

A generic property of families of Lagrangian systems

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Abstract

We prove that a generic Lagrangian has finitely many minimizing measures for every cohomology class.

1. Introduction

Let M be a compact boundaryless smooth manifold.

Let \mathbb{T} be either the group $(\mathbb{R}/\mathbb{Z}, +)$ or the trivial group $(\{0\}, +)$.

A *Tonelli Lagrangian* is a C^2 function $L : \mathbb{T} \times TM \rightarrow \mathbb{R}$ such that

- The restriction to each fiber of $\mathbb{T} \times TM \rightarrow \mathbb{T} \times M$ is a *convex* function.
- It is fiberwise *superlinear*:

$$\lim_{|\theta| \rightarrow +\infty} L(t, \theta)/|\theta| = +\infty, \quad (t, \theta) \in \mathbb{T} \times TM.$$

- The Euler-Lagrange equation

$$\frac{d}{dt} L_v = L_x$$

defines a *complete* flow $\varphi : \mathbb{R} \times (\mathbb{T} \times TM) \rightarrow \mathbb{T} \times TM$.

We say that a Tonelli Lagrangian L is *strong Tonelli* if $L + u$ is a Tonelli Lagrangian for each $u \in C^\infty(\mathbb{T} \times M, \mathbb{R})$. When $\mathbb{T} = \{0\}$ we say that the lagrangian is *autonomous*.

Let $\mathcal{P}(L)$ be the set of Borel probability measures on $\mathbb{T} \times TM$ which are invariant under the Euler-Lagrange flow φ . The action functional $A_L : \mathcal{P}(L) \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined as

$$A_L(\mu) := \langle L, \mu \rangle := \int_{\mathbb{T} \times TM} L \, d\mu.$$

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**Partially supported by CONACYT-México grant # 46467-F.

The functional A_L is lower semi-continuous and the minimizers of A_L on $\mathcal{P}(L)$ are called *minimizing measures*. The ergodic components of a minimizing measure are also minimizing, and they are mutually singular, so that the set $\mathfrak{M}(L)$ of minimizing measures is a simplex whose extremal points are the ergodic minimizing measures.

In general, the simplex $\mathfrak{M}(L)$ may be of infinite dimension. The goal of the present paper is to prove that this is a very exceptional phenomenon. The first results in that direction were obtained by Mañé in [4]. His paper has been very influential to our work.

We say that a property is *generic* in the sense of Mañé if, for each strong Tonelli Lagrangian L , there exists a residual subset $\mathcal{O} \subset C^\infty(\mathbb{T} \times M, \mathbb{R})$ such that the property holds for all the Lagrangians $L - u, u \in \mathcal{O}$. A set is called residual if it is a countable intersection of open and dense sets. We recall which topology is used on $C^\infty(\mathbb{T} \times M, \mathbb{R})$. Denoting by $\|u\|_k$ the C^k -norm of a function $u : \mathbb{T} \times M \rightarrow \mathbb{R}$, define

$$\|u\|_\infty := \sum_{k \in \mathbb{N}} \frac{\arctan(\|u\|_k)}{2^k}.$$

Note that $\|\cdot\|_\infty$ is not a norm. Endow the space $C^\infty(\mathbb{T} \times M, \mathbb{R})$ with the translation-invariant metric $\|u - v\|_\infty$. This metric is complete, hence the Baire property holds: any residual subset of $C^\infty(\mathbb{T} \times M, \mathbb{R})$ is dense.

THEOREM 1. *Let A be a finite-dimensional convex family of strong Tonelli Lagrangians. Then there exists a residual subset \mathcal{O} of $C^\infty(\mathbb{T} \times M, \mathbb{R})$ such that,*

$$u \in \mathcal{O}, \quad L \in A \quad \implies \quad \dim \mathfrak{M}(L - u) \leq \dim A.$$

In other words, there exist at most $1 + \dim A$ ergodic minimizing measures of $L - u$.

The main result of Mañé in [4] is that having a unique minimizing measure is a generic property. This corresponds to the case where A is a point in our statement. Our generalization of Mañé's result is motivated by the following construction due to John Mather:

We can view a 1-form on M as a function on TM which is linear on the fibers. If λ is closed, the Euler-Lagrange equation of the Lagrangian $L - \lambda$ is the same as that of L . However, the minimizing measures of $L - \lambda$, are not the same as the minimizing measures of L . Mather proves in [5] that the set $\mathfrak{M}(L - \lambda)$ of minimizing measures of the lagrangian $L - \lambda$ depends only on the cohomology class c of λ . If $c \in H^1(M, \mathbb{R})$ we write $\mathfrak{M}(L - c) := \mathfrak{M}(L - \lambda)$, where λ is a closed form of cohomology c .

It turns out that important applications of Mather theory, such as the existence of orbits wandering in phase space, require understanding not only of the set $\mathfrak{M}(L)$ of minimizing measures for a fixed or generic cohomology classes

but of the set of all Mather minimizing measures for every $c \in H^1(M, L)$. The following corollaries are crucial for these applications.

COROLLARY 2. *The following property is generic in the sense of Mañé: For all $c \in H^1(M, \mathbb{R})$, there are at most $1 + \dim H^1(M, \mathbb{R})$ ergodic minimizing measures of $L - c$.*

We say that a property is of infinite codimension if, for each finite-dimensional convex family A of strong Tonelli Lagrangians, there exists a residual subset \mathcal{O} in $C^\infty(\mathbb{T} \times M, \mathbb{R})$ such that none of the Lagrangians $L - u$, $L \in A$, $u \in \mathcal{O}$ satisfy the property.

COROLLARY 3. *The following property is of infinite codimension: There exists $c \in H^1(M, \mathbb{R})$, such that $L - c$ has infinitely many ergodic minimizing measures.*

Another important issue concerning variational methods for Arnold diffusion questions is the total disconnectedness of the quotient Aubry set. John Mather proves in [7, § 3] that the quotient Aubry set $\overline{\mathcal{A}}$ of any Tonelli Lagrangian on $\mathbb{T} \times TM$ with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\dim M \leq 2$ (or with $\mathbb{T} = \{0\}$ and $\dim M \leq 3$) is totally disconnected. See [7] for its definition.

The elements of the quotient Aubry set are called *static classes*. They are disjoint subsets of $\mathbb{T} \times TM$ and each static class supports at least one ergodic minimizing measure. We then get

COROLLARY 4. *The following property is generic in the sense of Mañé: For all $c \in H^1(M, \mathbb{R})$ the quotient Aubry set $\overline{\mathcal{A}}_c$ of $L - c$ has at most $1 + \dim H^1(M, \mathbb{R})$ elements.*

2. Abstract results

Assume that we are given

- Three topological vector spaces E, F, G .
- A continuous linear map $\pi : F \rightarrow G$.
- A bilinear pairing $\langle u, \nu \rangle : E \times G \rightarrow \mathbb{R}$.
- Two metrizable convex compact subsets $H \subset F$ and $K \subset G$ such that $\pi(H) \subset K$.

Suppose that

- (i) The map

$$E \times K \ni (u, \nu) \longmapsto \langle u, \nu \rangle$$

is continuous.

We will also denote $\langle u, \pi(\mu) \rangle$ by $\langle u, \mu \rangle$ when $\mu \in H$. Observe that each element $u \in E$ gives rise to a linear functional on F

$$F \ni \mu \longmapsto \langle u, \mu \rangle$$

which is continuous on H . We shall denote by H^* the set of affine and continuous functions on H and use the same symbol u for an element of E and for the element $\mu \longmapsto \langle u, \mu \rangle$ of H^* which is associated to it.

- (ii) The compact K is separated by E . This means that, if η and ν are two different points of K , then there exists a point u in E such that $\langle u, \eta \rangle \neq \langle u, \nu \rangle \neq 0$.

Note that the topology on K is then the weak topology associated to E . A sequence η_n of elements of K converges to η if and only if we have $\langle u, \eta_n \rangle \rightarrow \langle u, \eta \rangle$ for each $u \in E$. We shall, for notational conveniences, fix once and for all a metric d on K .

- (iii) E is a Frechet space. It means that E is a topological vector space whose topology is defined by a translation-invariant metric, and that E is complete for this metric.

Note then that E has the Baire property. We say that a subset is residual if it is a countable intersection of open and dense sets. The Baire property says that any residual subset of E is dense.

Given $L \in H^*$ denote by

$$M_H(L) := \arg \min_H L$$

the set of points $\mu \in H$ which minimize $L|_H$, and by $M_K(L)$ the image $\pi(M_H(L))$. These are compact convex subsets of H and K .

Our main abstract result is:

THEOREM 5. *For every finite-dimensional affine subspace A of H^* , there exists a residual subset $\mathcal{O}(A) \subset E$ such that, for all $u \in \mathcal{O}(A)$ and all $L \in A$, we have*

$$(1) \quad \dim M_K(L - u) \leq \dim A.$$

Proof. We define the ε -neighborhood V_ε of a subset V of K as the union of all the open balls in K which have radius ε and are centered in V . Given a subset $D \subset A$, a positive number ε , and a positive integer k , denote by $\mathcal{O}(D, \varepsilon, k) \subset E$ the set of points $u \in E$ such that, for each $L \in D$, the convex set $M_K(L - u)$ is contained in the ε -neighborhood of some k -dimensional convex subset of K .

We shall prove that the theorem holds with

$$\mathcal{O}(A) = \bigcap_{\varepsilon > 0} \mathcal{O}(A, \varepsilon, \dim A).$$

If u belongs to $\mathcal{O}(A)$, then (1) holds for every $L \in A$. Otherwise, for some $L \in A$, the convex set $M_K(L - u)$ would contain a ball of dimension $\dim A + 1$, and, if ε is small enough, such a ball is not contained in the ε -neighborhood of any convex set of dimension $\dim A$.

So we have to prove that $\mathcal{O}(A)$ is residual. In view of the Baire property, it is enough to check that, for any compact subset $D \subset A$ and any positive ε , the set $\mathcal{O}(D, \varepsilon, \dim A)$ is open and dense. We shall prove in 2.1 that it is open, and in 2.2 that it is dense. \square

2.1. *Open.* We prove that, for any $k \in \mathbb{Z}^+$, $\varepsilon > 0$ and any compact $D \subset A$, the set $\mathcal{O}(D, \varepsilon, k) \subset E$ is open. We need a Lemma.

LEMMA 6. *The set-valued map $(L, u) \mapsto M_H(L - u)$ is upper semi-continuous on $A \times E$. This means that for any open subset U of H , the set*

$$\{(L, u) \in A \times E : M_H(L - u) \subset U\} \subset A \times E$$

is open in $A \times E$. Consequently, the set-valued map $(L, u) \mapsto M_K(L - u)$ is also upper semi-continuous.

Proof. This is a standard consequence of the continuity of the map

$$A \times E \times H \ni (L, u, \mu) \mapsto (L - u)(\mu) = L(\mu) - \langle u, \mu \rangle. \quad \square$$

Now let u_0 be a point of $\mathcal{O}(D, \varepsilon, k)$. For each $L \in D$, there exists a k -dimensional convex set $V \subset K$ such that $M_K(L - u_0) \subset V_\varepsilon$. In other words, the open sets of the form

$$\{(L, u) \in D \times E : M_H(L - u) \subset V_\varepsilon\} \subset D \times E,$$

where V is some k -dimensional convex subset of K , cover the compact set $D \times \{u_0\}$. So there exists a finite subcovering of $D \times \{u_0\}$ by open sets of the form $\Omega_i \times U_i$, where Ω_i is an open set in A and $U_i \subset \mathcal{O}(\Omega_i, \varepsilon, k)$ is an open set in E containing u_0 . We conclude that the open set $\bigcap U_i$ is contained in $\mathcal{O}(D, \varepsilon, k)$, and contains u_0 . This ends the proof. \square

2.2. *Dense.* We prove the density of $\mathcal{O}(A, \varepsilon, \dim A)$ in E for $\varepsilon > 0$. Let w be a point in E . We want to prove that w is in the closure of $\mathcal{O}(A, \varepsilon, \dim A)$.

LEMMA 7. *There exists an integer m and a continuous map*

$$T_m = (w_1, \dots, w_m) : K \longrightarrow \mathbb{R}^m,$$

with $w_i \in E$ such that

$$(2) \quad \forall x \in \mathbb{R}^m \quad \text{diam } T_m^{-1}(x) < \varepsilon,$$

where the diameter is taken for the distance d on K .

Proof. In $K \times K$, to each element $w \in E$ we associate the open set

$$U_w = \{(\eta, \mu) \in K \times K : \langle w, \eta - \mu \rangle \neq 0\}.$$

Since E separates K , the open sets $U_w, w \in E$ cover the complement of the diagonal in $K \times K$. Since this complement is open in the separable metrizable set $K \times K$, we can extract a countable subcovering from this covering. So we have a sequence U_{w_k} , with $w_k \in E$, which covers the complement of the diagonal in $K \times K$. This amounts to say that the sequence w_k separates K . Defining $T_m = (w_1, \dots, w_m)$, we have to prove that (2) holds for m large enough. Otherwise, we would have two sequences η_m and μ_m in K such that

$$T_m(\mu_m) = T_m(\eta_m) \quad \text{and} \quad d(\mu_m, \eta_m) \geq \varepsilon.$$

By extracting a subsequence, we can assume that the sequences μ_m and η_m have different limits μ and η , which satisfy $d(\eta, \mu) \geq \varepsilon$. Take m large enough, so that $T_m(\eta) \neq T_m(\mu)$. Such a value of m exists because the linear forms w_k separate K . We have that

$$T_m(\mu_k) = T_m(\eta_k) \quad \text{for} \quad k \geq m.$$

Hence at the limit $T_m(\eta) = T_m(\mu)$. This is a contradiction. □

Define the function $F_m : A \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ as

$$F_m(L, x) := \min_{\substack{\mu \in H \\ T_m \circ \pi(\mu) = x}} (L - w)(\mu),$$

when $x \in T_m(\pi(H))$ and $F_m(L, x) = +\infty$ if $x \in \mathbb{R}^m \setminus T_m(\pi(H))$. For $y = (y_1, \dots, y_m) \in \mathbb{R}^m$, let

$$M_m(L, y) := \arg \min_{x \in \mathbb{R}^m} [F_m(L, x) - y \cdot x] \subset \mathbb{R}^m$$

be the set of points which minimize the function $x \mapsto F_m(L, x) - y \cdot x$. We have that

$$M_K(L - w - y_1 w_1 - \dots - y_m w_m) \subset T_m^{-1}(M_m(L, y)).$$

Let

$$\mathcal{O}_m(A, \dim A) := \{y \in \mathbb{R}^m \mid \forall L \in A : \dim M_m(L, y) \leq \dim A\}.$$

From Lemma 7 it follows that

$$y \in \mathcal{O}_m(A, \dim A) \implies w + y_1 w_1 + \dots + y_m w_m \in \mathcal{O}(A, \varepsilon, \dim A).$$

Therefore, in order to prove that w is in the closure of $\mathcal{O}(A, \varepsilon, \dim A)$, it is enough to prove that 0 is in the closure of $\mathcal{O}_m(A, \dim A)$, which follows from the next proposition.

PROPOSITION 8. *The set $\mathcal{O}_m(A, \dim A)$ is dense in \mathbb{R}^m .*

Proof. Consider the Legendre transform of F_m with respect to the second variable,

$$\begin{aligned} G_m(L, y) &= \max_{x \in \mathbb{R}^m} y \cdot x - F_m(L, x) \\ &= \max_{\mu \in H} \langle w + y_1 w_1 + \dots + y_m w_m, \mu \rangle - L(\mu). \end{aligned}$$

It follows from this second expression that the function G_m is convex and finite-valued, hence continuous on $A \times \mathbb{R}^m$.

Consider the set $\tilde{\Sigma} \subset A \times \mathbb{R}^m$ of points (L, y) such that $\dim \partial G_m(L, y) \geq \dim A + 1$, where ∂G_m is the subdifferential of G_m . It is known, see the appendix, that this set has Hausdorff dimension at most

$$(m + \dim A) - (\dim A + 1) = m - 1.$$

Consequently, the projection Σ of the set $\tilde{\Sigma}$ on the second factor \mathbb{R}^m also has Hausdorff dimension at most $m - 1$. Therefore, the complement of Σ is dense in \mathbb{R}^m . So it is enough to prove that

$$y \notin \Sigma \implies \forall L \in A : \dim M_m(L, y) \leq \dim A.$$

Since we know by definition of Σ that $\dim \partial G_m(L, y) \leq \dim A$, it is enough to observe that

$$\dim M_m(L, y) \leq \dim \partial G_m(L, y).$$

The last inequality follows from the fact that the set $M_m(L, y)$ is the subdifferential of the convex function

$$\mathbb{R}^m \ni z \longmapsto G_m(L, z)$$

at the point y . □

3. Application to Lagrangian dynamics

Let C be the set of continuous functions $f : \mathbb{T} \times TM \rightarrow \mathbb{R}$ with linear growth, i.e.

$$\|f\|_\ell := \sup_{(t, \theta) \in \mathbb{T} \times TM} \frac{|f(t, \theta)|}{1 + |\theta|} < +\infty,$$

endowed with the norm $\|\cdot\|_\ell$.

We apply Theorem 5 to the following setting:

- $F = C^*$ is the vector space of continuous linear functionals $\mu : C \rightarrow \mathbb{R}$ provided with the weak- \star topology. Recall that

$$\lim_n \mu_n = \mu \iff \lim_n \mu_n(f) = \mu(f), \quad \forall f \in C.$$

- $E = C^\infty(\mathbb{T} \times M, \mathbb{R})$ provided with the C^∞ topology.
- G is the vector space of finite Borel signed measures on $\mathbb{T} \times M$, or equivalently the set of continuous linear forms on $C^0(\mathbb{T} \times M, \mathbb{R})$, provided with the weak- \star topology.
- The pairing $E \times G \rightarrow \mathbb{R}$ is given by integration:

$$\langle u, \nu \rangle = \int_{\mathbb{T} \times M} u \, d\nu.$$

- The continuous linear map $\pi : F \rightarrow G$ is induced by the projection $\mathbb{T} \times TM \rightarrow \mathbb{T} \times M$.
- The compact $K \subset G$ is the set of Borel probability measures on $\mathbb{T} \times M$, provided with the weak- \star topology. Observe that K is separated by E .
- The compact $H_n \subset F$ is the set of holonomic probability measures which are supported on

$$B_n := \{(t, \theta) \in \mathbb{T} \times TM \mid |\theta| \leq n\}.$$

Holonomic probabilities are defined as follows: Given a C^1 curve $\gamma : \mathbb{R} \rightarrow M$ of period $T \in \mathbb{N}$ define the element μ_γ of F by

$$\langle f, \mu_\gamma \rangle = \frac{1}{T} \int_0^T f(s, \gamma(s), \dot{\gamma}(s)) \, ds$$

for each $f \in C$. Let

$$\Gamma := \{\mu_\gamma \mid \gamma \in C^1(\mathbb{R}, M) \text{ is periodic of integral period}\} \subset F.$$

The set \mathcal{H} of holonomic probabilities is the closure of Γ in F . One can see that \mathcal{H} is convex (cf. Mañé [4, Prop. 1.1(a)]). The elements μ of \mathcal{H} satisfy $\langle 1, \mu \rangle = 1$ therefore we have $\pi(\mathcal{H}) \subset K$.

Note that each Tonelli Lagrangian L gives rise to an element of H_n^* .

Let $\mathfrak{M}(L)$ be the set of minimizing measures for L and let $\text{supp } \mathfrak{M}(L)$ be the union of their supports. Recalling that we have defined $M_{H_n}(L)$ as the set of measures $\mu \in H_n$ which minimize the action $\int L \, d\mu$ on H_n , we have:

LEMMA 9. *If L is a Tonelli lagrangian then there exists $n \in \mathbb{N}$ such that*

$$\dim \pi(M_{H_n}(L)) = \dim \mathfrak{M}(L).$$

Proof. Birkhoff theorem implies that $\mathfrak{M}(L) \subset \mathcal{H}$ (cf. Mañé [4, Prop. 1.1.(b)]). In [5, Prop. 4, p. 185] Mather proves that $\text{supp } \mathfrak{M}(L)$ is compact, therefore $\mathfrak{M}(L) \subset H_n$ for some $n \in \mathbb{N}$.

In [4, §1] Mañé proves that minimizing measures are also all the minimizers of the action functional $A_L(\mu) = \int L \, d\mu$ on the set of holonomic measures, therefore $\mathfrak{M}(L) = M_{H_n}(L)$ for some $n \in \mathbb{N}$.

In [5, Th. 2, p. 186] Mather proves that the restriction $\text{supp } \mathfrak{M}(L) \rightarrow M$ of the projection $TM \rightarrow M$ is injective. Therefore the linear map $\pi : \mathfrak{M}(L) \rightarrow G$ is injective, so that $\dim \pi(M_{H_n}(L)) = \dim \pi(\mathfrak{M}(L)) = \dim \mathfrak{M}(L)$. \square

Proof of Theorem 1. Given $n \in \mathbb{N}$ apply Theorem 5 and obtain a residual subset $\mathcal{O}_n(A) \subset E$ such that

$$L \in A, \quad u \in \mathcal{O}_n(A) \implies \dim \pi(M_{H_n}(L - u)) \leq \dim A.$$

Let $\mathcal{O}(A) = \bigcap_n \mathcal{O}_n(A)$. By the Baire property $\mathcal{O}(A)$ is residual. We have that

$$L \in A, \quad u \in \mathcal{O}(A), \quad n \in \mathbb{N} \implies \dim \pi(M_{H_n}(L - u)) \leq \dim A.$$

Then by Lemma 9, $\dim \mathfrak{M}(L - u) \leq \dim A$ for all $L \in A$ and all $u \in \mathcal{O}(A)$. \square

Appendix A. Convex functions

Given a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $x \in \mathbb{R}^n$, define its subdifferential as

$$\partial f(x) := \{ \ell : \mathbb{R}^n \rightarrow \mathbb{R} \text{ linear} \mid f(y) \geq f(x) + \ell(y - x), \forall y \in \mathbb{R}^n \}.$$

Then the sets $\partial f(x) \subset \mathbb{R}^n$ are convex. If $k \in \mathbb{N}$, let

$$\Sigma_k(f) := \{ x \in \mathbb{R}^n \mid \dim \partial f(x) \geq k \}.$$

The following result is standard.

PROPOSITION 10. *If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function then for all $0 \leq k \leq n$ the Hausdorff dimension $HD(\Sigma_k(f)) \leq n - k$.*

We recall here an elegant proof due to Ambrosio and Alberti; see [1]. Note that much more can be said on the structure of Σ_k , see [2], [9] for example.

By adding $|x|^2$ if necessary (which does not change Σ_k) we can assume that f is superlinear and that

$$(3) \quad f(y) \geq f(x) + \ell(y - x) + \frac{1}{2} |y - x|^2 \quad \forall x, y \in \mathbb{R}^n, \quad \forall \ell \in \partial f(x).$$

LEMMA 11. $\ell \in \partial f(x), \quad \ell' \in \partial f(x') \implies |x - x'| \leq \|\ell - \ell'\|.$

Proof. From inequality (3) we have that

$$\begin{aligned} f(x') &\geq f(x) + \ell(x' - x) + \frac{1}{2} |x' - x|^2, \\ f(x) &\geq f(x') + \ell'(x - x') + \frac{1}{2} |x - x'|^2. \end{aligned}$$

Then

$$(4) \quad 0 \geq (\ell' - \ell)(x - x') + |x - x'|^2$$

$$(5) \quad \|\ell - \ell'\| |x - x'| \geq (\ell - \ell')(x - x') \geq |x - x'|^2.$$

Therefore $\|\ell - \ell'\| \geq |x - x'|$. \square

Since f is superlinear, the subdifferential ∂f is surjective and we have:

COROLLARY 12. *There exists a Lipschitz function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that*

$$\ell \in \partial f(x) \implies x = F(\ell).$$

Proof of Proposition 10. Let A_k be a set with $HD(A_k) = n - k$ such that A_k intersects any convex subset of dimension k . For example,

$$A_k = \{x \in \mathbb{R}^n \mid x \text{ has at least } k \text{ rational coordinates}\}.$$

Observe that

$$x \in \Sigma_k \implies \partial f(x) \text{ intersects } A_k \implies x \in F(A_k).$$

Therefore $\Sigma_k \subset F(A_k)$. Since F is Lipschitz, we have that $HD(\Sigma_k) \leq HD(A_k) = n - k$. \square

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(Received February 22, 2006)