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SHORT NOTE

Correction to "On the linear combination of normal and Laplace random variables", by Nadarajah, S., *Computational Statistics*, 2006, 21, 63–71

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Abstract In the above mentioned paper, some errors were found in the expressions given for the distribution of a linear combination of Normal and Laplace random variables, Z, given in formulae (3, Theorem 1), (6), and (7) that can lead to obtaining negative values for the mentioned distribution. The corrected versions for these expressions are presented here. In addition, the density function of Z is also provided.

Keywords Complementary error function · Density function of linear combinations of Laplace and Normal variables · Laplace distribution · Normal distribution

1 Introduction

Nadarajah (2006) provides expressions for the exact distribution of $Z = \alpha X + \beta Y$ where X and Y are independent random variables distributed as Normal and Laplace, respectively. The corresponding density functions are

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \tag{1}$$

and

$$f_Y(y) = \frac{1}{2\varphi} \exp\left(-\frac{|y-\lambda|}{\varphi}\right), \tag{2}$$

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where σ and φ are scale parameters, therefore positive, and μ and λ are location parameters, thus $\mu, \lambda \in \mathbb{R}$. To derive his Theorem 1 where the expression for the cdf $F(z)$ is given, Nadarajah uses a formula, presented in Prudnikov et al. (1986, 2.8.9.1, p. 110) which is also given in his Lemma 1, for calculating integrals of the form

$$\int_0^\infty \exp(-px)\,\mathrm{erfc}\,(cx+b)\mathrm{d}x, \tag{3}$$

where $p > 0$, and erfc is the complementary error function

$$\mathrm{erfc}\,(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt.$$

The crucial point is that the formula provided in Prudnikov et al. is only valid for $c > 0$. When c is negative, some limits of integration must be modified in a change of variable required in the derivation of this formula that lead to a different expression. Furthermore, if the corrections presented here are not considered, the original expression of Theorem 1 in Nadarajah (2006) can provide negative results for $F(z)$. As an example of this situation, consider the case of $\alpha = 5$, $\beta = -5$, $\mu = 0$, $\sigma = \lambda = \varphi = 1$ that yields a negative value of $F(z = 0) = -9.065$, while the correct value is 0.741.

2 Correction

The central correction is contained in the appropriate calculation of the integral (3). The following lemma addresses this.

Lemma 1 *The following integral can be expressed as*:

$$\int_0^\infty \exp(-px)\,\mathrm{erfc}(cx+b)\mathrm{d}x = \frac{1}{p}\mathrm{erfc}(b) + \begin{cases} -\frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\mathrm{erfc}\left(b+\frac{p}{2c}\right), & \text{if } c>0, \\ \frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\mathrm{erfc}\left(-b-\frac{p}{2c}\right), & \text{if } c<0. \end{cases} \tag{4}$$

Proof The detailed proof is given in the Appendix. □

Note that the first row of (4) is a special case of Prudnikov's formula for $c > 0$. The second row of (4) constitutes the correction presented here.

Therefore, depending on whether the value of c is positive or negative, the corresponding row in (4) must be considered when solving Nadarajah's two integrals given in his formula (5). This formula can be derived from Proposition 6.1.12 in

Laha and Rohatgi (1979) and was rewritten here in a convenient way in order to facilitate obtaining the correct version of Theorem 1,

$$F(z) = \frac{1}{4\varphi}\left\{\int_0^\infty \exp\left(-\frac{w}{\varphi}\right)\operatorname{erfc}\left[\left(\frac{\beta}{\alpha}\right)\frac{1}{\sqrt{2}\sigma}w + \frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}\right]dw \right. \tag{5}$$

$$\left. + \int_0^\infty \exp\left(-\frac{w}{\varphi}\right)\operatorname{erfc}\left[-\left(\frac{\beta}{\alpha}\right)\frac{1}{\sqrt{2}\sigma}w + \frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}\right]dw\right\}. \tag{6}$$

Let

$$p{=}1/\varphi > 0, \quad b{=}\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}, \; c_1 = \left(\frac{\beta}{\alpha}\right)\left(\frac{1}{\sqrt{2}\sigma}\right), \text{ and } c_2{=}-\left(\frac{\beta}{\alpha}\right)\left(\frac{1}{\sqrt{2}\sigma}\right).$$

Note that p is always positive because φ is a scale parameter. Note also that the sign of c_1 and c_2 is determined by the sign of α/β. Without loss of generality, α will always be considered positive.

The following version of Theorem 1 is thus obtained.

Theorem 1 *Suppose X and Y are Normal* (μ, σ^2) *and Laplace* (λ, φ) *distributed, respectively. Then, the cdf of* $Z = \alpha X + \beta Y$ *can be expressed depending on the sign of* α/β *as follows*:

If $\alpha/\beta > 0$, *then*

$$\begin{aligned} F(z) = \frac{1}{4}\Bigg[&2\operatorname{erfc}\left(\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}\right) \\ &- \exp\left(\frac{\beta\lambda+\alpha\mu-z}{\beta\varphi} + \frac{\alpha^2\sigma^2}{2\beta^2\varphi^2}\right)\operatorname{erfc}\left(\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma} + \frac{\alpha\sigma}{\sqrt{2}\beta\varphi}\right) \\ &+ \exp\left(-\frac{\beta\lambda+\alpha\mu-z}{\beta\varphi} + \frac{\alpha^2\sigma^2}{2\beta^2\varphi^2}\right)\operatorname{erfc}\left(-\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma} + \frac{\alpha\sigma}{\sqrt{2}\beta\varphi}\right)\Bigg]. \end{aligned} \tag{7}$$

If $\alpha/\beta < 0$, *then*

$$\begin{aligned} F(z) = \frac{1}{4}\Bigg[&2\operatorname{erfc}\left(\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}\right) \\ &+ \exp\left(\frac{\beta\lambda+\alpha\mu-z}{\varphi\beta} + \frac{\alpha^2\sigma^2}{2\beta^2\varphi^2}\right)\operatorname{erfc}\left(-\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma} - \frac{\alpha\sigma}{\sqrt{2}\beta\varphi}\right) \\ &- \exp\left(-\frac{\beta\lambda+\alpha\mu-z}{\beta\varphi} + \frac{\alpha^2\sigma^2}{2\beta^2\varphi^2}\right)\operatorname{erfc}\left(\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma} - \frac{\alpha\sigma}{\sqrt{2}\beta\varphi}\right)\Bigg]. \end{aligned} \tag{8}$$

Proof The proof is a direct application of (4) to the integrals in (5) and (6). For solving the first integral (5), the values of (p, b, c_1) will be used, but depending on the sign of c_1, either the first or second row of (4) must be used. For solving the second integral

(6) the values of (p, b, c_2) will be used, but again, depending on the sign of c_2 one or the other row of (4) must be used. □

Also, Corollaries 1 and 2 should be corrected as follows:

Corollary 1 *Suppose that X and Y are distributed according to* (1) *and* (2)*, respectively. Then the cdf of* $Z = X + Y$ *can be expressed as*:

$$F(z) = \frac{1}{4}\left[2\,\mathrm{erfc}\left(\frac{\lambda+\mu-z}{\sqrt{2}\sigma}\right) - \exp\left(\frac{\lambda+\mu-z}{\varphi}+\frac{\sigma^2}{2\varphi^2}\right)\mathrm{erfc}\left(\frac{\lambda+\mu-z}{\sqrt{2}\sigma}+\frac{\sigma}{\sqrt{2}\varphi}\right)\right.$$
$$\left. + \exp\left(-\frac{\lambda+\mu-z}{\varphi}+\frac{\sigma^2}{2\varphi^2}\right)\mathrm{erfc}\left(-\frac{\lambda+\mu-z}{\sqrt{2}\sigma}+\frac{\sigma}{\sqrt{2}\varphi}\right)\right].$$

Proof The proof is a direct application of (7) in Theorem 1 in the case of $\alpha = \beta = 1$, $(\alpha/\beta) > 0$. □

Corollary 2 *Suppose that X and Y are distributed according to (1) and (2), respectively. Then the cdf of* $Z = X - Y$ *can be expressed as*:

$$F(z) = \frac{1}{4}\left[2\,\mathrm{erfc}\left(\frac{-\lambda+\mu-z}{\sqrt{2}\sigma}\right)\right.$$
$$+ \exp\left(-\frac{-\lambda+\mu-z}{\varphi}+\frac{\sigma^2}{2\varphi^2}\right)\mathrm{erfc}\left(-\frac{-\lambda+\mu-z}{\sqrt{2}\sigma}+\frac{\sigma}{\sqrt{2}\varphi}\right)$$
$$\left. - \exp\left(\frac{-\lambda+\mu-z}{\varphi}+\frac{\sigma^2}{2\varphi^2}\right)\mathrm{erfc}\left(\frac{-\lambda+\mu-z}{\sqrt{2}\sigma}+\frac{\sigma}{\sqrt{2}\varphi}\right)\right].$$

Proof The proof is a direct application of (8) in Theorem 1 in the case of $\alpha = 1$, $\beta = -1$, with $(\alpha/\beta) < 0$. □

Finally, the density function of Z is presented in the following theorem.

Theorem 2 *Suppose X and Y are Normal* (μ, σ^2) *and Laplace* (λ, ϕ) *distributed, respectively. Then, the density function of* $Z = \alpha X + \beta Y$ *can be expressed depending on the sign of* α/β *as follows*:

If $\alpha/\beta > 0$, *then*

$$f(z) = \frac{1}{4\beta\varphi}\exp\left(\frac{\alpha^2\sigma^2}{2\beta^2\varphi^2}\right)\left[\exp\left(\frac{\beta\lambda+\alpha\mu-z}{\beta\varphi}\right)\mathrm{erfc}\left(\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}+\frac{\alpha\sigma}{\sqrt{2}\beta\varphi}\right)\right.$$
$$\left. + \exp\left(-\frac{\beta\lambda+\alpha\mu-z}{\beta\varphi}\right)\mathrm{erfc}\left(-\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}+\frac{\alpha\sigma}{\sqrt{2}\beta\varphi}\right)\right].$$

If $\alpha/\beta < 0$, *then*

$$f(z) = -\frac{1}{4\beta\varphi}\exp\left(\frac{\alpha^2\sigma^2}{2\beta^2\varphi^2}\right)\left\{\exp\left(\frac{\beta\lambda+\alpha\mu-z}{\beta\varphi}\right)\mathrm{erfc}\left(-\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}-\frac{\alpha\sigma}{\sqrt{2}\beta\varphi}\right)\right.$$
$$\left. + \exp\left(-\frac{\beta\lambda+\alpha\mu-z}{\beta\varphi}\right)\mathrm{erfc}\left(\frac{\beta\lambda+\alpha\mu-z}{\sqrt{2}\alpha\sigma}-\frac{\alpha\sigma}{\sqrt{2}\beta\varphi}\right)\right\}.$$

Proof Noting that

$$\frac{d}{\mathrm{d}x}\operatorname{erfc}(ax+d) = -\frac{2a}{\sqrt{\pi}}\exp\left[-(ax+d)^2\right], \tag{9}$$

the expression for the density function of Z is obtained by differentiating the distribution $F(z)$ in (7) and (8) in Theorem 1 with respect to z. □

Appendix

The integral (3) can be solved by integration by parts and considering formula (9).
Therefore,

$$\int_0^\infty \exp(-px)\operatorname{erfc}(cx+b)\mathrm{d}x = \lim_{x\to\infty}\left[-\frac{1}{p}\exp(-px)\operatorname{erfc}(cx+b)\right] + \frac{1}{p}\operatorname{erfc}(b) - \frac{2c}{p\sqrt{\pi}}\int_0^\infty \exp\left[-(cx+b)^2 - px\right]\mathrm{d}x. \tag{10}$$

The limit in the first term in the right side above, is zero since $p > 0$ and

$$\lim_{x\to\infty}\operatorname{erfc}(cx) = \begin{cases} 0, & \text{if } c > 0, \\ 2, & \text{if } c < 0. \end{cases}$$

The integral in the last term in the right side of (10) yields different results depending on the sign of c as will be shown below. This integral can be expressed as

$$\int_0^\infty \exp\left[-(cx+b)^2 - px\right]\mathrm{d}x = \exp\left(\frac{bp}{c} + \frac{p^2}{4c^2}\right)\int_0^\infty \exp\left[-\left(cx+b+\frac{p}{2c}\right)^2\right]\mathrm{d}x.$$

Making the following change of variable in this last integral, $w = cx + b + p/(2c)$, yields different limits of integration depending on the sign of c :

$$w \in \begin{cases} \left(b+\frac{p}{2c}, \infty\right), & \text{if } c > 0, \\ \left(b+\frac{p}{2c}, -\infty\right), & \text{if } c < 0. \end{cases} \tag{11}$$

The more general formula (2.8.9.1) of Prudnikov et al. (1986, p. 110) provides the correct result for the integral (3) but only for positive values of c. Nadarajah (2006) apparently overlooked the fact that for $c < 0$ a different result had to be used, noting the different limits of integration given in (11) that must be considered when $c < 0$.

Therefore (10) can be solved as

$$\int_0^\infty \exp(-px)\,\mathrm{erfc}(cx+b)\mathrm{d}x$$

$$= \frac{1}{p}\,\mathrm{erfc}(b) - \frac{2c}{p\sqrt{\pi}} \int_0^\infty \exp\left[-\,(cx+b)^2 - px\right]\mathrm{d}x$$

$$= \frac{1}{p}\,\mathrm{erfc}\,(b) + \begin{cases} -\frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\frac{2}{\sqrt{\pi}}\int_{b+p/(2c)}^{\infty}\exp\left(-w^2\right)dw, & \text{if } c>0, \\ \frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\frac{2}{\sqrt{\pi}}\int_{-\infty}^{b+p/(2c)}\exp\left(-w^2\right)dw, & \text{if } c<0, \end{cases}$$

$$= \frac{1}{p}\,\mathrm{erfc}\,(b) + \begin{cases} -\frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\mathrm{erfc}\left(b+\frac{p}{2c}\right), & \text{if } c>0, \\ \frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\mathrm{erfc}\left(-b-\frac{p}{2c}\right), & \text{if } c<0, \end{cases} \tag{12}$$

noting that

$$\frac{2}{\sqrt{\pi}}\int_{-\infty}^{b+p/(2c)}\exp\left(-w^2\right)dw = 2-\mathrm{erfc}\left(b+\frac{p}{2c}\right) = \mathrm{erfc}\left(-b-\frac{p}{2c}\right).$$

The expression (12) was given in (4) and is the formula that should be used to obtain the correct version of Theorem 1 provided here.

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