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Some limits related to random iterations of a lamplighter group

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Abstract

We obtain limits for random iterations of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$, which refer to the state of a lamp (light "on" or "off"), and the number of changes in a set of lamps.

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1. Introduction

Random walks and Markov chains on wreath products, and random wreath products have been studied by several authors (e.g., Cartwright, 1988; Evans, 2001; Fill and Schoolfield, 2001; Lyons et al., 1996; Peres and Revelle, 2004; Pittet and Saloff-Coste, 2002; Revelle, 2003; Schoolfield, 2002). Some examples of wreath products appear under the name of lamplighter groups in the random walk literature. On the other hand, random iterations of mappings, in particular random iterations of groups, have been investigated extensively (e.g., Ådahl et al., 2003; Diaconis and Freedman, 1999; Gorostiza, 1973a; Guivarc'h, 2000; Guivarc'h et al., 1977; Hennion and Hervé, 2004; Rachev and Yukich, 1991; Wu and Woodroofe, 2000; and references therein), but we have not found results related to random iterations of wreath products.

In this note we obtain two limits for some random iterations of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$. The first result says that if the lamplighter performs a random walk on $\mathbb{Z}$, changing the state of the lamp at each visited site with probability $q > 0$, then in the long run the light at each site spends equal amounts of time on and off. However, this is a straightforward consequence of elementary theory of Markov chains. A more interesting result is obtained by extending the sum on $\mathbb{Z}_2$ to $\mathbb{Z}^+$ (but note that $\mathbb{Z}^+ \wr \mathbb{Z}$ is not a group). In this way we have a limit for the number of changes in a given set of lamps. This limit is the same for the number of visits of a random walk to a given set of sites, where each visit is observed with probability q. These results are simple,

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but they suggest that problems of this type for more general classes of random iterations of wreath products may be interesting.

2. Random iterations of wreath products

We refer to Pittet and Saloff-Coste (2002) for background on wreath products and a description of lamplighter groups. Let K and H be finitely generated groups, and $\mathbf{K}_H$ the class of functions $k : H \to K$ with bounded support. The wreath product $K \wr H$ is the semidirect product $K \wr H = \mathbf{K}_H \rtimes H$ with

$$(k,h)(k',h') = (k\tau_h k', hh'), \quad k,k' \in \mathbf{K}_H, \;\; h,h' \in H,$$

where $k\tau_h k'(\ell) = k(\ell)k'(h^{-1}\ell)$, $\ell \in H$. Hence

$$(k_1,h_1)\dots(k_n,h_n) = (\{k_1(\ell)k_2(h_1^{-1}\ell)\dots k_n(h_{n-1}^{-1}\dots h_1^{-1}\ell), \ell \in H\}, h_1\dots h_n), \tag{2.1}$$

and

$$(k_n,h_n)\dots(k_1,h_1) = (\{k_n(\ell)k_{n-1}(h_n^{-1}\ell)\dots k_1(h_2^{-1}\dots h_n^{-1}\ell), \ell \in H\}, h_n\dots h_1). \tag{2.2}$$

If H and K are Abelian, (2.1) and (2.2) are given by

$$(k_1,h_1)\dots(k_n,h_n) = \left(\left\{\sum_{j=1}^{n} k_j\left(\ell - \sum_{i=1}^{j-1} h_i\right), \ell \in H\right\}, \sum_{i=1}^{n} h_i\right), \tag{2.3}$$

and

$$(k_n,h_n)\dots(k_1,h_1) = \left(\left\{\sum_{j=1}^{n} k_j\left(\ell - \sum_{i=j+1}^{n} h_i\right), \ell \in H\right\}, \sum_{i=1}^{n} h_i\right), \tag{2.4}$$

where, obviously, $\sum_{i=1}^{j-1} h_i = 0$ for $j = 1$ and $\sum_{i=j+1}^{n} h_i = 0$ for $j = n$.

Let $(k_n,h_n), n = 1,2,\dots$ be a sequence of independent, identically distributed (i.i.d.) random elements of $K \wr H$. The problem is how (2.1) and (2.2) behave as $n \to \infty$. An analogous problem that has been studied is random iterations of isometries of Euclidean space $\mathbb{R}^d$ (represented by the group $E(d) = \mathbb{R}^d \rtimes \mathrm{O}(d)$, but in this case the groups K and H are not finitely generated). The first component of (2.1) obeys a central limit theorem (Ådahl et al., 2003; Gorostiza, 1973a; Rachev and Yukich, 1991; and references therein). With the order (2.2) there is also a central limit theorem, but the limit is different (Gorostiza, 1973b).

We consider here a simple type of random iterations of the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$. The lamplighter performs a random walk on $\mathbb{Z}$, changing the state of the lamp (light "on" or "off") at each visited site with probability $q > 0$. (In a more general model, the lamplighter can make changes at the same time on a finite set of lamps related to its current position, according to some probability distribution.) Note that (2.3) and (2.4) have the same distribution. We will work with (2.3).

3. Limits

Let $S_n = \sum_{i=1}^{n} h_i, \; n \geqslant 1, \; S_0 = 0$, denote the random walk on $\mathbb{Z}$ which corresponds to the second component in (2.3). S_n represents the position of the lamplighter at the nth step. We assume that S_n is recurrent and satisfies the assumptions for the local central limit theorem (e.g., Breiman, 1968), with $Eh = 0$ and $Eh^2 = \sigma^2$. Limit theorems for S_n are classical. In order to simplify calculations, we assume that k_j and h_j are independent for each j, $k_j(\ell) = 0$ for all $\ell \neq 0$, and

$$k_j(0) = \begin{cases} 1 & \text{with probability } q, \\ 0 & \text{with probability } 1-q, \end{cases} \tag{3.1}$$

$0 < q \leqslant 1$.

We consider the first component in (2.3),

$$Y_n(\ell) = \sum_{j=1}^{n} k_j(\ell - S_{j-1}), \quad \ell \in \mathbb{Z}. \tag{3.2}$$

Thus, at each visited site the lamplighter changes the state of the lamp with probability q and leaves it as it is with probability $1-q$, and then moves to another site. $Y_n(\ell)$ represents the state of the lamp at ℓ after the first $n-1$ steps of the lamplighter.

Proposition 1. *For each* $\ell \in \mathbb{Z}$ *and any* $q>0$,

$$\lim_{n\to\infty} P[Y_n(\ell)=0] = \lim_{n\to\infty} P[Y_n(\ell)=1] = \frac{1}{2}. \tag{3.3}$$

Proof. Fix ℓ and $k_1(\ell)=a$. Let

$$N_n(\ell) = \#\{j, 0 \leqslant j \leqslant n-1 : S_j = \ell\}.$$

Then, from (3.2) we have (shifting the summation index to simplify notation, by the i.i.d. assumption)

$$\begin{aligned} P[Y_n(\ell)=0] &= P\left[a + \sum_{j=1}^{n-1} k_{j+1}(\ell - S_j) = 0\right] \\ &= \sum_{r=0}^{n-1} P\left[a + \sum_{j=1}^{n-1} k_{j+1}(\ell - S_j) = 0 \,\middle|\, N_n(\ell) = r\right] P[N_n(\ell)=r] \\ &= \sum_{r=0}^{n-1} P\left[a + \sum_{j=1}^{r} k_j(\ell) = 0\right] P[N_n(\ell)=r]. \end{aligned}$$

By (3.1), $(\sum_{j=1}^{r} k_j(\ell))_{r=1,2,\ldots}$ is a Markov chain on $\{0,1\}$ with transition probabilities $P(0\to 1) = P(1\to 0) = q$. Hence, by elementary theory (convergence to the stationary distribution), for any a we have

$$\lim_{r\to\infty} P\left[a + \sum_{j=1}^{r} k_j(\ell) = 0\right] = \frac{1}{2}.$$

On the other hand, by recurrence of S_j, for any r,

$$\lim_{n\to\infty} P[N_n(\ell)=r] = 0.$$

Then (3.3) follows easily. $\square$

This result means that in the long run the light at each site has equal probabilities of being on or off, independently of q.

For the second limit we regard Y_n as a random measure on $\mathbb{Z}$:

$$Y_n(I) = \sum_{\ell\in I}\sum_{j=1}^{n} k_j(\ell - S_{j-1}) = \sum_{j=1}^{n}\sum_{\ell\in I} k_j(\ell - S_{j-1}) = \sum_{j=1}^{n} k_j(I - S_{j-1}), \tag{3.4}$$

where I is a bounded set in $\mathbb{Z}$, and the sum of values of the k_j is now on $\mathbb{Z}^+$. Hence $Y_n(I)$ represents the number of times the lamplighter changes states of any lamps in I in the first $n-1$ steps. $Y_n(I)$ is also a randomly observed occupation time of I by the first $n-1$ steps of the random walk S_j, where each visit to I is registered with probability q and not registered with probability $1-q$. If $q=1$, $Y_n(I)$ is an ordinary occupation time, and the result coincides with the known one for that case (Breiman, 1968). Intuitively, $Y_n(I)$ is expected to behave asymptotically like the occupation time of I by the random walk S_j, multiplied by q, and this is confirmed by the result.

Proposition 2. *For any bounded set I of $\mathbb{Z}$ any $q>0$,*

$$\frac{1}{\sqrt{n}} Y_n(I) \Rightarrow \frac{q|I|}{\sigma\sqrt{2}} X \quad \textit{as } n \to \infty, \tag{3.5}$$

where the random variable X has the truncated normal distribution with density function

$$f(x) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-x^2/4}, \quad x \geqslant 0.$$

($\Rightarrow$ *denotes convergence in distribution and $|I|$ is the cardinality of I*).

Proof. We compute the Laplace transform of $Y_n(I)/\sqrt{n}$ from (3.4) using the i.i.d. assumption:

$$E \exp\left\{-u \frac{1}{\sqrt{n}} Y_n(I)\right\} = A_n(u) B_n(u),$$

where

$$A_n(u) = E \exp\left\{-u \frac{1}{\sqrt{n}} k_1(I)\right\}, \quad B_n(u) = E \exp\left\{-u \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} k_{j+1}(I - S_j)\right\}$$

(the summation index has been shifted by the i.i.d. assumption as above). We have

$$A_n(u) = \begin{cases} 1, & 0 \notin I, \\ \mathrm{e}^{-u/\sqrt{n}} q + 1 - q, & 0 \in I. \end{cases}$$

Let

$$N_n(I) = \#\{j, 0 \leqslant j \leqslant n-1 : S_j \in I\},$$

then

$$\begin{aligned} B_n(u) &= \sum_{r=0}^{n-1} E\left[\exp\left\{-u \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} k_{j+1}(I - S_j)\right\} \middle| N_n(I) = r\right] P[N_n(I) = r] \\ &= \sum_{r=0}^{n-1} \left(E \exp\left\{-u \frac{1}{\sqrt{n}} k_1(0)\right\}\right)^r P[N_n(I) = r] \\ &= \sum_{r=0}^{n-1} \left(\mathrm{e}^{-u/\sqrt{n}} q + 1 - q\right)^r P[N_n(I) = r] \\ &= \sum_{r=0}^{n-1} \sum_{k=0}^{r} \binom{r}{k} \mathrm{e}^{-uk/\sqrt{n}} \left(\frac{q}{1-q}\right)^k (1-q)^r P[N_n(I) = r]. \end{aligned}$$

Next we compute the mth moment of $Y_n(I)/\sqrt{n}$:

$$\begin{aligned} E\left(\frac{1}{\sqrt{n}} Y_n(I)\right)^m &= (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}u^m} A_n(u) B_n(u)\bigg|_{u=0} \\ &= (-1)^m \sum_{j=0}^{m} \binom{m}{j} \frac{\mathrm{d}^j}{\mathrm{d}u^j} A_n(u)\bigg|_{u=0} \frac{\mathrm{d}^{m-j}}{\mathrm{d}u^{m-j}} B_n(u)\bigg|_{u=0}, \end{aligned} \tag{3.6}$$

$$\frac{\mathrm{d}^j}{\mathrm{d}u^j} A_n(u)\bigg|_{u=0} = \begin{cases} 1, & j = 0, \\ 0, & j > 0,\ 0 \notin I, \\ \dfrac{(-1)^j}{n^{j/2}} q, & j > 0,\ 0 \in I, \end{cases}$$

$$\frac{d^{m-j}}{du^{m-j}} B_n(u)\bigg|_{u=0} = \frac{(-1)^{m-j}}{n^{(m-j)/2}} \sum_{r=0}^{n-1} \sum_{k=0}^{r} \binom{r}{k} k^{m-j} \left(\frac{q}{1-q}\right)^k (1-q)^r P[N_n(I)=r]. \tag{3.7}$$

We will use the occupation time theorem for $N_n(I)$:

$$\lim_{n\to\infty} E\left(\frac{N_n(I)}{\sqrt{n}}\right)^m = \left(\frac{\sqrt{2}|I|}{\sigma}\right)^m \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi}} \tag{3.8}$$

(see Breiman, 1968, note that $2^m\Gamma((m+1)/2)/\sqrt{\pi} = m!/\Gamma(1+m/2)$).

The sum $\sum_{k=0}^{r}\binom{r}{k}k^{m-j}(q/(1-q))^k$ in (3.7) can be written as a sum of terms, each containing a power of r, and only the term with the highest power of r, which is r^{m-j}, obeys (3.8) and will contribute to the limit. In order to verify this assertion we need the following combinatorial result.

Consider the sum

$$\sum_{k=0}^{r} \binom{r}{k} k^s p^k, \tag{3.9}$$

where s is an integer $\geqslant 1$ and $p>0$. Using the formula

$$\sum_{k=0}^{r} \binom{r}{k} k^s p^k = rp \sum_{i=0}^{s-1} \binom{s-1}{i} \sum_{k=0}^{r-1} \binom{r-1}{k} k^i p^k,$$

it can be shown by induction that the dominating term in (3.9), i.e., the one containing the highest power of r, is

$$r^s p^s (1+p)^{r-s}.$$

Using this with $s = m-j \geqslant 1$ and $p = q/(1-q)$ in (3.7), we see that the dominating term there is

$$\frac{(-1)^{m-j} q^{m-j}}{n^{(m-j)/2}} \sum_{r=0}^{n-1} r^{m-j} P[N_n(I)=r] = (-1)^{m-j} q^{m-j} E\left(\frac{N_n(I)}{\sqrt{n}}\right)^{m-j}.$$

Hence the dominating term in (3.6) is

$$q^m E\left(\frac{N_n(I)}{\sqrt{n}}\right)^m, \quad 0 \notin I,$$

$$q^m E\left(\frac{N_n(I)}{\sqrt{n}}\right)^m + \sum_{j=1}^{m} \binom{m}{j} q^{m-j+1} \frac{1}{n^{j/2}} E\left(\frac{N_n(I)}{\sqrt{n}}\right)^{m-j}, \quad 0 \in I.$$

Therefore, by (3.8),

$$\lim_{n\to\infty} E\left(\frac{1}{\sqrt{n}} Y_n(I)\right)^m = \left(\frac{q\sqrt{2}|I|}{\sigma}\right)^m \frac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{\pi}} \tag{3.10}$$

for each $m = 1, 2, \ldots$.

The numbers on the right-hand side of (3.10) are the mth moments of the random variable on the right-hand side of (3.5), so (3.5) is proved by the method of moments. □

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