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LETTER

Letter to the Editor

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On a paper by Nadarajah and Kotz (Statistical Methods and Applications 15: 151–158, 2006)

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The formula presented by Prudnikov et al. (1986, 2.8.9.1., p. 110) for solving real integrals of the form $\int_0^\infty \exp(-px)\,\mathrm{erfc}(cx+b)dx$ cannot be applied as stated when $c < 0$. When c is negative, some limits of integration must be modified in a change of variable required in the calculation of this integral. Nadarajah and Kotz (2006) use this formula in a case where c is negative (the first integral in their formula 6) to derive their Theorem 1 which is consequently in error.

The correct results are the following:

Lemma 1 *For a real positive p, and $c, b \in \mathbb{R}$,*

$$\int_0^\infty \exp(-px)\,\mathrm{erfc}(cx+b)dx = \frac{1}{p}\,\mathrm{erfc}(b) + \begin{cases} -\frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p}{4c^2}\right)\mathrm{erfc}\left(b+\frac{p}{2c}\right), & \text{if } c > 0, \\ \frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p}{4c^2}\right)\mathrm{erfc}\left(-b-\frac{p}{2c}\right), & \text{if } c < 0. \end{cases} \tag{1}$$

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Proof The integral can be calculated by integration by parts:

$$\int_0^\infty \exp(-px)\,\mathrm{erfc}(cx+b)dx = \lim_{x\to\infty}\left[-\frac{1}{p}\exp(-px)\,\mathrm{erfc}\,(cx+b)\right] + \frac{1}{p}\,\mathrm{erfc}\,(b) - \frac{2c}{p\sqrt{\pi}}\int_0^\infty \exp\left[-(cx+b)^2 - px\right]dx.$$

Since $p > 0$, the limit in the first-term in the right side is zero. The integral in the third term can be expressed as

$$\int_0^\infty \exp\left[-(cx+b)^2 - px\right]dx = \exp\left(\frac{bp}{c} + \frac{p^2}{4c^2}\right)\int_0^\infty \exp\left[-\left(cx+b+\frac{p}{2c}\right)^2\right]dx.$$

A change of variable, $w = cx + b + p/(2c)$, yields different limits of integration depending on the sign of c :

$$w \in \begin{cases} \left(b+\frac{p}{2c},\infty\right), & \text{if } c > 0, \\ \left(b+\frac{p}{2c},-\infty\right), & \text{if } c < 0. \end{cases}$$

Therefore

$$\int_0^\infty \exp(-px)\,\mathrm{erfc}(cx+b)dx$$

$$= \frac{1}{p}\,\mathrm{erfc}\,(b) + \begin{cases} -\frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\frac{2}{\sqrt{\pi}}\int_{b+p/(2c)}^{\infty} \exp\left(-w^2\right)dw, & \text{if } c > 0, \\ \frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\frac{2}{\sqrt{\pi}}\int_{-\infty}^{b+p/(2c)} \exp\left(-w^2\right)dw, & \text{if } c < 0, \end{cases}$$

$$= \frac{1}{p}\,\mathrm{erfc}\,(b) + \begin{cases} -\frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\mathrm{erfc}\left(b+\frac{p}{2c}\right), & \text{if } c > 0, \\ \frac{1}{p}\exp\left(\frac{bp}{c}+\frac{p^2}{4c^2}\right)\mathrm{erfc}\left(-b-\frac{p}{2c}\right), & \text{if } c < 0, \end{cases}$$

noting that

$$\frac{2}{\sqrt{\pi}}\int_{-\infty}^{b+p/(2c)} \exp\left(-w^2\right)dw = 2 - \mathrm{erfc}\left(b+\frac{p}{2c}\right) = \mathrm{erfc}\left(-b-\frac{p}{2c}\right).$$

□

Theorem 1 *The cdf of* $Z = |X/Y|$ *can be expressed as follows,*

$$F(z) = \begin{cases} \frac{1}{2}\left[\exp\left(-\frac{\mu\lambda}{z} + \frac{\lambda^2\sigma^2}{2z^2}\right)\operatorname{erfc}\left(-\frac{\mu}{\sqrt{2}\sigma} + \frac{\lambda\sigma}{\sqrt{2}z}\right)\right. & \\ \left. + \exp\left(\frac{\mu\lambda}{z} + \frac{\lambda^2\sigma^2}{2z^2}\right)\operatorname{erfc}\left(\frac{\mu}{\sqrt{2}\sigma} + \frac{\lambda\sigma}{\sqrt{2}z}\right)\right], & \text{if } z > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

Proof The proof is a direct application of (1) to formula 6 of Nadarajah and Kotz (2006). □

Theorem 2 *The pdf of* $Z = |X/Y|$ *can be expressed as follows,*

$$f(z) = \begin{cases} \frac{\sqrt{2}\lambda\sigma}{\sqrt{\pi}z^2}\exp\left(-\frac{\mu^2}{2\sigma^2}\right) & \\ +\frac{1}{2}\left(\frac{\mu\lambda}{z^2} - \frac{\lambda^2\sigma^2}{z^3}\right)\exp\left(-\frac{\mu\lambda}{z} + \frac{\lambda^2\sigma^2}{2z^2}\right)\operatorname{erfc}\left(-\frac{\mu}{\sqrt{2}\sigma} + \frac{\lambda\sigma}{\sqrt{2}z}\right) & \\ -\frac{1}{2}\left(\frac{\mu\lambda}{z^2} + \frac{\lambda^2\sigma^2}{z^3}\right)\exp\left(\frac{\mu\lambda}{z} + \frac{\lambda^2\sigma^2}{2z^2}\right)\operatorname{erfc}\left(\frac{\mu}{\sqrt{2}\sigma} + \frac{\lambda\sigma}{\sqrt{2}z}\right), & \text{if } z > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Proof The pdf is obtained by differentiating with respect to z the expression given in (2), taking into account that

$$\frac{d}{dz}\operatorname{erfc}\left(\frac{a}{z} + b\right) = \frac{2a}{\sqrt{\pi}z^2}\exp\left[-\left(\frac{a}{z} + b\right)^2\right].$$

□

Reply

by Saralees Nadarajah, School of Mathematics, University of Manchester, UK

The letter to the editor by Diaz-Frances and Montoya concerns Theorem 1 of Nadarajah and Kotz (2006). The proof of this theorem uses the result:

$$\int_0^\infty \exp(-px)\operatorname{erfc}(cx+b)dx = \frac{1}{p}\operatorname{erfc}(b) - \frac{1}{p}\exp\left(\frac{p^2+4pbc}{4c^2}\right)\operatorname{erfc}\left(b + \frac{p}{2c}\right), \tag{3}$$

which is a particular case of Eq. (2.8.9.1) in Prudnikov et al. (1986). However, (3) is valid only if $Re\ p > 0$ and $|arg\ c| < \pi/4$. The calculations in Nadarajah and Kotz (2006) have ignored the second of these two conditions. The condition $|arg\ c| < \pi/4$ is not satisfied by some of the calculations leading to Theorem 1 in Nadarajah and Kotz (2006). As a result Theorem 1 is not correct in the stated form. We thank Diaz-Frances and Montoya for pointing out this mistake and for deriving the correct version of Theorem 1 in Nadarajah and Kotz (2006).

We would like to mention, however, that the proof of Lemma 1 in Diaz-Frances and Montoya can be simplified greatly if one notes $\operatorname{erfc}(x) + \operatorname{erfc}(-x) = 2$. Note that if $c < 0$ then one can write

$$\int_0^\infty \exp(-px) \operatorname{erfc}(cx + b) dx = \int_0^\infty \exp(-px) \{2 - \operatorname{erfc}(-cx - b)\}\, dx$$
$$= \frac{2}{p} - \int_0^\infty \exp(-px) \operatorname{erfc}(-cx - b)\, dx \qquad (4)$$

and apply (3) for the integral in (4).

References

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