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The discrete Szëgo kernel

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Of concern on this paper are complex-valued functions defined on the *integer lattice* (i.e. the set $\mathbb{Z} \times i\mathbb{Z}$) which are discrete analytic according to the definition given by Ferrand. In particular, we will study a Hilbert space consisting of the boundary values of discrete analytic functions defined on a finite simply connected union of unit squares of the integer lattice (a *simple region*), which is a discrete version of the *Szëgo space*. We will prove that this space admits a reproducing kernel, the discrete Szëgo kernel and will develop a general method to construct it. To sum up, the main merit of this paper is to present by means of an orthogonal projection operator a way to select among boundary values, those that can be extended to an analytic continuation.

Keywords: discrete Szëgo kernel; discrete analytic functions; Szëgo space; Hilbert space

AMS Subject Classification: 30G25; 30C40; 32A25; 31C20

1. Introduction

Discrete analytic functions were introduced by Ferrand for the case of the usual square lattice in the 40s and were studied during the last fifty years in a more general context and setting by Duffin, Zeilberger and Dym, Lovász, Mercat and others. These investigations have centred around a programme which attempts to extend many important properties of the 'usual' analytic functions to discrete 'world'. Almost all the results can be extended to the discrete setting if previously one finds the appropriate discrete version of the notion under study.

On the other hand, in some problems of the theory of integrable systems arises the question of investigating discrete kernel functions, for instance, recently a discrete Green function has received some attention.

In this paper, we consider one more discrete kernel function: the Szëgo kernel for a simple region. Let $\mathbf{R_s}$ be a simple region. As in the classic case, two basic spaces of discrete functions would be definable as well. Consequently, we define the spaces $\mathbf{L^2}(\partial\mathbf{R_s})$ and $\mathbf{L}^2\mathbf{A}(\partial\mathbf{R_s})$. The first of which consisting of all discrete functions that are $\mathbf{L}^2$ over $\partial\mathbf{R_s}$ (i.e. the set of all discrete functions defined over $\partial\mathbf{R_s}$), while the second space is the subspace of $\mathbf{L^2}(\partial\mathbf{R_s})$ of all discrete functions which can be extended to an analytic discrete function in $\mathbf{R_s}$. We show that $\mathbf{L^2A}(\partial\mathbf{R_s})$ makes sense because such an extension to $\mathbf{R_s}$ is unique as it will be seen below. Next, we achieve the result about the existence of reproducing kernel associated to $\mathbf{L^2A}(\partial\mathbf{R_s})$. The essential tool in the construction of a

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discrete Szëgo kernel for a simple region turns out to be a simple description of the geometry of these regions which allows us to introduce and study the concept of order of a simple region. Simple regions are those regions for which a discrete Szëgo kernel can be obtained. The notion of a simple region is very natural and produces two disjoint subsets of $\partial\mathbf{R}_s$ from which we 'read' all the necessary information to construct the discrete Szëgo kernel.

2. Reviews of two related topics

The content of this section is divided into two main parts. In each of these we will give a brief introduction to two topics which we try to relate to this paper. The first part deals with the theory of discrete analytic functions as presented by Duffin [5]. The second part is concerned about the theory of reproducing kernels, and it is based upon the ideas developed by Aronsajn [1].

We do not intend to give a complete exposition of the subjects, but instead we wish for the reader to recall some of the main ideas and results from each theory. Most of the proofs will be omitted. We must encourage the interested reader to search for the references below for a more detailed approach on the matter.

2.1 Discrete analytic functions

In this part, we briefly survey the theory of discrete analytic functions following the ideas introduced by Duffin in his article *Basic properties of discrete analytic functions* [5]. We will denote by i the imaginary unit and for any complex number z, we will write $\bar{z}$ for its complex conjugate.

In this paper, we are concerned with the theory of complex-valued functions defined on the subset of the complex plane $\mathcal{M} = \mathbb{Z} \times \mathrm{i}\mathbb{Z}$, called the *integer lattice*. For any $z_0 \in \mathcal{M}$, we say that the unit square with corners at $\{z_0, z_0 + 1, z_0 + 1 + \mathrm{i}, z_0 + \mathrm{i}\}$ is the *unit square associated to* or *at* z_0.

We recall that a function $f : \mathcal{M} \to \mathbb{C}$ is *discrete analytic* in the unit square associated to z_0 if

$$\frac{f(z_0 + 1 + \mathrm{i}) - f(z_0)}{1 + \mathrm{i}} = \frac{f(z_0 + \mathrm{i}) - f(z_0 + 1)}{\mathrm{i} - 1}. \tag{1}$$

Such functions were first introduced by Ferrand (Lelong) [6] in 1944. Many interesting properties of discrete analytic functions were obtained later by Duffin [5] which are discrete versions of well-known results in the theory of functions of a complex variable. A function which is discrete analytic in every unit square of $\mathcal{M}$ is a *discrete entire function*. In the following, when there is no risk of confusion, the word 'discrete' will be omitted.

We define the following concepts:

1. A *region* of the integer lattice is the union of unit squares.
2. A *simple region* of the integer lattice is a finite union of unit squares such that
   (a) it is simply connected when considering the unit squares as full blocks with its boundary and interior included.
   (b) its boundary is a simple closed (polygonal) curve, composed of edges of unit squares.
3. The points on the boundary of a region are called *boundary points*, and all other points are called *interior points*.

Figure 1 shows an example of a simple region, where the interior points are z_0 and z_1 and all the other points are boundary points. From now on, whenever we have fixed a point $z_0 \in \mathcal{M}$, we will assume that $z_1 = z_0 + 1, z_2 = z_0 + 1 + \mathrm{i}$ and $z_3 = z_0 + \mathrm{i}$. Also, in the following we will use the notation $f_j := f(z_j)$.

Let z_0 be a point in the integer lattice. Duffin defined a discrete analytic function in the square associated to z_0 as follows

$$f_0 + \mathrm{i}f_1 + \mathrm{i}^2 f_2 + \mathrm{i}^3 f_3 = 0. \tag{2}$$

It is easy to see that the definition given by Duffin is equivalent to that of Ferrand. In all the paper, we shall employ the definition as it was used by Duffin.

Next, we introduce the translation operators X and Y, over the set of functions defined on $\mathcal{M}$ as follows

$$X^n f(z) = f(z+n), \quad Y^m f(z) = f(z+\mathrm{i}m) \quad \text{for all } \ z \in \mathcal{M},\ m,\ n \in \mathbb{Z}.$$

Defining the operator L by

$$L = I + \mathrm{i}X - XY - \mathrm{i}Y,$$

we have $Lf(z_0) = f_0 + \mathrm{i}f_1 - f_2 - \mathrm{i}f_3$. Hence, the discrete analytic condition can be written as $Lf(z_0) = 0$.

Let $f : \mathcal{M} \to \mathbb{C}$ be a discrete analytic function in the unit square at z_0 and put $u := \Re(f)$ and $v := \Im(f)$. Then,

$$0 = Lf(z_0) = f_0 + \mathrm{i}f_1 - f_2 - \mathrm{i}f_3 = u_0 + \mathrm{i}v_0 + \mathrm{i}(u_1 + \mathrm{i}v_1) - (u_2 + \mathrm{i}v_2) - \mathrm{i}(u_3 + \mathrm{i}v_3)$$

$$= u_0 - v_1 - u_2 + v_3 + \mathrm{i}(v_0 + u_1 - v_2 - u_3)$$

Thus, we deduce the discrete version of the **Cauchy–Riemann equations**

$$u_2 - u_0 = v_3 - v_1 \quad \text{and} \quad u_3 - u_1 = v_0 - v_2.$$

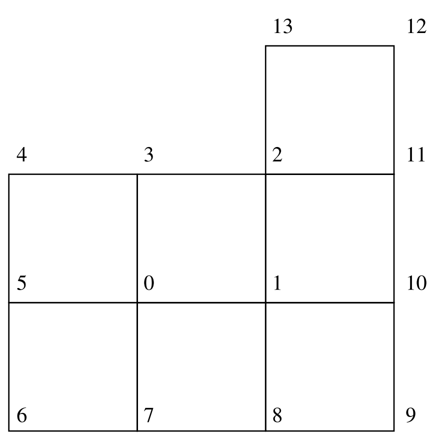

Figure 1. Simple region.

If the operator L' is given by

$$L' = I - iX^{-1} - Y^{-1}X^{-1} + iY^{-1},$$

we define the *laplacian operator* $D = -L'L$,

$$Df(z_0) = -L'Lf(z_0) = -LL'f(z_0) = f_2 + f_4 + f_6 + f_8 - 4f_0. \tag{3}$$

We say that $f : \mathcal{M} \to \mathbb{C}$ is *discrete harmonic* at a point $z_0 \in \mathcal{M}$ if $Df(z_0) = 0$. And from (3) we deduce that if f is discrete analytic in a region $\mathbf{R}$, then f is discrete harmonic in the interior of $\mathbf{R}$.

We recall the following very important result.

Theorem 1 (The Maximum Principle). *Let $\mathbf{R} \subset \mathcal{M}$ a finite region and $f : \mathbf{R} \to \mathbb{C}$ a harmonic function in the interior of $\mathbf{R}$. Then*

$$\max_{z\in\mathbf{R}} |f(z)| = \max_{z\in\partial\mathbf{R}} |f(z)|.$$

i.e. $|f|$ takes its maximum at the boundary of $\mathbf{R}$.

Here, we give a proof of this statement, more in the spirit of this work.

Proof. Asume that f takes its maximum at an interior point z_0 of $\mathbf{R}$. Since f is harmonic at z_0 we have

$$0 \le 4|f_0| - (|f_2| + |f_4| + |f_6| + |f_8|) \le 4|f_0| - |f_2 + f_4 + f_6 + f_8|$$
$$\le |f_2 + f_4 + f_6 + f_8 - 4f_0| = |Df(z_0)| = 0.$$

Hence, $|f_0| = |f_j|$, $j = 2, 4, 6, 8$; in particular, $|f|$ takes its maximum at $z_2 = z_0 + 1 + i$. If $z_2 \in \partial\mathbf{R}$, the result follows. If this is not the case, then by an analog argument $|f|$ reaches its maximum at $z_2 = z_0 + 2 + 2i$. This process must end in a point of the boundary of $\mathbf{R}$, since it is a finite region. This completes the proof. □

Now, We give some information about the discrete theory of integration due to Duffin.

Let $a = z_0, z_1, \ldots, z_m = b$ be a chain of lattice points ($|z_{j+1} - z_j| = 1$, $j = 1, 2, \ldots, m$). For a given function $f : \mathcal{M} \to \mathbb{C}$, we define the *line integral* by

$$\int_a^b f\,dz = \frac{1}{2}\sum_{n=1}^{m} (f_n + f_{n-1})(z_n - z_{n-1}).$$

In the case where we have a closed chain C ($a = b$), it can be shown that

$$\int_C f\,dz = \frac{1}{2}\sum_{n=1}^{m} f_n(z_{n+1} - z_{n-1}).$$

Thus, when C is a unit square we get

$$\int_C f\,dz = \frac{1}{2}[(1 - i)f_0 + (1 + i)f_1 + (i - 1)f_2 + (-1 - i)f_3] = \frac{(1 - i)}{2} Lf(z_0)$$

The proof of the following fact (not given here) is due to Duffin.

Let $\mathbf{R_s}$ be a simple region. Then

$$\int_{\partial \mathbf{R_s}} f \mathrm{d}z = \frac{(1-\mathrm{i})}{2} \sum_{\mathbf{R_s}} Lf, \tag{4}$$

where $\sum_{\mathbf{R_s}}$ denotes the sum over all the unit squares in $\mathbf{R_s}$.

If we compare the expression (4) with the usual case, it is natural to say that $(1+\mathrm{i})/(4\pi)Lf(z)$ is the discrete version of the residue. Although probably for the sake of simplicity, Duffin defined the residue of f at the square associated to z as $Lf(z)$. Also, from the previous proposition it follows that if f es analytic in the simple region $\mathbf{R_s}$, then

$$\int_{\mathbf{R_s}} f \mathrm{d}z = 0.$$

Another interesting result obtained by Duffin deals with the definition of the derivative and it is stated in the theorem below. First, for a given function f we define its *dual function*, denoted by f^-, as

$$f^-(x+\mathrm{i}y) = (-1)^{(x+y)}\bar{f}(x+\mathrm{i}y). \tag{5}$$

Then we have,

Let F be an analytic function in a region $\mathbf{R} \subset \mathcal{M}$. Let $a, b \in \mathbf{R}$ and $k \in \mathbb{C}$ be a constant. Then

$$f(z) = \left\{ 4 \int_b^z F^- \mathrm{d}z + k \right\}^-$$

is analytic in $\mathbf{R}$, and

$$F(z) = \int_a^z f \mathrm{d}z + F(a).$$

All integration paths are assumed to be in $\mathbf{R}$.

The function f given in (5) is called the *derivative* of F, while the function k^- defined by $k^-(x+\mathrm{i}y) = (-1)^{(x+y)}\bar{k}$ is said to be a *biconstant*. Hence, the derivative of a discrete analytic function is unique up to an arbitrary biconstant.

2.2 Reproducing kernels

Kernels functions (for instance, the Green and the Neumann functions) are of wide applicability in physics and mathematics in particular in function theory, partial differential equations, etc. A natural way to construct a kernel function is to express it in terms of a complete orthonormal system. In this case the kernel function is called reproducing kernel. The reproducing kernels are of importance in various fields of physics and mathematics because they make it possible to solve numerically some boundary value and mapping problems.

We begin this subsection by recalling the definition of a reproducing kernel:

DEFINITION 2. *Let $[H, \langle\cdot,\cdot\rangle_H]$ be a Hilbert space whose elements are complex-valued functions defined on a set E. A reproducing kernel for H is a function $K : E \times E \rightarrow \mathbb{C}$ such that*

- *for every $y \in E$, $K(x, y) \in H$ as a function of x and*

- *for every function $f \in H$, we say that K reproduces this function in the sense that*

$$\forall y \in E, \quad f(y) = \langle f(\cdot), K(\cdot, y)\rangle_H.$$

Let us recall a little of the theory of reproducing kernel. It is well known that a reproducing kernel, if it exists, it is uniquely determined by the Hilbert space H. On the other hand, if $K(x, y)$ is a reproducing kernel for the Hilbert space H whose elements are complex-valued functions over a set E, then the following properties hold:

1. $K(x, y) = \langle K(\cdot, y), K(\cdot, x)\rangle$ for all $x, y \in E$.
2. $K(x, y) = \overline{K(y, x)}$ for all $x, y \in E$.
3. $K(y, y) \geq 0$, $\forall y \in E$; and $K(y, y) = 0 \Leftrightarrow K(x, y) = 0$ for all $x \in E$.

It is obvious that the question arise naturally of determining if, given a Hilbert space composed by complex-valued functions over a set E, it exists a reproducing kernel associated to this Hilbert space. The answer can be stated in the following form **(The Existence Criterion)**:

Let H be a Hilbert space composed by complex-valued functions over a set E. Then, H admits a reproducing kernel $K(x, y)$ if and only if for every $y \in E$, $f \mapsto f(y)$ is a bounded linear functional over H.

Definition 3. A positive matrix *defined over a set E is a hermitian function $K : E \times E \rightarrow \mathbb{C}$ such that for every finite set $\{u_j\}_{j=1}^n \subset E$ and arbitrary complex numbers $\{c_j\}_{j=1}^n \subset \mathbb{C}$, we have*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} K(x_k, x_j) \geq 0.$$

The following results will be enounced without a proof. Details can be found in Ref. [1]: Every reproducing kernel is a positive matrix. Moreover, if $K(x, y)$ is a reproducing kernel, then $|K(x, y)|^2 \leq K(x, x)K(y, y)$. On the other hand, let $K(x, y)$ be a positive matrix over a set E. Then it exists a unique Hilbert space H, consisting of complex functions defined on E, that admits K as a reproducing kernel.

Finally we have,

Theorem 4. *Let H be a Hilbert space and $H_K \subset H$ a closed subspace of H. If H_K admits a reproducing kernel $K(x, y)$, then the map $\Pi : H \rightarrow H$ given by*

$$(\Pi f)(y) = \langle f(\cdot), K(\cdot, y)\rangle \quad \text{with } f \in H$$

defines an orthogonal projection operator over H_K.

Now, we present the proof of this result for the sake of completeness of this paper.

Proof. The Hilbert space H can be written as a direct sum $H = H_K \oplus H_K^{\perp}$. That is, every function $f \in H$ has a unique representation as $f = f_K + f_K^{\perp}$, with $f_K \in H_K$ and $f_K^{\perp} \in H_K^{\perp}$. Thus,

$$\begin{aligned}(\Pi f)(y) &= \langle f(\cdot), K(\cdot, y)\rangle = \left\langle f_K + f_K^{\perp}, K(\cdot, y)\right\rangle = \langle f_K, K(\cdot, y)\rangle + \left\langle f_K^{\perp}, K(\cdot, y)\right\rangle \\ &= \langle f_K, K(\cdot, y)\rangle = f_K(y).\end{aligned}$$

Then, $\Pi f = f_K$, i.e. Π is an orthogonal projection. $\square$

The usual Szëgo kernel [7].

Let Ω be a bounded domain of the complex plane such that its boundary is a smooth curve. Let $\mathcal{H}^2(\partial\Omega)$ be the closed subspace of the Hilbert space $\mathcal{L}^2(\partial\Omega)$ consisting of the boundary values of holomorphic functions in Ω. That is

$$\mathcal{H}^2(\partial\Omega) = \left\{\{u \in \mathcal{L}^2(\partial\Omega) | \exists \mathrm{U} : \Omega \to \mathbb{C} \text{ holomorphic with } \mathrm{U}|_{\partial\Omega} = u\right\},$$

which we will call the *Szëgo space* of Ω.

There is a reproducing kernel, called the *Szëgo kernel*, naturally associated to Ω which arises from considering the orthogonal projection $K : \mathcal{L}^2(\partial\Omega) \to \mathcal{H}^2(\partial\Omega)$.

3. The discrete Szëgo kernel

In this section, we will discuss the results obtained concerning the discrete version of the Szëgo kernel. First, we introduce the appropriate $\mathbf{L}^2$ space of functions and the corresponding closed subspace $\mathbf{L^2A}$ that will be the discrete version of the Szëgo space. Then we will study some geometric aspects of simple regions of the grid $\mathcal{M}$. Finally, the fundamental results of this article will be proved.

3.1 Preliminaries

In what follows $\mathbf{R_s}$ will denote a simple region of the grid. Let us fix an arbitrary point $z_1 \in \partial\mathbf{R_s}$. Since the boundary of $\mathbf{R_s}$ is a closed polygonal curve, we can find a closed chain of grid points $\{z_1, z_2, \dots, z_{k+1} = z_1\}$ passing through every vertex of $\partial\mathbf{R_s}$. We will identify the semi-closed chain $\{z_1, z_2, \dots, z_k\}$, which is a way of enlisting the points of $\partial\mathbf{R_s}$, with the boundary of $\mathbf{R_s}$ from here on.

Definition 5. *We define the following space of functions*

$$\mathbf{L^2}(\partial\mathbf{R_s}) := \left\{f : \partial\mathbf{R_s} \to \mathbb{C} : \int_{\partial\mathbf{R_s}} |f(z)|^2 |\mathrm{d}z| < \infty\right\} = \{f : \partial\mathbf{R_s} \to \mathbb{C}\}.$$

with the scalar product

$$\langle f, g\rangle_{L^2} = \int_{\partial\mathbf{R_s}} f(z)\overline{g(z)}|\mathrm{d}z| = \frac{1}{2}\sum_{j=1}^{k} f(z_j)\overline{g(z_j)}|z_{j+1} - z_{j-1}|,$$

where $z_0 = z_k$ y $z_{k+1} = z_1$.

In this definition we introduce a notion of line integral based on the one given by Duffin, only that we consider $|\mathrm{d}z|$ instead of $\mathrm{d}z$.

Since $\partial\mathbf{R_s} = \{z_1, z_2, \dots, z_k\}$ is a finite set, we can think of a function $f \in \mathbf{L^2}(\partial\mathbf{R_s})$ as a complex k-vector

$$f = (f_1, f_2, \dots, f_k), \quad \text{where } f_j = f(z_j),\ j = 1, 2, \dots, k.$$

Thus, we can identify the space $[\mathbf{L^2}(\partial\mathbf{R_s}), \langle\cdot,\cdot\rangle_{L^2}]$ with the Hilbert space $[\mathbb{C}^k, \langle\cdot,\cdot\rangle_{L^2}]$. Therefore, $[\mathbf{L^2}(\partial\mathbf{R_s}), \langle\cdot,\cdot\rangle_{L^2}]$ is a Hilbert space of dimension k. In fact, if

$f, g \in \mathbb{C}^k = \mathbf{L^2}(\partial \mathbf{R_s})$, then $\langle f, g \rangle_{L^2} = f^t Ag$, where A is the diagonal positive matrix

$$A_{ii} = \frac{|z_{i+1} - z_{i-1}|}{2} \in \left\{1, \frac{1}{\sqrt{2}}\right\}.$$

We will denote by Φ_j $(j = 1, 2, \ldots, k)$ the *canonic functions*

$$\begin{cases} \Phi_j : \partial \mathbf{R_s} \to \mathbb{C} \\ \Phi_j(z_i) = \delta_{ij}. \end{cases}$$

which are equivalent to the canonic vectors $\{e_j\}_{j=1}^k$ of $\mathbb{C}^k$.

Definition 6. *We define the discrete Szëgo space, denoted by* $\mathbf{L}^2\mathbf{A}(\partial \mathbf{R_s})$*, as follows*

$$\mathbf{L^2A}(\partial \mathbf{R_s}) = \{f \in \mathbf{L^2}(\partial \mathbf{R_s}) | \exists F : \mathbf{R_s} \to \mathbb{C} \text{ analytic, with } F|_{\partial \mathbf{R_s}} = f\}.$$

That is, $\mathbf{L^2A}(\partial \mathbf{R_s})$ *consists of the functions defined on* $\partial \mathbf{R_s}$ *which can be extended to an analytic function on the interior of* $\mathbf{R_s}$.

Proposition 7. *Let* $f \in \mathbf{L^2A}(\partial \mathbf{R_s})$*, then its analytic extension to* $\mathbf{R_s}$ *is unique.*

Proof. Let us suppose that F and G are analytic extensions of f to $\mathbf{R_s}$. Since F − G is analytic on $\mathbf{R_s}$, it is consequently harmonic. Thus, by the Maximum Principle

$$\max_{\mathbf{R_s}} |F - G| = \max_{\partial \mathbf{R_s}} |F - G| = \max_{\partial \mathbf{R_s}} |f - f| = 0$$

Therefore $F = G$, and the proposition follows. □

Since the set of discrete analytic functions on $\mathbf{R_s}$ is a linear variety and since $\mathbf{L^2}(\partial \mathbf{R_s})$ is finite-dimensional, the subspace $\mathbf{L^2A}(\partial \mathbf{R_s})$ is closed and hence a Hilbert space itself.

Proposition 8. *The Hilbert space* $\mathbf{L^2A}(\partial \mathbf{R_s})$ *admits a reproducing kernel* (*the discrete Szëgo kernel*).

Proof. Let $z_j \in \partial \mathbf{R_s}$, with $j \in \{1, 2, \ldots, k\}$. We define the linear functional

$$\varphi_j : \mathbf{L^2A}(\partial \mathbf{R_s}) \to \mathbb{C} \quad \varphi_j(f) = f(z_j).$$

Then

$$|\varphi_j(f)|^2 = |f(z_j)|^2 \leq 2|f(z_j)|^2 |z_{j+1} - z_{j-1}| \leq \sum_{i=1}^{k} 2|f(z_i)|^2 |z_{i+1} - z_{i-1}| = 4\|f\|^2,$$

that is, $|\varphi_j(f)| \leq 2\|f\|$ for every $f \in \mathbf{L^2A}(\partial \mathbf{R_s})$. Thus, φ_j is a bounded linear functional for every $j \in \{1, 2, \ldots, k\}$. By the existence criterion, it follows that $\mathbf{L^2A}(\partial \mathbf{R_s})$ admits a reproducing kernel. □

3.2 A basis for the discrete Szëgo space $L^2A(\partial R_s)$

Now we will introduce a concept that will be useful in the study of the geometry of simple regions.

DEFINITION 9. *If a finite region* $\mathbf{R}$ *consists of k unit squares, we say that the region has order k and we write* $o(\mathbf{R}) = k$.

LEMMA 10. *Let* $\mathbf{R_s}$ *be a simple region. Then,* $\mathrm{card}(\partial \mathbf{R_s}) = 2n$ *for some* $n \in \mathbb{N}$.

Proof. Note that on a bipartite graph every cycle is of even length. □

In this section, we will determine the discrete Szëgo kernel for an arbitrary simple region $\mathbf{R_s}$. By means of the equation of discrete analiticity (2) given by Duffin, we will find a subset Ω of the boundary of the simple region such that for every function $\hat{f} : \Omega \to \mathbb{C}$, it exists a unique function $f \in \mathbf{L^2A}(\partial \mathbf{R_s})$, with $f|_\Omega = \hat{f}$. The values of f over $\partial \mathbf{R_s} \backslash \Omega$ will be linear combinations of its values on Ω. That is, for every $z \in \Delta := \partial \mathbf{R_s} \backslash \Omega$, there exist complex numbers $\{\alpha_\omega(z)\}_{\omega \in \Omega}$, independent of f, such that

$$f(z) = \sum_{\omega \in \Omega} \alpha_\omega(z) f(\omega). \tag{6}$$

Such a set $\Omega \subset \partial \mathbf{R_s}$ will be said to satisfy the *spanning condition* on the simple region $\mathbf{R_s}$.

We will then show how to use the linear relations of (6) to construct a basis for the Szëgo space, and by means of this basis, we will obtain the Szëgo kernel. Finally, we will show the Szëgo orthogonal projection.

Really, the case to prefer a given set Ω of Cauchy data to another is a matter of debate. For instance, we can take half + 1 of boundary vertices as Ω. In fact, as we can see the non-upper-right boundary vertices is such a 'good half plus one'.

3.2.1 Examples

Before introducing a general method to construct the set Ω defined above, let us consider the following examples.

Example 1. Let us consider a simple region $\mathbf{R_s}$ of order 1 (Figure 2), and let $f \in \mathbf{L^2A}(\partial \mathbf{R_s})$ be an arbitrary function. Then, f is analytic on the sole unit square of $\mathbf{R_s}$; that is, $f_1 + \mathrm{i}f_2 - f_3 - \mathrm{i}f_4 = 0$. Then we can write any of the function's values, say f_3, as a linear

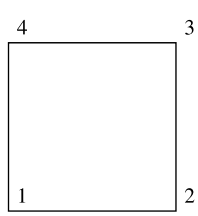

Figure 2. Simple region of order 1.

combination of the other three values:

$$f_3 = f_1 + \mathrm{i}f_2 - \mathrm{i}f_4. \tag{7}$$

In this way, the value f_3 is uniquely determined by this linear combination, whose coeficients do not depend on the function, but only on the region and on Duffin's equation. Thus, a function defined arbitrarily on the vertices z_1, z_2 and z_4 has a unique analytic extension to $\partial\mathbf{R_s}$ ($= \mathbf{R_s}$, in this case) which is defined at z_3 by (7). Thus the set $\Omega = \{z_1, z_2, z_4\}$ clearly satisfies the spanning condition.

Note that in this case we could have chosen any other triplet of vertices to define Ω, so that it is not unique in general. Also, it is clear that Ω cannot contain less than three elements if the analytic extension is to be unique, neither it can consist of the four vertices because not every function defined on a square is discrete analytic there.

Since there are several sets satisfying the spanning condition for a given simple region, we will give a general method to choose one that will only depend on the region. Following the idea suggested by the simple example above we propose the following method.

Method to find Ω: Let $\mathbf{R_s}$ be a simple region of the integer lattice and let $V \equiv V(\mathbf{R_s})$ be its set of vertices. We define the set $\Delta \subset V$ consisting of the upper-right vertices of all the unit squares of $\mathbf{R_s}$. We set $\Omega(\mathbf{R_s}) := V \backslash \Delta$.

It is obvious that a vertex v of a simple region is an interior vertex if and only if the four unit squares of the integer lattice of which it is a vertex belong to the simple region. From the method described above we see that the set $\Omega(\mathbf{R_s})$ consists of all vertices in $\mathbf{R_s}$ that are not a right-upper vertex of any square in the region. Hence, we have the desired property $\Omega(\mathbf{R_s}) \subset \partial\mathbf{R_s}$.

As one can see that the dimension of the Szëgo space is the half the number of boundary points plus one (see the subsection 3.3 for more detail), indeed any subset of the boundary with the right cardinal without upper-right boundary vertices will do when the boundary is simply connected, it is simply discrete Hodge theory. Therefore, the demonstration of the fact that the non-upper-right boundary vertices satisfy the spanning condition is simply stating that their number is $((|\partial\mathbf{R_s}|)/2) + 1$. We do not present a proof of this fact here which can be seen in Ref. [2] where the notion of discrete integrability is put onto firm grounds, although it does not talk about reproducing kernels, which is the topic here.

Note that, even if the subset of boundary points satisfying the spanning condition can be chosen in several ways, one cannot choose ANY subset with cardinal 'half $+1$'. For instance, consider the simple region consisting of two squares, as the one shown in Example 2 (Figure 3). It has 6 boundary points, so the Szëgo space has dimension $6/2 + 1 = 4$. However, we cannot choose the four vertices of one of the unit squares of the simple region to be in our subset of the boundary, since the analyticity may not be

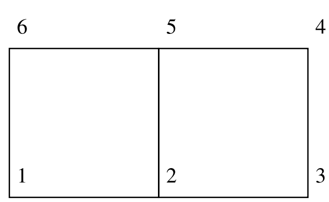

Figure 3. Simple region of order 2.

satisfied in that square and the spanning condition will not hold. This justifies our introduction in this paper of the non-upper-right corner choice of the vertices, even if there is no special reason to choose this method from others that also work.

Now, we would like incorporate two more examples to justify our method.

In the following examples we will show that, in addition, the set $\Omega(\mathbf{R_s})$ satisfies the spanning condition.

Example 2. Let us consider a simple region $\mathbf{R_s}$ of order 2, as the one shown in Figure 3. Let $f \in \mathbf{L^2A}(\partial\mathbf{R_s})$ be an arbitrary function. Since $\mathbf{R_s}$ has no interior points, then f is analytic on both squares of the region (there is no 'analytic extension' to the interior of $\mathbf{R_s}$).

Following our method we find that $\Omega(\mathbf{R_s}) = \{z_1, z_2, z_3, z_6\}$, consisting of the vertices not being upper-right corners of unit squares of $\mathbf{R_s}$. We apply Duffin's equation on the left square of the region, to obtain the value of f at z_5 as a linear combination of the values of f in the vertices of $\Omega(\mathbf{R_s})$:

$$f_5 = f_1 + \mathrm{i}f_2 - \mathrm{i}f_6. \tag{8}$$

Now, we use Duffin's equation in the right square of the region and isolate the value of f at z_4:

$$f_4 = f_2 + \mathrm{i}f_3 - \mathrm{i}f_5. \tag{9}$$

Replacing the value f_5 in Equation (9) by the one found in (8) we obtain

$$f_4 = f_2 + \mathrm{i}f_3 - \mathrm{i}(f_1 + \mathrm{i}f_2 - \mathrm{i}f_6) = -\mathrm{i}f_1 + 2f_2 + \mathrm{i}f_3 - f_6. \tag{10}$$

In this way we are able to represent the values of f on $\Delta = \{z_4, z_5\}$ as a linear combination of its values on Ω. Clearly, a function defined arbitrarily on Ω has a unique extension to all of $\partial\mathbf{R_s} = \mathbf{R_s}$ defined by (8) and (10); thus, the set Ω satisfies the spanning condition.

As it can be seen from the examples above, for rectangular regions our method always puts in Ω the boundary vertices found at the very left column of the region and those found on the lowest row. However, for more general regions the result is slightly different, as it will be shown by the following example.

Example 3. Let $\mathbf{R_s}$ be a simple region of order 9, as the one appearing in Figure 4. Note that the boundary points are $\partial\mathbf{R_s} = \{z_1, z_2, \ldots, z_{14}\}$, while the interior points of the region are z_{15}, z_{16} and z_{17}.

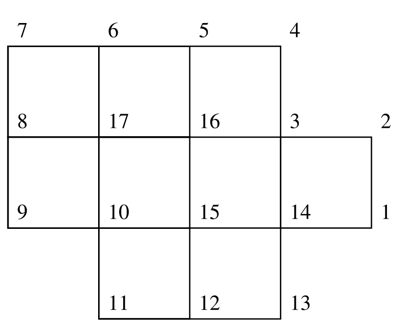

Figure 4. Simple region of order 9.

As before, we define $\Omega \equiv \Omega(\mathbf{R_s})$ as the set of points not being upper-right corners of unit squares of the region, $\Omega = \{z_1, z_7, z_8, \ldots, z_{13}\}$.

Let $f \in \mathbf{L^2A}(\partial\mathbf{R_s})$ be arbitrary. Since f has a unique analytic extension to the interior of $\mathbf{R_s}$, we denote this extension by the same letter f without risk of confusion. We now look for the linear combinations of the values of f on Ω that will determine those on $\Delta = \partial\mathbf{R_s}\backslash\Omega$. We use Duffin's equation at lowest-left unit square of $\mathbf{R_s}$ to isolate the value of f at the upper-right vertex in terms of the other three vertices. Then, we do the same at the next unit square to the right, substituting the linear combination obtained in the previous step. We continue in this way through all the lower unit squares of the region. In this example we have, for the lower-left unit square

$$f_{15} = f_{11} + \mathrm{i}f_{12} - \mathrm{i}f_{10},$$

then to the right of this square we get

$$f_{14} = f_{12} + \mathrm{i}f_{13} - \mathrm{i}f_{15} = f_{12} + \mathrm{i}f_{13} - \mathrm{i}(f_{11} + \mathrm{i}f_{12} - \mathrm{i}f_{10}) = -f_{10} - \mathrm{i}f_{11} + 2f_{12} + \mathrm{i}f_{13}.$$

So we have found the linear combination of the values of f on Ω that determine uniquely its value f_{14}. Of course this linear combination depends only of the region and of Ω; the coefficients being given by Duffin's equation and recursive substitution.

We continue this process in the next row up, from left to right. Using Duffin's equation at each square of this row and subsituting all the values of f (in terms of its values on Ω) that have been found in previous steps. Then we proceed to the next row up, until we have used Duffin's equation once on every unit square of the region. In this way we find the linear combinations desired, and the set Ω satisfies the spanning condition.

In the previous example, we describe the process to obtain the linear combinations of the values of f on Ω that determine uniquely its values on Δ. Note that we apply Duffin's equation once at every unit square of the region. Thus, if we define arbitrarily a function $\hat{f} : \Omega \to \mathbb{C}$ and we extend it to a function $f : \partial\mathbf{R_s} \to \mathbb{C}$ defined at Δ by the linear combinations obtained, we have that $f \in \mathbf{L^2A}(\partial\mathbf{R_s})$ automatically.

3.2.2 Building a basis for $\boldsymbol{L^2A}(\partial\boldsymbol{R_s})$

In this section, we will use the results obtained above to build a basis for the Szëgo space associated to a given simple region $\mathbf{R_s}$. The spanning property of the set $\Omega(\mathbf{R_s})$ will be fundamental in this task. Only first we are compelled to introduce a simple notation to simplify our discussion.

Let $\mathbf{R_s}$ be a fixed simple region of the integer lattice and let $\partial\mathbf{R_s} = \{z_1, z_2, \ldots, z_k\}$ be its boundary (semi-closed chain). We define the set $\Omega(\mathbf{R_s}) \equiv \Omega \subset \partial\mathbf{R_s}$ as explained before and we set $\Delta := \partial\mathbf{R_s}\backslash\Omega$. We introduce the next notation for the disjoint set of indices:

$$I_\Omega = \{j \in \mathbb{N} | z_j \in \Omega\}$$
$$I_\Delta = \{j \in \mathbb{N} | z_j \in \Delta\},$$

Suppose that $f \in \mathbf{L^2A}(\partial\mathbf{R_s})$ is a given function. Then the values of f on Δ are given by a linear combination of its values on Ω, say

$$f_i = \sum_{j \in I_\Omega} \alpha_{ij} f_j, \quad \text{for each } \mathrm{i} \in I_\Delta$$

where the coefficients α_{ij} depend only of the simple region (and the method used to define Ω) and not of the function $f \in \mathbf{L^2A}(\partial\mathbf{R_s})$. We write f in terms of the canonic basis of the Hilbert space $\mathbf{L^2}(\partial\mathbf{R_s})$, $\{\Phi_j\}_{j=1}^k$ as follows:

$$f = \sum_{j=1}^{k} f_j\Phi_j = \sum_{j\in I_\Omega} f_j\Phi_j + \sum_{i\in I_\Delta} f_i\Phi_i = \sum_{j\in I_\Omega} f_j\Phi_j + \sum_{i\in I_\Delta}\left(\sum_{j\in I_\Omega}\alpha_{ij}f_j\right)\Phi_i \tag{11}$$

Now, let us define the following elements of $\mathbf{L^2}(\partial\mathbf{R_s})$:

$$\omega_j = \Phi_j + \sum_{i\in I_\Delta}\alpha_{ij}\Phi_i, \quad j \in I_\Omega$$

The set $\{\omega_j\}_{j\in I_\Omega}$ is linearly independent because the canonic function Φ_j appears only in ω_j for each $j \in I_\Omega$. On the other hand, we have that

$$\sum_{j\in I_\Omega} f_j\omega_j = \sum_{j\in I_\Omega} f_j\left[\Phi_j + \sum_{i\in I_\Delta}\alpha_{ij}\Phi_i\right] = \sum_{j\in I_\Omega} f_j\Phi_j + \sum_{i\in I_\Delta}\left(\sum_{j\in I_\Omega}\alpha_{ij}f_j\right)\Phi_i = f,$$

so that the functions $\{\omega_j\}_{j\in I_\Omega}$ generate the subspace $\mathbf{L^2A}(\partial\mathbf{R_s})$. Hence these functions form a basis of the Szëgo space associated to $\mathbf{R_s}$. It follows that every subset of the boundary of $\mathbf{R_s}$ that satisfies the spanning condition has the same cardinality, since we can define a basis for $\mathbf{L^2A}(\partial\mathbf{R_s})$ indexed by its elements. This cardinality is of course the dimension of the Szëgo space.

Now we are in the position of finding the Szëgo kernel for any given simple region $\mathbf{R_s}$. What we do is apply to the basis $\{\omega_j\}_{j\in I_\Omega}$ defined above the Gram–Schmidt orthonormalization process. Where the scalar product is not the usual dot product of $\mathbb{C}^k$ but the one described in Definition 5. After this process we will obtain an orthonormal basis $\{u_j\}_{j\in I_\Omega}$, and from the general theory of reproducing kernels we simply write

$$K_s(x,y) = \sum_{j\in I_\Omega} u_i(x)\overline{u_j(y)},$$

which is the Szëgo kernel for the simple region $\mathbf{R_s}$. Now, it is clear that the map $\Pi_s : \mathbf{L^2}(\partial\mathbf{R_s}) \to \mathbf{L^2}(\partial\mathbf{R_s})$ given by

$$(\Pi_s f)(y) = \langle f(\cdot), K_s(\cdot, y)\rangle \quad \text{with } f \in \mathbf{L}^2(\partial\mathbf{R_s})$$

defines the Szëgo orthogonal projection operator over $\mathbf{L^2A}(\partial\mathbf{R_s})$.

3.3 About the dimension of the Szëgo space

Note that if $\mathbf{R_s}$ is a simple region then, $\dim(\mathbf{L^2A}(\partial\mathbf{R_s})) = \mathrm{i}(\mathbf{R_s}) + 1$. In fact, as it is well known discrete analytic functions decompose onto independent harmonic functions on the even graph and on its odd dual graph (see [8]). Analyticity associates to a given harmonic function on the even graph a harmonic function on the dual graph, unique up to an additive constant. Therefore the dimension of the Szëgo space should be half the number of boundary points plus one, and this is true in any combinatorics, not only in the flat square case.

We recommend to read the interesting papers [3,4] where certain pairing of two harmonic functions is presented in terms of 'response of an electrical network' but the idea is essentially the same, in this case the set of compatible tensions and currents was pinpointed.

Finally, we believe that this kernel could be used and generalized to 'less' rigid lattices (we are thinking in random tiling domain). Also we hope to find the relation between the discrete Szëgo kernel and the Jordan algebras.

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References

[1] N. Aronszajn, *La théorie générale des noyaux reproduisants et ses applications, première partie*, Proc. Cambridge Philos. Soc. 39 (1944), p. 133.
[2] A. I. Bobenko, *Discrete differential geometry, integrability as consistency*, Lecture Notes in Phys. 644 (2004), pp. 85–110.
[3] I. Colin de Verdiere, *Reseaux lectriques planaires*, Comment. Math. Helv. 69 (1994), pp. 351–374.
[4] I. Colin de Verdiere, I. Gitler, and D. Vertigan, *Reseaux électriques planaires II*, Comment. Math. Helv. 71 (1996), pp. 144–167.
[5] R. J. Duffin, *Basic properties of discrete analytic functions*, Duke Math. J. 23 (1956), pp. 335–363.
[6] J. Ferrand (Lelong), *Fonctions préharmoniques et fonctions préholomorphes*, Bull. Sci. Math. 68 (1944), pp. 152–180, second series.
[7] N. Kerzman and E. M. Stein, *The Cauchy kernel, the Szëgo kernel, and the Riemann mapping function*, Math. Ann. 236 (1978), pp. 85–93.
[8] L. Lovász, *Discrete analytic functions: An exposition*, manuscript (2000). Available at http://research.microsoft.com/users/lovasz/analytic.pdf,
[9] C. Mercat, *Discrete polynomials and discrete holomorphic aproximation*, (arXiv:math-ph/0206041 v1).
[10] C. Mercat, *Exponentials form a basis for discrete holomorphic functions*, Bull. Soc. math. France 132(2) (2004), pp. 305–326.
[11] S. Saitoh, *Theory of Reproducing Kernels and its Applications*. Longman Scientific and Technical, England, 1988.
[12] D. Zeilberger and H. Dym, *Further properties of discrete analytic functions*, J. Math. Anal. Appl. 58 (1977), pp. 405–418.