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An Algebraic Formula for the Index of a Vector Field on an Isolated Complete Intersection Singularity

AN ALGEBRAIC FORMULA FOR THE INDEX OF A VECTOR FIELD ON AN ISOLATED COMPLETE INTERSECTION SINGULARITY

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ABSTRACT. — Let $(V, 0)$ be a germ of a complete intersection variety in $\mathbb{C}^{n+k}$, $n > 0$, having an isolated singularity at 0 and X be the germ of a holomorphic vector field having an isolated zero at 0 and tangent to V. We show that in this case the homological index and the GSV-index coincide. In the case when the zero of X is also isolated in the ambient space $\mathbb{C}^{n+k}$ we give a formula for the homological index in terms of local linear algebra.

RÉSUMÉ. — Soit $(V, 0)$ un germe d'intersection complète dans $\mathbb{C}^{n+k}$, $n > 0$, avec singularité isolée en 0 et soit X un germe de champs de vecteurs holomorphes en $\mathbb{C}^{n+k}$ tangents à V et qui a une singularité isolée dans V en 0. Nous montrons que dans ce cas l'indice homologique et l'indice GSV coïncident. Dans le cas où le zéro de X est aussi isolé dans l'espace ambiant $\mathbb{C}^{n+k}$, nous donnons une formule pour l'indice homologique en terme de l'algèbre linéaire.

Introduction

An isolated singular point (zero) p of a vector field on $\mathbb{C}^n$ has an index. It can be defined as the degree of the map $X/\|X\|$ from a small sphere around the point p to the unit sphere. If the vector field is holomorphic, then the index can also be defined as the dimension of a certain algebra: If, in local coordinates centred at the point p,

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i},$$

then the index is equal to the dimension of the complex vector space $\mathcal{O}_{\mathbb{C}^n,0}/(X_1, \ldots, X_n)$, where $\mathcal{O}_{\mathbb{C}^n,0}$ is the ring of germs of holomorphic functions of n variables and $(X_1, \ldots, X_n)$ is the ideal generated by the components of the vector field X.

Now let X be a vector field tangent to the germ $(V, 0)$ of a complex analytic variety of pure dimension n with an isolated singularity at 0 such that X has an isolated singular point at 0 as well. Then one can try to generalize the two notions of the index mentioned above. If V is a complete intersection, then one can still define an index of X at 0 as the degree of a certain map. This is done in [4, 11] and this index is called the GSV-index $\mathrm{Ind}_{\mathrm{GSV}}(X; V, 0)$. If, more generally, V is a complex analytic variety of pure dimension n, but X is a holomorphic vector field, then the third author proposed a generalization of the algebraic index, which is called the *homological index* [2]. This is defined as follows: Consider the sheaves Ω_V^j of germs of differential j-forms on V, $j = 0, \ldots, n$. Contraction by the vector field X defines a complex

$$\Omega_V : 0 \leftarrow \mathcal{O}_V \xleftarrow{X} \Omega_V^1 \xleftarrow{X} \Omega_V^2 \xleftarrow{X} \ldots \xleftarrow{X} \Omega_V^n \leftarrow 0$$

Since both $(V, 0)$ and X have isolated singularities, this complex has finite dimensional homology groups. The *homological index* $\mathrm{Ind}_{\mathrm{hom}}(X; V, 0)$ of the vector field X at 0 is defined as the Euler characteristic of this complex

$$\mathrm{Ind}_{\mathrm{hom}}(X; V, 0) := \chi(\Omega_V) = \sum_{i=0}^{n} (-1)^i \dim H_i(\Omega_V).$$

In [2] it was shown that the homological index and the GSV-index differ only by a constant depending on the germ of the variety $(V, 0)$ but not on the vector field X. In the case when V is a hypersurface in $\mathbb{C}^{n+1}$ it was shown in that paper that this constant is equal to zero. Moreover, in this case an algebraic formula for the homological index under the additional hypothesis that X has an isolated singularity in $\mathbb{C}^{n+1}$ was given.

In this paper we consider the case when $(V, 0)$ is the germ of a complete intersection variety having an isolated singularity at 0 defined by the vanishing of the germs of holomorphic functions $f_1, \ldots, f_k$ in $\mathbb{C}^{n+k}$, $n > 0$, and X the germ of a holomorphic vector field on $\mathbb{C}^{n+k}$ tangent to V and having on V an isolated zero at 0. We show that in this case the homological index and the GSV-index coincide (Theorem 2.4). In the case when the zero of X is also isolated in the ambient space $\mathbb{C}^{n+k}$ we give a formula for the homological index in terms of local linear algebra (Corollary 4.2). When V is a hypersurface we recover the formula of [2].

Our method of proof is as follows. The tangency condition can be expressed by an anticommutative square of finite free $\mathcal{O}$-modules. Given such a square we construct a double complex which we call the Gobelin[1]. It turns out that the Gobelin is weaved from Koszul-complexes and complexes introduced by Buchsbaum and Eisenbud. Using their results and the first spectral sequence of the Gobelin we prove that one can cut the Gobelin to obtain a finite free resolution of the complex Ω_V. It follows that the Euler characteristic of the cut Gobelin is equal to the homological index of V. We then construct a deformation V_λ of V to its Milnor fiber and a family of vector fields X_λ tangent to V_λ. Applying the Gobelin construction to this family yields a situation in which we can apply the results of [3] and [2] and hence conclude that the Euler characteristics of the family of cut Gobelins is independent of λ. It follows that the GSV-Index and the homological index agree.

The second spectral sequence of the Gobelin provides formulae for the homology groups of the Gobelin in terms of local linear algebra.

The paper is organized as follows. We first define the Gobelin double complex $\mathcal{G}$ and show in Theorem 1.6 that the vertical complexes of the Gobelin are resolutions up to a certain column. In Section 2 we apply this to the situation that X is a holomorphic vector field tangent to a germ of an n-dimensional complete intersection singularity $(V, 0)$ in $\mathbb{C}^N$ with X and V both having an isolated singularity at $0 \in \mathbb{C}^N$. We show that the homological index of X at 0 is the Euler characteristic of the total complex of the subcomplex of the Gobelin consisting of the first $n+1$ columns of $\mathcal{G}$ (Proposition 2.1). We derive from this that the homological index coincides with the GSV-index. In order to compute the homological index we show in Section 3 that the total complex of the Gobelin is quasi-isomorphic to the total complex of a simpler double complex which we call the small Gobelin. Here we need the fact that X has an isolated singularity in the ambient space $\mathbb{C}^N$. In Section 4 we derive from this algebraic formulae for the homological index. The main formula is contained in Corollary 4.2.

We wish to thank the referee for useful comments, in particular for pointing out an error in our previous calculation of Example 4.4.

1. The Gobelin Double Complex

Throughout this paper we fix a field $\mathbb{K}$ and consider a local Noetherian $\mathbb{K}$-algebra $\mathcal{O}$. All tensor products will be over $\mathbb{K}$, unless otherwise specified.

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[1] A Gobelin is a richly embroidered French wall tapestry.

If φ is a matrix with entries in $\mathcal{O}$, we denote by $I_\ell(\varphi)$ the ideal of its $\ell \times \ell$ minors. A complex has *length* r if it contains $r + 1$ non-zero elements. A complex is called a *resolution* if its first non zero term is in degree 0 and it has zero homology in positive degrees.

In this section we will develop the technical tools needed to prove our results.

1.1. Construction of the Gobelin

Consider a matrix identity over $\mathcal{O}$

$$(1.1) \quad \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1N} \\ \vdots & \ddots & \vdots \\ \varphi_{\ell 1} & \cdots & \varphi_{\ell N} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{\ell 1} & \cdots & c_{\ell k} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix}$$

and write this equation as

$$(1.2) \qquad\qquad \varphi X = cf.$$

Let F, G and H be finite dimensional $\mathbb{K}$-vector spaces of dimensions N, ℓ, and k respectively. Then the equation (1.2) gives rise to the anticommutative square

$$(1.3) \quad \begin{array}{ccc} \mathcal{O} & \xrightarrow{-f} & H \otimes \mathcal{O} \\ X \downarrow & & \downarrow c \\ F \otimes \mathcal{O} & \xrightarrow{\varphi} & G \otimes \mathcal{O} \end{array}$$

Let $\mathbb{P}^{\ell-1}$ denote the projective space $\mathrm{Proj}(G)$ and $\mathcal{O}_{\mathbb{P}^{\ell-1}}(1)$ the sheaf of hyperplane sections on $\mathbb{P}^{\ell-1}$. Let $s_1, \ldots, s_\ell$ be a basis of its global sections, $s := (s_1, \ldots, s_\ell)$, $\widetilde{\mathcal{O}} := \mathcal{O} \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}$ and $\widetilde{\mathcal{O}}(m) := \mathcal{O} \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}(1)^{\otimes m}$. We tensor the diagram (1.3) with the sheaf $\mathcal{O}_{\mathbb{P}^{\ell-1}}$ and continue at the right bottom of the square with the tensor product of the natural morphism

$$s \cdot\ : G \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}} \longrightarrow \mathcal{O}_{\mathbb{P}^{\ell-1}}(1)$$

with $\mathcal{O}$ to obtain the following anticommutative square of $\widetilde{\mathcal{O}}$-sheaves on $\mathbb{P}^{\ell-1}$:

$$(1.4) \quad \begin{array}{ccc} \widetilde{\mathcal{O}} & \xrightarrow{-f} & H \otimes \widetilde{\mathcal{O}} \\ X \downarrow & & \downarrow s \cdot c \\ F \otimes \widetilde{\mathcal{O}} & \xrightarrow{s \cdot \varphi} & \widetilde{\mathcal{O}}(1) \end{array}$$

Since going around the square gives a 1×1 matrix, we may transpose the upper part of the square and obtain the anticommutative square

$$(1.5) \quad \begin{array}{ccc} \widetilde{\mathcal{O}} & \xrightarrow{(s \cdot c)^t} & H^* \otimes \widetilde{\mathcal{O}}(1) \\ X \downarrow & & \downarrow -f^t \\ F \otimes \widetilde{\mathcal{O}} & \xrightarrow{s \cdot \varphi} & \widetilde{\mathcal{O}}(1) \end{array} \quad .$$

The total complex associated to this square is the syzygy

$$(1.6) \qquad \widetilde{\mathcal{O}} \xrightarrow{\left(\begin{smallmatrix} X \\ (s \cdot c)^t \end{smallmatrix}\right)} [F \otimes \widetilde{\mathcal{O}}] \oplus [H^* \otimes \widetilde{\mathcal{O}}(1)] \xrightarrow{(s \cdot \varphi, -f^t)} \widetilde{\mathcal{O}}(1)$$

Using this and the notation

$$(1.7) \qquad \mathcal{V} := [F \otimes \widetilde{\mathcal{O}}] \oplus [H^* \otimes \widetilde{\mathcal{O}}(1)], \quad \mathcal{V}(m) := \mathcal{V} \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{O}}(m)$$

we consider upward vertical Koszul complexes $\mathcal{K}_{(s \cdot \varphi, -f^t)} \otimes \mathcal{O}(-k + i - 1)$ of length $N + k$,

$$(1.8) \quad 0 \leftarrow \mathcal{O}(-k + i) \leftarrow \mathcal{V}(-k + i - 1) \leftarrow \Lambda^2 \mathcal{V}(-k + i - 2) \leftarrow \cdots$$
$$\leftarrow \Lambda^{N+k} \mathcal{V}(-k + i - N - k) \leftarrow 0$$

whose arrows are given by contractions with $(s \cdot \varphi, -f^t)$ and rightward horizontal Koszul complexes $\mathcal{K}_{\left(\begin{smallmatrix} X \\ (s \cdot c)^t \end{smallmatrix}\right)} \otimes \mathcal{O}(-k + j)$ of length $N + k$,

$$(1.9) \quad 0 \to \mathcal{O}(-k + j) \to \mathcal{V}(-k + j) \to \Lambda^2 \mathcal{V}(-k + j) \to \cdots$$
$$\to \Lambda^{N+k} \mathcal{V}(-k + j) \to 0$$

with arrows given by exterior products with $\left(\begin{smallmatrix} X \\ (s \cdot c)^t \end{smallmatrix}\right)$. These complexes we weave into a bi-infinite double complex of sheaves $\widetilde{\mathcal{G}}$ on $\mathbb{P}^{\ell-1}$ (see Fig. 1.1 for an example). The anticommutativity of this double complex follows from the anticommutativity of exterior product and contraction in the exterior algebra and the fact that the syzygy (1.6) of this square is a complex.
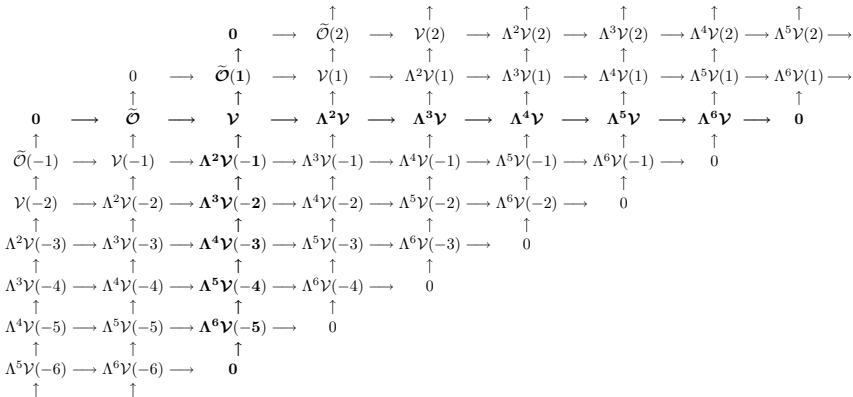

Figure 1.1. Part of the double complex $\widetilde{\mathcal{G}}$ for $N = 4$, $k = 2$ and $l = 2$. The generating Koszul complexes of (1.6) are typed in a darker tone.

DEFINITION 1.1. — *We define the* Gobelin double complex *of the anti-commutative square* (1.3)

$$\mathcal{G} := \Gamma(\widetilde{\mathcal{G}})^*$$

as the dual of the global section double complex of $\widetilde{\mathcal{G}}$. *The terms of the Gobelin are* $\mathcal{O}$-*modules*

$$\mathcal{G}_{i,j} = H^0(\mathbb{P}^{\ell-1}, \Lambda^{k+i-j}\mathcal{V}(-k+j))^*$$

$$= \bigoplus_{r+s=k+i-j} D_{-k+j+r}G^* \otimes \Lambda^r H \otimes \Lambda^s F^* \otimes \mathcal{O}$$

where $D_m G^* := H^0(\mathbb{P}^{\ell-1}, \mathcal{O}_{\mathbb{P}^{\ell-1}}(m))^*$ *is the homogeneous component of the divided power algebra of* $\mathbb{K}[x_1, \ldots, x_\ell]$ *of degree* m, *and the connecting maps are constructed using the matrices* f *and* φ *for the vertical strands and* X *and* c *for the horizontal ones (see Fig. 1.2 for an example).*
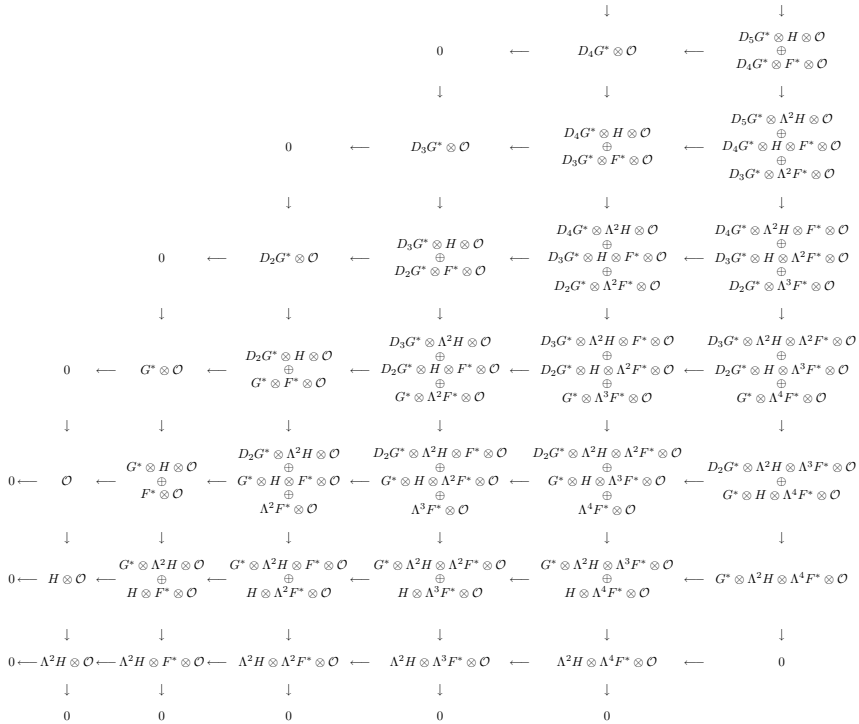

Figure 1.2. *The lower left hand part of the Gobelin for* $N = 4$, $k = 2$, $l = 2$ *beginning at* $(0,0)$.

LEMMA 1.2. — *Let $\mathcal{G}$ be the Gobelin constructed from the identity (1.1) over the local $\mathbb{K}$-algebra $\mathcal{O}$. We have:*

(1) *The Gobelin is a double complex of finite free $\mathcal{O}$-modules and $\mathcal{G}_{i,j}$ is non-zero only for $j \geqslant 0$, $i = \max\{0, j - k\}, \ldots, N + j$.*

(2) *The 0th row of the Gobelin is the complex*

$$0 \longleftarrow \Lambda^k H \otimes \mathcal{O} \longleftarrow \Lambda^k H \otimes F^* \otimes \mathcal{O} \longleftarrow \Lambda^k H \otimes \Lambda^2 F^* \otimes \mathcal{O} \longleftarrow \cdots$$

$$\longleftarrow \Lambda^k H \otimes \Lambda^N F^* \otimes \mathcal{O} \longleftarrow 0$$

with maps being contractions with X, i.e. the tensor product of $\Lambda^k H$ with the Koszul complex $\mathcal{K}_X$.

Proof. —

(1) Since the Gobelin is obtained as the vector space of global sections of the double complex of sheaves $\widetilde{\mathcal{G}}$ and then taking duals, the anticommutativity follows from the anticommutativity of $\tilde{\mathcal{G}}$. Furthermore the summands of

$$\mathcal{G}_{i,j} = \bigoplus_{r+s=k+i-j} D_{-k+j+r} G^* \otimes \Lambda^r H \otimes \Lambda^s F^* \otimes \mathcal{O}$$

are nonzero if and only if $0 \leqslant -k + j + r$ and $0 \leqslant r \leqslant k$ and $0 \leqslant s \leqslant N$.

(2) This follows from substituting $j = 0$ in the formula for $\mathcal{G}_{i,j}$. $\qquad\square$

1.2. The Vertical Complexes $\mathcal{A}^m$ of the Gobelin

Denote by $\mathcal{A}^m$ the complex obtained from the mth column of the Gobelin. The complexes $\mathcal{A}^m$ are the tensor product of two complexes, arising from the direct sum decomposition of the middle module in the syzygy (1.6). One is the part of the Buchsbaum-Eisenbud strand of the Koszul complex associated to the map

$$\varphi : F \otimes \mathcal{O} \longrightarrow G \otimes \mathcal{O}$$

above the splicing map and the other is the Koszul complex associated to the sequence $f_1, \ldots, f_k$. In this subsection we apply the Buchsbaum-Eisenbud and Koszul Theorems to describe the homologies of the vertical complexes in the Gobelin.

Denote by $\mathcal{K}_f$ the Koszul complex obtained from the morphism $-f^t : H^* \otimes \mathcal{O} \longrightarrow \mathcal{O}$:

$$0 \longleftarrow \mathcal{O} \longleftarrow H^* \otimes \mathcal{O} \longleftarrow \Lambda^2 H^* \otimes \mathcal{O} \longleftarrow \cdots \longleftarrow \Lambda^k H^* \otimes \mathcal{O} \longleftarrow 0$$

with morphisms contractions with $-f^t = (-f_1, \ldots, -f_k)$. It's 0-homology group is

$$H_0(\mathcal{K}_f) = \mathcal{O}_V := \mathcal{O}/(f_1, \ldots, f_k).$$

If $-f_1, \ldots, -f_k$ is a regular $\mathcal{O}$ sequence, then by Koszul's Theorem [1, Theorem 17.4, p. 424] $H_j(\mathcal{K}_f) = 0$ for $j > 0$. Let $\mathcal{K}_f^*$ be its dual

$$0 \longleftarrow \Lambda^k H \otimes \mathcal{O} \longleftarrow \Lambda^{k-1} H \otimes \mathcal{O} \longleftarrow \cdots \longleftarrow H \otimes \mathcal{O} \longleftarrow \mathcal{O} \longleftarrow 0$$

with morphisms contractions with $-f^t = (-f_1, \ldots, -f_k)$. It's 0-homology group is again

$$H_0(\mathcal{K}_f^*) = \mathcal{O}_V.$$

Next we consider the Koszul complex $\mathcal{K}_{s \cdot \varphi}$ of $\widetilde{\mathcal{O}}$-sheaves obtained from the morphism

$$s \cdot \varphi : F \otimes \widetilde{\mathcal{O}} \longrightarrow \widetilde{\mathcal{O}} \tag{1}$$

on $\mathbb{P}^{\ell-1}$ and tensor it with $\widetilde{\mathcal{O}}(m-1)$:

$$0 \leftarrow \widetilde{\mathcal{O}}(m) \leftarrow F \otimes \widetilde{\mathcal{O}}(m-1) \leftarrow \Lambda^2 F \otimes \widetilde{\mathcal{O}}(m-2) \leftarrow \cdots$$
$$\leftarrow \Lambda^N F \otimes \widetilde{\mathcal{O}}(m-N) \leftarrow 0$$

As in [1, A2.6.1, p. 591] we denote by $\mathcal{K}_{s \cdot \varphi}^m$ the complex of its global sections

$$0 \leftarrow S_m G \otimes \mathcal{O} \leftarrow S_{m-1} G \otimes F \otimes \mathcal{O} \leftarrow S_{m-2} G \otimes \Lambda^2 F \otimes \mathcal{O} \leftarrow \cdots$$
$$\leftarrow S_{m-N} G \otimes \Lambda^N F \otimes \mathcal{O} \leftarrow 0$$

where $S_m G = H^0(\mathbb{P}^{\ell-1}, \mathcal{O}_{\mathbb{P}^{\ell-1}}(m))$, and by $(\mathcal{K}_{s \cdot \varphi}^m)^*$ the dual complex of free $\mathcal{O}$-modules

$$0 \leftarrow D_{m-N} G^* \otimes \Lambda^N F^* \otimes \mathcal{O} \leftarrow \cdots$$
$$\leftarrow D_{m-1} G^* \otimes F^* \otimes \mathcal{O} \leftarrow D_m G^* \otimes \mathcal{O} \leftarrow 0$$

Note that for $m \leqslant N$ the first nonzero term of $(\mathcal{K}_{s \cdot \varphi}^m)^*$ is $D_0 G^* \otimes \Lambda^m F^* \otimes \mathcal{O}$. In this case the complexes $\mathcal{K}_{s \cdot \varphi}^m$ and $(\mathcal{K}_{s \cdot \varphi}^m)^*$ have length m. For $m \geqslant N$ the complex $(\mathcal{K}_{s \cdot \varphi}^m)^*$ has length N. For $m \leqslant N$ we have

$$H_0((\mathcal{K}_{s \cdot \varphi}^m)^*) = \frac{\Lambda^m F^* \otimes \mathcal{O}}{s \cdot \varphi(G^* \otimes \Lambda^{m-1} F^* \otimes \mathcal{O})}$$

$$H^0(\mathcal{A}^m) = \frac{\Lambda^m F^* \otimes \mathcal{O}}{s \cdot \varphi(G^* \otimes \Lambda^{m-1} F^* \otimes \mathcal{O}) + f(H^* \otimes \Lambda^m F^* \otimes \mathcal{O})}$$

LEMMA 1.3. — *For $m \geqslant 0$ we have*

(1) $\mathcal{A}^m = (\mathcal{K}_{s \cdot \varphi}^m)^* \otimes_{\mathcal{O}} \mathcal{K}_f^*$

(2) *If $f_1, \ldots, f_k$ is a regular $\mathcal{O}$-sequence then $\mathcal{A}^m$ is quasi-isomorphic to the complex $(\mathcal{K}_{s \cdot \varphi}^m)^* \otimes_{\mathcal{O}} \mathcal{O}_V$ and $H_j(\mathcal{A}^m) = 0$ for $j > \min(N, m)$.*

Proof. —

(1) Over $\mathbb{P}^{\ell-1}$, the m-th column $\mathcal{K}_{(s\cdot\varphi,-f^t)} \otimes \mathcal{O}(m-1)$ of the double-complex $\widetilde{\mathcal{G}}$ is a tensor product of two Koszul complexes, $[\mathcal{K}_{s\cdot\varphi} \otimes \widetilde{\mathcal{O}}(m-1)] \otimes_{\widetilde{\mathcal{O}}} \mathcal{K}_f$, due to the direct sum decomposition of $\mathcal{V}$ in (1.7). Since the second complex is independent of the variables of $\mathbb{P}^{\ell-1}$, the tensor product can be taken over $\mathcal{O}$. Taking global sections and then dualizing we obtain $\mathcal{A}^m = (\mathcal{K}_{s\cdot\varphi}^m)^* \otimes_{\mathcal{O}} \mathcal{K}_f^*$ as complexes.

(2) The homology of $\mathcal{A}^m = (\mathcal{K}_{s\cdot\varphi}^m)^* \otimes \mathcal{K}_f^*$ can be computed from the double complex where we put in the horizontal axis the complex $(\mathcal{K}_{s\cdot\varphi}^m)^*$ and on the vertical the complex $\mathcal{K}_f^*$. If we compute the spectral sequence where we do first the vertical homology, we obtain by Koszul's Theorem that the only homology group is at $j = 0$, where the homology complex is $(\mathcal{K}_{s\cdot\varphi}^m)^* \otimes \mathcal{O}_V$. Since the spectral sequence degenerates, we obtain that the homology of this complex computes the homology of the double complex. This proves the first statement. Now the second statement is immediate from this, since the complex $(\mathcal{K}_{s\cdot\varphi}^m)^* \otimes \mathcal{O}_V$ has length $\min\{N, m\}$ which is shorter than the complex $(\mathcal{K}_{s\cdot\varphi}^m)^* \otimes \mathcal{K}_f^*$, so the last homology groups of the larger complex $(\mathcal{K}_{s\cdot\varphi}^m)^* \otimes \mathcal{K}_f^* = \mathcal{A}^m$ vanish. $\square$

LEMMA 1.4. — *Assume that $n := N - \ell \geqslant 0$ and that the depth of $I_\ell(\varphi) = n + 1$, the greatest possible value, then we have:*

(1) *For $m = 0, \ldots, n + 1$ the complex $(\mathcal{K}_{s\cdot\varphi}^m)^*$ is a resolution of $H_0\big((\mathcal{K}_{s\cdot\varphi}^m)^*\big)$.*

(2) *If $f_1, \ldots, f_k$ is a regular sequence in $\mathcal{O}$ then $\mathcal{A}^m$ is a resolution for $m \leqslant n - k + 1$. For $m = n - k + 2, \ldots, n + 1$ we have $H_j(\mathcal{A}^m) = 0$ for $j > m - n + k - 1$.*

(3) *If in addition $\mathcal{O}$ is a local Cohen-Macaulay ring and $f_1, \ldots, f_r$ is an $\mathcal{O}/I_\ell(\varphi)$-regular sequence then $\mathcal{A}^m$ is a resolution for $m \leqslant n - k + r + 1$ and for $m = n - k + r + 2, \ldots, n + 1$ we have $H_j(\mathcal{A}^m) = 0$ for $j > m - n + k - r - 1$.*

Proof. —

(1) For $m = 0, \ldots, n$ we glue the complexes $\mathcal{K}_{s\cdot\varphi}^{n-m}$ on the left with $(\mathcal{K}_{s\cdot\varphi}^m)^*$ on the right using the splicing map $\varepsilon : \Lambda^m F^* \cong \Lambda^{n-m+\ell} F \longrightarrow \Lambda^{n-m} F$ which is contraction by $\Lambda^\ell \varphi^t$:

$$0 \longleftarrow \mathcal{K}_{s\cdot\varphi}^{n-m} \overset{\varepsilon}{\longleftarrow} (\mathcal{K}_{s\cdot\varphi}^m)^* \longleftarrow 0.$$

The complexes so obtained are called $\mathcal{C}^{n-m}$ in [1, A2.6]. There D. Eisenbud also defines $\mathcal{C}^{n-m} = \mathcal{K}_{s\cdot\varphi}^{n-m}$ for $m \leqslant -1$ and $\mathcal{C}^{n-m} =$

$(\mathcal{K}^m_{s \cdot \varphi})^*$ for $m \geqslant n + 1$. The length of the complexes $\mathcal{C}^{n-m}$ is $n + 1$ for $m = -1, \dots, n + 1$.

The Buchsbaum-Eisenbud Theorem [1, Theorem A2.10, p.594] applied to φ asserts that under our hypothesis the complex $\mathcal{C}^{n-m}$ is a free resolution of $H_0(\mathcal{C}^{n-m})$, for $m \leqslant n + 1$.

If we cut the complex $\mathcal{C}^{n-m}$ at the splicing map, we obtain that for $m = 0, \dots, n + 1$ the complex $(\mathcal{K}^m_{s \cdot \varphi})^*$ is a free resolution of its 0-homology module.

(2) For $m = 0, \dots, n + 1$ consider the double complex $\mathcal{C}^{n-m} \otimes \mathcal{K}^*_f$ with horizontal axis $i = 0, \dots, n + 1$ and vertical axis $j = 0, \dots, k$. Consider the spectral sequence where we first do vertical homology. By Koszul's Theorem we only have non-zero terms for $j = 0$, where the homology is $\mathcal{C}^{n-m} \otimes \mathcal{O}_V$. Hence the spectral sequence degenerates, the total complex $\mathcal{C}^{n-m} \otimes \mathcal{K}^*_f$ is quasi-isomorphic to $\mathcal{C}^{n-m} \otimes \mathcal{O}_V$ and the only non-zero homology groups of the total complex are in $j = 0, \dots, n + 1$.

Now we do the other spectral sequence, doing first the horizontal homology. Again by the Buchsbaum-Eisenbud Theorem we obtain that the only non-vanishing terms are in $i = 0$ where we obtain the homology complex $H_0(\mathcal{C}^{n-m}) \otimes \mathcal{K}^*_f$. So again the spectral sequence degenerates and $\mathcal{C}^{n-m} \otimes \mathcal{K}^*_f$ is quasi-isomorphic to $H_0(\mathcal{C}^{n-m}) \otimes \mathcal{K}^*_f$. Both spectral sequences together give $H_j(\mathcal{C}^{n-m} \otimes \mathcal{K}^*_f) = 0$ for $j > \min\{k, n + 1\}$.

If we cut the complex $\mathcal{C}^{n-m}$ at the splicing map ε, the right hand side is $(\mathcal{K}^m_{s \cdot \varphi})^*$. The double complexes $\mathcal{C}^{n-m} \otimes \mathcal{K}^*_f$ and $\mathcal{A}^m = (\mathcal{K}^m_{s \cdot \varphi})^* \otimes \mathcal{K}^*_f$ coincide on the columns to the right of the splicing map. Since $\mathcal{C}^{n-m}$ has a complex of length $n - m$ left of the splicing map, the 0-th column of the cut double complex is the $(n-m+1)$-st column of the complex $\mathcal{C}^{n-m}$.

Doing for both double complexes the vertical homology first, both spectral sequences degenerate, with $\mathcal{C}^{n-m} \otimes \mathcal{O}_V$ and $(\mathcal{K}^m_{s \cdot \varphi})^* \otimes \mathcal{O}_V$ respectively in the 0th row. Now, both of these complexes coincide to the right of the splicing map, so that we have

$$H_j(\mathcal{A}^m) = H_j((\mathcal{K}^m_{s \cdot \varphi})^* \otimes \mathcal{O}_V) = H_{j+n-m+1}(\mathcal{C}^{n-m} \otimes \mathcal{O}_V) \text{ for } j > 0.$$

Hence by the vanishing above, $\mathcal{A}^m$ is a resolution of $H_0(\mathcal{A}^m)$ if $n - m + 2 > k$, i.e. $m \leqslant n - k + 1$.

For $m = n - k + 2, \dots, n + 1$ we still have $H_j(\mathcal{A}^m) = 0$ for $n - m + 1 + j > k$, i.e. $j > m - n + k - 1$.

(3) Under the hypothesis that $\mathcal{O}$ is Cohen-Macaulay, we have by [1, Corollary A2.13, p. 599] that $H_0(\mathcal{C}^{n-m})$ is a maximal Cohen Macaulay $\mathcal{O}/I_\ell(\varphi)$-module for $m = 0, \ldots, n + 1$. The assumption that $f_1, \ldots, f_r$ is a $\mathcal{O}/I_\ell(\varphi)$-regular sequence implies that $f_1, \ldots, f_r$ is also a $H_0(\mathcal{C}^{n-m})$-regular sequence by [1, Proposition 21.9, p.529]. This then means that $H_i\big(H_0(\mathcal{C}^{n-m}) \otimes \mathcal{K}_f^*\big) = 0$ for $i > k - r$ by Koszul's Theorem. Repeating the argument at the end of Part 2 of this lemma, we obtain that $\mathcal{A}^m$ is a resolution for $n - m + 2 > k - r$ and that for $m = n - k + r + 2, \ldots, n + 1$ we have $H_j(\mathcal{A}^m) = 0$ for $j > m - n + k - r - 1$. $\qquad\square$

PROPOSITION 1.5. — *Let* $\mathrm{tot}(\mathcal{G})$ *be the total complex of the Gobelin* $\mathcal{G}$ *constructed from the identity* (1.1) *over the local* $\mathbb{K}$-*algebra* $\mathcal{O}$, *and*

$$H_0(\mathcal{A}) : 0 \longleftarrow H_0(\mathcal{A}^0) \longleftarrow H_0(\mathcal{A}^1) \longleftarrow \cdots \longleftarrow H_0(\mathcal{A}^N) \longleftarrow 0$$

the complex induced by taking vertical homology. Then one has the following statements:

(1) *If* $f_1, \ldots, f_k$ *is an* $\mathcal{O}$ *regular sequence,* $N \geqslant \ell$ *and the depth of* $I_\ell(\varphi) = N - \ell + 1$, *the greatest possible value, then*

$$H_i\big(\mathrm{tot}(\mathcal{G})\big) = H_i\big(H_0(\mathcal{A})\big) \text{ for } i \leqslant N - \ell - k + 1.$$

(2) *If in addition* $\mathcal{O}$ *is Cohen-Macaulay and* $f_1, \ldots, f_r$ *is an* $\mathcal{O}/I_\ell(\varphi)$-*regular sequence, then*

$$H_i(\mathrm{tot}(\mathcal{G})) = H_i(H_0(\mathcal{A})) \text{ for } i \leqslant N - \ell - k + r + 1.$$

Proof. — We look at the spectral sequence, where we do vertical homology first. By Lemma 1.4 the vertical strands $\mathcal{A}^m$ have nonzero homology only in step 0 for $m \leqslant N - \ell - k + 1$ and for $m \leqslant N - \ell - k + r + 1$ using the stronger hypotheses. Since this spectral sequence converges to the homology of the total Gobelin $\mathrm{tot}(\mathcal{G})$, this proves the claim. $\qquad\square$

THEOREM 1.6. — *In the situation of Proposition 1.5 Part 2 for* $i < N - \ell - k + r + 1$ *consider the finite double complexes* $\mathcal{G}^{\mathrm{cut}}_{\leqslant i} \subset \mathcal{G}$ *obtained by considering only the columns* $\mathcal{A}^m$ *for* $m = 0, \ldots, i$. *Then* $\mathrm{tot}(\mathcal{G}^{\mathrm{cut}}_{\leqslant i})$ *is quasi-isomorphic to*

$$H_0(\mathcal{A})_{\leqslant i} : 0 \longleftarrow H_0(\mathcal{A}^0) \longleftarrow H_0(\mathcal{A}^1) \longleftarrow \cdots \longleftarrow H_0(\mathcal{A}^i) \longleftarrow 0$$

Proof. — By Lemma 1.4 all vertical strands $\mathcal{A}^m$ of $\mathcal{G}^{\mathrm{cut}}_{\leqslant i}$ are resolutions. $\qquad\square$

2. Comparing Homological Index and GSV Index

Let V be a germ of a complete intersection variety having an isolated singularity at 0 defined by the vanishing of the germs of holomorphic functions $f_1, \ldots, f_k$ in $\mathbb{C}^N$, $N > k$, and X a germ of a holomorphic vector field having an isolated zero at 0 in $\mathbb{C}^N$ and tangent to V, i.e. $X(f) = c \cdot f$ where c is the $k \times k$ matrix of cofactors. If we denote by φ the Jacobi matrix describing the differential df of $f \colon \mathbb{C}^N \to \mathbb{C}^k$ we obtain the matrix equality $\varphi \cdot X = c \cdot f$. Denote by $\mathcal{G}$ the Gobelin constructed from this equality. Note that in this situation $F^* \otimes \mathcal{O} = \Omega^1_{\mathbb{C}^N}$ and $H = G$ are vector spaces of the same dimension k.

PROPOSITION 2.1. — *In this situation set $n := \dim V = N - k$ and let $\mathcal{G}^{\mathrm{cut}} := \mathcal{G}^{\mathrm{cut}}_{\leqslant n}$ be the subcomplex consisting of the first $n + 1$ columns of $\mathcal{G}$. Then*

$$\mathrm{Ind}_{\mathrm{hom}}(X; V, 0) = \chi(\mathrm{tot}(\mathcal{G}^{\mathrm{cut}})).$$

Proof. — We want to apply Theorem 1.6. For this let $\mathcal{O}$ be the ring of convergent power series in N variables. Since V is a complete intersection $f_1, \ldots, f_k$ is an $\mathcal{O}$-regular sequence. Now $I_k(\varphi)$ describes the critical locus C_f of f. Since V has an isolated singularity the image of C_f is a hypersurface in $\mathbb{C}^k$. The critical locus C_f has therefore codimension $N - k + 1$ in $\mathbb{C}^N$ and depth $I_k(\varphi) = N - k + 1$ has the maximal possible value. Moreover $\mathcal{O}$ is Cohen-Macaulay. Since V has an isolated singularity, the codimension of the singular locus in the critical locus is $k - 1$. After a holomorphic base change we can therefore assume that $f_1, \ldots, f_{k-1}$ is a regular sequence in $\mathcal{O}/I_k(\varphi)$. So we can apply Theorem 1.6 with $r = k - 1$ and obtain that $\mathcal{G}^{\mathrm{cut}}$ is quasi-isomorphic to

$$H_0(\mathcal{A})_{\leqslant n} \colon 0 \longleftarrow H_0(\mathcal{A}^0) \longleftarrow H_0(\mathcal{A}^1) \longleftarrow \cdots \longleftarrow H_0(\mathcal{A}^n) \longleftarrow 0$$

By Lemma 1.3 the vertical strand $\mathcal{A}^m$ in the Gobelin $\mathcal{G}$ is quasi-isomorphic to the complex $(\mathcal{K}^m_{s \cdot \varphi})^* \otimes \mathcal{O}_V$ which is equal to

$$0 \leftarrow \Omega^m \otimes \mathcal{O}_V \xleftarrow{\wedge df} \Omega^{m-1} \otimes \mathcal{O}_V \xleftarrow{\wedge df} \cdots$$

in our situation. We obtain

$$H_0(\mathcal{A}^m) = \frac{\Omega^m}{df \wedge \Omega^{m-1}} \otimes \mathcal{O}_V = \Omega^m_V$$

and an equality of complexes $\Omega_V = H_0(\mathcal{A})_{\leqslant n}$ since both are given by contraction with X. Since $\mathrm{Ind}_{\mathrm{hom}}(X; V, 0) := \chi(\Omega_V)$ the claim follows. $\square$

PROPOSITION 2.2. — *Let X_λ be a holomorphic family of germs of holo-morphic vector fields in $\mathbb{C}^{n+k}$ with isolated singularities and tangent to the complete intersections $V_\lambda := f^{-1}(\alpha(\lambda))$,*

$$d(f - \alpha(\lambda)).X_\lambda = c_\lambda.(f - \alpha(\lambda))$$

with 0 an isolated singularity for V_0, and the other V_λ smooth. Then we have

$$\mathrm{Ind}_{\mathrm{hom}}(X_0; V, 0) = \sum_{X_\lambda(p_{\lambda,j})=0} \mathrm{Ind}_{\mathrm{hom}}(X_\lambda; V_\lambda, p_{\lambda,j}) =$$

$$= \sum_{X_\lambda(p_{\lambda,j})=0} \mathrm{Ind}_{\mathrm{GSV}}(X_\lambda; V_\lambda, p_{\lambda,j}) = \mathrm{Ind}_{\mathrm{GSV}}(X_0; V_0, 0)$$

Proof. — Consider the family of cut Gobelins $\mathcal{G}_\lambda^{\mathrm{cut}}$ constructed for (X_λ, c_λ), choose a conveniently small ball U around 0 and denote the double complex of sheaves on U obtained from the Gobelins by $\mathcal{G}_{U,\lambda}^{\mathrm{cut}}$. The free $\mathcal{O}_U$ modules in the Gobelin are independent of λ and the morphisms are dependent on λ. By Proposition 2.1 the Euler characteristic of of $\mathrm{tot}(\mathcal{G}_0^{\mathrm{cut}})$ is the homological index of X_0 at 0.

Now for $\lambda \neq 0$, since V_λ is smooth, the complex has only non-zero homol-ogy at degree 0 and its dimension is equal to $\sum\limits_{X_\lambda(p_{\lambda,j})=0} \mathrm{Ind}_{\mathrm{GSV}}(X_\lambda; V_\lambda, p_{\lambda,j})$, since at smooth points the homological and the GSV-index coincide by Koszul's Theorem. Hence the Euler characteristic of $\mathrm{tot}(\mathcal{G}_\lambda^{\mathrm{cut}})$ equals the above sum, for $\lambda \neq 0$.

Now the family of holomorphic complexes of sheaves $\mathrm{tot}(\mathcal{G}_{U,\lambda}^{\mathrm{cut}})$ on U is formed by free sheaves on U having cohomology sheaves supported on $\{X_\lambda = 0\}$ and hence the projection of the supports to λ is a finite map. These are the hypothesis needed in the Theorem from [3], and we obtain as a conclusion that the Euler characteristics of the complexes coincide for 0 and for small values of λ. □

PROPOSITION 2.3. — *Let V_λ be a holomorphic family of complete in-tersection germs defined by $f_\lambda = (f_{1,\lambda}, \ldots, f_{k,\lambda})^t$ in $\mathbb{C}^N$, such that V_0 has an isolated singularity in 0. Then there exists a holomorphic family of holo-morphic vector fields X_λ tangent to V_λ such that for λ small enough X_λ has isolated singularities in $\mathbb{C}^N$ near 0.*

Proof. — Let $U \subset \mathbb{C}^N$ be an open neighbourhood of 0 where all compo-nents of f_λ are convergent for small λ. From now on we denote by V_λ and X_λ representatives in U.

A vector field X_λ is tangent to V_λ if there exists a $k \times k$ matrix c_λ such that

$$\varphi_\lambda X_\lambda = c_\lambda f_\lambda$$

with φ_λ the matrix of partial derivatives of f. Collecting the coefficients of X_λ and c_λ in one column, we obtain the matrix equation

$$\left(\varphi_\lambda \left| \begin{array}{ccc} f_\lambda^t & & \\ & \cdots & \\ & & f_\lambda^t \end{array} \right. \right) \begin{pmatrix} X_\lambda \\ -c_{\lambda,11} \\ -c_{\lambda,12} \\ \vdots \\ -c_{\lambda,kk} \end{pmatrix} = 0$$

We write $(\varphi_\lambda \otimes 1 \,|\, E_k \otimes f_\lambda^t)$ for the left-hand matrix. Note that we have

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

for matrices A, B, C, D of appropriate sizes. Using this we have

$$\left(\varphi_\lambda \otimes 1 \,\middle|\, E_k \otimes f_\lambda^t \right) \left(\begin{array}{c|c} E_N \otimes f_\lambda^t & \psi_\lambda \\ \hline -\varphi_\lambda \otimes E_k & 0 \end{array} \right) = 0$$

where ψ_λ is the $N \times \binom{N}{k+1}$-matrix from the Buchsbaum-Rim complex $\mathcal{C}^1(\varphi_\lambda)$ presenting the kernel of φ_λ. (This holds since in our situation φ_λ drops rank in expected codimension.) With

$$M_\lambda = \left(E_N \otimes f_\lambda^t \,\middle|\, \psi_\lambda \right)$$

and $w \in \mathbb{C}^{Nk+\binom{N}{k+1}} =: W$ we obtain a family $X_{\lambda,w} := M_\lambda \cdot w$ that is tangent to all V_λ. We now prove that there exists a $w \in W$ such that $X_{0,w}$ has isolated singularities in U. This is done by a dimension count. Consider the incidence variety

$$I = \{(a, w) \,|\, M_0(a)w = 0\} \subset U \times W$$

and the natural projections

$$I \xrightarrow{\;q\;} W$$
$$\big\downarrow{\scriptstyle p}$$
$$U$$

We have three cases. For $a \in U - V_0$ at least one component of $f_\lambda(a)$ is non zero and therefore $M_0(a)$ has full rank N, the codimension of $p^{-1}(a)$ is $N = \dim(U - V_0)$ and

$$\dim p^{-1}(U - V_0) = \dim W.$$

Let S be the singular set of V_0. If $a \in V_0 - S$ then $\varphi_0(a)$ has full rank k. By Buchsbaum-Rim ψ_0 also presents the kernel of φ_0 and therefore $\operatorname{rank} M_0(a) = \operatorname{rank} \psi_0(a) = N - k = \dim(V_0 - S)$. It follows that

$$\dim p^{-1}(V_0 - S) = \dim W.$$

Finally for $a \in S$ we do not know anything, but since S is finite we still have

$$\dim p^{-1}(S) \leqslant \dim W.$$

This proves that $\dim I = \dim W$ and therefore either q is not surjective (and the generic fiber empty) or the generic fiber of q is discrete. Now

$$q^{-1}(w) = \{a \in U \mid M_0(a)w = X_{0,w}(a) = 0\}$$

and consequently for generic w, $X_{0,w}$ has only isolated singularities in U. By semicontinuity this holds also for $X_{\lambda,w}$ with λ small enough. $\qquad \square$

Theorem 2.4. — *Let V be a holomorphic complete intersection germ defined by $f_\lambda = (f_{1,\lambda}, \ldots, f_{k,\lambda})^t$ in $\mathbb{C}^N$ with an isolated singularity in 0. If X is a holomorphic vector field tangent to V such that X has an isolated singularity at 0, then*

$$\operatorname{Ind}_{\mathrm{hom}}(X; V, 0) = \operatorname{Ind}_{\mathrm{GSV}}(X; V, 0)$$

Proof. — By [2] the difference

$$\operatorname{Ind}_{\mathrm{hom}}(X; V, 0) - \operatorname{Ind}_{\mathrm{GSV}}(X; V, 0)$$

is a constant that depends on V but not on X. By Proposition 2.2 and Proposition 2.3 there exists an X' such that this difference is zero. $\qquad \square$

3. The small Gobelin

In this section we will consider the spectral sequence that takes horizontal homology of the Gobelin first.

3.1. The Horizontal Complexes $\mathcal{B}^m$ of the Gobelin

Denote by $\mathcal{B}^m$ the complex obtained from the mth row of the Gobelin. The complexes $\mathcal{B}^m$ are the tensor product of two complexes, arising from the direct sum decomposition of the middle module in the syzygy (1.6). One is the part of the Buchsbaum-Eisenbud strand of the Koszul complex associated to the map

$$c \colon H \otimes \mathcal{O} \longrightarrow G \otimes \mathcal{O}$$

above the splicing map and the other is the Koszul complex associated to the sequence $X_1, \ldots, X_N$. In this subsection we apply the Buchsbaum-Eisenbud and Koszul Theorems to describe the homologies of the horizontal complexes in the Gobelin.

Denote by $\mathcal{K}_X$ the Koszul complex obtained from the morphism $X \colon \mathcal{O} \longrightarrow F \otimes \mathcal{O}$ and by $\mathcal{K}_X^*$ its dual:

$$0 \longleftarrow \mathcal{O} \longleftarrow F^* \otimes \mathcal{O} \longleftarrow \Lambda^2 F^* \otimes \mathcal{O} \longleftarrow \cdots \longleftarrow \Lambda^N F^* \otimes \mathcal{O} \longleftarrow 0$$

whose morphisms are contractions with $X^t = (X_1, \ldots, X_N)$. Its 0-homology group is

$$H_0(\mathcal{K}_X^*) = \mathbb{B} := \frac{\mathcal{O}}{(X_1, \ldots, X_N)}.$$

If $X_1, \ldots, X_N$ is a regular $\mathcal{O}$-sequence, then by Koszul's Theorem $H_j(\mathcal{K}_X^*) = 0$ for $j > 0$.

Consider the Koszul complex $\mathcal{K}_{(s \cdot c)^t}$ obtained from the morphism $(c \cdot s)^t \colon \widetilde{\mathcal{O}} \longrightarrow H^* \otimes \widetilde{\mathcal{O}}(1)$ tensored with $\widetilde{\mathcal{O}}(m-k)$:

$$0 \leftarrow \Lambda^k H^* \otimes \widetilde{\mathcal{O}}(m) \leftarrow \Lambda^{k-1} H^* \otimes \widetilde{\mathcal{O}}(m-1) \leftarrow \cdots \leftarrow \widetilde{\mathcal{O}}(m-k) \leftarrow 0.$$

We denote the complex of its global sections

$$0 \leftarrow S_m G \otimes \Lambda^k H^* \mathcal{O} \leftarrow S_{m-1} G \otimes \Lambda^{k-1} H^* \otimes \mathcal{O} \leftarrow \cdots$$
$$\leftarrow S_{m-k} G \otimes \mathcal{O} \leftarrow 0$$

by $\mathcal{K}_{(s \cdot c)^t}^m$ and by $(\mathcal{K}_{(s \cdot c)^t}^m)^*$ the dual complex

$$0 \leftarrow D_{m-k} G^* \otimes \mathcal{O} \leftarrow \cdots$$
$$\leftarrow D_{m-1} G^* \otimes \Lambda^{k-1} H \otimes \mathcal{O} \leftarrow D_m G^* \otimes \Lambda^k H \otimes \mathcal{O} \leftarrow 0$$

For $m \geqslant k$ these complexes have length k.

LEMMA 3.1. — *For all $m \geqslant 0$ we have*

(1) $\mathcal{B}^m = (\mathcal{K}_{(s \cdot c)^t}^m)^* \otimes_{\mathcal{O}} \mathcal{K}_X^*$

(2) *If $X_1, \ldots, X_N$ is a regular $\mathcal{O}$-sequence then the complex $\mathcal{B}^m$ is quasi-isomorphic to the complex $(\mathcal{K}_{(s \cdot c)^t}^m)^* \otimes_{\mathcal{O}} \mathbb{B}$ and $H_j(\mathcal{B}^m) = 0$ for $j > \min(k, m)$.*

Proof. — Same proof as for Lemma 1.3. □

3.2. The Small Gobelin

Over the ring $\mathbb{B} := \mathcal{O}/(X_1, \ldots, X_N)$ the identity $\varphi X = cf$ reduces to $cf = 0$. Using the notation $\widetilde{\mathbb{B}} := \mathbb{B} \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}$ and $\widetilde{\mathbb{B}}(1) := \mathbb{B} \otimes \mathcal{O}_{\mathbb{P}^{\ell-1}}(1)$ it gives rise to a smaller syzygy

$$\widetilde{\mathbb{B}} \xrightarrow{(s \cdot c)^t} H^* \otimes \widetilde{\mathbb{B}}(1) \xrightarrow{-f^t} \widetilde{\mathbb{B}}(1).$$

Weaving the two associated Koszul complexes, taking global sections and dualizing we obtain a smaller Gobelin, which we call $\mathcal{G}_{\mathbb{B}}$. Its colums are $\mathcal{A}_{\mathbb{B}}^m = D_m G^* \otimes \mathcal{K}_f^* \otimes \mathbb{B}$ and its rows are $\mathcal{B}_{\mathbb{B}}^m = (\mathcal{K}_{(s \cdot c)^t}^m)^* \otimes \mathbb{B}$.

THEOREM 3.2. — *If $X_1, \ldots, X_N$ is an $\mathcal{O}$-regular sequence, then the total Gobelin complexes $\mathrm{tot}(\mathcal{G})$ and $\mathrm{tot}(\mathcal{G}_{\mathbb{B}})$ are quasi-isomorphic.*

Proof. — By Part 2 of Lemma 3.1 the horizontal rows $\mathcal{B}_{\mathbb{B}}^m$ and $\mathcal{B}^m$ are quasi-isomorphic. Hence the horizontal homology of $\mathcal{G}$ and $\mathcal{G}_{\mathbb{B}}$ coincides. By construction the vertical maps between these homologies are also the same. Consequently the spectral sequence of $\mathcal{G}$ and of $\mathcal{G}_{\mathbb{B}}$ where we do the horizontal homology first shows that the two total complexes are quasi-isomorphic. □

Note that the small Gobelin $\mathcal{G}_{\mathbb{B}}$ is much simpler than the big Gobelin $\mathcal{G}$ (see Fig. 3.1).
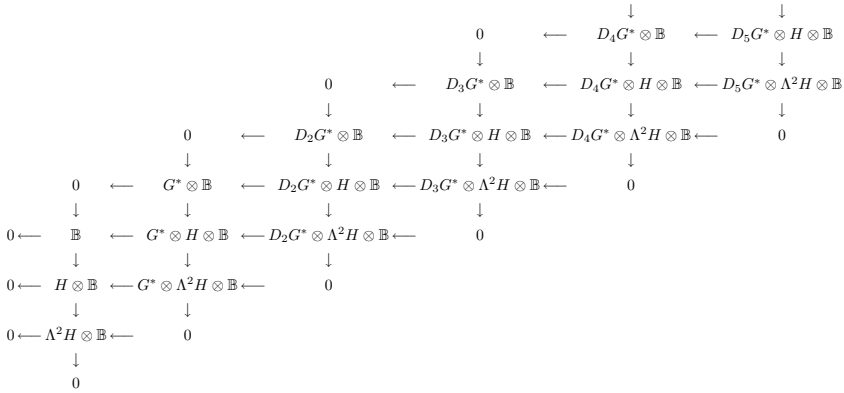

Figure 3.1. *The lower left hand part of the small Gobelin $\mathcal{G}_{\mathbb{B}}$ for $N = 4$, $k = 2$, $l = 2$ beginning at $(0,0)$.*

PROPOSITION 3.3. — *If $k = l$ then $\mathrm{rank}_{\mathbb{B}}(\mathrm{tot}\,\mathcal{G}_{\mathbb{B}})_i = \binom{k+i-1}{i}$.*

Proof. — We have

$$(\mathrm{tot}\,\mathcal{G}_{\mathbb{B}})_i = \bigoplus_{j=0}^{i} (\mathcal{G}_{\mathbb{B}})_{j,i-j} = \bigoplus_{j=0}^{i} D_j G^* \otimes \bigwedge^{k+j-(i-j)} H \cong \bigoplus_{j=0}^{i} D_j G^* \otimes \bigwedge^{i-2j} H$$

A basis of $D_j G^*$ is given by monomials x^α of degree j in k variables. A basis of $\bigwedge^{i-2j} H$ is given by square free monomials x^β of degree $i - 2j$ in k variables. Let S_i be the $\mathbb{B}$-module spanned by all monomials of degree i in k variables. We then have a natural map of free $\mathbb{B}$-modules:

$$\mu\colon \quad (\text{tot}\,\mathcal{G}_{\mathbb{B}})_i \quad \to \quad S_i$$
$$x^\alpha \otimes x^\beta \quad \mapsto \quad x^{2\alpha + \beta}$$

For an inverse of this map let x^α with $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a monomial of degree $i = |\alpha|$ in k variables. Consider the parity-function

$$\sigma(\alpha_l) = \begin{cases} 0 & \text{if } \alpha_l \text{ is odd} \\ 1 & \text{if } \alpha_l \text{ is odd} \end{cases}$$

Then $x^{\alpha - \sigma(\alpha)}$ has only even exponents while $x^{\sigma(\alpha)}$ is square free. The inverse of μ is therefore given by

$$\mu^{-1}\colon \quad S_i \quad \to \quad (\text{tot}\,\mathcal{G}_{\mathbb{B}})_i$$
$$x^\alpha \quad \mapsto \quad x^{\frac{1}{2}(\alpha - \sigma(\alpha))} \otimes x^{\sigma(\alpha)}.$$

It follows that the rank of $(\mathcal{G}_{\mathbb{B}})_i$ as $\mathbb{B}$-module is given by the number of monomials of degree i in k variables, i.e. $\binom{k+i-1}{i}$. □

4. Algebraic Formulae

In the geometric situation of Section 2 the ring $\mathbb{B} := \mathcal{O}/(X_1, \ldots, X_N)$ is a finite dimensional $\mathbb{K}$-vector space. Using the small Gobelin we can reduce the calculation of the homological index to linear algebra:

THEOREM 4.1. — *Let $(V, 0) \subset (\mathbb{C}^N, 0)$ be a complete intersection of dimension n defined by a regular sequence $f = (f_1, \ldots, f_k)$ of holomorphic function germs and $X = (X_1, \ldots, X_N)$ the germ of a holomorphic vector field tangent to V in $\mathbb{C}^N$. If V and X have an isolated singularity at 0 then*

$$H_i(\Omega_V) = \begin{cases} H_i(\text{tot}(\mathcal{G}_{\mathbb{B}})) & \text{for } i \leqslant n - 1 \\ \mathcal{O}_V / I_k(\varphi) & \text{for } i = n \end{cases}$$

Proof. — By Theorem 1.6 we have $H_i(\Omega_V) = H_i(\text{tot}(\mathcal{G}^{\text{cut}}_{\leqslant n}))$ for $i \leqslant n$. Since the columns $\mathcal{A}^m$, $m \leqslant n$, of $\mathcal{G}$ and $\mathcal{G}^{\text{cut}}_{\leqslant n}$ are resolutions, taking horizontal cohomology first shows that $H_i(\text{tot}(\mathcal{G}^{\text{cut}}_{\leqslant n})) = H_i(\text{tot}(\mathcal{G}))$ for $i \leqslant n-1$. Theorem 3.2 now gives $H_i(\text{tot}(\mathcal{G})) = H_i(\text{tot}(\mathcal{G}_{\mathbb{B}}))$.

For $i = n$ we denote by $T\Omega_V$ the subcomplex of torsion sheaves of Ω_V and by $\widetilde{\Omega}_V$ the quotient complex of torsion free sheaves. By [7] we have

$T\Omega_V^i = 0$ for $i < n$. Since X and V have an isolated singularity at 0 the last map of $\widetilde{\Omega}_V$ is injective and we have

$$H_n(\Omega_V) = T\Omega_V^n.$$

By [7, 8] we know $T\Omega_V^n = \mathcal{O}_V/I_k(\varphi)$. $\qquad\qquad\qquad\qquad\qquad\square$

The dimension of $\mathcal{O}_V/I_k(\varphi)$ is a numerical invariant of V and is the invariant τ' of Greuel [8].

COROLLARY 4.2. — *In the situation of Theorem 4.1 we have*

$$\mathrm{Ind}_{\mathrm{hom}}(X;V,0) = \left(\sum_{j=0}^{n-2}(-1)^j\binom{k-1+j}{j}\right)\dim\mathbb{B} + (-1)^{n-1}$$

$$\dim\mathrm{coker}\,\gamma_{n-1} + (-1)^n\tau'$$

where γ_{n-1} is the $(n-1)$-st map of $\mathrm{tot}(\mathcal{G}_\mathbb{B})$.

Proof. — Cut the total complex of the small Gobelin at the $(n-1)$-st map to obtain a shorter complex $\mathrm{tot}(\mathcal{G}_\mathbb{B})_{\leqslant n-1}$. We have on the one hand

$$\chi(\mathrm{tot}(\mathcal{G}_\mathbb{B})_{\leqslant n-1}) = \sum_{j=0}^{n-2}(-1)^j\dim H_j(\mathrm{tot}\,\mathcal{G}_\mathbb{B}) + (-1)^{n-1}\dim\ker\gamma_{n-2}$$

and on the other hand

$$\chi(\mathrm{tot}(\mathcal{G}_\mathbb{B})_{\leqslant n-1}) = \sum_{j=0}^{n-2}(-1)^j\mathrm{rank}\,\mathrm{tot}(\mathcal{G}_\mathbb{B})_j + (-1)^{n-1}\dim\mathrm{tot}(\mathcal{G}_\mathbb{B})_{n-1}.$$

Since

$$\dim H_{n-1}(\mathrm{tot}(\mathcal{G}_\mathbb{B})) = \dim\ker\gamma_{n-2} - \dim\mathrm{Im}\,\gamma_{n-1}$$

$$= \dim\ker\gamma_{n-2} - \dim\mathrm{tot}(\mathcal{G}_\mathbb{B})_{n-1} + \dim\mathrm{coker}\,\gamma_{n-1},$$

we obtain the desired formula by Proposition 3.3 and Theorem 4.1. $\qquad\square$

We will now apply this general formula in special situations.

4.1. Codimension 1 in $\mathbb{C}^N$

Here $k = 1$. The total complex of $\mathcal{G}_\mathbb{B}$ is then

$$0 \leftarrow \mathbb{B} \xleftarrow{f} \mathbb{B} \xleftarrow{c} \mathbb{B} \xleftarrow{f} \mathbb{B} \xleftarrow{c} \mathbb{B} \leftarrow \cdots$$

By Theorem 4.1 we have

$$H_i(\Omega_V) = \begin{cases} \mathbb{B}/(f) & \text{if } i = 0 \\ \mathrm{Ann}_\mathbb{B}(c)/(f) & \text{if } i \text{ is even} \\ \mathrm{Ann}_\mathbb{B}(f)/(c) & \text{if } i \text{ is odd} \end{cases}$$

and by Corollary 4.2

$$\operatorname{Ind}_{\text{hom}}(X; V, 0) = \begin{cases} \dim \mathbb{B} - \dim \mathbb{B}/(c) + \tau' & \text{for } n \text{ even} \\ \dim \mathbb{B}/(f) - \tau' & \text{for } n \text{ odd.} \end{cases}$$

We recover thus the results of [2].

4.2. Codimension 2 in $\mathbb{C}^N$

Here $k = 2$. The total complex of $\mathcal{G}_{\mathbb{B}}$ is in this case

$$0 \leftarrow \mathbb{B}^1 \xleftarrow{(f_1 \ f_2)} \mathbb{B}^2 \xleftarrow{\begin{pmatrix} -f_2 & c_{11} & c_{21} \\ f_1 & c_{12} & c_{22} \end{pmatrix}} \mathbb{B}^3 \xleftarrow{\begin{pmatrix} -c_{12} & -c_{22} & c_{11} & c_{21} \\ f_1 & 0 & f_2 & 0 \\ 0 & f_1 & 0 & f_2 \end{pmatrix}}$$

$$\mathbb{B}^4 \xleftarrow{\begin{pmatrix} -f_2 & 0 & c_{11} & c_{21} & 0 \\ 0 & -f_2 & 0 & c_{11} & c_{21} \\ f_1 & 0 & c_{12} & c_{22} & 0 \\ 0 & f_1 & 0 & c_{12} & c_{22} \end{pmatrix}} \mathbb{B}^5 \leftarrow \cdots$$

Or more generally

$$\cdots \leftarrow \mathbb{B}^{2i-1} \xleftarrow{\varphi_i} \mathbb{B}^{2i} \xleftarrow{\psi_i} \mathbb{B}^{2i+1} \leftarrow \cdots$$

with

$$\varphi_i = \begin{pmatrix} -c_{12} & -c_{22} & & & c_{11} & c_{21} \\ & \ddots & \ddots & & & \ddots & \ddots \\ & & -c_{12} & -c_{22} & & & c_{11} & c_{21} \\ \hline f_1 & & & & f_2 \\ & \ddots & & & & \ddots \\ & & \ddots & & & & \ddots \\ & & & f_1 & & & & f_2 \end{pmatrix}, \ \psi_i = \begin{pmatrix} -f_2 & & & c_{11} & c_{21} \\ & \ddots & & & \ddots & \ddots \\ & & -f_2 & & & c_{11} & c_{21} \\ \hline f_1 & & & -c_{12} & -c_{22} \\ & \ddots & & & \ddots & \ddots \\ & & f_1 & & & -c_{12} & -c_{22} \end{pmatrix}.$$

From Corollary 4.2 we obtain

$$\operatorname{Ind}_{\text{hom}}(X; V, 0) = \begin{cases} i(\dim \mathbb{B}) - \operatorname{coker} \psi_i + \tau' & \text{for } n = 2i \text{ even} \\ (1 - i)(\dim \mathbb{B}) + \operatorname{coker} \varphi_i - \tau' & \text{for } n = 2i - 1 \text{ odd.} \end{cases}$$

4.3. Complete Intersection Curves and Surfaces

If $k \geqslant 2$ the total complex of $\mathcal{G}_{\mathbb{B}}$ is

$$0 \leftarrow \mathbb{B} \xleftarrow[\gamma_0]{(f_1 \ f_2 \ \cdots \ f_k)} \mathbb{B}^k \xleftarrow[\gamma_1]{\begin{pmatrix} f_2 & 0 & f_3 & \cdots & 0 & 0 & f_k & c_{1,1} & \cdots & c_{k,1} \\ -f_1 & f_3 & 0 & \cdots & 0 & f_k & 0 & c_{1,2} & \cdots & c_{k,2} \\ 0 & -f_2 & -f_1 & \cdots & f_k & 0 & 0 & c_{1,3} & \cdots & c_{k,3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -f_3 & -f_2 & -f_1 & c_{1,k} & \cdots & c_{k,k} \end{pmatrix}} \mathbb{B}^{\binom{k}{2}+k} \leftarrow \cdots$$

If V is a complete intersection curve then

$$\mathrm{Ind}_{\mathrm{hom}}(X; V, 0) = \dim \mathbb{B}/(f) - \tau'.$$

This extends results of O. Klehn [10]. If V is a complete intersection surface we obtain

$$\mathrm{Ind}_{\mathrm{hom}}(X; V, 0) = \dim \mathbb{B} - \mathrm{coker}(\gamma_1) + \tau'$$

by Corollary 4.2.

4.4. Examples

We conclude with two explicit examples.

Example 4.3. — Consider the homogeneous complete intersection surface $V \subset \mathbb{C}^4$ defined by $x^2 + y^2 + zw = xy + z^2 + w^2 = 0$, and the vector field

$$X = \left(yz - z^2 - w^2, y^2 - xz + \frac{1}{2}yw, x^2 + yz + \frac{3}{4}zw, yw + \frac{1}{4}w^2 \right)$$

which is tangent to V with tangency relation

$$\varphi \cdot X = \begin{pmatrix} 2x & 2y & w & z \\ y & x & 2z & 2w \end{pmatrix} \cdot \begin{pmatrix} yz-z^2-w^2 \\ y^2-xz+\frac{1}{2}yw \\ x^2+yz+\frac{3}{4}zw \\ yw+\frac{1}{4}w^2 \end{pmatrix} = \begin{pmatrix} 2y+w & -2x \\ z & y+\frac{1}{2}w \end{pmatrix} \cdot \begin{pmatrix} x^2+y^2+zw \\ xy+z^2+w^2 \end{pmatrix} = c \cdot f.$$

The total complex of the small Gobelin $\mathcal{G}_{\mathbb{B}}$ constructed from this relation is

$$0 \leftarrow \mathbb{B}^1 \xleftarrow{\left(x^2+y^2+zw \ \ xy+z^2+w^2 \right)} \mathbb{B}^2 \xleftarrow{\begin{pmatrix} -xy-z^2-w^2 & -2y-w & -z \\ x^2+y^2+zw & 2x & -y-1/2w \end{pmatrix}} \mathbb{B}^3 \leftarrow \cdots$$

where $\mathbb{B} = \mathcal{O}/(X_1, \ldots, X_4)$ is a 16-dimensional vector space. We obtain $h_0(\Omega_V) = h_0(\mathcal{G}_{\mathbb{B}}) = 9$ and $h_1(\Omega_V) = h_1(\mathcal{G}_{\mathbb{B}}) = 4$. Since $\tau' = \dim \mathcal{O}_V / I_2(\varphi) = 7$ we have

$$\mathrm{Ind}_{\mathrm{GSV}}(X; V, 0) = \mathrm{Ind}_{\mathrm{hom}}(X; V, 0) = 9 - 4 + 7 = 12.$$

(The calculations for this example were done using the computer algebra system MACAULAY2 [6].)

Example 4.4. — Consider the complete intersection surface $V \subset \mathbb{C}^4$ defined by $x^3 + y^2 + zw = xy + z^2 + w^2 = 0$, and the vector field

$$X = \begin{pmatrix} 768xzw-192xw^2+4x^2-4xy-16zw+4w^2 \\ 24xzw+1152yzw-96xw^2-288yw^2+8xy-6y^2-2z^2+2w^2 \\ -12x^2w+960z^2w-288zw^2-192w^3+xy+7xz-5yz+z^2+w^2 \\ 48x^2w-48xyw+1152zw^2-288w^3-xy+8yz-z^2+5xw-7yw-w^2 \end{pmatrix}$$

which is tangent to V with tangency relation

$$\varphi \cdot X = \begin{pmatrix} 3x^2 & 2y & w & z \\ y & x & 2z & 2w \end{pmatrix} \cdot \begin{pmatrix} 768xzw - 192xw^2 + 4x^2 - 4xy - 16zw + 4w^2 \\ 24xzw + 1152yzw - 96xw^2 - 288yw^2 + 8xy - 6y^2 - 2z^2 + 2w^2 \\ -12x^2w + 960z^2w - 288zw^2 - 192w^3 + xy + 7xz - 5yz + z^2 + w^2 \\ 48x^2w - 48xyw + 1152zw^2 - 288w^3 - xy + 8yz - z^2 + 5xw - 7yw - w^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2304zw - 576w^2 + 12x - 12y & -192w^2 + 4y - z + w \\ 0 & 1920zw - 576w^2 + 12x - 10y + 2z - 2w \end{pmatrix} \cdot \begin{pmatrix} x^3 + y^2 + zw \\ xy + z^2 + w^2 \end{pmatrix} = c \cdot f.$$

Note that the equations for V are not weighted homogeneous. The total complex of the small Gobelin $\mathcal{G}_{\mathbb{B}}$ constructed from this relation is

$$0 \leftarrow \mathbb{B}^1 \xleftarrow{\; (\, x^3 + y^2 + zw \; \; xy + z^2 + w^2 \,)\;}$$

$$\mathbb{B}^2 \xleftarrow{\; \begin{pmatrix} xy + z^2 + w^2 & 2304zw - 576w^2 + 12x - 12y & 0 \\ -x^3 - y^2 - zw & -192w^2 + 4y - z + w & 1920zw - 576w^2 + 12x - 10y + 2z - 2w \end{pmatrix}\;}$$

$$\mathbb{B}^3 \leftarrow \cdots$$

where $\mathbb{B} = \mathcal{O}/(X_1, \ldots, X_4)$ is a 16-dimensional vector space. We obtain $h_0(\Omega_V) = h_0(\mathcal{G}_{\mathbb{B}}) = 9$ and $h_1(\Omega_V) = h_1(\mathcal{G}_{\mathbb{B}}) = 4$. Since $\tau' = \dim \mathcal{O}_V / I_2(\varphi) = 8$ we have

$$\mathrm{Ind}_{\mathrm{Hom}, V}(X) = 9 - 4 + 8 = 13.$$

The calculations for this example were done by the computer algebra system SINGULAR [9] using [5]. We are grateful to A. Frühbis-Krüger for her help.

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