Letter to the Editor

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On a paper by Nadarajah and Kotz (Statistical Methods and Applications 15: 151–158, 2006)

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The formula presented by Prudnikov et al. (1986, 2.8.9.1., p. 110) for solving real integrals of the form \( \int_0^\infty \exp(-px) \text{erfc}(cx + b)dx \) cannot be applied as stated when \( c < 0 \). When \( c \) is negative, some limits of integration must be modified in a change of variable required in the calculation of this integral. Nadarajah and Kotz (2006) use this formula in a case where \( c \) is negative (the first integral in their formula 6) to derive their Theorem 1 which is consequently in error.

The correct results are the following:

\[
\text{Lemma 1} \quad \text{For a real positive } p, \text{ and } c, b \in \mathbb{R},
\]

\[
\int_0^\infty \exp(-px) \text{erfc}(cx + b)dx = \frac{1}{p} \text{erfc}(b) + \begin{cases} -\frac{1}{p} \exp\left(\frac{bp}{c} + \frac{p}{4c^2}\right) \text{erfc}\left(b + \frac{p}{2c}\right), & \text{if } c > 0, \\ -\frac{1}{p} \exp\left(\frac{bp}{c} + \frac{p}{4c^2}\right) \text{erfc}\left(-b - \frac{p}{2c}\right), & \text{if } c < 0. \end{cases}
\]

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Proof The integral can be calculated by integration by parts:

\[
\int_0^\infty \exp(-px) \text{erfc}(cx + b)\,dx = \lim_{x \to \infty} \left[ -\frac{1}{p} \exp(-px) \text{erfc}(cx + b) \right] + \frac{1}{p} \text{erfc} (b) - \frac{2c}{p\sqrt{\pi}} \int_0^\infty \exp\left[ -(cx + b)^2 - px \right] \,dx.
\]

Since \( p > 0 \), the limit in the first-term in the right side is zero. The integral in the third term can be expressed as

\[
\int_0^\infty \exp\left[ -(cx + b)^2 - px \right] \,dx = \exp \left( \frac{bp}{c} + \frac{p^2}{4c^2} \right) \int_0^\infty \exp \left( -\left( cx + \frac{p}{2c} \right)^2 \right) \,dx.
\]

A change of variable, \( w = cx + b + p/(2c) \), yields different limits of integration depending on the sign of \( c \):

\[
w \in \begin{cases} 
(b + \frac{p}{2c}, \infty), & \text{if } c > 0, \\
(b + \frac{p}{2c}, -\infty), & \text{if } c < 0.
\end{cases}
\]

Therefore

\[
\int_0^\infty \exp(-px) \text{erfc}(cx + b)\,dx = \frac{1}{p} \text{erfc} (b) + \begin{cases} 
-\frac{1}{p} \exp \left( \frac{bp}{c} + \frac{p^2}{4c^2} \right) \frac{2}{\sqrt{\pi}} \int_{b+p/(2c)}^\infty \exp\left( -w^2 \right) \,dw, & \text{if } c > 0, \\
\frac{1}{p} \exp \left( \frac{bp}{c} + \frac{p^2}{4c^2} \right) \frac{2}{\sqrt{\pi}} \int_{-\infty}^{-b+p/(2c)} \exp\left( -w^2 \right) \,dw, & \text{if } c < 0,
\end{cases}
\]

\[
= \frac{1}{p} \text{erfc} (b) + \begin{cases} 
-\frac{1}{p} \exp \left( \frac{bp}{c} + \frac{p^2}{4c^2} \right) \text{erfc} \left( b + \frac{p}{2c} \right), & \text{if } c > 0, \\
\frac{1}{p} \exp \left( \frac{bp}{c} + \frac{p^2}{4c^2} \right) \text{erfc} \left( -b - \frac{p}{2c} \right), & \text{if } c < 0,
\end{cases}
\]

noting that

\[
\frac{2}{\sqrt{\pi}} \int_{-\infty}^{b+p/(2c)} \exp(-w^2) \,dw = 2 - \text{erfc} \left( b + \frac{p}{2c} \right) = \text{erfc} \left( -b - \frac{p}{2c} \right).
\]
Theorem 1  The cdf of \( Z = |X/Y| \) can be expressed as follows,

\[
F(z) = \begin{cases} 
\frac{1}{2} \left[ \exp \left( -\frac{\mu \lambda}{z} + \frac{\lambda^2 \sigma^2}{2z^2} \right) \text{erfc} \left( -\frac{\mu}{\sqrt{2}\sigma} + \frac{\lambda \sigma}{\sqrt{2}z} \right) 
+ \exp \left( \frac{\mu \lambda}{z} + \frac{\lambda^2 \sigma^2}{2z^2} \right) \text{erfc} \left( \frac{\mu}{\sqrt{2}\sigma} + \frac{\lambda \sigma}{\sqrt{2}z} \right) \right], & \text{if } z > 0, \\
0, & \text{otherwise.}
\end{cases}
\]

Proof  The proof is a direct application of (1) to formula 6 of Nadarajah and Kotz (2006). \( \square \)

Theorem 2  The pdf of \( Z = |X/Y| \) can be expressed as follows,

\[
f(z) = \begin{cases} 
\sqrt{\frac{2\lambda \sigma}{\pi z^2}} \exp \left( -\frac{\mu^2}{2\sigma^2} \right) 
+ \frac{1}{2} \left( \frac{\mu \lambda}{z^2} - \frac{\lambda^2 \sigma^2}{z^3} \right) \exp \left( -\frac{\mu \lambda}{z} + \frac{\lambda^2 \sigma^2}{2z^2} \right) \text{erfc} \left( -\frac{\mu}{\sqrt{2}\sigma} + \frac{\lambda \sigma}{\sqrt{2}z} \right) 
- \frac{1}{2} \left( \frac{\mu \lambda}{z^2} + \frac{\lambda^2 \sigma^2}{z^3} \right) \exp \left( \frac{\mu \lambda}{z} + \frac{\lambda^2 \sigma^2}{2z^2} \right) \text{erfc} \left( \frac{\mu}{\sqrt{2}\sigma} + \frac{\lambda \sigma}{\sqrt{2}z} \right), & \text{if } z > 0, \\
0, & \text{otherwise.}
\end{cases}
\]

Proof  The pdf is obtained by differentiating with respect to \( z \) the expression given in (2), taking into account that

\[
\frac{d}{dz} \text{erfc} \left( \frac{a}{z} + b \right) = \frac{2a}{\sqrt{\pi} z^2} \exp \left[ -\left( \frac{a}{z} + b \right)^2 \right].
\]

\( \square \)

Reply

by Saralees Nadarajah, School of Mathematics, University of Manchester, UK

The letter to the editor by Diaz-Frances and Montoya concerns Theorem 1 of Nadarajah and Kotz (2006). The proof of this theorem uses the result:

\[
\int_0^\infty \exp(-px) \text{erfc}(cx + b) \, dx = \frac{1}{p} \text{erfc}(b) - \frac{1}{p} \exp \left( \frac{p^2 + 4pbc}{4c^2} \right) \text{erfc} \left( b + \frac{p}{2c} \right),
\]

(3)
which is a particular case of Eq. (2.8.9.1) in Prudnikov et al. (1986). However, (3) is valid only if $\text{Re } \rho > 0$ and $|\text{arg } c| < \pi/4$. The calculations in Nadarajah and Kotz (2006) have ignored the second of these two conditions. The condition $|\text{arg } c| < \pi/4$ is not satisfied by some of the calculations leading to Theorem 1 in Nadarajah and Kotz (2006). As a result Theorem 1 is not correct in the stated form. We thank Diaz-Frances and Montoya for pointing out this mistake and for deriving the correct version of Theorem 1 in Nadarajah and Kotz (2006).

We would like to mention, however, that the proof of Lemma 1 in Diaz-Frances and Montoya can be simplified greatly if one notes $\text{erfc}(x) + \text{erfc}(-x) = 2$. Note that if $c < 0$ then one can write

$$
\int_0^\infty \exp(-px) \text{erfc}(cx + b) \, dx = \int_0^\infty \exp(-px) \{2 - \text{erfc}(-cx - b)\} \, dx
$$

$$
= \frac{2}{p} - \int_0^\infty \exp(-px) \text{erfc}(-cx - b) \, dx \quad (4)
$$

and apply (3) for the integral in (4).

References
