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CONTRACTIVE APPROXIMATIONS FOR THE VARADHAN'S FUNCTION ON A FINITE MARKOV CHAIN*

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Abstract. This work concerns Markov chains with finite state space. Given a real-valued cost function on the state space, the corresponding Varadhan's function, measuring the exponential growth rate of the aggregated costs, is characterized as the unique limit of the fixed points of a family of contraction operators, a conclusion that does not involve any condition on the transition law.

Key words. risk-sensitive average cost, decreasing function along trajectories, Poisson equation, closed set

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1. Introduction.

Let $\{X_t\}$ be a Markov chain with finite state space S and transition matrix $P = [p_{xy}]$. Given a positive number λ and a function $C\colon S \to \mathbf{R}$, interpreted as a running cost, define the function

$$J(x) := \limsup_{n\to\infty} \frac{1}{\lambda n} \log \left(\mathbf{E}_x \left[\exp\left\{ \lambda \sum_{t=0}^{n-1} C(X_t) \right\} \right] \right), \qquad x \in S, \tag{1.1}$$

which measures the (largest expected) exponential growth rate of the aggregated costs; the number λ is interpreted as the (constant) "risk sensitivity" of the observer (see [1], [3], [8]) and, following Fleming and McEneaney [7], $J(\cdot)$ will be referred to as the Varadhan function associated to C and λ. Now let the cost function C be fixed and assume that $\{X_t\}$ is an aperiodic chain for which the whole state space is a communicating class; i.e., for all $x, y \in S$ there exists a nonnegative integer $n = n(x, y)$ such that $\mathbf{P}_x\{X_n = y\} > 0$. In this case, the Perron–Frobenius theory of positive matrices (see [12], [4]) yields that $J(\cdot)$ is constant and its value γ is such that $e^{\lambda\gamma}$ is the largest eigenvalue of the matrix $[e^{\lambda C(x)} p_{xy}]$. Moreover, γ is a unique number for which there exists a function $h\colon S \to \mathbf{R}$ satisfying the following Poisson equation:

$$e^{\lambda[\gamma + h(x)]} = \mathbf{E}_x \left[e^{\lambda[C(X_0) + h(X_1)]} \right], \qquad x \in S; \tag{1.2}$$

this result can be traced back, at least, to Howard and Matheson [9] and still holds if the aperiodicity condition is suppressed (see, for instance, [2]); also, it is interesting to observe that when this Poisson equation is satisfied, the limit superior in (1.1) can be replaced by limit. However, this characterization of Varadhan's function, which is valid regardless of the positive value of λ, no longer holds when the underlying Markov chain is not communicating as described above. Indeed, under the unichain assumption that $\{X_t\}$ has only one recurrent class, it was recently shown in [1] and [8] that, if the class of transient states is nonempty, then for a given cost function C it can be ensured that Varadhan's function is constant only if λ *is small enough*, a phenomenon that reflects the following fact: When λ is sufficiently large, the costs incurred at transient states, which can be visited only at "early stages," have a definite influence on the value of $J(\cdot)$; see Remark 4.1(ii). Moreover, an example was given in [3] which shows that, even if $J(\cdot)$ takes a single value, it is not generally determined

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via (1.2). Rather, under the above unichain condition, it was proved in [3] that Varadhan's function is characterized as the optimal value of a convex minimization problem in a finite-dimensional Euclidean space, so that $J(\cdot)$ *cannot be generally determined by solving a single equation.*

The main objective of this paper is to provide an alternative characterization of Varadhan's function in terms of a family $\{V_\alpha\}$ of functions on the state space. For each $\alpha \in (0,1)$, V_α is a unique fixed point of a contraction operator T_α so that V_α can be determined by solving a single equation, and the main result, stated as Theorem 2.1 in what follows, establishes that $J(\cdot)$ is a unique limit point of $\{(1-\alpha)\,V_\alpha\}$ as α increase to 1, extending a known result in (risk-neutral) dynamic programming; see, for instance, [10] or [11]. The argument used to establish this result allows one to establish an interesting conclusion in Remark 4.1(i), namely, that although Varadhan's function is not generally characterized by a single equation, $J(\cdot)$ is *always* determined by a system of local (or reduced) Poisson equations which are similar to (1.2).

The main difference with other characterizations of $J(\cdot)$ already available is that the approximation result in Theorem 2.1 does not involve any condition on the communication-recurrence structure of the underlying Markov chain. As a by-product of the analysis performed below, it will be shown that the limit superior in the equality (1.1) defining $J(\cdot)$ can always be replaced by limit.

The organization of the paper is as follows. In section 2 the contraction operators T_α, $\alpha \in (0,1)$, are introduced, and the main approximation result is stated as Theorem 2.1 in terms of the corresponding fixed points $\{V_\alpha\}$. Next, the necessary technical preliminaries to establish this result are presented in section 3 and, finally, the main result is proved in section 4.

Notation. The set of nonnegative integers is denoted by $\mathbf{N}$, whereas $\mathcal{B}(S)$ stands for the class of all real-valued functions defined on the state space S. Given $C \in \mathcal{B}(S)$,

$$\|C\| := \max\big\{|C(x)| \,|\, x \in S\big\}$$

is the corresponding maximum norm.

2. Contractive operators and main result.

The approximation result for Varadhan's function in (1.1) involves the following collection of operators on the space $\mathcal{B}(S)$, which has been previously used to analyze, under strong communication-recurrence conditions, the risk-sensitive average index for *controlled* Markov chains; see, for instance, [2] and [6] for the discrete case under complete and partial observations, respectively, and [5] for models over Borel spaces. The formulation of operator T_α in (2.1) below is similar to the definition of the discounted operator in (risk-neutral) dynamic programming (see [10], [11]).

DEFINITION 2.1. *For each* $\alpha \in (0,1)$, *define the operator* $T_\alpha\colon \mathcal{B}(S) \to \mathcal{B}(S)$ *as follows*:

$$T_\alpha W(x) = C(x) + \frac{1}{\lambda}\log\sum_y p_{xy}e^{\lambda\alpha W(y)}, \qquad W \in \mathcal{B}(S), \quad x \in S. \tag{2.1}$$

The basic contractive property of T_α is stated in the following lemma.

LEMMA 2.1. *For each* $\alpha \in (0,1)$, *assertions* (i)–(iii) *hold.*

(i) T_α *is a contraction operator on* $\mathcal{B}(S)$ *with coefficient* α, *i.e.*,

$$\|T_\alpha V - T_\alpha W\| \leqq \alpha\|V - W\|, \qquad V, W \in \mathcal{B}(S).$$

(ii) *There exists a unique function* $V_\alpha = \mathcal{B}(S)$ *such that*

$$T_\alpha V_\alpha = V_\alpha. \tag{2.2}$$

(iii) *Moreover, this fixed point* V_α *satisfies the inequality* $\|(1-\alpha)\,V_\alpha\| \leqq \|C\|$.

Proof. Let $\alpha \in (0,1)$ be arbitrary but fixed.

(i) Given $V, W \in \mathcal{B}(S)$, the inequality

$$C(x) + \alpha V(y) \leqq C(x) + \alpha W(y) + \alpha\|V - W\|$$

is always valid, so that

$$\exp\left[\lambda(C(x) + \alpha V(y))\right] \leqq \exp\left[\alpha\lambda\|V - W\| + \lambda\big(C(x) + \alpha W(y)\big)\right].$$

Therefore, (2.1) yields that for every $x \in S$,

$$\begin{aligned} e^{\lambda[T_\alpha V](x)} &= \sum_y p_{xy} e^{\lambda(C(x)+\alpha V(y))} \leqq e^{\alpha\lambda\|V-W\|}\left[\sum_y p_{xy} e^{\lambda(C(x)+\alpha W(y))}\right] \\ &= e^{\alpha\lambda\|V-W\|+\lambda[T_\alpha W](x)}, \end{aligned}$$

and then $[T_\alpha V](x) \leqq [T_\alpha W](x) + \alpha\|V - W\|$. By interchanging the roles of V and W, this leads to $|[T_\alpha V](x) - [T_\alpha W](x)| \leqq \alpha\|V - W\|$, and the conclusion follows, since $x \in S$ is arbitrary.

(ii) The existence of a unique fixed point $V_\alpha \in \mathcal{B}(S)$ follows from part (i).

(iii) Observe that (2.1) with $W \equiv 0$ yields that $T_\alpha 0 = C$, and from part (i) it follows that

$$\|V_\alpha - C\| = \|T_\alpha V_\alpha - T_\alpha 0\| \leqq \alpha\|V_\alpha - 0\| = \alpha\|V_\alpha\|;$$

since $\|V_\alpha\| - \|C\| \leqq \|V_\alpha - C\|$ this implies that $(1-\alpha)\|V_\alpha\| \leqq \|C\|$. Lemma 2.1 is proved.

DEFINITION 2.2. *Given $\alpha \in (0,1)$, let $V_\alpha \in \mathcal{B}(S)$ be a unique fixed point of T_α and define $g_\alpha\colon S \to \mathbf{R}$ as follows*:

$$g_\alpha(x) := (1-\alpha)\, V_\alpha(x), \qquad x \in S. \tag{2.3}$$

Combining (2.1) and (2.2), we see that, for each $\alpha \in (0,1)$, the fixed point V_α is characterized by the equation

$$e^{\lambda V_\alpha(x)} = \mathbf{E}_x\left[e^{\lambda[C(X_0)+\alpha V_\alpha(X_1)]}\right], \qquad x \in S, \tag{2.4}$$

and given a fixed state z, straightforward calculations using (2.3) lead to

$$e^{\lambda[g_\alpha(z)+h_\alpha(x)]} = \mathbf{E}_x\left[e^{\lambda[C(X_0)+\alpha h_\alpha(X_1)]}\right], \qquad x \in S, \tag{2.5}$$

where $h_\alpha(x) := V_\alpha(x) - V_\alpha(z)$ for every state x. Suppose now that Varadhan's function assumes the single value γ and that the pair $(\gamma, h(\cdot))$ satisfies (1.2). Comparing this latter equation with (2.5), it follows that, *for α close to* 1, $(g_\alpha(z), h_\alpha(\cdot))$ is an "approximate solution" to (1.2), and it might be expected that the difference between $g_\alpha(z)$ and γ is small; since $z \in S$ is arbitrary and $J(\cdot) \equiv \gamma$, this would lead to the conclusion that $g_\alpha(\cdot)$ converges to $J(\cdot)$ when α increases to 1. The main result of this note, stated as Theorem 2.1, establishes that this conclusion is true, even if the Poisson equation (1.2) does not admit a solution, in particular, even if $J(\cdot)$ is not constant; also, according to the following result, the limit superior in the definition of Varadhan's function can be replaced by a limit.

THEOREM 2.1. *For each $\alpha \in (0,1)$, let $g_\alpha(\cdot)$ be as in Definition* 2.1. *In this case*,

$$\lim_{\alpha \nearrow 1} g_\alpha(x) = J(x), \qquad x \in S, \tag{2.6}$$

and, *moreover*, *for every $x \in S$*,

$$J(x) = \lim_{n\to\infty} \frac{1}{\lambda n} \log\left(\mathbf{E}_x\left[\exp\left\{\lambda \sum_{t=0}^{n-1} C(X_t)\right\}\right]\right). \tag{2.7}$$

This theorem extends a result obtained in [2], where, assuming that the whole state space is a communicating class, convergence (2.6) was established. As already noted, $J(\cdot)$ is not generally determined by a single equation, and the main advantage of the above approximation result is that finding $g_\alpha(\cdot)$ always reduces to solving (2.4). The proof of Theorem 2.1 will be given in section 4.

3. Technical preliminaries. This section contains the basic tools that will be used to establish Theorem 2.1. First, notice that Lemma 2.1(iii) and Definition 2.2 together yield that $\|g_\alpha\| \leqq \|C\|$ so that all the limit points of the family $\{g_\alpha \,|\, \alpha \in (0,1)\}$ are finite. Throughout the remainder, $g \in \mathcal{B}(S)$ is a fixed limit point of $\{g_\alpha\}$ as α increases to 1, and the sequence $\{\alpha_k\} \subset (0,1)$ satisfies the following:

$$\alpha_k \nearrow 1 \quad \text{as} \quad k \to \infty \quad \text{and} \quad \lim_{k\to\infty} g_{\alpha_k}(x) = g(x), \qquad x \in S. \tag{3.1}$$

The following result is the first step in relating $g(\cdot)$ to Varadhan's function.

THEOREM 3.1. (i) *For each* $x \in S$,

$$g(x) = \max\big\{g(y) \mid p_{xy} > 0\big\}.$$

Consequently,

(ii) *with probability* 1, *the sequence* $\{g(X_t)\}$ *decreases along trajectories. More precisely, for each positive integer* n *and* $x \in S$,

$$g(X_n) \leqq g(X_{n-1}) \leqq \cdots \leqq g(X_1) \leqq g(X_0) = g(x) \quad \mathbf{P}_x\text{-}a.s.$$

Proof. (i) Combining (2.4) and Definition 2.2, it is not difficult to see that

$$e^{\lambda g_{\alpha_k}(x)} = e^{\lambda(1-\alpha_k)\,C(x)}\left(\mathbf{E}_x\left[\exp\left\{\frac{\lambda\alpha_k}{1-\alpha_k}\, g_{\alpha_k}(X_1)\right\}\right]\right)^{1-\alpha_k}, \qquad x \in S. \tag{3.2}$$

Set

$$\varepsilon_k := \big\|g_{\alpha_k}(\cdot) - g(\cdot)\big\|, \tag{3.3}$$

and notice that

$$\lim_{k\to\infty} \varepsilon_k = 0 \tag{3.4}$$

by (3.1) and the finiteness of S. Observe now that for every $k \in \mathbf{N}$ and $x \in S$, (3.3) allows one to write

$$\begin{aligned}\exp\left\{-\frac{\lambda\alpha_k\varepsilon_k}{1-\alpha_k}\right\}\mathbf{E}_x\left[\exp\left\{\frac{\lambda\alpha_k}{1-\alpha_k}\, g(X_1)\right\}\right] &\leqq \mathbf{E}_x\left[\exp\left\{\frac{\lambda\alpha_k}{1-\alpha_k}\, g_{\alpha_k}(X_1)\right\}\right]\\ &\leqq \exp\left\{\frac{\lambda\alpha_k\varepsilon_k}{1-\alpha_k}\right\}\mathbf{E}_x\left[\exp\left\{\frac{\lambda\alpha_k}{1-\alpha_k}\, g(X_1)\right\}\right],\end{aligned}$$

and hence,

$$\begin{aligned}e^{-\lambda\alpha_k\varepsilon_k} &= e^{\lambda(1-\alpha_k)\,C(x)}\left(\mathbf{E}_x\left[\exp\left\{\frac{\lambda\alpha_k}{1-\alpha_k}\, g(X_1)\right\}\right]\right)^{1-\alpha_k}\\ &\leqq \left(\mathbf{E}_x\left[\exp\left\{\frac{\lambda\alpha_k}{1-\alpha_k}\, g_{\alpha_k}(X_1)\right\}\right]\right)^{1-\alpha_k} \leqq e^{\lambda\alpha_k\varepsilon_k}\left(\mathbf{E}_x\left[\exp\left\{\frac{\lambda\alpha_k}{1-\alpha_k}\, g(X_1)\right\}\right]\right)^{1-\alpha_k}.\end{aligned} \tag{3.5}$$

Next, write $(\mathbf{E}_x[e^{\lambda\alpha_k g(X_1)/(1-\alpha_k)}])^{1-\alpha_k} = (\mathbf{E}_x[e^{\lambda p_k g(X_1)}])^{(a_k/p_k)}$, where $p_k := \alpha_k/(1-\alpha_k)$. Since $\alpha_k \nearrow 1$, it follows that $p_k \nearrow \infty$, so that

$$\lim_{k\to\infty}\big(\mathbf{E}_x[e^{\lambda p_k g(X_1)}]\big)^{1/p_k} = \max\big\{e^{\lambda g(y)} \mid p_{xy} > 0\big\},$$

and then

$$\begin{aligned}\lim_{k\to\infty}\left(\mathbf{E}_x\left[\exp\left\{\frac{\lambda\alpha_k}{1-\alpha_k}\, g(X_1)\right\}\right]\right)^{1-\alpha_k} &= \lim_{k\to\infty}\big(\mathbf{E}_x[e^{\lambda p_k g(X_1)}]\big)^{(a_k/p_k)}\\ &= \max\big\{e^{\lambda g(y)} \mid p_{xy} > 0\big\}.\end{aligned}$$

Combining this latter convergence with (3.4) and (3.5), it follows that for every state x,

$$\lim_{k\to\infty}\left(\mathbf{E}_x\left[\exp\left\{\frac{\lambda\alpha_k}{1-\alpha_k}\,g_{\alpha_k}(X_1)\right\}\right]\right)^{1-\alpha_k} = \max\left\{e^{\lambda g(y)}\mid p_{xy}>0\right\} = e^{\lambda\max\{g(y)\,|\,p_{xy}>0\}},$$

where the second equality used that the exponential function is increasing, and together with (3.1), taking the limit as k goes to ∞ in (3.2), this leads to

$$e^{\lambda g(x)} = e^{\lambda\max\{g(y)\,|\,p_{xy}>0\}},\qquad x\in S,$$

and part (i) follows.

(ii) Given $x\in S$ and $t\in\mathbf{N}$, notice that $g(X_t)=\max\{g(y)\,|\,p_{X_t y}>0\}$, by part (i), and this yields that

$$\mathbf{P}\big[g(X_{t+1})\leqq g(X_t)\mid X_t\big]=1$$

so that $\mathbf{P}_x\{g(X_{t+1})\leqq g(X_t)\}=1$, and the conclusion follows, since $x\in S$ and $t\in\mathbf{N}$ are arbitrary. Theorem 3.1 is proved.

Theorem 3.1 will now be used to relate Varadhan's function with the limit point $g(\cdot)$. First, let $k\in\mathbf{N}$ be fixed. Combining (2.4) and Definition 2.2, it follows that for every $x\in S$,

$$e^{\lambda g_{\alpha_k}(x)+\lambda\alpha_k V_{\alpha_k}(x)} = e^{\lambda C(x)}\mathbf{E}_x\big[e^{\lambda\alpha_k V_{\alpha_k}(X_1)}\big]$$

and an induction argument yields that, for every positive integer n,

$$e^{\lambda\alpha_k V_{\alpha_k}(x)} = \mathbf{E}_x\left[\exp\left\{\lambda\sum_{t=0}^{n-1}\big[C(X_t)-g_{\alpha_k}(X_t)\big]+\lambda\alpha_k V_{\alpha_k}(X_n)\right\}\right],\qquad x\in S. \tag{3.6}$$

THEOREM 3.2. *Let γ_0 be the minimum value of the limit point $g(\cdot)$ in* (2.3). *The following assertions* (i) *and* (ii) *are valid*:

(i) $g(\cdot)\geqq J(\cdot)$;

(ii) *let x be such that $g(x)=\gamma_0$; in this case, $g(x)=J(x)$ and* (2.7) *holds for this state x.*

Proof. Let $x\in S$ and let the positive integers n and k be arbitrary, and observe that, with the notation in (3.3), the following holds $\mathbf{P}_x$-a.s.:

$$-n\varepsilon_k+\sum_{t=0}^{n-1}\big[C(X_t)-g(X_t)\big]\leqq\sum_{t=0}^{n-1}\big[C(X_t)-g_{\alpha_k}(X_t)\big]\leqq n\varepsilon_k+\sum_{t=0}^{n-1}\big[C(X_t)-g(X_t)\big]. \tag{3.7}$$

(i) By Theorem 3.1(ii), the inequality $\sum_{t=0}^{n-1}[C(X_t)-g(X_t)]\geqq\sum_{t=0}^{n-1}[C(X_t)-ng(x)]$ holds with probability 1 with respect to $\mathbf{P}_x$, so that the left-hand inequality in the above displayed relation yields

$$\sum_{t=0}^{n-1}\big[C(X_t)-g_{\alpha_k}(X_t)\big]\geqq -n\big(\varepsilon_k+g(x)\big)+\sum_{t=0}^{n-1}C(X)\quad \mathbf{P}_x\text{-a.s.}$$

Combining this with (3.6), it follows that

$$e^{\lambda\alpha_k V_{\alpha_k}(x)}\geqq e^{-\lambda n(\varepsilon_k+g(x))}\mathbf{E}_x\left[\exp\left\{\lambda\sum_{t=0}^{n-1}C(X_t)+\lambda\alpha_k V_{\alpha_k}(X_n)\right\}\right]$$

and then

$$\frac{2\alpha_k\|V_{\alpha_k}\|}{n}+\varepsilon_k+g(x)\geqq\frac{1}{\lambda n}\,\log\left(\mathbf{E}_x\left[\exp\left\{\lambda\sum_{t=0}^{n-1}C(X_t)\right\}\right]\right).$$

Taking limit superior as n goes to ∞, this inequality and (1.1) together imply that $\varepsilon_k + g(x) \geqq J(x)$; from this point (3.4) yields that $g(x) \geqq J(x)$, showing that $g(\cdot)$ is an upper bound of $J(\cdot)$, since $x \in S$ is arbitrary.

(ii) Let n and k be positive integers and suppose that $x \in S$ satisfies $g(x) = \gamma_0$. In this case $\sum_{t=0}^{n-1}[C(X_t) - g(X_t)] \leqq \sum_{t=0}^{n-1} C(X_t) - n\gamma_0$, since γ_0 is the minimum value of $g(\cdot)$, so that the right-hand inequality in (3.7) yields that

$$\sum_{t=0}^{n-1} \left[C(X_t) - g_{\alpha_k}(X_t)\right] \leqq n(\varepsilon_k - \gamma_0) + \sum_{t=0}^{n-1} C(X_t),$$

and, via (3.6), it follows that

$$e^{\lambda \alpha_k V_{\alpha_k}(x)} \leqq e^{-\lambda n(\varepsilon_k - \gamma_0)} \mathbf{E}_x \left[\exp\left\{ \lambda \sum_{t=0}^{n-1} C(X_t) + \lambda \alpha_k V_{\alpha_k}(X_n) \right\} \right].$$

Therefore,

$$e^{-2\lambda\alpha_k \|V_{\alpha_k}\| + n\lambda(\gamma_0 - \varepsilon_k)} \leqq \mathbf{E}_x \left[\exp\left\{ \lambda \sum_{t=0}^{n-1} C(X_t) \right\} \right]$$

and then

$$\frac{-2\alpha_k \|V_{\alpha_k}\|}{n} + \gamma_0 - \varepsilon_k \leqq \frac{1}{\lambda n} \log \left(\mathbf{E}_x \left[\exp\left\{ \lambda \sum_{t=0}^{n-1} C(X_t) \right\} \right] \right).$$

Taking limit inferior as n tends to ∞ in both sides of this inequality, it follows that

$$\gamma_0 - \varepsilon_k \leqq \liminf_{n\to\infty} \frac{1}{\lambda n} \log \left(\mathbf{E}_x \left[\exp\left\{ \lambda \sum_{t=0}^{n-1} C(X_t) \right\} \right] \right)$$

which, letting k increase to ∞ and using (3.4), leads to

$$\gamma_0 \leqq \liminf_{n\to\infty} \frac{1}{\lambda n} \log \left(\mathbf{E}_x \left[\exp\left\{ \lambda \sum_{t=0}^{n-1} C(X_t) \right\} \right] \right).$$

To conclude, observe that part (i) and (1.1) allow one to write

$$\limsup_{n\to\infty} \frac{1}{\lambda n} \log \left(\mathbf{E}_x \left[\exp\left\{ \lambda \sum_{t=0}^{n-1} C(X_t) \right\} \right] \right) = J(x) \leqq g(x) = \gamma_0;$$

combining the last two displayed relations, it follows that $J(x) = \gamma_0 = g(x)$ and that (2.7) holds. Theorem 3.2 is proved.

4. Proof of the main theorem.

The preliminaries in the previous section will now be used to prove Theorem 2.1. Let γ_i, $i = 0, 1, \ldots, d$, be the different values of the function $g(\cdot)$ in (3.1) arranged in increasing order:

$$\gamma_0 < \gamma_1 < \cdots < \gamma_d. \tag{4.1}$$

Next, define the sets G_i by

$$G_i := \left\{ x \in S \mid g(x) = \gamma_i \right\}, \qquad i = 0, 1, \ldots, d, \tag{4.2}$$

and notice that

$$S = \bigcup_{i=0}^{d} G_i. \tag{4.3}$$

Proof of Theorem 2.1. For each $i = 0, 1, \ldots, d$, consider the following proposition:

$\mathbb{P}_i$: *For each* $x \in G_i$, $\displaystyle\lim_{n\to\infty} \frac{1}{\lambda n} \log \left(\mathbf{E}_x \left[\exp\left\{ \lambda \sum_{t=0}^{n-1} C(X_t) \right\} \right] \right) = J(x) = g(x).$

Since $g(\cdot)$ is an arbitrary limit point of the family $\{g_\alpha\}$ as $\alpha \nearrow 1$, the conclusion of Theorem 2.1 is equivalent to the truth of $\mathbb{P}_i$ for every $i = 0, 1, \ldots, d$; see (4.3). Now observe that $\mathbb{P}_0$ holds by Theorem 3.2(ii), and to establish the desired conclusion it is sufficient to prove that $\mathbb{P}_i$ is valid when $d \geqq i > 0$. With this in mind, suppose that d is positive, let i be a positive integer less than or equal to d, assume that $\mathbb{P}_k$ holds for $0 \leqq k < i$, and define

$$\Gamma_i = \bigcup_{r=0}^{i-1} G_r. \tag{4.4}$$

With this notation, Theorem 3.1(ii) and (4.1)–(4.3) together imply that $G_i \cup \Gamma_i$ is a closed set, and from (2.4) it follows that, for every $k \in \mathbf{N}$ and $x \in G_i$,

$$e^{\lambda V_{\alpha_k}(x)} = e^{\lambda C(x)}\left[\sum_{y\in G_i} p_{xy} e^{\lambda \alpha_k V_{\alpha_k}(y)} + \sum_{y\in\Gamma_i} p_{xy} e^{\lambda\alpha_k V_{\alpha_k}(y)}\right].$$

Now let $z \in G_i$ be arbitrary but fixed and notice that using Definition 2.1, the above equation is equivalent to

(4.5)
$$e^{\lambda g_{\alpha_k}(x)+\alpha_k\lambda h_{\alpha_k}(x)} = e^{\lambda C(x)}\left[\sum_{y\in G_i} p_{xy} e^{\lambda\alpha_k h_{\alpha_k}(y)} + \sum_{y\in\Gamma_i} p_{xy} e^{\lambda\alpha_k h_{\alpha_k}(y)}\right], \qquad x \in G_i,$$

where for each $x \in G_i \cup \Gamma_i$,

$$h_{\alpha_k}(x) = V_{\alpha_k}(x) - V_{\alpha_k}(z). \tag{4.6}$$

Without loss of generality it can be assumed, taking a subsequence if necessary, that in addition to (3.1) the following limits exist:

$$\lim_{k\to\infty} h_k(x) =: h(x) \in [-\infty, \infty], \qquad x \in G_i \bigcup \Gamma_i. \tag{4.7}$$

Next, let $y \in \Gamma_i$ be arbitrary, and notice that (4.1), (4.2), and (4.4) yield $g(y) < \gamma_i = g(x)$ so that $\lim_{k\to\infty}(1-\alpha_k)\, h_{\alpha_k}(y) = \lim_{k\to\infty}(1-\alpha_k)\,[V_{\alpha_k}(y) - V_{\alpha_k}(z)] = \lim_{k\to\infty}[g_{\alpha_k}(y) - g_{\alpha_k}(z)] = g(y) - g(z) < 0$, and then $h(y) = -\infty$ when $y \in \Gamma_i$. Using this fact, after taking the limit as k goes to ∞ in (4.5) it follows, via (3.1) and (4.7), that

$$e^{\lambda g(x)+\lambda h(x)} = e^{\lambda C(x)} \sum_{y\in G_i} p_{xy} e^{\lambda h(y)}, \qquad x \in G_i. \tag{4.8}$$

To continue, define the set A by

$$A = \big\{x \in G_i \mid h(x) > -\infty\big\} \tag{4.9}$$

and notice that $e^{\lambda h(y)} = 0$ when $y \in G_i \setminus A$, so that (4.8) is equivalent to

$$e^{\lambda g(x)+\lambda h(x)} = e^{\lambda C(x)} \sum_{y\in A} p_{xy} e^{\lambda h(y)}, \qquad x \in G_i. \tag{4.10}$$

Next, define the set B as follows:

$$B = \big\{x \in A \mid h(x) < \infty\big\} \tag{4.11}$$

and observe the following facts:

(a) State z belongs to G_i and $h(z) = 0$, by (4.6) and (4.7), and then $z \in B$, by (4.9) and (4.11).

(b) Let $x \in B$ be arbitrary so that $h(x)$ is finite. In this case the left-hand side of (4.10) is finite, and then so is the right-hand side. Therefore, if $p_{xy} > 0$ for some $y \in A$, so that $h(y) > -\infty$, it follows that $e^{\lambda h(y)} < \infty$, and then $h(y) < \infty$ and hence $y \in B$. In short, if $x \in B$ and $p_{xy} > 0$ for some $y \in A$, then $y \in B$.

Notice that (b), (4.10), and equality $g(z) = g(x)$ for $x \in B$ allow one to write

$$e^{\lambda g(x)+\lambda h(x)} = e^{\lambda C(x)} \sum_{y\in B} p_{xy} e^{\lambda h(y)}, \qquad x \in B, \tag{4.12}$$

and defining the exit time $T := \min\{n > 0 \mid X_n \notin B\}$, this equation is equivalent to

$$e^{\lambda g(x)+\lambda h(x)} = \mathbf{E}_x\left[e^{\lambda C(X_0)+\lambda h(X_1)} I[T > 1]\right], \qquad x \in B,$$

which, via an induction argument, implies that for every positive integer n and $x \in B$, $e^{n\lambda g(z)+\lambda h(x)} = \mathbf{E}_x[\exp\{\lambda \sum_{t=0}^{n-1} C(X_t)\} e^{\lambda h(X_n)} I[T > n]]$. Since state z lies in B, by fact (a) established above, it follows that

$$e^{n\lambda g(x)+\lambda h(x)} = \mathbf{E}_x\left[\exp\left\{\lambda \sum_{t=0}^{n-1} C(X_t)\right\} e^{\lambda h(X_n)}\right];$$

recalling that the state space is finite, setting $b := \max\{|h(x)| \mid x \in B\}$, (4.9), and (4.11) yields that $b < \infty$, whereas the above displayed inequality implies

$$e^{n\lambda g(x)-2\lambda b} \leqq \mathbf{E}_z\left[\exp\left\{\lambda \sum_{t=0}^{n-1} C(X_t)\right\}\right]$$

so that

$$g(z) - \frac{2b}{n} \leqq \frac{1}{\lambda n} \log\left(\mathbf{E}_z\left[\exp\left\{\lambda \sum_{t=0}^{n-1} C(X_t)\right\}\right]\right).$$

Taking limit inferior as $n \to \infty$ on both sides of this relation, it follows that

$$g(z) \leqq \liminf_{n\to\infty} \frac{1}{\lambda n} \log\left(\mathbf{E}_z\left[\exp\left\{\lambda \sum_{t=0}^{n-1} C(X_t)\right\}\right]\right).$$

Combining this inequality with (1.1) and Theorem 3.2(i) it follows that

$$g(z) = J(z) = \lim_{n\to\infty} \frac{1}{\lambda n} \log\left(\mathbf{E}_z\left[\exp\left\{\lambda \sum_{t=0}^{n-1} C(X_t)\right\}\right]\right);$$

since $z \in G_i$ is arbitrary, this establishes that $\mathbb{P}_i$ holds for a positive integer $i \leqq d$ and, as already noted, this completes the proof. Theorem 2.1 is proved.

Remark 4.1.

(i) It is interesting to observe the following consequence of the proof of Theorem 2.1. Given $z \in G_i$ with $i > 0$, it has been shown that there exists $B \subset G_i$ such that (4.12) holds, where the function $h(\cdot)$ in (4.9) is finite on the set B, and it is not difficult to see that this conclusion can be extended to the case $i = 0$. Since $g_\alpha(x) = g_\alpha(z) + (1-\alpha)\, h_\alpha(x)$, it follows that $J(x) = g(x) = g(z)$ for each $x \in B$, where the first equality follows from Theorem 2.1. Therefore, the following conclusion can be established: *For each* $z \in S$, *there exists* $B_z \equiv B \subset S$ *such that* (a) $z \in B$; (b) $J(\cdot)$ *takes a single value, say* γ_B, *on the set* B; *and, moreover,* (c) *there exists* $h_B\colon B \to \mathbf{R}$ *such that the pair* $(\gamma_B, h_B(\cdot))$ *satisfies the "local Poisson equation"*

$$e^{\lambda\gamma_B+\lambda h_B(x)} = e^{\lambda C(x)} \sum_{y\in B} p_{xy} e^{\lambda h_B(y)}, \qquad x \in B = B_z. \tag{4.13}$$

Thus, although Varadhan's function cannot generally be characterized by a single Poisson equation, $J(\cdot)$ can *always* be determined by solving the local equations (4.13) for all possible sets B_z. If z is recurrent, it is not difficult to see that B_z can be chosen as the recurrent class containing z (see [1], [2], [3], [8]) but, to the best of the authors' knowledge, providing a practical way to find B_z when z is transient is currently an interesting open problem.

(ii) On the other hand, assume now that the Markov chain has a single recurrent class but the class of transient states in nonempty. In this case, it is possible to have that $J(\cdot)$

is not constant and assumes values γ_i, $i = 0, 1, \ldots, d$, where $d > 0$ (see (4.1)), and for each $i > 0$ the set G_i in (4.2) is contained in the class of transient states [3]. In this case, when $i > 0$, for each $z \in G_i$ the set $B_z \equiv B$ in (4.13) is contained in G_i and, consequently, each $x \in B$ is transient so that (4.13) shows that $J(z) = \gamma_B$ is completely determined by the behavior of $\{C(X_t)\}$ on the set of transient states. As already noted, this fact establishes an interesting contrast with the classical (risk-neutral) average cost.

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