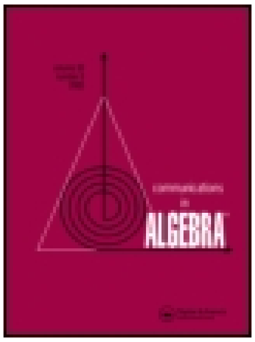

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QUASI-JORDAN ALGEBRAS

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In this article we introduce a new algebraic structure of Jordan type and we show several examples. This new structure, called "quasi-Jordan algebras," appears in the study of the product

$$x \triangleleft y := \frac{1}{2}(x \dashv y + y \vdash x),$$

where x, y are elements in a dialgebra $(D, \dashv, \vdash)$. The quasi-Jordan algebras are a generalization of Jordan algebras where the commutative law is changed by a quasi-commutative identity and a special form of the Jordan identity is retained. We show a few results about the relationship between Jordan algebras and quasi-Jordan algebras. Also, we compare quasi-Jordan algebras with some structures. In particular, we find a special relation with Leibniz algebras. We attach a quasi-Jordan algebra to any ad-nilpotent element of index of nilpotence at most 3 in a Leibniz algebra.

Key Words: Dialgebras; Jordan algebras; Leibniz algebras.

2000 Mathematics Subject Classification: 17C50; 17A32.

INTRODUCTION

There are three strongly related classes of algebras: associative, Jordan, and Lie algebras. It is known that any associative algebra A becomes a Jordan algebra A^+ under the symmetric product (Jordan product) $x \bullet y := \frac{1}{2}(xy + yx)$ and becomes a Lie algebra under the skew-symmetric product (Lie bracket) $[x, y] := xy - yx$. Moreover, we know from the works of Tits, Kantor, and Koecher that Jordan algebras can be imbedded into Lie algebras (see Kantor, 1964; Koecher, 1967; Tits, 1962). In particular, for any Jordan algebra J there is a Lie algebra $L(J)$ such that J is a subspace of $L(J)$ and the product of J can be expressed in terms of the bracket in $L(J)$ (see Kantor, 1964; Koecher, 1967; Tits, 1962).

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It is known that the universal enveloping algebra of a Lie algebra has the structure of an associative algebra. More recently, Loday introduced the notion of *Leibniz algebras* (see Loday, 1993), which is a generalization of Lie algebras. Loday also showed that the relationship between Lie algebras and associative algebras can be translated into an analogous relationship between Leibniz algebras and the so-called *dialgebras* (see Loday, 2001) which are a generalization of associative algebras. In particular, Loday showed that any dialgebra $(D, \dashv, \vdash)$ becomes a Leibniz algebra D_{Leib} under the Leibniz bracket $[x, y] := x \dashv y - y \vdash x$ and the universal enveloping algebra of a Leibniz algebra has the structure of a dialgebra (see Loday, 2001 or Loday and Pirashvili, 1993).

Our aim is to discover a new generalization of Jordan algebras. This new structure, called *quasi-Jordan algebra*, is noncommutative in general although it is not in general equivalent to a noncommutative Jordan algebra (see Bremner et al., 2008, Section 6) and satisfies a special Jordan identity. The quasi-Jordan algebras appear in the study of the product

$$x \triangleleft y := \frac{1}{2}(x \dashv y + y \vdash x),$$

where x and y are elements in a dialgebra D over a field K of characteristic different from 2. They also appear associated to other products defined over Jordan bimodules and vector spaces of linear transformations.

In the first section, we study the algebra D^+, where D is a dialgebra. After the natural construction of the symmetrization D^+, we give the definition of quasi-Jordan algebras and show some examples.

In the second section we present other examples and show several properties. In particular, in this section we study the relationship between quasi-Jordan algebras and noncommutative Jordan algebras.

In the last section we define the concept of a *Q-Jordan element* in Leibniz algebras and we show how it is possible to attach a quasi-Jordan algebra to any Q-Jordan element of a Leibniz algebra (over a field of characteristic different from 2 and 3). This result follows a construction given by Fernández, García, and Gómez (see Fernández López et al., 2007).

The fundamental problem that we would like to study in a future article is the imbedding of quasi-Jordan algebras into Leibniz algebras.

1. ALGEBRAIC STRUCTURES OF JORDAN TYPE GENERATED BY DIALGEBRAS

Around 1990, Loday introduced the notions of *Leibniz algebras* and *diassociative algebras* (*dialgebras*) (see Loday, 2001).

Definition 1. A *Leibniz algebra* over a field K is a K-vector space L equipped with a binary operation, called *Leibniz bracket*, $[\cdot, \cdot] : L \times L \to L$, which satisfies the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \qquad \text{for all } x, y, z \in L. \tag{L}$$

If the bracket is skew-symmetric, then L is a Lie algebra.

Example 2. Let L be a Lie algebra and let M be a L-module with action $M \times L \to M$, $(m, x) \mapsto mx$. Let $f : M \to L$ be a L-equivariant linear map, i.e.,

$$f(mx) = [f(m), x], \qquad \text{for all } m \in M \quad \text{and} \quad x \in L.$$

Then one can put a Leibniz structure on M as follows:

$$[m, n]' := mf(n), \qquad \text{for all } m, n \in M.$$

Additionally, the map f defines a homomorphism between Leibniz algebras, since

$$f([m, n]') = f(mf(n)) = [f(m), f(n)].$$

Example 3. Let (A, d) be a differential associative algebra (so $d(ab) = da\, b + a\, db$ and $d(a\, db) = da\, db = d(da\, b)$). Define the bracket on A by the formula

$$[a, b] := a\, db - db\, a$$

The vector space A equipped with this bracket is a Leibniz algebra.

It follows from the Leibniz identity (L) that in any Leibniz algebra we have

$$[x, [y, y]] = 0, \qquad [x, [y, z]] + [x, [z, y]] = 0.$$

Let L be a Leibniz algebra. Let L^{ann} be the subspace of L spanned by elements of the form $[x, x]$, $x \in L$. For any $x, y \in L$, we define

$$ann(x, y) := [x, y] + [y, x] \in L^{ann}(L).$$

If we take the set

$$Z^r(L) = \{z \in L \mid [x, z] = 0, \forall x \in L\},$$

we obtain that:

1. $L^{ann} \subset Z^r(L)$;
2. L^{ann} and $Z^r(L)$ are two-sided ideals of L;
3. $[Z^r(L), L] \subset L^{ann}$.

The quotient of the Leibniz algebra L by the ideal L^{ann} gives a Lie algebra denoted by L_{Lie}. Moreover, the ideal L^{ann} is the smallest two-sided ideal of L such that L/L^{ann} is a Lie algebra. The quotient map $\pi : L \to L_{Lie}$ is a homomorphism of Leibniz algebras. Besides π is universal with respect to all homomorphisms from L to another Lie algebra L', i.e., the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\pi"] \arrow[dr] & L_{Lie} \arrow[d] \\ & L'. \end{tikzcd}$$

Since $L^{ann} \subset Z^r(L)$, we see that

$$L^{Lie} := L/Z^r(L)$$

is also a Lie algebra. Thus by definition one has a central extension of Lie algebras

$$0 \to L^a \to L_{Lie} \to L^{Lie} \to 0,$$

where $L^a = Z^r(L)/L^{ann}$.

The Leibniz algebras are in fact right Leibniz algebras. For the opposite structure (left Leibniz algebras), the left Leibniz identity is

$$[[x, y]', z]' = [y, [x, z]']' - [x, [y, z]']'. \tag{L'}$$

Definition 4. A *dialgebra* over a field K is a K-vector space D equipped with two bilinear associative products

$$\dashv : D \times D \to D,$$
$$\vdash : D \times D \to D$$

satisfying the identities

$$x \dashv (y \dashv z) = x \dashv (y \vdash z), \tag{D1}$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{D2}$$
$$(x \vdash y) \vdash z = (x \dashv y) \vdash z. \tag{D3}$$

Observe that the analog of formula (D2), but with the product symbols pointing outward, is not valid in general in dialgebras: $(x \dashv y) \vdash z \neq x \dashv (y \vdash z)$.

A morphism of dialgebras from D to D' is a linear map $f : D \to D'$ such that

$$f(x \dashv y) = f(x) \dashv f(y) \qquad \text{and} \qquad f(x \vdash y) = f(x) \vdash f(y),$$

for all x, y in D.

A bar-unit in D is an element e in D such that

$$x \dashv e = x = e \vdash x, \qquad \text{for all } x \in D.$$

A bar-unit needs not to be unique. The subset of bar-units of D is called its *Halo*. A unital dialgebra is a dialgebra with a specified bar-unit e. The problem of adding a bar-unit to dialgebras remains open.

Observe that if a dialgebra has an element ϵ which satisfies $\epsilon \dashv x = x$ for any $x \in D$, then $\dashv = \vdash$ and D is an associative algebra with unit ϵ.

Example 5. If A is an associative algebra, then the formula $x \dashv y = xy = x \vdash y$ defines a structure of dialgebra on A.

Example 6. If (A, d) is a differential associative algebra, then the formulas $x \dashv y = x\, dy$ and $x \vdash y = dx\, y$ define a structure of dialgebra on A.

The next example of dialgebra was studied in Felipe et al. (2005).

Example 7. Let V be a vector space and fix $\varphi \in V'$ (the algebraic dual). Then one can define a dialgebra structure on V by setting $x \dashv y = \varphi(y)x$ and $x \vdash y = \varphi(x)y$, denoted by V_φ. If $\varphi \neq 0$, then V_φ is a dialgebra with nontrivial bar-units. Moreover, its halo is an affine space modeled after the subspace $Ker\,\varphi$.

If D is a dialgebra and we define the bracket $[\cdot,\cdot] : D \times D \to D$ by

$$[x, y] := x \dashv y - y \vdash x, \qquad \text{for all } x, y \in D,$$

then $(D, [\cdot,\cdot])$ is a Leibniz algebra. Moreover, Loday showed that the following diagram is commutative:

$$\begin{array}{ccc} Dias & \xrightarrow{-} & Leib \\ \uparrow & & \uparrow \\ As & \xrightarrow{-} & Lie \end{array}$$

where *Dias*, *As*, *Lie*, and *Leib* denote, respectively, the categories of dialgebras, associative, Lie, and Leibniz algebras (see Loday, 2001).

If we translate the quasi-multiplication (Jordan product) to the dialgebra framework, we obtain a new algebraic structure of Jordan type. Let D be a dialgebra over a field K of characteristic different from 2. We define the product $\lhd : D \times D \to D$ by

$$x \lhd y := \frac{1}{2}(x \dashv y + y \vdash x), \tag{$\lhd$1}$$

for all $x, y \in D$.

Simple calculations show that the product $\lhd$ satisfies the identities

$$x \lhd (y \lhd z) = x \lhd (z \lhd y) \tag{QJ1}$$

$$(y \lhd x) \lhd x^2 = (y \lhd x^2) \lhd x \tag{QJ2}$$

$$x^2 \lhd (x \lhd y) = x \lhd (x^2 \lhd y), \tag{QJ3}$$

but the product $\lhd$ is noncommutative in general.

Note 8. Chapoton introduced the notion of *commutative dialgebra*. A dialgebra D is commutative if the Leibniz algebra D_{Leib} has trivial product (i.e., if $x \dashv y = y \vdash x$, for all x, y in D (see Chapoton, 2001)). In this case, if D is a commutative dialgebra, $(D, \lhd)$ is associative and satisfies the identity

$$x \lhd (y \lhd z) = x \lhd (z \lhd y), \qquad \text{for all } x, y, z \in D.$$

These algebras are called *Perm algebras* (see Chapoton, 2001, p. 105).

If D is a unital dialgebra, with a specific bar-unit e, we have that $x \triangleleft e = x$, for all x in D. This implies that e is a right unit for the algebra $(D, \triangleleft)$. In this case we have by (QJ2) and (QJ3) that

$$x^2 \triangleleft x = x \triangleleft x^2 \tag{1}$$

and

$$x^2 \triangleleft x^2 = (x^2 \triangleleft x) \triangleleft x, \tag{2}$$

for all x, y in D.

Then all algebras $(D, \triangleleft)$ that satisfy the identities (QJ1), (QJ2), and (QJ3) with right unit e defined over a field of characteristic zero are power-associative. If D can be embedded in a unital dialgebra (1) is satisfied.

In this part we are going to introduce the definition of quasi-Jordan algebra.

Definition 9. A *quasi-Jordan algebra* is a vector space $\Im$ over a field K of characteristic different from 2 equipped with a bilinear product $\triangleleft : \Im \times \Im \to \Im$ that satisfies

$$x \triangleleft (y \triangleleft z) = x \triangleleft (z \triangleleft y) \qquad \text{(right commutativity)} \tag{QJ1}$$

$$(y \triangleleft x) \triangleleft x^2 = (y \triangleleft x^2) \triangleleft x \qquad \text{(right Jordan identity),} \tag{QJ2}$$

for all $x, y, z \in \Im$, where $x^2 = x \triangleleft x$.

Note that in terms of the left and right multiplicative maps L_x and R_x, defined for $x \in \Im$ by $L_x(y) = x \triangleleft y$ and $R_x(y) = y \triangleleft x$, for all $y \in \Im$, the identities (QJ1) and (QJ2) are equivalent to

$$L_x L_y = L_x R_y \tag{QJ1*}$$

$$R_x R_{x^2} = R_{x^2} R_x \tag{QJ2*}$$

There is an analogous structure if we define a product $\triangleright : \Im \times \Im \to \Im$ by $x \triangleright y := y \triangleleft x$, for all $x, y \in \Im$. This product satisfies the identities

$$(x \triangleright y) \triangleright z = (y \triangleright x) \triangleright z \qquad \text{(left commutativity)} \tag{QJ1$'$}$$

$$x^2 \triangleright (x \triangleright y) = x \triangleright (x^2 \triangleright y), \qquad \text{(left Jordan identity)} \tag{QJ2$'$}$$

for all $x, y \in \Im$, where $x^2 = x \triangleright x$.

We will only consider the right quasi-Jordan algebras.

Note 10. The Jordan and Perm algebras are obvious examples of quasi-Jordan algebras.

2. EXAMPLES AND PROPERTIES OF QUASI-JORDAN ALGEBRAS

In this section we show other examples of quasi-Jordan algebras and we prove some properties. First, we recall the definitions of Jordan algebra and Jordan bimodule.

Definition 11. Let J be a vector space over a field K of characteristic different from 2. We say that J is a *Jordan algebra* over J if there is a product $\bullet : J \times J \to J$ that satisfies the identities

$$a \bullet b = b \bullet a \tag{3}$$

and

$$a^2 \bullet (b \bullet a) = (a^2 \bullet b) \bullet a, \tag{4}$$

for all $a, b \in J$, where $a^2 = a \bullet a$.

Definition 12. Let J be a Jordan algebra and let M be a vector space over the same field as J. Then M is a *Jordan bimodule* for J in case there are two bilinear compositions $(m, a) \mapsto ma$ and $(m, a) \mapsto am$, for all $m \in M$ and $a \in J$, satisfying

$$ma = am \tag{5}$$

and

$$(a^2, m, a) = (a^2, b, m) + 2(ma, b, a) = 0, \tag{6}$$

for all $m \in M$ and $a, b \in J$, where (a, b, c) denotes the associator.

Example 13. Let J be a Jordan algebra and let M be a Jordan bimodule. A linear map $f : M \to J$ is called J-*equivariant* over M if $f(am) = a \bullet f(m)$, for all $m \in M$ and $a \in J$. If f is a J-equivariant map over M, then we define the product $\triangleleft : M \times M \to M$ by

$$m \triangleleft n = f(n)m, \qquad \text{for all } m, n \in M. \tag{$\triangleleft$2}$$

The product $\triangleleft$ satisfies the identities

$$m \triangleleft (n \triangleleft s) = m \triangleleft (s \triangleleft n) \tag{QJ1}$$

$$(n \triangleleft m) \triangleleft m^2 = (n \triangleleft m^2) \triangleleft m \tag{QJ2}$$

$$m \triangleleft (n \triangleleft m^2) + 2(m^2 \triangleleft n) \triangleleft m = (m \triangleleft n) \triangleleft m^2 + 2m^2 \triangleleft (n \triangleleft m), \tag{QJ4}$$

for all $m, n, s \in M$, but it is, in general, not commutative.

If we compare the products $\triangleleft$ defined by ($\triangleleft$1) and ($\triangleleft$2), these products satisfy the identities (QJ1), (QJ2), and (QJ4).

Another way to define a product over Jordan algebras and Jordan bimodules is the following: over the vector space $J \times M$, where J is a Jordan algebra and M is a Jordan bimodule over J, we define the product $\triangleleft$ by

$$(a, m) \triangleleft (b, n) = (ab, mb), \qquad \text{for all } a, b \in J \quad \text{and} \quad m, n \in M.$$

For simplicity we write $(a, m)b = (ab, mb)$. This product is a particular case of the product defined by ($\triangleleft$2). If we define the projection map $\pi_J : J \times M \to J$ by $\pi_J(a, m) = a$, then $\pi_J((a, m)b) = ab = a\pi_J(b, n)$ and this is equivalent to

$$\pi_J((a, m) \triangleleft (b, n)) = \pi_J(a, m) \bullet \pi_J(b, n),$$

for all $a, b \in J$ and $m, n \in M$.

We have that $J \times M$ is a Jordan bimodule over J with bilinear compositions defined by $a(b, n) = (ab, an)$ and $(b, m)a = (ba, ma)$. The map π_J defined over $J \times (J \times M)$ is J-equivariant since

$$\pi_J((a, m)b) = \pi_J(a, m) \bullet b, \qquad \text{for all } b \in J \quad \text{and} \quad (a, m) \in J \times M.$$

Now, we are going to construct a quasi-Jordan algebra with respect to a vector space and its Jordan algebra of linear transformations.

Example 14. Let V be a vector space over a field K with a characteristic different from 2 and let $gl^+(V)$ be a Jordan algebra of linear transformations over V with a product defined by

$$A \bullet B = \frac{1}{2}(AB + BA),$$

where AB denotes the composition of the maps A and B. We consider the vector space $gl^+(V) \times V$ and we define the product $\triangleleft : (gl^+(V) \times V) \times (gl^+(V) \times V) \to gl^+(V) \times V$ by

$$(A, u) \triangleleft (B, v) = (A \bullet B, Bu), \tag{$\triangleleft$4}$$

for all $A, B \in gl(V)$ and $u, v \in V$. This product satisfies the identities

$$(A, u) \triangleleft ((B, v) \triangleleft (C, w)) = (A, u) \triangleleft ((C, w) \triangleleft (B, v)) \tag{QJ1}$$

and

$$((B, v) \triangleleft (A, u)) \triangleleft (A, u)^2 = ((B, v) \triangleleft (A, u)^2) \triangleleft (A, u), \tag{QJ2}$$

for all $A, B \in gl^+(V)$ and $u, v \in V$, where $(A, u)^2 = (A, u) \triangleleft (A, u)$.

This algebra is power-associative and (Id, v), where Id denotes the identity map over V, is a right unit which is not a left unit.

Following Kinyon and Weinstein's ideas (see Kinyon and Weinstein, 2001), in this part we will obtain Jordan algebras from quasi-Jordan algebras and we show a universal property about homomorphisms from quasi-Jordan algebras to Jordan algebras.

Let $\Im$ be a quasi-Jordan algebra. A subspace $I \subset \Im$ is called *left* (resp., *right*) *ideal* if for any $a \in I$ and $x \in \Im$ we have $a \lhd x \in I$ (resp., $x \lhd a \in I$). If I is both left and right ideal, then I is called a *two-sided ideal*.

For a quasi-Jordan algebra $\Im$ we put

$$Z^r(\Im) = \{z \in \Im \mid x \lhd z = 0, \forall x \in \Im\}.$$

We denote by $\Im^{ann}$ the subspace of $\Im$ spanned by elements of the form $x \lhd y - y \lhd x$, with $x, y \in \Im$. We have that $\Im$ is a Jordan algebra if and only if $\Im^{ann} = \{0\}$.

It follows from the right commutativity (QJ1) that in any quasi-Jordan algebra we have

$$x \lhd (y \lhd z - z \lhd y) = 0.$$

The last identity implies

$$\Im^{ann} \subset Z^r(\Im).$$

Lemma 15. *Let $\Im$ be a quasi-Jordan algebra. Then $\Im^{ann}$ and $Z^r(\Im)$ are two-sided ideals of $\Im$. Moreover,*

$$(Z^r(\Im) \lhd \Im) \subset \Im^{ann}.$$

Proof. Because $\Im^{ann} \subset Z^r(\Im)$ and $\Im \lhd Z^r(\Im) = 0$ it suffices to show the last inclusion. For all $z \in Z^r(\Im)$ and $x \in \Im$ by definition we have $x \lhd z = 0$ and

$$z \lhd x = z \lhd x - x \lhd z \in \Im^{ann}.$$

Then $Z^r(\Im) \lhd \Im \subset \Im^{ann}$ and $\Im^{ann}, Z^r(\Im)$ are two-sided ideals of $\Im$. □

Let $(\Im, \lhd)$ be a quasi-Jordan algebra. If we consider the quotient algebra $\Im_{Jor} := \Im/\Im^{ann}$ we see that $\Im_{Jor}$ is a Jordan algebra. Moreover, the ideal $\Im^{ann}$ is the smallest two-sided ideal in $\Im$ such that $\Im/\Im^{ann}$ is a Jordan algebra. In effect, let I be any two-sided ideal of $\Im$ such that $\Im/I$ is a Jordan algebra, then $x \lhd y - y \lhd x + I = I$ and this implies that $\Im^{ann} \subset I$. The quotient map $\pi : \Im \to \Im_{Jor}$ is a homomorphism of quasi-Jordan algebras. Besides π is universal with respect to all homomorphisms from $\Im$ to another Jordan algebra J, this is equivalent to the fact that following diagram commutes:

$$\begin{tikzcd} \Im \arrow[r, "\pi"] \arrow[dr] & \Im_{Jor} \arrow[d] \\ & J. \end{tikzcd}$$

A *right unit* in a quasi-Jordan algebra $\Im$ is an element e in $\Im$ such that $x \lhd e = x$, for all $x \in \Im$.

Let $\Im$ be a quasi-Jordan algebra, if there is an element ϵ in $\Im$ such that $\epsilon \lhd x = x$, then $\Im$ is a classical Jordan algebra and ϵ is a unit. For this reason we only consider right units over quasi-Jordan algebras.

It is possible to attach a unit to any Jordan algebra, but in quasi-Jordan algebras the problem of attaching a right unit is an open problem. Additionally, the right units in quasi-Jordan algebras are not unique (see Examples 16 and 18).

We denote by $U_r(\Im)$ the set of all right units of a quasi-Jordan algebra $\Im$. A *right unital quasi-Jordan algebra* is a quasi-Jordan algebra with a specified right unit e.

Example 16. Let V be a vector space and fix $\varphi \in V'$ with $\varphi \neq 0$. We define the product $\lhd : V \times V \to V$ by $x \lhd y = \varphi(y)x$, for all $x, y \in V$. Then (V, φ) is a quasi-Jordan algebra and all elements x in V such that $\varphi(x) \neq 0$ define a right unit $x/\varphi(x)$. Moreover, $U_r(V)$ is an affine space modeled after the $Ker\, \varphi$.

We will show the following characterization of the ideal $\Im^{ann}$ and the set $U_r(\Im)$ of all right units.

Lemma 17. *Let $\Im$ be a right unital quasi-Jordan algebra, with a specific right unit e. Then*

$$\Im^{ann} = Z^r(\Im),$$

$$\Im^{ann} = \{x \in \Im \mid e \lhd x = 0\},$$

and

$$U_r(\Im) = \{x + e \mid x \in \Im^{ann}\}.$$

Proof. For all $z \in Z^r(\Im)$, we have that $z = z \lhd e - e \lhd z \in \Im^{ann}$, i.e., $\Im^{ann} = Z^r(\Im)$. It is clear that $\Im^{ann} \subset \{x \in \Im \mid e \lhd x = 0\}$. Suppose that $x \in \Im$ satisfies $e \lhd x = 0$, then $x = x \lhd e - e \lhd x \in \Im^{ann}$. Finally, we have that $e + x \in U_r(\Im)$, for all $x \in \Im^{ann}$. Conversely, for any $e' \in U_r(\Im)$, we have $e' - e = e' \lhd e - e \lhd e' \in \Im^{ann}$. □

Example 18. Let V be a 2-dimensional vector space with base $\{e_1, e_2\}$. If we define the product $\lhd : V \times V \to V$ with respect to e_1 and e_2 by $e_i \lhd e_j = e_i$, for $i = 1, 2$, and extend the product to V by linearity, we have that $(V, \lhd)$ is a noncommutative quasi-Jordan algebra.

Now, if we consider the symmetric product $x \bullet y = x \lhd y + y \lhd x$, for all $x, y \in V$, then $(V, \bullet)$ is a Jordan algebra.

Example 19. Let J be a Jordan algebra and let M be a Jordan bimodule such that the identity $(a, b, am) = 0$ is not satisfied. Then the quasi-Jordan algebra $(J \times M, \lhd)$, with product $\lhd$ defined by ($\lhd$3), is not a Jordan algebra with respect to the symmetric product

$$(a, m) \bullet (b, n) = \frac{1}{2}((a, m) \lhd (b, n) + (b, n) \lhd (a, m))$$

In this point it is important to recall the definition of a noncommutative Jordan algebra and see the relation with quasi-Jordan algebras (see Bremner et al., 2008, Section 6). A *noncommutative Jordan algebra* is a vector space J_n over a field K of characteristic different from 2 equipped with a product $\cdot : J_n \times J_n \to J_n$ satisfying the flexible law and the Jordan identity, i.e.,

$$x \cdot (y \cdot x) = (x \cdot y) \cdot x \tag{7}$$

$$x^2 \cdot (y \cdot x) = (x^2 \cdot y) \cdot x, \tag{8}$$

for all $x, y \in J_n$. The following lemma gives necessary and sufficient conditions for an algebra to be a noncommutative Jordan algebra (see Bremner et al., 2008, Section 6, Fact 4).

Lemma 20. *An algebra A is a noncommutative Jordan algebra if and only if it is flexible (satisfies the flexible law) and the corresponding plus-algebra A^+ is a Jordan algebra ($A^+ = (A, \bullet)$ is a Jordan algebra, with $x \bullet y = \frac{1}{2}(x \cdot y + y \cdot x)$).*

Remark 21. The previous lemma and the last example imply that there are quasi-Jordan algebras which are not noncommutative Jordan algebras. Moreover, there exist noncommutative Jordan algebras which are not quasi-Jordan algebras. In effect, let A be an associative and noncommutative algebra over a field K (characteristic $\neq 2$) and let $a \in K$ with $a \neq \frac{1}{2}$. We define a new product on A as

$$x \bullet_a y = axy + (1-a)yx$$

and we denote the resulting algebra by A^a. The algebra A^a is a noncommutative Jordan algebra, but it is not a quasi-Jordan algebra.

Remark 22. Let $\mathfrak{I}$ be a quasi-Jordan algebra. If we define the associator $(x, y, z) := x \lhd (y \lhd z) - (x \lhd y) \lhd z$, then $(\mathfrak{I}, [\cdot,\cdot], (\cdot,\cdot,\cdot))$ is an Akivis algebra (see Bremner et al., 2008, Section 8).

In future works we will search to find more specific relations between (commutative and noncommutative) Jordan algebras and quasi-Jordan algebras. Furthermore, we will search for relations between quasi-Jordan algebras and Leibniz algebras.

3. QUASI-JORDAN ALGEBRAS GENERATED BY Q-JORDAN ELEMENTS IN LEIBNIZ ALGEBRAS

In this section we show that it is possible to attach a quasi-Jordan algebra to some elements of a Leibniz algebra L.

This construction translates the results obtained by Fernández López et al. (2007, Theorem 2.4) to the context of Leibniz algebras and quasi-Jordan algebras.

We begin by introducing the definition of an adjoint map in Leibniz algebras and we show some properties.

Definition 23. Let L be a Leibniz algebra. For all $x \in L$, we define the *adjoint* map $ad_x : L \to L$ by $ad_x y = [y, x]$, for all $y \in L$. Additionally, the Leibniz identity implies that ad_x is a derivation over L, since $ad_x[y, z] = [ad_x y, z] + [y, ad_x z]$, for all $y, z \in L$.

Remark 24. The map $ad : L \to gl(L)$, $x \mapsto ad_x$, where $gl(L)$ is the Lie algebra of linear maps over L with a Lie bracket $[T, S] = TS - ST$, is an antihomomorphism of Leibniz algebras,

$$ad_{[x,y]} = [ad_y, ad_x], \qquad \text{for all } x, y \in L. \tag{9}$$

The set $ad(L) = \{ad_x \mid x \in L\}$ with the bracket defined by $[ad_x, ad_y] := ad_x ad_y - ad_y ad_x$ turns out to be a Lie algebra, in particular it is a Lie subalgebra of $gl(L)$.

Notation 25. We will use capital letters to denote the adjoint maps (the elements of $ad(L)$): $X = ad_x$, $Y = ad_y$, etcetera. In this notation the last identity has the form

$$ad_{[x,y]} = [Y, X]. \tag{10}$$

Definition 26. Let L be a Leibniz algebra and let x be an element in L. We say that x is an *ad-nilpotent element* if there is a positive integer m such that $ad_x^m = 0$. For any ad-nilpotent element $x \in L$ we define the *ad-nilpotence index* of x as the positive integer m such that $ad_x^m = 0$ and $ad_x^{m-1} \neq 0$.

The elements in a Leibniz algebra that satisfy the following definition are the central objects in the construction of quasi-Jordan algebras from Leibniz algebras.

Definition 27. We say that an element x in a Leibniz algebra L is a *Q-Jordan element* if x is an ad-nilpotent element of index at most 3.

Example 28. A natural example of Jordan elements are the zero-square elements in a dialgebra. Let D be a dialgebra and let L be the Leibniz algebra D_{Leib}. For any $x, y \in L$, we have:

1. $ad_x^2(y) = y \dashv (x \dashv x) - 2(x \vdash y) \dashv x + (x \vdash x) \vdash y$;
2. $ad_x^3(y) = y \dashv x_\dashv^3 - 3(x \vdash y) \dashv (x \dashv x) + 3(x \vdash x) \vdash (y \dashv x) + x_\vdash^3 \dashv y$,

where $x_\dashv^3 = x \dashv (x \dashv x)$ and $x_\vdash^3 = (x \vdash x) \vdash x$.

Thus, if x is an element in D such that $x \dashv x = 0$ or $x \vdash x = 0$ then $ad_x^3(y) = 0$, since $(x \vdash y) \dashv (x \dashv x) = (x \vdash y) \dashv (x \vdash x)$ and $(x \vdash x) \vdash (y \dashv x) = (x \dashv x) \vdash (y \dashv x)$. This implies that x is a Q-Jordan element in the Leibniz algebra D^-. Note also $ad_x^2(L) = x \vdash L \dashv x$.

Example 29. Let L be Leibniz algebra with basis $\{a, b, c, d, e\}$ defined by

$$\begin{array}{llll} [a, b] = c & [a, c] = -2a & [b, a] = -c & [b, c] = 2b, \\ [c, a] = 2a & [c, b] = -2b & [d, b] = e & [d, c] = -d, \\ [e, a] = d & [e, c] = e, & & \end{array}$$

where the omitted products are equal to zero. It is not difficult to see that a, b are Q-Jordan elements in L, L^{ann} is the subspace generate by $\{d, e\}$ and L_{Lie} is isomorphic to the Lie algebra sl_2.

The next example of Leibniz algebras appeared in the work Liu (2006) on simple Leibniz algebra with Lie factor sl_2.

Example 30. Let L be a Leibniz algebra over a field K (of characteristic zero) with basis $\{h, e, f, u, v, w\}$ defined by

$$[h, e] = 2e + 2\alpha u \quad [h, f] = -2f + \beta w \quad [e, h] = -2e \quad [e, f] = h + \alpha v,$$
$$[f, h] = 2f \quad [f, e] = -h - \beta v \quad [u, h] = -2u \quad [u, f] = -v,$$
$$[v, e] = -2u \quad [v, f] = -w \quad [w, h] = 2w \quad [w, e] = -2v,$$

where the omitted products are equal to zero and α, β are two fixed elements of the field K.

We have that e is a Q-Jordan element in L, L^{ann} is the subspace generated by $\{u, v, w\}$ and $Ker_L(x)$ is the subspace generated by $\{e, h, u, v\}$.

We are going to recall two results due to Konstrikin (see Benkart, 1977) and Benkart and Isaacs (1977) about ad-nilpotent elements in Lie algebras.

Theorem 31 (Konstrikin). *Let $\mathfrak{g}$ be a Lie algebra and let a be a nonzero element in $\mathfrak{g}$ such that $ad_a^m = 0$, for $m \geq 4$. If $\mathfrak{g}$ is n-torsion free for all $n \leq m$, then*

$$\left(ad_{ad_a^{m-1}(c)}\right)^{m-1} = 0, \quad \textit{for all } c \in \mathfrak{g}.$$

Therefore $\mathfrak{g}$ contains a nonzero ad-nilpotent element of index at most 3.

Theorem 32. *Any nonzero finite dimensional Lie algebra over an algebraically closed field of arbitrary characteristic necessarily contains a nonzero ad-nilpotent element and therefore a nonzero ad-nilpotent element of index at most* 3.

This result is trivial in Leibniz algebras that are non-Lie algebras, because $ad_z = 0$ for all $z \in L^{ann}$ and $L^{ann} \neq \{0\}$.

Conjecture 33. We conjecture that the last two theorems are true in the context of Leibniz algebras for nontrivial elements (i.e., $x \in L$ such that $x \notin L^{ann}$).

Lemma 34. *Let L be a Leibniz algebra. Then for all positive integers n we have*

$$AD_X^n(Y) = ad_{X^n(y)}, \quad \textit{for all } x, y \in L, \tag{11}$$

where $AD_X : ad(L) \to ad(L)$, $Y \mapsto [X, Y]$ for all $X, Y \in ad(L)$, is the adjoint map over $ad(L)$.

Proof. If $n = 1$, by (10) we have

$$AD_X(Y) = [X, Y] = ad_{[y,x]} = ad_{ad_x(y)} = ad_{X(y)}.$$

We suppose that the property is true for $n = k$, this is

$$AD_X^k(Y) = ad_{X^k(y)}, \qquad \text{for all } x, y \in L.$$

Because $X^{k+1} = X(X^k(y)) = [X^k(y), x]$, then

$$\begin{aligned} AD_X^{k+1}(Y) &= AD_X(AD_X^k(Y)) = AD_X(ad_{X^k(y)}) \\ &= [X, ad_{X^k(y)}] = [ad_x, ad_{X^k(y)}] \\ &= ad_{[X^k(y),x]} = ad_{X^{k+1}(y)} \end{aligned}$$

and the result is true for $n = k + 1$. □

Now, we are going to show that it is possible to obtain a quasi-Jordan algebra from any Q-Jordan element in a Leibniz algebra. Throughout this section we will be dealing with Leibniz algebras over a field K containing $1/6$ (K containing the elements $1/2$ and $1/3$). Following the ideas of Lemma 2.3 in Fernández López et al. (2007), we have the next result for ad-nilpotent elements of index at most 3 in Leibniz algebras.

Lemma 35. *Let x be an ad-nilpotent element of index at most 3 of a Leibniz algebra L. For any $a, b \in L$ and $\alpha \in K$, we have:*

1. $X^2AX = XAX^2$;
2. $X^2AX^2 = 0$;
3. $X^2A^2XAX^2 = X^2AXA^2X^2$;
4. $[X^2(a), X(b)] = -[X(a), X^2(b)]$;
5. $ad_x^2([a, [b, x]]) = [X(a), X^2(b)]$;
6. $X^2 ad_{[a,X^2(b)]} = ad_{[X^2(a),b]} X^2$;
7. $ad_{X^2(a)} ad_{X^2(b)} = X^2ABX^2$;
8. αx, $ad_x^2(a)$ *are Q-Jordan elements in L,*

where $A = ad_a$ and $B = ad_b$.

Proof. 1. Because $X^3(a) = 0$, then $ad_{X^3(a)} = 0$. By (11), we have

$$\begin{aligned} 0 = ad_{X^3(a)} &= AD_X^3(A) \\ &= [X, [X, [X, A]]] \\ &= X^3A - 3X^2AX + 3XAX^2 - AX^3 \\ &= 3(XAX^2 - X^2AX), \end{aligned}$$

which proves 1, since L is 3-torsion free.

2. From 1 we have that $X^2AX = XAX^2$. Then multiplying on the right side by X we obtain 2.

3. By 2 we have

$$
\begin{aligned}
0 &= X^2([[[X, A], A], A])X^2 \\
&= X^2(XA^3 - 3AXA^2 + 3A^2XA - A^3X)X^2 \\
&= 3(X^2A^2XAX^2 - X^2AXA^2X^2).
\end{aligned}
$$

4. From $X^3 = 0$, using the Leibniz identity we get

$$
\begin{aligned}
0 = X^3([a, b]) &= [[[[a, b], x], x], x] \\
&= [[[[a, x], b], x], x] + [[[a, [b, x]], x], x] \\
&= [[[[a, x], x], b], x] + 2[[[a, x], [b, x]], x] + [[a, [[b, x], x]], x].
\end{aligned}
$$

The Leibniz identity implies that

$$
\begin{aligned}
0 &= 3([[[a, x], x], [b, x]] + [[a, x], [[b, x], x]]) \\
&= 3([X^2(a), X(b)] + [X(a), X^2(b)]).
\end{aligned}
$$

Then $[X^2(a), X(b)] = -[X(a), X^2(b)]$, because L is 3-torsion free.

5. From the Leibniz identity and 4, we have

$$
\begin{aligned}
ad_x^2([a, [b, x]]) &= [[[a, [b, x]], x], x] \\
&= [[[a, x], [b, x]], x] + [[a, [[b, x], x]], x] \\
&= [[[a, x], x], [b, x]] + 2[[a, x], [[b, x], x]].
\end{aligned}
$$

From the definition of X, we obtain

$$
\begin{aligned}
ad_x^2([a, [b, x]]) &= [X^2(a), X(b)] + 2[X(a), X^2(b)] \\
&= -[X(a), X^2(b)] + 2[X(a), X^2(b)] \\
&= [X(a), X^2(b)].
\end{aligned}
$$

6. Since $ad_{[a,X^2(b)]} = [[X, [X, B]], A]$, we get

$$
\begin{aligned}
X^2ad_{[a,X^2(b)]} &= X^2((X^2B - 2XBX + BX^2)A - A(X^2B - 2XBX + BX^2)) \\
&= 2X^2AXBX - X^2ABX^2 \\
&= 2XAXBX^2 - X^2ABX^2 \\
&= [B, [X, [X, A]]]X^2 \\
&= ad_{[X^2(a),b]}X^2.
\end{aligned}
$$

7. From (11) and 2, we have

$$\begin{aligned} ad_{X^2(a)}ad_{X^2(b)} &= [X,[X,A]][X,[X,B]] \\ &= (X^2A - 2XAX + AX^2)(X^2B - 2XBX + BX^2) \\ &= -2X^2AXBX + X^2ABX^2 + 4XAX^2BX - 2XAXBX^2 \\ &= X^2ABX^2. \end{aligned}$$

8. $ad^3_{\alpha x} = \alpha^3 ad^3_x$ shows that αx is a Jordan element. Using 2 and 7, we get

$$\begin{aligned} ad^3_{X^2(a)} = ad_{X^2(a)}ad^2_{X^2(a)} &= [X,[X,A]]X^2A^2X^2 \\ &= (X^2A - 2XAX + AX^2)X^2A^2X^2 \\ &= 0, \end{aligned}$$

so $ad^2_x(a)$ is a Q-Jordan element. □

Theorem 36. *Let L be a Leibniz algebra and let x be a Q-Jordan element of L. Then L with the new product defined by*

$$a \lhd b := \frac{1}{2}[a,[b,x]]$$

is a nonassociative algebra, denoted by $L^{(x)}$, such that

$$Ker_L(x) := \{a \in L \mid X^2(a) = 0\}$$

is an ideal of $L^{(x)}$.

Proof. Let $a \in Ker_L(x)$ and let $b \in L$. Using 4 and 5 in the previous lemma, we get

$$\begin{aligned} &X^2([b,[a,x]]) = [X(b), X^2(a)] = 0, \\ &X^2([a,[b,x]]) = [X(a), X^2(b)] = -[X^2(a), X(b)] = 0, \end{aligned}$$

since $X^2(a) = 0$. Therefore $a \lhd b$ and $b \lhd a$ are in $Ker_L(x)$. □

In the following theorem we are going to see that it is possible to attach a quasi-Jordan algebra L_x to any Q-Jordan element x in a Leibniz algebra L. This result is Theorem 2.4 in Fernández López et al. (2007) for Leibniz algebras.

Theorem 37. *Let L be a Leibniz algebra and let x be a Q-Jordan element of L. Then $L_x := L^{(x)}/Ker_L(x)$ is a quasi-Jordan algebra.*

Proof. Let $a, b \in L$ and $\overline{a}$ denotes the coset of a with respect to $ker_L(x)$.

We have

$$c \lhd (b \lhd a) = \frac{1}{4}[c,[[b,[a,x]],x]] \quad \text{and} \quad c \lhd (a \lhd b) = \frac{1}{4}[c,[[a,[b,x]],x]].$$

Since

$$\begin{aligned} ad_x^2([c, [[b, [a, x]], x]]) &= [X(c), X^2([b, [a, x]])] \\ &= [X(c), X^2([[b, a], x] - [[b, x], a])] \\ &= [X(c), X^2(-[[b, x], a])], \end{aligned}$$

by 4 in Lemma 35 we have

$$\begin{aligned} ad_x^2([c, [[b, [a, x]], x]]) &= -[X^2(c), X(-[[b, x], a])] \\ &= -[X^2(c), [-[[b, x], a], x]] \\ &= -[X^2(c), [[a, [b, x]], x]] \\ &= -[X^2(c), X([a, [b, x]])]. \end{aligned}$$

Using 4 from Lemma 35, we obtain

$$\begin{aligned} ad_x^2([c, [[b, [a, x]], x]]) &= [X(c), X^2([a, [b, x]])] \\ &= ad_x^2([c, [[a, [b, x]], x]]). \end{aligned}$$

Then $c \triangleleft (b \triangleleft a) - c \triangleleft (a \triangleleft b) \in Ker_L(x)$, this is $\bar{c} \triangleleft (\bar{a} \triangleleft \bar{b}) = \bar{c} \triangleleft (\bar{b} \triangleleft \bar{a})$. We will verify the Jordan identity. Let $a, b \in L$ and put $w := [[a, [a, x]], x]$. Then

$$8(\bar{b} \triangleleft \bar{a}^2) \triangleleft \bar{a} = \overline{[[b, [[a, [a, x]], x]], [a, x]]} = \overline{[[b, w], [a, x]]}$$

and

$$\begin{aligned} 8(\bar{b} \triangleleft \bar{a}) \triangleleft \bar{a}^2 &= \overline{[[b, [a, x]], [[a, [a, x]], x]]} = \overline{[[b, [a, x]], w]} \\ &= \overline{[[b, w], [a, x]]} + \overline{[b, [[a, x], w]]} \\ &= 8(\bar{b} \triangleleft \bar{a}^2) \triangleleft \bar{a} + \overline{[b, [[a, x], w]]} \end{aligned}$$

Thus we only need to verify that $[b, [[a, x], w]]$ is in $Ker_L(x)$. In effect, because $[b, [[a, x], w]] = ad_{[[a,x],w]}(b)$, then

$$\begin{aligned} ad_x^2 ad_{[[a,x],w]} &= X^2[W, [X, A]] \\ &= X^2WXA - X^2WAX + X^2AXW, \end{aligned}$$

since $X^3 = 0$, and

$$ad_w = W = X^2A^2 - 2XAXA + 2AXAX - A^2X^2.$$

From

$$\begin{aligned} X^2WAX &= 0 \\ -X^2WAX &= -2X^2AXAXAX + X^2A^2X^2AX \\ X^2AXW &= 2X^2AXAXAX - X^2AXA^2X^2, \end{aligned}$$

we have $ad_x^2 ad_{[[a,x],w]} = 0$, i.e., $X^2([b, [[a, x], w]]) = 0$, for all $a, b \in L$. □

In Example 36, we have $\bar{u} \lhd \bar{f} = \bar{f}$ and $\bar{f} \lhd \bar{u} = \bar{0}$, then L_e is not commutative. Therefore, L_x is a noncommutative algebra in general.

Remark 38. Let L be a Leibniz algebra and let x be a Q-Jordan element in L. Then L_x is a Jordan algebra if and only if $[a, [b, x]] - [b, [a, x]] \in Ker_L(x)$, for all $a, b \in L$. In particular, if L is a Lie algebra then L_x is a Jordan algebra.

Definition 39. For any Q-Jordan element x of a Leibniz algebra L, the quasi-Jordan algebra L_x we have just introduced will be called the *quasi-Jordan algebra of* L at x.

Note that for the quasi-Jordan algebra of L at x, we have $L_x^{ann} \subseteq \overline{L^{ann}}$ and $\overline{Z^r(L)} \subseteq Z^r(L_x)$, because

$$\begin{aligned} \bar{a} \lhd \bar{b} &= \overline{[a, [b, x]] - [b, [a, x]]} = \overline{[a, [b, x]] - [[b, a], x] + [[b, x], a]} \\ &= \overline{[a, [b, x]] + [[b, x], a]} \in \overline{L^{ann}} \end{aligned}$$

and

$$[a, [b, x]] = [a, -[x, b]] = 0, \qquad \text{for all } b \in Z^r(L).$$

Therefore, if $L^{ann} \subseteq Ker_L(x)$, then L_x is a Jordan algebra.

Moreover, if L is a Leibniz algebra, then $[a, b] = 0$, for any $a \in L$ and $b \in L^{ann}$, and this implies that Leibniz algebras cannot be nondegenerate in the classical sense, because all elements in L^{ann} are absolute zero divisors of L. Therefore we introduce the following definition.

Definition 40. An element x in a Leibniz algebra L is called an *absolute zero divisor of* L if $ad_x^2 = 0$. A Leibniz algebra L is said to be *nondegenerate* if the absolute zero divisors in L are elements of L^{ann}, i.e., if $ad_x^2 = 0$, for $x \in L$, then $x \in L^{ann}$.

It should be noted that the above definition agrees with the definition of nondegenerate Lie algebra, since $L^{ann} = \{0\}$ in this case.

Additionally, if L is a nondegenerate Leibniz algebra, then $L^{ann} = Z^r(L)$, because $ad_z = 0$ for all $z \in Z^r(L)$.

The following lemma shows that if $[[y, x], x] - 2x \in L^{ann}$, then there are right units in L_x.

Lemma 41. *Let L be a Leibniz algebra and let x be a Jordan element in L. If y is an element in L such that $[[y,x],x]-2x \in L^{ann}$, then $\bar{y}$ is a right unit in L_x. Moreover, $\bar{z}+\bar{y}$ is a right unit in L_x, for all $z \in Z^r(L)$.*

Proof. For all $a \in L$, we have

$$\begin{aligned} X^2([a,[y,x]]) &= [X(a), X^2(y)] \\ &= [[a,x],[[y,x],x]] = 2[[a,x],x] = 2X^2(a), \end{aligned}$$

since $0 = [b, X^2(y) - 2x] = [b, X^2(y)] - [b, 2x]$, for all $b \in L$. Therefore, $X^2([a,[y,x]] - 2a) = 0$ and this is equivalent to $\bar{a} \triangleleft \bar{y} = \bar{a}$, for all $\bar{a} \in L_x$. The identities $[a,z] = 0$, for all $z \in Z^r(L)$, and $[a,[b,c]] = [a, -[c,b]]$, for all $a,b,c \in L$, implies that $\bar{z}+\bar{y}$ is a right unit in L_x, for all $z \in Z^r(L)$. □

The previous lemma shows that if $\bar{y}$ is a right unit of L_x, then $y \notin Z^r(L)$.

The next lemma proves that the existence of the right unit is equivalent to $[[y,x],x]-2x \in L^{ann}$, for some $y \in L$ and for x a Q-Jordan element in a nondegenerate Leibniz algebra L.

Lemma 42. *Let L be a nondegenerate Leibniz algebra and let x be a Jordan element of L. Then $\bar{y}$ is a right unit for the quasi-Jordan algebra L_x if and only if $[[y,x],x] - 2x \in L^{ann}$.*

Proof. Suppose that $y \in L$ which $[[y,x],x]-2x \in L^{ann}$, then $\bar{y}$ is a right unit of L_x by the last lemma.

Suppose, conversely, that $\bar{y}$ is a right unit for L_x. Put $z := X^2(y) - 2x$. For all $a \in L$,

$$\begin{aligned} [[a,z],z] &= [[a,X^2(y)],X^2(y)] - 2[[a,x],X^2(y)] - 2X^2([a,[y,x]] - 2a) \\ &= [[a,X^2(y)] - 2[a,x],X^2(y)] \\ &= [[a,X^2(y)-2x],X^2(y)] \\ &= [[a,z],z] + 2[[a,z],x]. \end{aligned}$$

Then $[[a,z],x] = 0$, for all $a \in L$, since L is 2-torsion free. As $[z,x] = -2[x,x]$, we have

$$\begin{aligned} 0 = [[a,z],x] &= [[a,x],z] + [a,[z,x]] \\ &= [[a,x],z] - 2[a,[x,x]] = [[a,x],z]. \end{aligned}$$

Finally, the Leibniz identity and the identities $[[a,z],x] = 0 = [[a,x],z]$ imply

$$\begin{aligned} [[a,z],z] &= [[a,X^2(y)],z] - 2[[a,x],z] \\ &= [[a,[[y,x],x],z] \\ &= [[[a,[y,x]],x],z] - [[[a,x],[y,x]],z] \\ &= -[[[a,x],z],[y,x]] - [[a,x],[[y,x],z]] = 0. \end{aligned}$$

Therefore $z \in L^{ann}$, because L is nondegenerate. □

Remark 43. The previous lemmas show that if there is an right unit $\bar{y}$ in L_x, then $\overline{Z^r(L)} \subset L_x^{ann}$ and $\overline{Z^r(L)} + \bar{y} \subset U_r(L_x)$.

We are going to define a special operator over quasi-Jordan algebras. This operator agrees with the U-operator over Jordan algebras (see McCrimmon, 2004, Section I.4.1).

Definition 44. Let $\mathfrak{I}$ be a quasi-Jordan algebra and let R_a be a right multiplicative map by a over $\mathfrak{I}$ ($R_a : \mathfrak{I} \to \mathfrak{I}$, $b \mapsto b \triangleleft a$). For all $a \in \mathfrak{I}$ we define the U-operator $U_a : \mathfrak{I} \to \mathfrak{I}$ by

$$U_a = 2R_a^2 - R_{a^2}, \qquad \text{where } a^2 = a \triangleleft a. \tag{12}$$

We will show a special formula for the U-operator over L_x. The next lemma is Lemma 2.4 in Fernández López et al. (2007), for Leibniz algebras.

Lemma 45. *Let x be a Q-Jordan element of a Leibniz algebra L. Then the quasi-Jordan algebra L_x of L at x has a U-operator given by*

$$U_{\bar{a}}\bar{b} = \frac{1}{4}\overline{A^2X^2(b)}, \qquad \textit{for all } \bar{a}, \bar{b} \in L_x. \tag{13}$$

Proof. For all $c \in L$ we have $[c, x] \in Ker_L(x)$, since $X^2([c, x]) = X^3(c) = 0$. Then $\overline{[c, x]} = \bar{0}$ and

$$\begin{aligned}
\overline{A^2X^2(b)} &= \overline{[[[[b, x], x], a], a]} \\
&= \overline{[[[[b, x], a], x], a]} + \overline{[[[b, x], [x, a]], a]} \\
&= \overline{[[[[b, x], a], a], x]} + 2\overline{[[[b, x], a], [x, a]]} + \overline{[[b, x], [[x, a], a]]} \\
&= 2\overline{[[[b, a], x], [x, a]]} + 2\overline{[[b, [x, a]], [x, a]]} + \overline{[[b, [[x, a], a]], x]} \\
&\quad + \overline{[b, [x, [[x, a], a]]]} \\
&= 2\overline{[[[b, a], x], [x, a]]} + 2\overline{[[b, [a, x]], [a, x]]} - \overline{[b, [[a, [a, x]], x]]} \\
&= 4\left(2(\bar{b} \triangleleft \bar{a})\bar{a} - \bar{b} \triangleleft \bar{a}^2\right) \\
&= 4U_{\bar{a}}\bar{b},
\end{aligned}$$

since $\overline{[[[b, a], x], [x, a]]} = \bar{0}$, because

$$\begin{aligned}
X^2([X([b, a]), [x, a]]) &= -X^2([X([b, a]), [a, x]]) \\
&= -[X(X([b, a]), X^2(a)] \\
&= [X^2(X([b, a]), X(a)] \\
&= [X^3([b, a]), X(a)] \\
&= 0,
\end{aligned}$$

for all $b, a \in L$. $\square$

Let $\Im$ be a quasi-Jordan algebra. Since $\Im \lhd \Im^{ann} = 0$, quasi-Jordan algebras cannot be nondegenerate in the classical sense, because all elements in $\Im^{ann}$ are absolute zero divisors of $\Im$. Therefore we introduce the following generalization of the definition of nondegenerate Jordan algebra (see Jacobson, 1968, p. 155).

Definition 46. Let $\Im$ be a quasi-Jordan algebra. An element a in $\Im$ is called an *absolute zero divisor* of $\Im$ if $U_a = 0$. A quasi-Jordan algebra $\Im$ is said to be *nondegenerate* if the absolute zero divisors of $\Im$ are elements of $\Im^{ann}$, i.e., $U_a = 0$, for $a \in \Im$, implies $a \in \Im^{ann}$.

It should be noted that the above definition agrees with the definition of nondegenerate Jordan algebra, since $\Im^{ann} = \{0\}$ in this case.

If $\Im$ is a nondegenerate quasi-Jordan algebra, the last definition implies $\Im^{ann} = Z^r(\Im)$, because $U_z = 0$, for all $z \in Z^r(\Im)$.

Lemma 47. *Let L be a nondegenerate Leibniz algebra and let x be a Q-Jordan element of L. If $\bar{a} \in L_x$ is an absolute zero divisor of L_x, then $\bar{a} \in Z^r(L_x)$.*

Proof. Let $\bar{a}$ be an absolute zero divisor in $Jor(L_x)$. Then $U_{\bar{a}}\bar{b} = \bar{0}$ for every $\bar{b} \in L_x$. By Lemma 44 we have $0 = ad_x^2 ad_a^2 ad_x^2(b) = X^2A^2X^2(b) = ad^2_{X^2(a)}(b)$ and therefore $(ad_x^2(a))_L \in L^{ann}$, since L is nondegenerate.

By 5 in Lemma 35, $X^2([b,[a,x]]) = [X^2(b), X^2(a)] = 0$, for any $b \in L$. Therefore $\bar{b} \lhd \bar{a} = \bar{0}$, for all $b \in L$, and this implies $\bar{a} \in Z^r(L_x)$. $\square$

Now, we will characterize the Q-Jordan elements in Leibniz algebras by inner ideals. First, we define inner ideals in Leibniz algebras.

Definition 48. Let L be a Leibniz algebra. A vector subspace B of L is an *inner ideal* if $[[L,B],B] \subseteq B$. Clearly, any ideal I of L is an inner ideal. Moreover, subideals of L are also inner ideals.

An *abelian inner ideal* is an inner ideal B which is also an abelian subalgebra, this is $[B,B] = 0$.

According to the last definition, we have the following characterization for Jordan elements in Leibniz algebras.

Lemma 49. *Let L be a Leibniz algebra and let $x \in L$. The following conditions are equivalent*:

1. $ad_x^3 = 0$, *with* $x \notin L^{ann}$;
2. $x \in B$, *for B an abelian inner ideal such that* $B \nsubseteq L^{ann}$.

Proof. We suppose that $ad_x^3 = 0$ $(X^3(L) = 0)$ and $x \notin L^{ann}$. First, we are going to show that $X^2(L)$ is an abelian inner ideal of L. It suffices to show $YZ(L) \subseteq X^2(L)$, for $Y = ad_y$ and $Z = ad_z$, where $y = X^2(v)$ and $z = X^2(w)$. This equality is 7 in Lemma 35. This implies that $YZ(L) \subseteq X^2(L)$.

On the other hand,

$$[y,z] = Z(y) = (X^2W - 2XWX + WX^2)(X^2(v)) = 0$$

therefore $X^2(L)$ is an abelian inner ideal of L. Then $Fx + X^2(L)$ is an abelian inner ideal of L, where F is the field over which L is defined. Because $ad_x \neq 0$, then $x \notin L^{ann}$.

Now, we suppose that $x \in B$, for B an abelian inner ideal of L such that $B \nsubseteq L^{ann}$. Then for $x \in B$ such that $x \notin L^{ann}$, we have $ad_x^3(L) = [[[L, x], x], x] \subseteq [B, x] = 0$. □

Let L be a Leibniz algebra. A nonzero element $x \in L$ is called *von Neumann regular* if $X^3 = 0$ and $x \in X^2(L)$. It is clear if $x \in L$ is von Neumann regular, then $x \notin Z^r(L)$ and x is a Q-Jordan element

The following lemma is Lemma 2.7 in Fernández López et al. (2007) for Leibniz algebras with the same proof.

Lemma 50. *Let x be a Q-Jordan element of L.*

1. *If I is an ideal of L and $x \in I$ is von Neumann regular, then both quasi-Jordan algebras I_x and L_x agree.*
2. *If $L = I \oplus J$ is a direct sum of ideals and $x = i + j$ with respect to this decomposition, then $L_x \cong I_i \times J_j$.*
3. *For any inner ideal B of L, $B_x := (B/Ker_L(x) \cap B, \lhd)$ is a subalgebra of L_x.*

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