adapted from Lamberton and Lapeyre (1996, Lemma 7.2.2) and will be omitted.

Lemma 2.1 *Let $\phi(t, x, y) : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that, for each $y \in \mathbb{R}$, the function $(t, x) \to \phi(t, x, y)$ is continuous and let $(X_t^\pi)_{t \geq 0}$ be the*

right continuous process defined in (1.1) for an admissible strategy π. Assume that for every t ∈ [0, T]

$$E\left(\int_0^t ds \int_{\mathbb{R}} \phi^2(s, X_s^\pi, y)\nu(dy)\right) < \infty.$$

Then the process $(M_t)_{t\geq 0}$ defined by

$$M_t = \sum_{j=1}^{N_t} \phi\left(\tau_j, X_{\tau_j}^\pi, Y_j\right) - \lambda \int_0^t ds \int_{\mathbb{R}} \phi\left(s, X_s^\pi, y\right) \nu(dy),$$

where $\tau_n = \inf\{t \geq 0, \ N_t = n\}$, is a square integrable martingale and

$$M_t^2 - \lambda \int_0^t ds \int_{\mathbb{R}} \phi^2\left(s, X_s^\pi, y\right) \nu(dy),$$

is a martingale.

Proof of Theorem 2.1 Given $\pi \in \mathcal{A}$, $0 \leq s < T$ and $x \in \mathbb{R}$, Ito's formula implies that, for any $r \in [s, T)$,

$$V\left(r, X_r^{s,x,\pi}\right) = V(s, x) + \int_s^r \frac{\partial V}{\partial t}\left(t, X_{t-}^{s,x,\pi}\right) dt$$

$$+ \int_s^r \frac{\partial V}{\partial x}\left(t, X_{t-}^{s,x,\pi}\right) \left\{c + (a - \eta)\pi_{t-} + \eta X_{t-}^{s,x,\pi}\right\} dt$$

$$+ \frac{\sigma^2}{2} \int_s^r \frac{\partial^2 V}{\partial x^2}\left(t, X_{t-}^{s,x,\pi}\right) \pi_{t-}^2 dt$$

$$+ \sum_{s \leq t \leq r} \left[V\left(t, X_t^{s,x,\pi}\right) - V\left(t, X_{t-}^{s,x,\pi}\right)\right] + \int_s^r \frac{\partial V}{\partial x}\left(t, X_{t-}^{s,x,\pi}\right) \sigma \pi_{t-} dB_t$$

$$= V(s, x) + \int_s^r \frac{\partial V}{\partial t}\left(t, X_{t-}^{s,x,\pi}\right) dt$$

$$+ \int_s^r \frac{\partial V}{\partial x}\left(t, X_{t-}^{s,x,\pi}\right) \left\{c + \eta X_{t-}^{s,x,\pi} + (a - \eta)\pi_{t-}\right\} dt$$

$$+ \frac{\sigma^2}{2} \int_s^r \frac{\partial^2 V}{\partial x^2}\left(t, X_{t-}^{s,x,\pi}\right) \pi_{t-}^2 dt$$

$$+ \lambda \int_s^r \int_{\mathbb{R}+} \left[V\left(t, X_{t-}^{s,x,\pi} - y\right) - V\left(t, X_{t-}^{s,x,\pi}\right) \right] \nu(dy)dt$$

$$+ \left[\sum_{s \leq t \leq r} \left[V\left(t, X_t^{s,x,\pi}\right) - V\left(t, X_{t-}^{s,x,\pi}\right) \right] \right.$$

$$\left. - \lambda \int_s^r \int_{\mathbb{R}+} \left[V\left(t, X_{t-}^{s,x,\pi} - y\right) - V\left(t, X_{t-}^{s,x,\pi}\right) \right] \nu(dy)dt \right]$$

$$+ \int_s^r \frac{\partial V}{\partial x}\left(t, X_{t-}^{s,x,\pi}\right) \sigma \pi_{t-} dB_t. \tag{2.6}$$

The last term is a martingale because it is an stochastic integral with respect to Brownian motion and the integrand satisfies (2.5). Moreover, defining

$$\phi(t, x, y) := V(t, x - y) - V(t, x), \tag{2.7}$$

and the stopping times as $(\tau_j)_{j \geq 1}$, we have that

$$\sum_{j=N_s+1}^{N_r} \phi\left(\tau_j, X_{\tau_j^-}^{s,x,\pi}, Y_j\right) = \sum_{s \leq t \leq r} \left(V\left(t, X_t^{s,x,\pi}\right) - V\left(t, X_{t-}^{s,x,\pi}\right) \right).$$

From the previous lemma, the term

$$\left[\sum_{t \leq r} \left[V\left(t, X_t^{\pi}\right) - V\left(t, X_{t-}^{\pi}\right) \right] - \lambda \int_0^r \int_{\mathbb{R}+} \left[V\left(t, X_{t-}^{\pi} - y\right) - V\left(t, X_{t-}^{\pi}\right) \right] \nu(dy)dt \right]$$

is a martingale. Now, using the hypotheses that V solves the HJB equation (2.3) and since the probability of a jump at a fixed time t is zero, taking expectations on both sides of (2.6), it follows that

$$\mathbf{E} V\left(r, X_r^{s,x,\pi}\right) = V(s, x) + \mathbf{E} \int_s^r A^{\pi} V\left(t, X_t^{s,x,\pi}\right) dt$$

$$\leq V(s, x) + \mathbf{E} \int_s^r \sup_{\pi \in \mathbb{R}} A^{\pi} V\left(t, X_t^{\pi}\right) dt$$

$$= V(s, x), \tag{2.8}$$

where A^{π} is the operator

$$A^{\pi}(V(t,x)) = \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)(c + \pi_t(a - \eta) + \eta x)$$
$$+ \frac{\sigma^2}{2}\pi_t^2 \frac{\partial^2 V}{\partial x^2}(t,x) + \lambda \int_R [V(t,x-y) - V(t,x)]\nu(dy). \quad (2.9)$$

Letting $r = s$, we get

$$W(s,x) \le V(s,x).$$

The assumptions stated in the second part of the Theorem imply that π^* is an admissible feedback control. Repeating the above calculations with $\pi_t = \pi^*(t, X_{t-}^{\pi^*})$, it follows that inequality (2.8) becomes an equality. Hence,

$$V(s,x) = \mathbf{E}\left[U\left(X_T^{s,x,\pi^*}\right)\right] \le W(s,x),$$

and together with the first part implies that

$$V(s,x) = W(s,x).$$

When $s = T$ the theorem follows directly from the terminal condition $V(T,x) = U(T,x)$.   □

Remark 2.1 In the next section a closed form solution to the HJB equation (2.3) will be found if the insurer has an exponential risk preference. Also, an estimate for the ruin probability when the optimal investment strategy is followed will be obtained.

3 Explicit solutions for exponential utility function

In this section we will obtain an explicit solution to the HJB equation (2.3) when the utility function is of exponential type, i.e.

$$U(x) = -e^{-\gamma x}.$$

Also, using the verification theorem proved in the previous section, an explicit optimal strategy π^* will be found.

In view of the form of the utility function and the dynamics of the wealth process X_t^{π}, we propose as solution to the HJB equation the function

$$W(t,x) = -K_t \exp\{-\gamma x e^{\eta(T-t)}\}, \quad (3.10)$$

where the function K_t will be defined below.

From the definition of $W(t, x)$ we have that

$$\frac{\partial W}{\partial t} = \{-K'_t - K_t[\gamma x \eta e^{\eta(T-t)}]\} \exp\{-\gamma x e^{\eta(T-t)}\}, \qquad (3.11)$$

$$\frac{\partial W}{\partial x} = K_t \exp\{-\gamma x e^{\eta(T-t)}\}\gamma e^{\eta(T-t)}, \qquad (3.12)$$

$$\frac{\partial^2 W}{\partial x^2} = -K_t \gamma^2 e^{2\eta(T-t)} \exp\{-\gamma x e^{\eta(T-t)}\} \qquad (3.13)$$

and

$$\lambda \int_R [W(x-y) - W(x)]\nu(dy)$$

$$= -K_t \lambda \exp\{-\gamma x e^{\eta(T-t)}\} \left[\int_{\mathbb{R}} [\exp\{\gamma y e^{\eta(T-t)}\} - 1]\nu(dy)\right]. \qquad (3.14)$$

Substituting expressions (3.11–3.14) in (2.3), we obtain

$$-K'_t - K_t \gamma x \eta e^{\eta(T-t)} + K_t(c + \eta x)\gamma e^{\eta(T-t)}$$

$$+ \max_\pi \left\{-\frac{1}{2}\sigma^2 \pi^2 K_t \gamma^2 e^{2\eta(T-t)} + K_t \gamma(a-\eta)\pi e^{\eta(T-t)}\right\}$$

$$-\lambda K_t \int_{\mathbb{R}} [\exp\{\gamma y e^{\eta(T-t)}\} - 1]\nu(dy)$$

$$= -K'_t + K_t c\gamma e^{\eta(T-t)}$$

$$+ \max_\pi \left\{K_t e^{\eta(T-t)}\left[-\frac{1}{2}\sigma^2 \pi^2 \gamma^2 e^{\eta(T-t)} + \gamma(a-\eta)\pi\right]\right\}$$

$$-\lambda K_t \int_{\mathbb{R}} [\exp\{\gamma y e^{\eta(T-t)}\} - 1]\nu(dy), \qquad (3.15)$$

and the maximum in the last expression is attained at

$$\pi^*(t, x) = \frac{a - \eta}{\gamma \sigma^2} e^{-\eta(T-t)}.$$

Substituting π^* in (3.15), we obtain the following first order differential equation for K_t:

$$K'_t - K_t \left[\frac{1}{2}\frac{(a-\eta)^2}{\sigma^2} - \lambda\beta_t + c\gamma e^{\eta(T-t)}\right] = 0, \qquad (3.16)$$

where

$$\beta_t := \int_{\mathbb{R}} [\exp\{\gamma y e^{\eta(T-t)}\} - 1]\nu(dy).$$

In view of (3.10), the terminal condition $W(T, x) = -e^{-\gamma x}$ will be satisfied when $K_T = 1$. Hence, the solution of the ODE (3.16) is given by

$$K_t = \exp \left\{ -\frac{1}{2}\frac{(a - \eta)^2}{\sigma^2}(T - t) + \frac{c\gamma}{\eta}[1 - e^{\eta(T-t)}] + \lambda \int_t^T \beta_s ds \right\}.$$

Remark 3.1 When the interest rate η is zero, β_t is independent of t. In this case its constant value is denoted by β.

Theorem 3.1 *Assume that*

$$\int_{\mathbb{R}} \exp\{4\gamma y e^{\eta T}\}\nu(dy) < \infty.$$

Then, the value function defined in (1.2) has the form

$$W(t, x) = -\exp\left\{-\frac{1}{2}\frac{(a-\eta)^2}{\sigma^2}(T-t) + \frac{c\gamma}{\eta}[1 - e^{\eta(T-t)}] + \lambda \int_t^T \beta_s ds\right\}$$

$$\cdot \exp\{-\gamma x e^{\eta(T-t)}\}, \tag{3.17}$$

and

$$\pi^*(t, x) = \frac{a - \eta}{\gamma\sigma^2}e^{-\eta(T-t)}$$

is an optimal strategy.

   In particular, when $\eta = 0$ we have that

$$W(t, x) = -\exp\left\{-\frac{1}{2}\frac{a}{\sigma^2}(T - t) + c\gamma(T - t) + \lambda\beta(T - t)\right\}e^{-\gamma x} \tag{3.18}$$

and

$$\pi^*(t, x) = \frac{a}{\gamma\sigma^2}.$$

Proof We have checked already that the function $W(t, x)$ defined in (3.10) solves the HJB equation (2.3). Now, we would like to apply Theorem 2.1 and, in order to do that, we shall verify first that the needed assumptions are satisfied. Let $\pi \in \mathcal{A}$ be an admissible strategy. Next we get an estimate which yields the first condition (2.4)

when $\eta = 0$. Observe that substituting the definitions of $W(t, x)$ and X_t^π we get

$$\mathbf{E} \int_{\mathbb{R}} |W(t, X_{t-}^\pi - y) - W(t, X_{t-}^\pi)|^2 \nu(dy) = K_t^2 \mathbf{E} \exp\{-2\gamma X_{t-}^\pi\} \int_{\mathbb{R}} [e^{\gamma y} - 1]^2 \nu(dy)$$

$$= K_t^2 \int_{\mathbb{R}} [e^{\gamma y} - 1]^2 \nu(dy) \mathbf{E} \exp\{-2\gamma X_{t-}^\pi\} \tag{3.19}$$

$$= K_t^2 \int_{\mathbb{R}} [e^{\gamma y} - 1]^2 \nu(dy) e^{-2\gamma(x+ct)}$$

$$\cdot \mathbf{E} \exp\left\{-2\gamma a \int_0^t \pi_r dr - 2\gamma\sigma \int_0^t \pi_r dB_r + 2\gamma \sum_{j=1}^{N_{t-}} Y_j\right\} \tag{3.20}$$

Now, define the following exponential martingale

$$L_t = \exp\left\{-4\gamma\sigma \int_0^t \pi_r dB_r - \frac{1}{2}(4\gamma)^2 \sigma^2 \int_0^T \pi_r^2 dr\right\}.$$

Notice that Novikov's condition is satisfied in view of that strategy π is bounded (by constant A_π), according with the definition of admissible strategies. Then, from the independence of N_t and Y_j, $j = 1, 2, \ldots$, and using Holder's inequality and the form of the moment generating function,

$$\mathbf{E} \exp\left\{-2\gamma a \int_0^t \pi_r dr - 2\gamma\sigma \int_0^t \pi_r dB_r + 2\gamma \sum_{j=1}^{N_{t-}} Y_j\right\}$$

$$= \mathbf{E} \exp\left\{\int_0^t \left[-2\gamma a\pi_r + 8\gamma^2\sigma^2\pi_r^2\right] dr\right\}$$

$$\cdot L_t^{\frac{1}{2}} \exp\left\{4\gamma \sum_{j=1}^{N_{t-}} Y_j\right\}^{1/2}$$

$$\leq e^{[2\gamma|a|A_\pi + 8\gamma^2\sigma^2 A_\pi^2]T} \{\mathbf{E}L_t\}^{1/2} \left\{\mathbf{E} \exp\left\{4\gamma \sum_{j=1}^{N_{t-}} Y_j\right\}\right\}^{1/2}$$

$$= e^{[2\gamma|a|A_\pi + 8\gamma^2\sigma^2 A_\pi^2]T} \exp\left\{\frac{\lambda t}{2}\left[\int_{\mathbb{R}} e^{4\gamma y}\nu(dy) - 1\right]\right\}$$

$$< \infty.$$

In order to prove the second condition (2.5), notice that

$$\mathbf{E}[\pi_t W_x(t, X_{t-}^\pi)]^2 \le A_\pi^2 \gamma^2 \mathbf{E} e^{-2\gamma X_{t-}^\pi}. \tag{3.21}$$

Then, using the same arguments given after (3.19), it follows that the right-hand side of (3.21) is finite. Now, once we have verified the hypotheses of Theorem 2.1, it can be applied to derive the results of the theorem.

When the interest rate η is non-zero, we apply the following argument. Given X_t^π the unique solution of (1.1), set

$$\tilde{X}_t^\pi := e^{\eta(T-t)} X_t^\pi, \quad \tilde{c}_t := e^{\eta(T-t)} c,$$
$$\tilde{\nu}(dy \times dt) := e^{\eta(T-t)} \nu(dy \times dt), \quad \tilde{S}_t = e^{\eta(T-t)} S_t,$$

where $\nu(dy \times dt)$ is the random Poisson measure associated with the Poisson process $(N_t)_{t\ge 0}$ and the distribution $\nu(dy)$ of the random variables $(Y_j)_{j\ge 1}$. Then, $\tilde{X}_t^\pi$ solves the equation

$$\tilde{X}_t^\pi = x + \int_0^t [(a - \eta)\pi_r + \tilde{c}_r] dr + \int_0^t \sigma \pi_r dB_r - \int_0^t \int_{\mathbb{R}} y \tilde{\nu}(dy \times dr),$$

which corresponds to the case when the interest rate is zero, with drift $a - \eta$. Hence, the results can be derived from the first part of the proof. $\square$

4 Ruin probability

In this section we shall estimate the ruin probability when the insurer follows the optimal strategy obtained in the previous section, and we show our results include those of Gaier et al. (2003).

Recall that the wealth process associated with the optimal investment strategy π^* is given by

$$X_t^* = x + ct - \sum_{j=1}^{N_t} Y_j + \int_0^t \frac{(a - \eta)^2}{\gamma \sigma^2} e^{-\eta(T-r)} dr$$

$$+ \int_0^t \eta X_r^* dr + \int_0^t \frac{(a - \eta)}{\gamma \sigma} e^{-\eta(T-r)} dB_r, \quad \text{for } \eta \ge 0. \tag{4.22}$$

On the other hand, it is clear that

$$P[X_s^* \le 0, \text{ for some } s \in [0, t]] = P\left[\sup_{s \in [0,t]} -X_s^* \ge 0\right]. \tag{4.23}$$

The upper bound for the ruin probability will be proved with the aid of the following result, which is a particular case of Lemma 3.1 in Denis et al. (2005).

Lemma 4.1 *Let $(H_t)_{t \geq 0}$ be the process defined by:*

$$H_t = z + \int_0^t \alpha_s \, dB_s + \int_0^t b_s \, ds + \sum_{i=1}^{N_t} Y_i', \quad t \geq 0. \tag{4.24}$$

where

1. *$(B_t)_{t \geq 0}$ is a standard Brownian motion,*
2. *The process $(\alpha_t)_{t \geq 0}$ is predictable,*
3. *$(b_t)_{t \geq 0}$ is adapted,*
4. *The last term is a compound Poisson process, i.e. $(N_t)_{t \geq 0}$ is a Poisson process with intensity $\lambda > 0$, independent of the jump sizes $(Y_i')_{i \geq 1}$; both are independent of $(B_t)_{t \geq 0}$.*

Assume that

(i) *$E(\int_0^t \alpha_s^2 \, ds) < \infty$ for all $t > 0$,*
(ii) *$(b_t)_{t \geq 0}$ is integrable,*
(iii) *The law of the non-negative, non-degenerate i. i. d. random variables $(Y_i')_{i \geq 1}$ admits a Laplace transform $L(r)$ for $0 < r < K \leq \infty$,*
(iv) *There exist $0 < \delta < K$ and a constant $M_t(\delta) \geq 0$ such that for all $s \in [0, t]$,*

$$\delta \int_0^s b_u \, du + \frac{\delta^2}{2} \int_0^s \alpha_u^2 \, du + \lambda s (L(\delta) - 1) \leq M_t(\delta), \quad a.e. \tag{4.25}$$

Then, for each $\delta > 0$ such that $M_t(\delta) \geq 0$ we have

$$P\left[\sup_{s \leq t} H_s \geq 0 \right] \leq e^{\delta z + M_t(\delta)}. \tag{4.26}$$

Proof We have

$$P\left(\sup_{s \leq t} H_s \geq 0 \right) = P\left(\delta \sup_{s \leq t} H_s \geq 0 \right)$$

$$\leq P\left(\sup_{0 \leq s \leq t} \mathbb{M}_s \geq \exp\left(-z\delta - M_t(\delta) \right) \right),$$

where $(\mathbb{M}_t)_{t \geq 0}$ is the martingale defined by

$$\mathbb{M}_s = \exp\left(\delta \left[\int_0^s \alpha_u \, dB_u + \sum_{i=1}^{N_t} Y_i' \right] - \frac{\delta^2}{2} \int_0^s \alpha_u^2 \, du - \lambda s (L(\delta) - 1) \right).$$

$(\mathbb{M}_t)_{t\geq 0}$ is a martingale since it is the product of a continuous martingale and a purely discontinuous one. The maximal inequality for exponential martingales gives the result.   □

Observe that inequality (4.26) can be useful if $\delta z + M_t(\delta) < 0$, as will be the case when we obtain a bound for the Ruin Probability in the following Proposition.

Proposition 4.1 *Assume that*

1. *The law of the random variables $(Y_i)_{i\geq 1}$ admits a Laplace transform $L(r)$ for $0 < r < K \leq \infty$, if $K < \infty$, $\lim_{r\to K} L(r) = \infty$, and the following safety loading condition is satisfied*

$$\left(c + \frac{(a-\eta)^2}{\gamma\sigma^2}\right)e^{-\eta T} - \lambda\theta > 0, \quad \text{if } \eta \geq 0, \tag{4.27}$$

where $E[Y_1] = \theta$. Then, the ruin probability satisfies

$$P\left[\sup_{s\leq t} -X_s^* \geq 0\right] \leq e^{-\delta^* z},$$

where δ^* is the positive root of the equation:

$$h_\eta(\delta) = -\delta\left(c + \frac{(a-\eta)^2}{\gamma\sigma^2}\right)e^{-\eta T} + \frac{\delta^2}{2}\frac{(a-\eta)^2}{\gamma^2\sigma^2}e^{-2\eta T} + \lambda(L(\delta) - 1) = 0. \tag{4.28}$$

2. *In addition, if $\eta = 0$, and $\frac{\delta^1}{2} < \gamma < \frac{1}{\theta}$, where δ^1 is the root of the equation*

$$h^1(\delta) = -\delta c + \lambda(L(\delta) - 1) = 0, \tag{4.29}$$

then

$$\delta^1 < \delta^*. \tag{4.30}$$

Proof 1.   It is clear that for $\eta > 0$, we can not apply Lemma 4.1 to the process $-X_t^*, t \geq 0$. We will use two auxiliary processes Z_t, Z_t^1, $t \geq 0$ defined by:

$$Z_t = X_t^* e^{-\eta t}, \tag{4.31}$$

$$Z_t^1 = x + \left(c + \frac{(a-\eta)^2}{\gamma\sigma^2}\right)e^{-\eta T}t - \sum_{i=1}^{N_t} Y_i + \int_0^t \frac{a-\eta}{\gamma\sigma}e^{-\eta T}dB_s. \tag{4.32}$$

By the integration by parts formula we have

$$
Z_t = x + \int_0^t e^{-\eta r} c \, dr - \sum_{j=1}^{N_t} e^{-\eta \tau_j} Y_j + \int_0^t e^{-\eta r} \frac{(a - \eta)^2}{\gamma \sigma^2} e^{-\eta(T-r)} dr
$$

$$
+ \int_0^t \eta e^{-\eta r} X_r^* dr + \int_0^t e^{-\eta r} \frac{a - \eta}{\gamma \sigma} e^{-\eta(T-r)} dB_r - \int_0^t \eta e^{-\eta r} X_r^* dr
$$

$$
= x + \int_0^t e^{-\eta r} c \, dr - \sum_{j=1}^{N_t} e^{-\eta \tau_j} Y_j + \int_0^t \frac{(a - \eta)^2}{\gamma \sigma^2} e^{-\eta T} dr
$$

$$
+ \int_0^t \frac{a - \eta}{\gamma \sigma} e^{-\eta T} dB_r. \tag{4.33}
$$

It follows that

$$
-Z_t \leq -Z_t^1, \quad t \geq 0
$$

and since

$$
X_t^* \leq 0 \text{ if and only if } Z_t \leq 0,
$$

then

$$
P \left[\sup_{0 \leq s \leq t} -X_s^* \geq 0 \right] = P \left[\sup_{0 \leq s \leq t} -Z_s \geq 0 \right] \leq P \left[\sup_{0 \leq s \leq t} -Z_s^1 \geq 0 \right].
$$

For each $\delta \geq 0$ let $M_t(\delta) = t h_\eta(\delta)$. Note that

$$
\lim_{\delta \to K} h_\eta(\delta) = \infty,
$$

since if $K < \infty$ by hypothesis $\lim_{\delta \to K} L(\delta) = \infty$, and if $K = \infty$, we have a positive quadratic term. Then, there exists $\delta > 0$ such that $M_t(\delta) \geq 0$. Then, applying Lemma 4.1 to the process $-Z_t^1$, we obtain

$$
P \left[\sup_{0 \leq s \leq t} -X_s^* \geq 0 \right] \leq P \left[\sup_{0 \leq s \leq t} -Z_s^1 \geq 0 \right] \leq e^{-\delta x + M_t(\delta)}.
$$

The existence of a positive root follows from the continuity of $h_\eta(\delta)$ and the fact that $h_\eta(\delta) < 0$ in a neighborhood of 0 since for $\delta > 0$,

$$\frac{h_\eta(\delta)}{\delta} = -\left(c + \frac{(a-\eta)^2}{\gamma\sigma^2}\right)e^{-\eta T}$$
$$+ \frac{\delta}{2}\frac{(a-\eta)^2}{\gamma^2\sigma^2}e^{-2\eta T} + \frac{\lambda(L(\delta)-1)}{\delta}.$$

From the safety loading hypothesis (4.27) we have:

$$\lim_{\delta\to 0}\frac{h_\eta(\delta)}{\delta} = -\left(c + \frac{(a-\eta)^2}{\gamma\sigma^2}\right)e^{-\eta T} + \lambda\theta < 0.$$

Then, there exists $\delta^* > 0$ (the root of Eq. 4.28) such that

$$P\left[\sup_{0\le s\le T} -X_s^*\right] \le e^{-\delta^* x}.$$

If $\eta = 0$, we can apply directly Lemma 4.1 to the process X_t^*.

2. Finally, to prove the second part of the Proposition we only need to verify that $\delta^1 < \delta^*$.

   Note that substituting δ^1 in Eq. (4.28) for $\eta = 0$, we obtain:

$$h_0(\delta^1) = -\delta^1\frac{a^2}{\gamma\sigma^2} + \frac{(\delta^1)^2}{2}\frac{a^2}{\gamma^2\sigma^2}.$$

Observing that $h_0(\delta^1) < 0$ if and only if $\frac{\delta^1}{2} < \gamma$, the result follows.

$\square$

Remark 4.1 Equation (4.29) corresponds to a Cramér Lundberg Process without investment, δ^1 is the Classical Lundberg parameter, and (4.30) says we can have a better exponential rate in the case with investment.

Remark 4.2 The case $\eta = 0$ is simpler than $\eta > 0$, but it is important in view of (4.30). Here,

$$X_t^* = x + ct - \sum_{i=1}^{N_t} Y_i + \int_0^t \frac{a^2}{\gamma\sigma^2}dr + \int_0^t \frac{a}{\gamma\sigma}dB_r, \tag{4.34}$$

the safety loading condition becomes

$$c + \frac{a^2}{\gamma\sigma^2} - \lambda\theta > 0, \quad \text{if } \eta = 0, \tag{4.35}$$

and

$$h_0(\delta) = -\delta\left\{c + \frac{a^2}{\gamma\sigma^2}\right\} + \frac{\delta^2}{2}\frac{a^2}{\gamma^2\sigma^2} + \lambda(L(\delta)-1) = 0. \tag{4.36}$$

Remark 4.3 For each $\gamma > 0$, let $\delta(\gamma)$ the root of $h_0(\delta)$ as in (4.28). (Recall that in this case $\eta = 0$.) If we let $M(\gamma) = \frac{a}{\gamma\sigma}$, then we have

$$0 = h_0(\delta(\gamma)) = h_0(\delta(\gamma), M(\gamma))$$
$$= -\delta(\gamma) \left(c + \frac{M(\gamma)a}{\sigma} \right) + \frac{\delta(\gamma)^2}{2} M(\gamma)^2 + \lambda[L(\delta(\gamma)) - 1].$$

Using the implicit function theorem, it can be shown that $\delta(\gamma)$ is a maximum when $\delta(\gamma) = \gamma$. In this case we obtained the same upper bound for the Ruin Probability as in Gaier, Grandits and Schachermayer Gaier et al. (2003), and the asymptotically optimal strategy obtained by Hipp and Schmidli Hipp and Schmidli (2004).

Proposition 4.2 *Assume that the random variables Y_i, $i \geq 1$ are exponentially distributed with mean θ and*

$$0 < \gamma < \frac{e^{-\eta T}}{\theta}. \tag{4.37}$$

Then

$$W(t, x) = -\exp \left\{ -\frac{1}{2} \frac{a - \eta}{\sigma^2} (T - t) + \frac{c\gamma}{\eta} [1 - e^{\eta(T-t)}] \right.$$
$$\left. -\frac{\lambda}{\eta} \log \left(\frac{1 - \gamma\theta}{1 - \gamma\theta e^{\eta(T-t)}} \right) \right\} \cdot \exp \{-\gamma x e^{\eta(T-t)}\}.$$

In particular, if $\eta = 0$, and

$$0 < \gamma < \frac{1}{\theta},$$

then

$$W(t, x) = -\exp \left\{ -\frac{1}{2} \frac{a}{\sigma^2} (T - t) + c\gamma(T - t) - \lambda \frac{\gamma\theta}{1 - \gamma\theta} (T - t) \right\} e^{-\gamma x}.$$

Proof In this case the function β_t is given by

$$\beta_t = \frac{1}{\theta} \int_0^\infty \left[\exp \{\gamma y e^{\eta(T-t)}\} - 1 \right] e^{-y\frac{1}{\theta}} dy.$$

Then β_t is finite if and only if

$$0 < \gamma < \frac{e^{-\eta(T-t)}}{\theta}, \quad \text{for all } t \in [0, T],$$

which is equivalent to expression (4.37).

Under this condition

$$\beta_t = \frac{1}{\eta} \frac{\theta \gamma \eta e^{\eta(T-t)}}{1 - \theta \gamma e^{\eta(T-t)}}$$
$$= \frac{1}{\eta} \frac{d \log}{dt} \left(1 - \gamma \theta e^{\eta(T-t)} \right),$$

and we get

$$\int_t^T \beta_s ds = \frac{1}{\eta} \log \left(\frac{1 - \gamma \theta}{1 - \gamma \theta e^{\eta(T-t)}} \right).$$

In particular if $\eta = 0$ we have

$$\beta_t = \frac{\gamma \theta}{1 - \gamma \theta},$$

$$\int_t^T \beta_s ds = \frac{\gamma \theta}{1 - \gamma \theta} (T - t).$$

The proof is complete.   □

In the exponential case, for $\eta = 0$, $h_0(\delta)$ becomes

$$h_0(\delta) = \frac{a^2 \theta}{2\gamma^2 \sigma^2} \delta^2 - \left(\left(c + \frac{a^2}{\gamma \sigma^2} \right) \theta + \frac{a^2}{2\gamma^2 \sigma^2} \right) \delta + \left(c + \frac{a^2}{\gamma^2 \sigma^2} - \theta \lambda \right). \quad (4.38)$$

For each $\gamma \in (0, 1/\theta)$ we obtain a positive root $\delta(\gamma)$ of h_0 of the form

$$\delta(\gamma) = \frac{c\sigma^2 \gamma^2}{a^2} + \gamma + \frac{1}{2\theta} + \sqrt{\left(\frac{c\sigma^2 \gamma^2}{a^2} + \gamma + \frac{1}{2\theta} \right)^2 - \frac{2}{\theta} \left(\frac{(c - \lambda \theta) \sigma^2 \gamma^2}{a^2} + 1 \right)}.$$

5 Numerical examples

In order to illustrate the behavior of the ruin probability for infinite horizon when the optimal strategy of investment $\pi_t = \frac{a}{\gamma \sigma^2}$ is applied, we present some numerical results for the case where the claims are exponentially distributed, with the parameter values used by Hipp and Plum (2000), and for different values of $\gamma \in (0, \frac{1}{\theta})$. The parameters have the following values: $a = \sigma = \theta = \lambda = 1$, $c = 2$, and $\eta = 0$.

Graph 1 shows how the root $\delta(\gamma)$ of $h_0(\delta)$ varies for different values of γ. For our data the Lundberg parameter for the classical case δ^1 is 0.5. The maximum value of δ^* is obtained at 0.640388 and for $\gamma \in (0.25, 0.9]$ the root is larger that 0.5.

Modified Lundberg parameter: δ(Y)
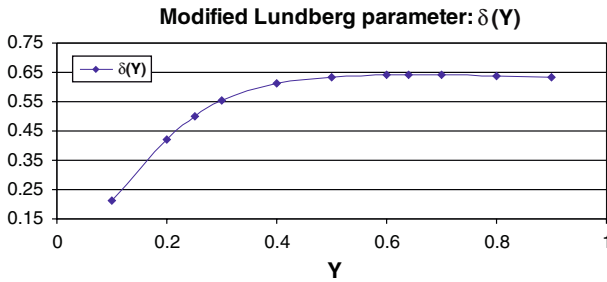
Graph 1

Let

$$Q_t = \sum_{i=1}^{N_t} Y_i - ct - \int_0^t \frac{a^2}{\gamma\sigma^2} dr - \int_0^t \frac{a}{\gamma\sigma} dB_r, \qquad (5.39)$$

denote the surplus; observe that $Q_t = x - X_t^*$. Let $\tau(x) = \inf_{0 \le t < \infty}\{t > 0 | Q_t > x\}$, we are interested in estimating

$$P[\tau(x) < \infty] = E(\mathbb{1}_{\tau(x)<\infty}).$$

We use a Monte Carlo method with importance sampling to estimate the ruin probability. These problems can be handled if we change the probability measure to one that increases the probability of occurrence of $\{\tau(x) < \infty\}$ (via importance sampling). Asmussen (2000, Chap. XI) used an exponential change of measure for the classical case. In our case, we propose the probability measure P^* obtained from P by the Radon–Nykodim derivative

$$\frac{dP^*}{dP} = e^{\delta Q_{\tau(x)} - \tau(x)h_0(\delta)},$$

where $h_0(\delta)$ is given by (4.38). If we choose as δ, the root δ^* of h_0, the calculation of the ruin probability reduces to

$$E(\mathbb{1}_{[\tau(x)<\infty]}) = E^*(e^{-\delta^* Q_{\tau(x)}} \mathbb{1}_{[\tau(x)<\infty]}).$$

With this method we obtain a considerable reduction of the variance (which implies that fewer paths are needed for the Monte Carlo simulation). The diffusion part of the process was approximated by the Euler method with N time steps. The very nature of Euler's scheme gives lower bounds for the ruin probabilities. When $\delta = \delta^*$ the estimation is optimal in an asymptotic sense, and for variance reduction, the variance is bounded by $e^{-2\delta^* x}$.

Graph 2 compares the probability of survival, for values of $z \in [0, 6]$ for $\gamma = 0.640388$, $\gamma = 0.3$ and when there is no investment. Observe that is smaller when there is no investment.

Survival Probability
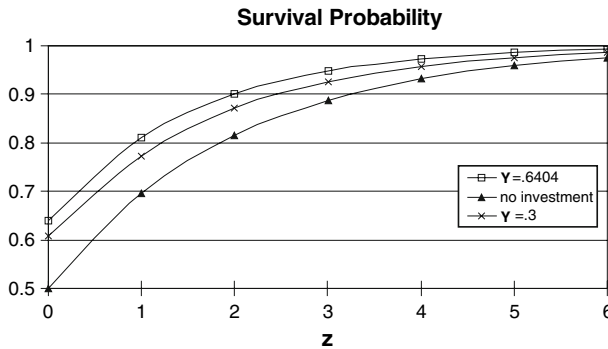
Graph 2

In the following table we show the 95% confidence intervals obtained numerically versus the bound $e^{-\delta^* x}$ (Proposition 4.1), for $T = 100$, $\gamma = 0.640388$, $\eta = 0$, and 100,000 paths. As before, the diffusion part of the process was approximated by the Euler method. Here we used $N = 100$ and 10,000 time steps. Note that the numerical scheme used, due to the discretization, can only give lower bounds for the Ruin Probability.

x	$N = 100$	Upper bound	$N = 10,000$
1	[0.1928,0.1947]	0.5270	[0.2374,0.2397]
2	[0.1022,0.1032]	0.2778	[0.1170,0.1182]
3	[0.0541,0.0547]	0.1464	[0.0619,0.0625]
4	[0.0287,0.029]	0.0772	[0.0323,0.0325]
5	[0.0151,0.0152]	0.0470	[0.0171,0.0173]
6	[0.0079,0.0080]	0.0214	[0.0091,0.0092]
7	[0.00418,0.00422]	0.0113	[0.00450,0.00454]
8	[0.00221,0.00223]	0.0057	[0.00253,0.00255]
9	[0.00117,0.00118]	0.0031	[0.00132,0.00133]
10	[0.00061,0.00062]	0.0016	[0.00071,0.00072]

The table shows that as N is larger the approximation of the Ruin Probability improves by increasing, and it is always lower than the upper bound, as expected.

6 Concluding remarks

We have presented an explicit solution to the HJB equation associated to the optimal investment problem of an insurance company when the utility function is of exponential type with parameter $\gamma > 0$. An optimal solution was derived from the previous equation and exponential bounds for the ruin probability were found.

For the case without investment in a bond we can compare our results to other in the literature: we obtained an optimal value of γ that gives us the same upper bound for the Ruin Probability as in Gaier et al. (2003), and the asymptotically optimal strategy obtained by Hipp and Schmidli (2004).

We can conclude that there is a deep relationship between maximizing the exponential utility function and minimizing the Probability of Ruin, supporting Ferguson's conjecture.

Numerical simulations were used to compare the survival probability for different values of γ. Finally, we compared numerical estimations of the ruin probability with the upper bound obtained.

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