International Journal of Mathematics
Vol. 19, No. 7 (2008) 777–799

NON-EMPTINESS OF MODULI SPACES OF COHERENT SYSTEMS

L. BRAMBILA-PAZ
CIMAT, Apdo. Postal 402, C.P. 36240
Guanajuato, Gto, México
lebp@cimat.mx

Received 31 August 2006
Revised 29 May 2007

Let X be a general smooth projective algebraic curve of genus $g \geq 2$ over $\mathbb{C}$. We prove that the moduli space $G(\alpha : n, d, k)$ of α-stable coherent systems of type (n, d, k) over X is empty if $k > n$ and the Brill–Noether number $\beta := \beta(n, d, n+1) = \beta(1, d, n+1) = g - (n+1)(n - d + g) < 0$. Moreover, if $0 \leq \beta < g$ or $\beta = g, n \not| g$ and for some $\alpha > 0$, $G(\alpha : n, d, k) \neq \emptyset$ then $G(\alpha : n, d, k) \neq \emptyset$ for all $\alpha > 0$ and $G(\alpha : n, d, k) = G(\alpha' : n, d, k)$ for all $\alpha, \alpha' > 0$ and the generic element is generated. In particular, $G(\alpha : n, d, n+1) \neq \emptyset$ if $0 \leq \beta \leq g$ and $\alpha > 0$. Moreover, if $\beta > 0$ $G(\alpha : n, d, n+1)$ is smooth and irreducible of dimension $\beta(1, d, n+1)$. We define a dual span of a generically generated coherent system. We assume $d < g + n_1 \leq g + n_2$ and prove that for all $\alpha > 0$, $G(\alpha : n_1, d, n_1 + n_2) \neq \emptyset$ if and only if $G(\alpha : n_2, d, n_1 + n_2) \neq \emptyset$. For $g = 2$, we describe $G(\alpha : 2, d, k)$ for $k > n$.

Keywords: Coherent systems; moduli space; stability; Brill-Noether

Mathematics Subject Classification 2000: 14H60, 14J60

1. Introduction

Let X be a smooth projective algebraic curve of genus $g \geq 2$ over $\mathbb{C}$. A coherent system over X of type (n, d, k) is a pair (E, V) where E is a vector bundle over X of rank n, degree d and V a linear subspace of $H^0(X, E)$ of dimension k.

A notion of stability for coherent systems was introduced in [12, 15, 11]. The definition of stability depends on a real parameter α, which corresponds to the choice of linearization of a group action. The coherent systems are also "augmented bundles" (see [2]) and are related with the existence of solutions of orthogonal vortex equations, where the parameter α appears in a natural way.

For any $\alpha \in \mathbb{R}$ denote by $G(\alpha : n, d, k)$ (respectively $\widetilde{G}(\alpha : n, d, k)$) the moduli space of α-stable (respectively α-semistable) coherent systems of type (n, d, k). From the definition of α-stability, one can see that in order to have α-stable coherent systems with $k \geq 1$, we need $\alpha > 0$. The expected dimension of $G(\alpha : n, d, k)$ is the Brill–Noether number $\beta(n, d, k) := n^2(g-1) + 1 - k(k - d + n(g-1))$. Note that if $k > n$, $\beta(n,d,k) = \beta(k-n,d,k)$. We denote by β the Brill–Noether number $\beta(n,d,n+1) = \beta(1,d,n+1) = g - (n+1)(n-d+g)$.

Basic properties of $G(\alpha : n,d,k)$ have been proved in [12, 11, 15] and particular cases in [8, 3, 5]. More general results can be found in [4, 10, 2]. Most of the detailed results known are for $k \leq n$. It is our purpose here to study the case $k > n$.

In [4, Proposition 4.6], it was proved that, for $k \geq n$, there exists α_L such that $G(\alpha : n,d,k) = G(\alpha' : n,d,k)$ if $\alpha, \alpha' > \alpha_L$. Denote this moduli space by $G_L(n,d,k)$.

For any (n,d,k), define $U(n,d,k)$ and $U^s(n,d,k)$ as

$$U(n,d,k) := \{(E,V) : (E,V) \in G_L(n,d,k) \text{ and } E \text{ is stable}\}$$

and

$$U^s(n,d,k) := \{(E,V) : (E,V) \text{ is of type } (n,d,k) \text{ and is } \alpha\text{-stable for all } \alpha > 0\}.$$

We prove the following (see Theorem 3.9).

Theorem 1. *Let X be general, $\beta < g$ or $\beta = g, n \nmid g$ and $k > n$. Then*

(1) *if $\beta < 0$, $G(\alpha : n,d,k) = \emptyset$ for all $\alpha > 0$;*
(2) *if for some $\alpha > 0, G(\alpha : n,d,k) \neq \emptyset$, then $G(\alpha : n,d,k) \neq \emptyset$ for all $\alpha > 0$;*
(3) *$G(\alpha : n,d,k) = G(\alpha' : n,d,k)$ for all $\alpha, \alpha' > 0$ i.e. $\alpha_L = 0$;*
(4) *$(E,V) \in G(\alpha : n,d,k)$ if and only if (E,V) is generically generated and $H^0(I_E^*) = 0$, where I_E is the image of the evaluation map $V \otimes \mathcal{O} \to E$;*
(5) *if for some $\alpha > 0$, $G(\alpha : n,d,k) \neq \emptyset$, then $U(n,d,k) = G(\alpha : n,d,k)$.*

Note that the results of Theorem 1 deal with the moduli spaces of coherent systems of type (n,d,k) whereas β refers to $(n,d,n+1)$. Moreover, if $\beta(n,d,n+1) \leq g$, $\beta(n,d,k) < 0$ for $k > n+1$.

If $\alpha_L = 0$, denote $G_L(n,d,k)$ by $G(n,d,k)$. In particular, $\mathcal{G}_d^{k-1} := G(1,d,k)$. For $k = n+1$, we have (see Theorem 4.3).

Theorem 2. *Let X be general and $\beta := \beta(n,d,n+1) \leq g$. Then*

(1) *$G(\alpha : n,d,n+1) \neq \emptyset$ if and only if $\beta \geq 0$;*
(2) *if $\beta \geq 0$, then $G(n,d,n+1) := G(\alpha : n,d,n+1) = G(\alpha' : n,d,n+1)$ for all $\alpha, \alpha' > 0$ and $\alpha_L = 0$;*
(3) *if $\beta > 0$, then $G(n,d,n+1)$ is smooth and irreducible of dimension β and the generic element is generated;*
(4) *$U^s(n,d,n+1) = G(n,d,n+1)$ and is birationally equivalent to $\mathcal{G}_d^n$;*
(5) *if $\beta = 0$, $G(n,d,n+1) \cong \mathcal{G}_d^n$ and the number of points of $G(n,d,n+1)$ is*

$$g! \prod_{i=0}^{n} \frac{i!}{(g-d+n+i)!}\,.$$

Moreover, (see Theorem 4.7)

Theorem 3. *If X is general and $g \geq n^2 - 1$, then for any degree $d \geq g + n - \frac{g}{n+1}$*

(1) $G(\alpha : n, d, n+1) \neq \emptyset$ *for all* $\alpha > 0$;
(2) $U(n, d, n+1) \neq \emptyset$ *and is smooth and irreducible.*

As was pointed out in [3, 4], coherent systems are related with Brill–Noether theory. Let $B(n, d, k)$ (respectively $\widetilde{B}(n, d, k)$) be the Brill–Noether locus defined by stable (respectively semistable) vector bundles of rank n, degree d and $\dim H^0(X, E) \geq k$. It is well known that for "small" α, (E, V) α-stable implies E semistable and E stable implies (E, V) α-stable. The approach to study the Brill–Noether loci in [4] is to describe $G(\alpha : n, d, k)$, usually for "large" α, and through "flips" obtain information of $G(\alpha : n, d, k)$ for smaller α.

In our case, i.e. $\beta < g$ or $\beta = g, n \not| g$ and $k > n$, it is enough to know the non-emptiness for one α to obtain non-emptiness for all α. Moreover, there are no "flips".

In [16], it was proved that if X is general and $g \geq \beta(n, d, n+1) \geq 0$, $B(n, d, n+1)$ is non-empty and has a component of the correct dimension. From the above results of coherent systems, we have (see Corollary 4.5)

Corollary 4. *If X is general and $g \geq \beta \geq 0$, $B(n, d, n+1)$ is irreducible if $\beta > 0$ and $G(n, d, n+1)$ is a desingularization of (the closure of) the Brill–Noether locus $B(n, d, n+1)$. Moreover, the natural map $\phi : G(\alpha : n, d, n+1) \to \widetilde{B}(n, d, n+1)$ is an isomorphism on the complement of the singular locus of $B(n, d, n+1) \subset \widetilde{B}(n, d, n+1)$.*

Actually, [4, Conditions 11.3] are satisfied in this case and hence the results in [4, Sec. 11] hold.

Besides the known relation between coherent systems and Brill–Noether theory, our results on $G(n, d, n+1)$ can be related with other problems. Given a generated linear system (L, V), we have the natural map

$$\phi_V : X \to \mathbb{P}(V^*).$$

In particular, if L has degree d and dim $V = n + 1$, we have (see Theorem 4.8).

Theorem 5. *Let X be general, $0 \leq \beta(n, d, n+1)$ and $T\mathbb{P}$ the tangent bundle of $\mathbb{P}(V^*)$. If $\beta < g$ or $\beta = g$ and $n \not| g$, then $\phi_V^*(T\mathbb{P})$ is stable. If either $g \geq n^2 - 1$ or $\beta = g$, $n|g$ and g and n are not both equal to 2, then there exist linear systems (L, V) such that $\phi_V^*(T\mathbb{P})$ is stable.*

We define a dual span of a generically generated coherent system (see Definition 5.3) and denote by $D(E, V) = (D(E)_\ell, V^*)$ a dual span of (E, V). If I_E is the image of the evaluation map $V \otimes \mathcal{O} \to E$ we prove (see Theorems 5.7 and 5.13)

Theorem 6. *Let X be a general curve of genus g and $d < g + n_1 \leq g + n_2$, then for all $\alpha > 0, G(\alpha : n_1, d, n_1 + n_2) \neq \emptyset$ if and only if $G(\alpha : n_2, d, n_1 + n_2) \neq \emptyset$.*

Theorem 7. *Let $(E,V) \in G(\alpha : n_1, d, n_1 + n_2)$. If either of the Petri maps of (I_E, V) or $(I_{D(E)_\ell}, V^*)$ is injective, then*

(1) *$G(\alpha : n_1, d, n_1+n_2)$ is smooth of dimension $\beta(n_1, d, n_1+n_2)$ in a neighbourhood of (E,V).*
(2) *$G(\alpha : n_2, d, n_1+n_2)$ is smooth of dimension $\beta(n_2, d, n_1+n_2)$ in a neighbourhood of the dual span $D(E,V)$.*

Denote by $G_0(n,d,k)$ the moduli space $G(\alpha : n, d, k)$ for "small" values of α (see Remark 2.2 (2)). For $n = 2$, we have (see Theorem 6.1).

Theorem 8. *Let X be general, $s \geq 3$ and $d < s+2g-\frac{4g}{s+2}$. If $G_0(2, d, 2+s)$ is non-empty then $G(\alpha : 2, d, 2+s)$ is non-empty for all $\alpha > 0$. Moreover, $U(2, d, 2+s) \neq \emptyset$.*

For $n = 2$ and $g = 2$, from the above results and the Riemann–Roch theorem, we know that

(1) if $d < 4$ and $k \geq 3$, $G(\alpha : 2, d, k) = \emptyset$ for all $\alpha > 0$;
(2) if $d = 5$ and $k > 3$, $U(2, d, k) = \emptyset$ and $G_0(2, 5, k) = \emptyset$;
(3) if $d \geq 6$ and $k = 3, 4$, $G(\alpha : 2, d, k) \neq \emptyset$ for all $\alpha > 0$. Moreover, $U(2, d, k) \neq \emptyset$;
(4) if $d \geq 6$ and $k > d - 2$, $U(2, d, k) = \emptyset$ and $G_0(2, d, k) = \emptyset$.

In particular for $d = 4, 5$, we have (see Theorems 6.11–6.13)

Theorem 9.

(1) *$U(2, 4, k) = \emptyset$ for $k \geq 3$.*
(2) *$G_0(2, 4, k) = \emptyset$ for $k \geq 5$.*
(3) *$G(\alpha : 2, 4, 3) \neq \emptyset$ for all $\alpha > 0$.*
(4) *$U^s(2, 4, 3) \cong G_L(2, 4, 3) \cong Pic^4(X)$.*

Theorem 10.

(1) *$\widetilde{G}_0(2, 4, 4) = \{(K \oplus K, H^0(K \oplus K))\}$.*
(2) *$G_0(2, 4, 4) = \emptyset$.*
(3) *$\widetilde{G}(\alpha : 2, 4, 4) \neq \emptyset$ for all $\alpha > 0$.*

Theorem 11.

(1) *$G_0(2, 5, 3) \neq \emptyset$.*
(2) *$U(2, 5, 3) \neq \emptyset$.*
(3) *$U(2, 5, 3) \neq G_0(2, 5, 3)$.*

Notation

We will denote by K the canonical bundle over X, by I_E the image of the evaluation map $V \otimes \mathcal{O} \to E$, $H^i(X,E)$ by $H^i(E)$, $\dim H^i(X,E)$ by $h^i(E)$, the rank of E by n_E, the degree of E by d_E and $\det(E)$ by L_E. By a general curve, we mean a Petri curve i.e. the Petri map

$$H^0(L) \otimes H^0(L^* \otimes K) \to H^0(K)$$

is injective for every line bundle L over X.

2. General Results

Let X be an irreducible smooth projective curve over $\mathbb{C}$ of genus $g \geq 2$. For any $\alpha \in \mathbb{R}$, define the α-slope of the coherent system (E, V) of type (n, d, k) as

$$\mu_\alpha(E, V) := \mu(E) + \alpha \frac{k}{n},$$

where $\mu(E) := d/n$ is the slope of the vector bundle E. A coherent subsystem $(F, W) \subseteq (E, V)$ is a coherent system such that $F \subseteq E$ and $W \subseteq V \cap H^0(F)$. For any $\alpha \in \mathbb{R}$, a coherent system (E, V) is α-stable (respectively α-semistable) if for all proper coherent subsystems (F, W),

$$\mu_\alpha(F, W) < \mu_\alpha(E, V) \quad (\text{respectively } \leq).$$

Denote the moduli space of α-stable (respectively α-semistable) coherent systems of type (n, d, k) by $G(\alpha : n, d, k)$ (respectively $\tilde{G}(\alpha : n, d, k)$) and by $\beta(n, d, k)$ the Brill–Noether number $\beta(n, d, k) := n^2(g-1) + 1 - k(k - d + n(g-1))$. From the infinitesimal study of the coherent systems (see [4, 10]), we have that

Proposition 2.1. *If $(E, V) \in G(\alpha : n, d, k)$, then $G(\alpha : n, d, k)$ is smooth of dimension $\beta(n, d, k)$ in a neighbourhood of (E, V) if and only if the Petri map $V \otimes H^0(E^* \otimes K) \to H^0(E \otimes E^* \otimes K)$ is injective. Moreover, $T_{(E,V)}G(\alpha : n, d, k) = \mathrm{Ext}^1((E, V), (E, V))$.*

If $B(n, d, k)$ (respectively $\widetilde{B}(n, d, k)$) is the Brill–Noether locus of stable (respectively semistable) vector bundles, then for "small" α, there is a natural map

$$\phi : G(\alpha : n, d, k) \to \widetilde{B}(n, d, k)$$

defined by $(E, V) \mapsto E$ that is injective over $B(n, d, k) - B(n, d, k+1)$.

Given a triple (n, d, k) denote by $C(n, d, k)$ the set

$$C(n, d, k) := \left\{ \alpha \in \mathbb{R} | 0 \leq \alpha = \frac{nd' - n'd}{n'k - nk'} \quad \text{with} \quad 0 \leq k' \leq k, 0 < n' \leq n, \right. \\ \left. \text{and} \quad nk' \neq n'k \right\}.$$

An element α in $C(n, d, k)$ is called a virtual critical point. The set $C(n, d, k)$ defines a partition of the interval $[0, \infty)$. With the natural order on $\mathbb{R}$, label the virtual critical points as α_i.

It is known (see [2, 4]) that

Remark 2.2. (1) If $(n, d, k) = 1$, then $G(\alpha : n, d, k) = \widetilde{G}(\alpha : n, d, k)$, for $\alpha \notin C(n, d, k)$.

(2) If $\alpha', \alpha'' \in (\alpha_i, \alpha_{i+1})$, then $G(\alpha' : n, d, k) = G(\alpha'' : n, d, k)$. Denote by $G_i(n, d, k)$ the moduli space $G(\alpha : n, d, k)$ for any $\alpha \in (\alpha_i, \alpha_{i+1})$.

(3) For $k \geq n$, there exists α_L such that for any $\alpha, \alpha' > \alpha_L$, $G(\alpha : n, d, k) = G(\alpha' : n, d, k)$. Denote by $G_L(n, d, k)$ the moduli space $G(\alpha : n, d, k)$ for $\alpha > \alpha_L$.
(4) Every irreducible component of $G_i(n, d, k)$ has dimension at least $\beta(n, d, k)$.

Remark 2.3. Let (E, V) be a coherent system of type (n, d, k). From the definition of α-stability and stability of a vector bundle, we have that

(1) if $(E, V) \in G(\alpha : n, d, k)$ and E is stable, then (E, V) is α'-stable for all $0 < \alpha' < \alpha$;
(2) if E is stable and for all coherent subsystems $(F, W) \subset (E, V)$, $\frac{\dim W}{n_F} \leq \frac{k}{n}$, then (E, V) is α-stable for all $\alpha > 0$;
(3) if E is semistable and for all coherent subsystems $(F, W) \subset (E, V)$, $\frac{\dim W}{n_F} < \frac{k}{n}$, then (E, V) is α-stable for all $\alpha > 0$;
(4) if E is semistable and for all coherent subsystems $(F, W) \subset (E, V)$, $\frac{\dim W}{n_F} \leq \frac{k}{n}$, then (E, V) is α-semistable for all $\alpha > 0$.

Let (E, V) be a coherent system of type (n, d, k) with $k > n$. We shall say that (E, V) (or E) is generically generated if the image I_E of the evaluation map $V \otimes \mathcal{O} \to E$ has rank n. That is, we have the exact sequence

$$0 \to I_E \to E \to \tau \to 0 \tag{2.1}$$

where τ is a torsion sheaf. We say that (E, V) (or E) is generated if $\tau = 0$; and strictly generically generated if $\tau \neq 0$.

Remark 2.4. Note that if (E, V) is generated with $H^0(E^*) = 0$, any quotient bundle Q is generated and $H^0(Q^*) = 0$.

We give a proposition that we will use in the following sections.

Proposition 2.5. *Let (E, V) be a generated coherent system of type (n, d, k) with E semistable and $k = n + s, s \geq 1$. If (F, W) is a coherent subsystem of (E, V),*

(1) $\dim W \leq n_F + s - 1$;
(2) *if* $\frac{(s-1)n}{s} < n_F, \mu_\alpha(F, W) < \mu_\alpha(E, V)$ *for all* $\alpha > 0$;
(3) *if* $\dim W \leq n_F$, $\mu_\alpha(F, W) < \mu_\alpha(E, V)$ *for all* $\alpha > 0$;
(4) *if* (E, V) *is of type* $(n, d, n + 1)$, *then it is* α*-stable for all* $\alpha > 0$.

Proof. Note that $d > 0$, so E^* is semistable of negative degree, hence $H^0(E^*) = 0$. Let (F, W) be a coherent subsystem of (E, V) and (Q, Z) the quotient coherent system. Since Q is generated and $H^0(Q^*) = 0$,

$$\dim(V) - \dim(H^0(F) \cap V) \geq n_Q + 1,$$

that is, $n_F + s - 1 \geq \dim(H^0(F) \cap V) \geq \dim W$.

If $\frac{(s-1)n}{s} < n_F$, $\frac{\dim(W)}{n_F} < \frac{\dim(V)}{n}$ and from Remark 2.3,

$$\mu_\alpha(F, W) < \mu_\alpha(E, V)$$

for all $\alpha > 0$. Similarly, for $\dim W \leq n_F$, $\mu_\alpha(F, W) < \mu_\alpha(E, V)$ for all $\alpha > 0$.

If $s = 1$, for all coherent subsystems (F, W), $n_F \geq \dim W$, therefore, from Remark 2.3, (E, V) is α-stable for all $\alpha > 0$. □

For any (n, d, k), define $U^s(n, d, k)$ and $U(n, d, k)$ as

$$U^s(n, d, k) := \{(E, V) : (E, V) \text{ is of type } (n, d, k) \text{ and is } \alpha\text{-stable for all } \alpha > 0\}; \tag{2.2}$$

and

$$U(n, d, k) := \{(E, V) : (E, V) \in G_L(n, d, k) \text{ and } E \text{ is stable}\}.$$

From Remark 2.3(1), we have that $U(n, d, k) \subset U^s(n, d, k)$. Note that $U^s(n, d, k)$ is embedded in $G_L(n, d, k)$. From the openness of α-stability, it follows that $U^s(n, d, k)$ is an open subset of $G_L(n, d, k)$. Moreover, if $(E, V) \in U^s(n, d, k)$, E is semistable.

Proposition 2.6. *If $d \geq n(2g - 1)$, $G(\alpha : n, d, n + 1) \neq \emptyset$ for all $\alpha > 0$. Moreover, $U(n, d, n + 1) \neq \emptyset$.*

Proof. If $d \geq n(2g - 1)$, every stable bundle E of rank n and degree d is generated and $h^0(E) \geq n + 1$. A generic subspace V of $H^0(E)$ of dimension $n + 1$ generates E. By Proposition 2.5(4), (E, V) is α-stable for all $\alpha > 0$. Hence $U(n, d, n + 1) \neq \emptyset$. □

Our aim is to prove that such coherent systems exist for smaller d.

3. Vector Bundles with Sections

In this section, we assume that X is a general curve and $k \geq n + 1$. We give three lemmas that we will use.

Lemma 3.1. *If F is generated and $H^0(F^*) = 0$, then $\mu(F) \geq 1 + \frac{g}{n_F+1}$.*

Proof. Recall from [14, Proposition 3.2] that if F is generated and $H^0(F^*) = 0$, then it is generated by a linear subspace $W \subseteq H^0(F)$ of dimension $n_F + 1$, and $h^0(\det(F)) \geq n_F + 1$. Moreover, the Brill–Noether theory for line bundles implies that

$$\beta(1, d_F, n_F + 1) = g - (n_F + 1)(n_F - d_F + g) \geq 0,$$

that is,

$$\mu(F) \geq 1 + \frac{g}{n_F + 1}. \tag{3.1}$$

□

Lemma 3.2. *Let E be a vector bundle such that $d_E \leq n_E + g$. If F is a vector bundle of rank $n_F < n_E$ that is generically generated and $H^0(I_F^*) = 0$, then $\mu(F) \geq \mu(E)$. Moreover, $\mu(F) = \mu(E)$ is possible only if $n_F = n_E - 1$.*

Proof. By hypothesis, $\mu(E) \leq \frac{g}{n_E} + 1$. If $\mu(I_F) \leq \mu(F) < \mu(E)$, then from Lemma 3.1, we get a contradiction. □

Corollary 3.3. *If E is a semistable bundle with $d_E < n_E + g$ or $d_E = g + n_E$, $n_E \not| g$, then E cannot have a proper generically generated subbundle F with $H^0(I_F^*) = 0$.*

Proof. Suppose that $F \subset E$ is generically generated with $H^0(I_F^*) = 0$. From the semistability and Lemma 3.2, $n_F = n_E - 1$ and $d_E = g + n_E$. But then E/F is a line bundle and $\mu(F) = \mu(E) = \mu(E/F)$, which is a contradiction if $d_E = g + n_E$, $n_E \not| g$. □

Lemma 3.4. *If F is generated by a subspace W of dimension* $\dim W \geq n_F + 1$, *then either $H^0(F^*) = 0$ or there is a subbundle G with $n_G < n_F$ that is generated and $H^0(G^*) = 0$.*

Proof. If $H^0(F^*) \neq 0$, then $F \cong \mathcal{O}^s \oplus G$ where G is generated, $H^0(G^*) = 0$ and $1 \leq n_G < n_F$. □

For coherent systems of type (n, d, k) with $k \geq n + 1$, we have the following propositions.

Proposition 3.5. *Let (E, V) be a coherent system of type (n, d, k) with $d < n + g$ or $d = g + n, n \not| g$. Then E is stable if and only if (E, V) is generically generated and $H^0(I_E^*) = 0$. Moreover, if $d = g + n, n | g$ and (E, V) is generically generated with $H^0(I_E^*) = 0, E$ is semistable.*

Proof. Suppose E is stable. Then I_E is generated by V. If $H^0(I_E^*) = 0$, from Corollary 3.3, $n_{I_E} = n_E$. If $H^0(I_E^*) \neq 0$, from Lemma 3.4, and Corollary 3.3, we get a contradiction.

Now suppose (E, V) is generically generated with $H^0(I_E^*) = 0$. If E is not stable, let Q be a quotient bundle such that $\mu(Q) \leq \mu(E)$. We have the following diagram

$$\begin{array}{ccccccccc} 0 & \to & I_E & \to & E & \to & \tau & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Q_1 & \to & Q & \to & \tau' & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{3.2}$$

where Q_1 is a quotient bundle of I_E such that $\mu(Q_1) \leq \mu(Q)$, $n_{Q_1} = n_Q$ and since I_E is generated and $H^0(I_E^*) = 0$, Q_1 is generated and $H^0(Q_1^*) = 0$. Thus,

$$1 + \frac{g}{n_Q + 1} \leq \mu(Q_1) \leq \mu(Q) \leq \mu(E) = \frac{d}{n} \leq 1 + \frac{g}{n}. \tag{3.3}$$

If $n_Q + 1 < n$, we get a contradiction. If $n_Q + 1 = n$, $\mu(Q) = \mu(E)$ and hence E is semistable. But, in that case, there exists a line bundle L_0 such that $Q \cong E/L_0$ and $\mu(E) = \mu(Q) = \mu(L_0)$. This will be a contradiction if $n \not| g$. Therefore E is stable. □

Proposition 3.6. *A generically generated coherent system (E, V) of type (n, d, k) with $d < g + n$ or $d = g + n, n \not| g$ and $H^0(I_E^*) = 0$ is α-stable for all $\alpha > 0$.*

Proof. From Proposition 3.5, E is stable. Let $(F, W) \subset (E, V)$ be a coherent subsystem of (E, V) with $n_F < n_E$. If $\dim(W) \geq n_F + 1$, the evaluation map defines a subbundle F', with $n_{F'} \leq n_F < n_E$ which is generically generated with $H^0(F'^*) = 0$. From Lemmas 3.4 and 3.2, $\mu(F') \geq \mu(E)$ which contradicts stability of E. Hence, $\dim W \leq n_F$ and from Remark 2.3, (E, V) is α-stable for all $\alpha > 0$. □

For $k = n + 1$, we have

Proposition 3.7. *A generically generated coherent system (E, V) of type $(n, d, n + 1)$ with $d \leq g + n$ and $H^0(I_E^*) = 0$ is α-stable for all $\alpha > 0$.*

Proof. From Proposition 3.5, E is semistable. Let (Q, W) be a proper quotient coherent system of (E, V). Then (Q, W) is generically generated. Moreover, since I_Q is a quotient of I_E, $H^0(I_Q^*) = 0$ and hence $\dim W \geq n_Q + 1$. So $\frac{n+1}{n} < \frac{\dim W}{n_Q}$ and the result follows from Remark 2.3(3). □

Conversely,

Proposition 3.8. *If (E, V) is an α-stable coherent system of type (n, d, k) with $d \leq g + n$, then (E, V) is generically generated and $H^0(I_E^*) = 0$. Moreover, E is semistable and stable if $d < n + g$ or $d = g + n, n \not| g$.*

Proof. Suppose that $I_E = \mathcal{O}^s \oplus G$ with $0 \leq s \leq n_{I_E} - 1$, G generated, $H^0(G^*) = 0$ and $\mu(G) \geq \frac{g}{n_G+1} + 1$. From the α-stability of (E, V) we have

$$\mu_\alpha(G, H^0(G) \cap V) < \mu_\alpha(E, V),$$

that is,

$$\alpha \left(\frac{k - s}{n_G} - \frac{k}{n} \right) < \mu(E) - \mu(G).$$

If $n_G < n$, then $\mu(E) - \mu(G) \leq \frac{g}{n} + 1 - (\frac{g}{n_G+1} + 1) \leq 0$, hence

$$\alpha \left(\frac{k - s}{n_G} - \frac{k}{n} \right) < 0$$

which is a contradiction since $s \leq n - n_G$. Hence $n_{I_E} = n$, (E, V) is generically generated and $H^0(I_E^*) = 0$. The last part follows from Proposition 3.5. □

From Propositions 3.5, 3.6 and 3.8, we have Theorem 1.

Theorem 3.9. *Let X be general, $\beta = \beta(n, d, n+1) < g$ or $\beta = g$, $n \not| g$ and $k \geq n+1$. Then*

(1) *if $\beta < 0, G(\alpha : n, d, k) = \emptyset$ for all $\alpha > 0$;*
(2) *if for some $\alpha > 0, G(\alpha : n, d, k) \neq \emptyset$, then $G(\alpha : n, d, k) \neq \emptyset$ for all $\alpha > 0$;*
(3) *$G(\alpha : n, d, k) = G(\alpha' : n, d, k)$ for all $\alpha, \alpha' > 0$ i.e. $\alpha_L = 0$;*
(4) *$(E, V) \in G(\alpha : n, d, k)$ if and only if (E, V) is generically generated and $H^0(I_E^*) = 0$;*
(5) *if for some $\alpha > 0$, $G(\alpha : n, d, k) \neq \emptyset$, then $U^s(n, d, k) = G(\alpha : n, d, k)$ and $U(n, d, k) \neq \emptyset$.*

Proof. Recall from the definition of β that $\beta(n, d, n + 1) = \beta(1, d, n + 1) = g - (n + 1)(n - d + g)$. Hence,

$$0 \leq \beta \Leftrightarrow \frac{g}{n+1} + 1 \leq \frac{d}{n}.$$

Moreover,

$$\beta \leq g \Leftrightarrow d \leq g + n.$$

If $(E, V) \in G(\alpha : n, d, k)$, E is generically generated and $H^0(I_E^*) = 0$ (see Proposition 3.8). Hence, by Lemma 3.1, $\mu(E) \geq \frac{g}{n+1} + 1$ i.e. $\beta(n, d, n + 1) \geq 0$. Parts (2)–(5) follow from Propositions 3.6 and 3.8. □

Corollary 3.10. *If $d < g + n$ and $g \leq n$, $G(\alpha : n, d, k) = \emptyset$ for all $\alpha > 0$ and $k \geq n + 1$.*

Proof. This follows from Theorem 3.9 since the Brill–Noether number is negative. □

4. Coherent Systems of Type $(n, d, n + 1)$

From Remark 2.2, we have that $G(\alpha : n, d, n + 1) = \widetilde{G}(\alpha : n, d, n + 1)$, for $\alpha \notin C(n, d, n + 1)$.

For $d \geq n(2g - 1)$, from Proposition 2.6, $U(n, d, n + 1) \neq \emptyset$. For small values of d we have the following proposition (see also [8, 16]).

Proposition 4.1. *If X is general and $0 \leq \beta \leq g$, then*

(1) *there exist generated coherent systems (E, V) with E semistable and, in particular, $U^s(n, d, n + 1) \neq \emptyset$;*
(2) *except when $g = n = 2$ and $d = 4$, there exist generated coherent systems (E, V) with E stable and, in particular, $U(n, d, n + 1) \neq \emptyset$.*

Proof. (1) The dimension of the subvariety consisting of line bundles L, for which L is not generated by a subspace $V \subset H^0(L)$ of dimension $n + 1$, has dimension

$g - (n+1)(n-(d-1)+g)+1 < \beta$, since they define a line bundle of degree $d-1$ with $n+1$ sections. Thus, from the Brill–Noether theory for line bundles, the set of generated line bundles L of degree d with $n+1 \leq \dim V \leq h^0(L)$ defines a non-empty open set of the Jacobian $J^d(X)$.

We have the following exact sequence.

$$0 \to E^* \to V \otimes \mathcal{O} \to L \to 0. \tag{4.1}$$

The coherent system (E, V^*) is generated and $H^0(E^*) = 0$. Hence, by Proposition 3.5, E is semistable and, by Proposition 2.5, (E, V^*) is α-stable for all $\alpha > 0$. So $U^s(n, d, n+1) \neq \emptyset$.

(2) If $d < g+n$ or if $d = g+n$ and $n \not| g$, the bundles E constructed in (1) are stable by Proposition 3.5; hence $U(n, d, n+1) \neq \emptyset$. If $d = g+n$ and $n|g$, and $g = an$ and $d = (a+1)n$, Butler [8] proves that E is stable unless L has the form $L \cong L'(Z)$ where Z is an effective divisor of degree $a+1$ and L' a line bundle with $h^0(L') = n$.

The Brill–Noether number $\beta(1, (a+1)(n-1), n) = 0$, hence there are finitely many choices for L'. The dimension of the family formed of the $L'(Z)$ has dimension $a+1$. Since $a+1 < an = g$, except for $g = n = 2$, we can find L lying outside this family. If $V \subset H^0(L)$ has dimension $n+1$ and generates L, then the kernel of the evaluation map

$$0 \to E^* \to V \otimes \mathcal{O} \to L \to 0,$$

together with the space V^*, defines the generated coherent system (E, V^*) with E stable. By Proposition 2.5, (E, V^*) is α-stable for all $\alpha > 0$, so $U(n, d, n+1) \neq \emptyset$.

□

Lemma 4.2. *Suppose that $(E, V) \in G(\alpha : n, d, n+1)$ is generically generated. Then $G(\alpha : n, d, n+1)$ is smooth of dimension β at (E, V).*

Proof. Let L denote the dual of the kernel of the evaluation map $V \otimes \mathcal{O} \to E$. The kernel of the Petri map

$$V \otimes H^0(E^* \otimes K) \to H^0(E \otimes E^* \otimes K) \tag{4.2}$$

is $H^0(L^* \otimes E^* \otimes K)$. Since E is generically generated from the dual of the exact sequence (2.1), we have

$$0 \to E^* \otimes L^* \otimes K \to I_E^* \otimes L^* \otimes K \to \tau \to 0. \tag{4.3}$$

However, since E is generically generated, I_E is generated and we have the following exact sequence

$$0 \to I_E^* \otimes L^* \otimes K \to V^* \otimes L^* \otimes K \to K \to 0. \tag{4.4}$$

The injectivity of the Petri map for line bundles gives $H^0(I_E^* \otimes L^* \otimes K) = 0$ and from (4.3), $H^0(E^* \otimes L^* \otimes K) = 0$. Therefore, $G(n, d, n+1)$ is smooth of dimension $\beta \geq 0$.

□

It is well known that for $n = 1$, the concept of stability is independent of α and $G(1, d, k) := G(\alpha : 1, d, k) = \mathcal{G}_d^{k-1}$, where $\mathcal{G}_d^{k-1}$ parameterizes linear series of degree d and dimension k ([1, Chap. 5]).

Therefore we have Theorem 2

Theorem 4.3. *Let X be general and $\beta = \beta(n, d, n+1) \leq g$. Then*

(1) $G(\alpha : n, d, n+1) \neq \emptyset$ *if and only if* $\beta \geq 0$;
(2) *if* $\beta \geq 0$, *then* $G(n, d, n+1) := G(\alpha : n, d, n+1) = G(\alpha' : n, d, n+1)$ *for all* $\alpha, \alpha' > 0$ *and* $\alpha_L = 0$;
(3) *if* $\beta > 0$, *then* $G(n, d, n+1)$ *is smooth and irreducible of dimension* β *and the generic element is generated*;
(4) $U^s(n, d, n+1) = G(n, d, n+1)$ *and is birationally equivalent to* $\mathcal{G}_d^n$;
(5) *if* $\beta = 0$ $G(n, d, n+1) \cong \mathcal{G}_d^n$ *and the number of points of* $G(n, d, n+1)$ *is*

$$g! \prod_{i=0}^{n} \frac{i!}{(g - d + n + i)!}.$$

Proof. (1) follows from Theorem 3.9(1) and Proposition 4.1.

(2) follows from Propositions 3.7 and 3.8.

For (3), smoothness follows from Proposition 3.8 and Lemma 4.2. Assume $\beta > 0$. The set of coherent systems $(E, V) \in G(\alpha : n, d, n+1)$ that are generated is parameterized by an irreducible variety and has dimension β (it is in correspondence with an open dense set in $B(1, d, n+1)$, which is irreducible). As in [4, Theorem 5.11], the irreducibility of $G(n, d, n+1)$ follows from the fact that the variety that parameterizes strictly generically generated coherent systems has dimension $< \beta$, so it cannot define a new component (see Remark 2.2). Hence, $G(n, d, n+1)$ is irreducible.

(4) follows from Proposition 3.8 and (3).

For (5), if $\beta = 0$, every $(E, V) \in G(n, d, n+1)$ is generated, hence $G(n, d, n+1) \cong \mathcal{G}_d^n$ which has cardinality

$$g! \prod_{i=0}^{n} \frac{i!}{(g - d + n + i)!}$$

(see [1, Chap. V, Theorem 4.4]). □

Remark 4.4. In our case, except when $g = n = 2$ and $d = 4$, [4, Conditions 11.3] are satisfied for $(n, d, n+1)$, i.e. $\beta(n, d, n+1) \leq n^2(g-1)$, $G_0(n, d, n+1)$ is irreducible and $B(n, d, n+1) \neq \emptyset$ and hence the results in [4] that assume Conditions 11.3 hold.

Corollary 4.5. *If X is a general curve and $0 \leq \beta(n, d, n+1) \leq g$, the Brill–Noether locus $B(n, d, n+1)$ is non-empty and irreducible except possibly when $g = n = 2$ and $d = 4$. Moreover, $G(\alpha : n, d, n+1)$ is a desingularization of (the closure of)*

$B(n,d,n+1)$. The natural map $\phi : G(\alpha : n,d,n+1) \to \widetilde{B}(n,d,n+1)$ is an isomorphism on $B(n,d,n+1) - B(n,d,n+2)$.

Note that the degree of the bundle E in such coherent systems satisfies the following inequalities

$$g + n - \frac{g}{n+1} \leq d \leq g + n. \tag{4.5}$$

Proposition 4.6. *If X is general and $g \geq n^2 - 1$, then, for any degree $d \geq g + n - \frac{g}{n+1}, U(n,d,n+1) \neq \emptyset$.*

Proof. From Proposition 4.1, there exist generated coherent systems (E, V) with E stable for $g+n-\frac{g}{n+1} \leq d \leq g+n$. Moreover they are α-stable for all $\alpha > 0$. Given such a coherent system (E, V) and an effective line bundle L, choose a section s of L and define the coherent system (E', V') as $E' := E \otimes L$ and V' the image of V in $H^0(E \otimes L)$ under the canonical inclusion $H^0(E) \hookrightarrow H^0(E \otimes L)$ induced by s. It is well known that E is stable if and only if E' is stable. Moreover, (see [15, Lemma 1.5]) (E, V) is α-stable if and only if (E', V') is α-stable.

Therefore, if $g \geq n^2 - 1$, the length of the interval $[\frac{g}{n+1}]$ is greater than or equal to $n - 1$, so after tensoring by an effective line bundle, we can obtain all the values of $d \geq g + n - \frac{g}{n+1}$. □

Moreover, from Theorem 4.3, Proposition 4.6, Lemma 4.2 and [4, Theorem 5.11], we have

Theorem 4.7. *If X is general and $g \geq n^2 - 1$, then for any degree $d \geq g+n-\frac{g}{n+1}$,*

(1) *$G(\alpha : n, d, n+1) \neq \emptyset$ for all $\alpha > 0$.*
(2) *$U^s(n, d, n+1) \neq \emptyset$ and is smooth and irreducible.*
(3) *$U(n, d, n+1) \neq \emptyset$ and is smooth and irreducible.*

Besides the known relation between coherent systems and Brill–Noether theory, our results on $G(n, d, n+1)$ can be related with other problems. Given a generated linear system (L, V), we have the natural map

$$\phi_V : X \to \mathbb{P}(V^*).$$

In particular, if L has degree d and $\dim V = n + 1$, we have

Theorem 4.8. *Let X be general, $0 \leq \beta(n, d, n+1)$ and $T\mathbb{P}$ the tangent bundle of $\mathbb{P}(V^*)$. If $\beta < g$ or $\beta = g$ and $n \not| g$, then $\phi_V^*(T\mathbb{P})$ is stable. If either $g \geq n^2 - 1$ or $\beta = g, n|g$ and g and n are not both equal to 2, then there exist linear systems (L, V) such that $\phi_V^*(T\mathbb{P})$ is stable.*

Proof. Under the hypothesis of the theorem, there exist generated linear systems (L, V). Denote by E the dual of the kernel of the evaluation map. Consider the dual Euler sequence

$$0 \to \Omega^1_{\mathbb{P}}(1) \to V \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(1) \to 0 \tag{4.6}$$

where $\Omega^1_{\mathbb{P}} = T\mathbb{P}^*$.

From the pull-back of (4.6), we have that $E \otimes L \cong \phi_V^*(T\mathbb{P})$ (see [9]). Recall that if E is stable, $E \otimes L$ is stable.

If $\beta < g$ or $\beta = g$ and $n \not| g$, all such E are stable by the proof of Proposition 4.1. If $\beta = g$, $n|g$ and g and n are not both equal to 2, some such E are stable, again by the proof of Proposition 4.1. Finally, if $g \geq n^2 - 1$, $U(n, d, n+1)$ is non-empty and irreducible by Theorem 4.7 and its generic element (E, V^*) is generated by the proof of [4, Theorem 5.11]. Now define (L, V) by dualizing the evaluation sequence of (E, V^*). □

5. Dual Span

For a generated coherent system (E, V) of type (n, d, k) with $H^0(E^*) = 0$, denote by $D(E)$ the dual of the kernel of the evaluation map, that is, we have the following exact sequences

$$0 \to D(E)^* \to V \otimes \mathcal{O} \to E \to 0 \tag{5.1}$$

and

$$0 \to E^* \to V^* \otimes \mathcal{O} \to D(E) \to 0. \tag{5.2}$$

In [4, 5.4], the coherent system $(D(E), V^*)$ is called the dual span of (E, V). Note that $(D(E), V^*)$ is a generated coherent system of type $(k - n, d, k)$. We will define the dual span for generically generated coherent systems.

Let (E, V) be a generically generated coherent system of type (n, d, k) with $H^0(I_E^*) = 0$. From [4, Proposition 4.4], we have the exact sequence

$$0 \to N \to V \otimes \mathcal{O} \to E \to \tau \to 0 \tag{5.3}$$

with $H^0(N) = 0$ and τ a torsion sheaf of length ℓ. From (5.3), we have the exact sequences

$$0 \to N \to V \otimes \mathcal{O} \to I_E \to 0 \tag{5.4}$$

and

$$0 \to I_E \to E \to \tau \to 0. \tag{5.5}$$

Lemma 5.1. *$N = D(I_E)^*$.*

Proof. The coherent system (I_E, V) is generated. From (5.4), $N = D(I_E)^*$. □

Remark 5.2. If (E, V) is generically generated and $H^0(I_E^*) = 0$, from (5.3) and Lemma 5.1, we have the sequences

$$0 \to D(I_E)^* \to V \otimes \mathcal{O} \to E \to \tau \to 0 \tag{5.6}$$

and

$$0 \to D(I_E)^* \to V \otimes \mathcal{O} \to I_E \to 0. \tag{5.7}$$

Moreover, $(D(I_E), V^*)$ is the dual span of (I_E, V).

Let

$$0 \to D(I_E) \to D(E)_\ell \to \tau' \to 0 \tag{5.8}$$

be an elementary transformation of $D(I_E)$ with τ' a torsion sheaf of length ℓ. The subspace $V^* \subset H^0(D(I_E))$ defines a subspace V' in $H^0(D(E)_\ell)$, which we identify with V^*.

Definition 5.3. Let (E, V) be a generically generated coherent system of type (n, d, k) with $H^0(I_E^*) = 0$. A dual span of (E, V), denoted by $D(E, V)$, is an elementary transformation $(D(E)_\ell, V^*)$ of $(D(I_E), V^*)$ of length ℓ where $\ell = d_E - d_{I_E}$.

Remark 5.4. (1) If (E, V) is strictly generically generated, then the family of dual spans associated to (E, V) has dimension at most $\ell n - 1$.

(2) If (E, V) is generated, there is a unique dual span given by $(D(E), V^*)$.

(3) If (E, V) is a generically generated coherent system of type (n, d, k), $(D(I_E), V^*)$ is a generated coherent system of type $(k - n, d - \ell, k)$.

(4) $D(E, V)$ is a coherent system of type $(k - n, d, k)$.

(5) The image of the evaluation map $V^* \otimes \mathcal{O} \to D(E)_\ell$ is $D(I_E)$.

Proposition 5.5. *Let (E, V) be a coherent systems of type (n, d, k). If (E, V) is generically generated with $H^0(I_E^*) = 0$, then a dual span $D(E, V) = (D(E)_\ell, V^*)$ is generically generated. Moreover, $H^0(I^*_{D(E)_\ell}) = 0$.*

Proof. The proposition follows from the definition of a dual span, since $(D(I_E), V^*)$ is generated and $I_{D(E)_\ell} = D(I_E)$. □

Remark 5.6. Note, from the definition of a dual span, that (E, V) is a dual span of $D(E, V) = (D(E)_\ell, V^*)$.

Theorem 5.7. *Let X be a general curve of genus g and $d < g + n_1 \leq g + n_2$, then for all $\alpha > 0$, $G(\alpha : n_1, d, n_1 + n_2) \neq \emptyset$ if and only if $G(\alpha : n_2, d, n_1 + n_2) \neq \emptyset$.*

Proof. Let $(E, V) \in G(\alpha : n_i, d, n_1 + n_2)$ for $i = 1, 2$. From Proposition 3.8, (E, V) is generically generated and $H^0(I_E^*) = 0$. From Proposition 5.5, a dual span $D(E, V) = (D(E)_\ell, V^*)$ is generically generated with $H^0(I^*_{D(E)_\ell}) = 0$ and from Proposition 3.6, it is α-stable for all $\alpha > 0$. □

For any (n, d, k), define G_g as

$$G_g(n, d, k) := \{(E, V) : (E, V) \text{ is of type } (n, d, k) \text{ and it is generated with } H^0(E^*) = 0\}.$$

Corollary 5.8. *If $d < g+n_1 \leq g+n_2$, then $G_g(n_i, d, n_1+n_2) \subset U(n_i, d, n_1+n_2)$ for $i = 1, 2$. Moreover, for $i = 1, 2$, $G_g(n_i, d, n_1 + n_2)$ is open and $G_g(n_1, d, n_1 + n_2) \cong G_g(n_2, d, n_1 + n_2)$.*

Proof. From Proposition 3.6, $(E, V) \in G_g(n_i, d, n_1 + n_2)$ is α-stable for all $\alpha > 0$ and from Proposition 3.5, E is stable. The dual span correspondence for generated coherent systems gives the isomorphism. □

To prove Theorem 7, we give four lemmas that we will use.

Lemma 5.9. *Let (E, V) be a generated coherent system. The Petri map of (E, V) is injective if and only if the Petri map of $D(E, V)$ is injective.*

Proof. Since (E, V) is generated, $D(E, V) = (D(E), V^*)$. We have the following exact sequences

$$0 \to D(E)^* \to V \otimes \mathcal{O} \to E \to 0, \tag{5.9}$$

and

$$0 \to E^* \to V^* \otimes \mathcal{O} \to D(E) \to 0. \tag{5.10}$$

The lemma follows from the cohomology sequences

$$0 \to H^0(D(E)^* \otimes E^* \otimes K) \to V \otimes H^0(E^* \otimes K) \xrightarrow{\psi} H^0(E \otimes E^* \otimes K) \cdots \tag{5.11}$$

and

$$\begin{aligned} 0 \to H^0(E^* \otimes D(E)^* \otimes K) &\to V \otimes H^0(D(E)^* \otimes K) \\ &\xrightarrow{\phi} H^0(D(E) \otimes D(E)^* \otimes K) \cdots \end{aligned} \tag{5.12}$$

since ϕ is injective if and only if ψ is injective. □

Lemma 5.10. *Let (E, V) be strictly generically generated. If the Petri map of (I_E, V) is injective, the Petri map of (E, V) is injective.*

Proof. The lemma follows from the cohomology sequences

$$0 \to H^0(D(I_E)^* \otimes I_E^* \otimes K) \to V \otimes H^0(I_E^* \otimes K) \xrightarrow{\psi} H^0(I_E \otimes I_E^* \otimes K) \cdots \tag{5.13}$$

and

$$0 \to H^0(D(I_E)^* \otimes E^* \otimes K) \to V \otimes H^0(E^* \otimes K) \xrightarrow{\psi} H^0(E \otimes E^* \otimes K) \cdots \tag{5.14}$$

and the cohomology of the exact sequence

$$0 \to E^* \otimes D(I_E)^* \otimes K \to I_E^* \otimes D(I_E)^* \otimes K \to \tau \to 0. \tag{5.15}$$

□

Let (E, V) be a generically generated coherent system. From Proposition 5.5, a dual span $D(E, V) = (D(E)_\ell, V^*)$ is generically generated. Hence, from Remark 5.2, we have the sequence

$$0 \to I_E^* \to V^* \otimes \mathcal{O} \to D(E)_\ell \to \tau \to 0. \tag{5.16}$$

Lemma 5.11. *The Petri map of (I_E, V) is injective if and only if the Petri map of $(I_{D(E)_\ell}, V^*)$ is injective.*

Proof. The lemma follows at once from Lemma 5.9 since $I_{D(E)_\ell} = D(I_E)$. □

Lemma 5.12. *If the Petri map of (I_E, V) is injective, the Petri map of a dual span $D(E,V) = (D(E)_\ell, V^*)$ is injective.*

Proof. From (5.16), the kernel of the Petri map of $(D(E)_\ell, V^*)$ is $H^0(I_E^* \otimes D(E)_\ell^* \otimes K)$.

From the exact sequence (5.8), we obtain the following exact sequence

$$0 \to D(E)_\ell^* \otimes I_E^* \otimes K \to D(I_E)^* \otimes I_E^* \otimes K \to \tau \to 0. \tag{5.17}$$

The kernel of the Petri map for (I_E, V) is $H^0(D(I_E)^* \otimes I_E^* \otimes K)$. Hence, if $H^0(D(I_E)^* \otimes I_E^* \otimes K) = 0$, $H^0(I_E^* \otimes D(E)_\ell^* \otimes K) = 0$. □

We now have Theorem 7.

Theorem 5.13. *Let $(E, V) \in G(\alpha : n_1, d, n_1 + n_2)$. If either of the Petri maps of (I_E, V) or $(I_{D(E)_\ell}, V^*)$ is injective, then*

(1) *$G(\alpha : n_1, d, n_1+n_2)$ is smooth of dimension $\beta(n_1, d, n_1+n_2)$ in a neighbourhood of (E, V).*
(2) *$G(\alpha : n_2, d, n_1+n_2)$ is smooth of dimension $\beta(n_2, d, n_1+n_2)$ in a neighbourhood of the dual span $D(E, V)$.*

Proof. If the Petri map of (I_E, V) is injective, from Lemmas 5.9, 5.10 and 5.12, the Petri maps of (E, V) and $D(E, V)$ are injective. From Proposition, 2.1 $G(\alpha : n_i, d, n_1 + n_2)$, $i = 1, 2$ respectively is smooth of dimension $\beta(n_i, d, n_1 + n_2)$ in a neighbourhood of (E, V) and of $D(E, V)$, respectively.

If the Petri map of (I_{D_E}, V) is injective, again from Lemmas 5.9 and 5.10, the Petri map of $D(E, V)$ is injective. From Lemma 5.11, the Petri map of (I_E, V) is injective and, as above, the Petri map of (E, V) is injective. Hence, $G(\alpha : n_i, d, n_1 + n_2)$, $i = 1, 2$ respectively is smooth of dimension $\beta(n_i, d, n_1+n_2)$ in a neighbourhood of $D(E, V)$ and of (E, V), respectively. □

Remark 5.14. Theorems 5.7 and 5.13 apply for any $\alpha > 0$. Since $d < g + n_1$ the bundles in $G(\alpha : n_1, d, n_1 + n_2)$ are stable (see Proposition 3.8). Hence, we have similar results for the Brill–Noether loci $B(n_1, d, n_1 + n_2)$ and $B(n_2, d, n_1 + n_2)$.

6. Rank 2 and Genus 2

In this section, we will consider the case $n = 2$ and then $g = 2$.

From Proposition 4.6, we have that for a general curve and $g \geq 3$, $G(\alpha; 2, d, 3) \neq \emptyset$ for all $\alpha > 0$ and $U(2, d, 3) \neq \emptyset$ for $d \geq \frac{2g}{3} + 2$. For $k > 4$, we have the following theorem.

Theorem 6.1. *Let X be general, $s \geq 3$ and $d < s + 2g - \frac{4g}{s+2}$. If $G_0(2, d, 2+s)$ is non-empty, then $G(\alpha : 2, d, 2+s)$ is non-empty for all $\alpha > 0$. Moreover, $U^s(2, d, 2+s) \neq \emptyset$.*

Proof. Let $(E, V) \in G_0(2, d, 2+s)$. Hence, E is semistable.

Let $r_s := \lceil \frac{2+s}{2} \rceil$ and (F, W) a coherent subsystem of (E, V) with $n_F = 1$. If $\dim W \geq r_s$, the Brill–Noether number $\beta(1, d_F, r_s) \geq 0$, that is, $d_F \geq r_s + g - 1 - \frac{g}{r_s}$. But then

$$d_F \geq r_s + g - 1 - \frac{g}{r_s} > \frac{d}{2},$$

which is a contradiction since E is semistable. Therefore, for any coherent subsystem (F, W), $\dim W < \frac{2+s}{2}$ and, from Remark 2.3(3), (E, V) is α-stable for all $\alpha > 0$. Therefore, $G(\alpha : 2, d, 2+s) \neq \emptyset$ for all $\alpha > 0$ and $U^s(2, d, 2+s) \neq \emptyset$. □

Let X be any curve. From Proposition 2.5, we have that any generated coherent system (E, V) of type $(n, d, n+1)$ with E stable is α-stable for all $\alpha > 0$. For $n = 2$, we have (see [4, Theorem 9.2] for general curve)

Proposition 6.2. *Let X be any curve. If $G_0(2, d, 4) \neq \emptyset$ and there exists a generated coherent system $(E, V) \in G_0(2, d, 4)$, then $G(\alpha : 2, d, 4) \neq \emptyset$ for all $\alpha > 0$ and $U^s(2, d, 4) \neq \emptyset$. Moreover, if E is stable, $U(2, d, 4) \neq \emptyset$.*

Proof. Let (F, W) be a coherent subsystem of (E, V) with $n_F = 1$. From Proposition 2.5, $\dim W \leq 2$. If $\dim W = 2$, since $(E, V) \in G_0(2, d, 4)$, $d_F < \mu(E)$. From Remark 2.3, (E, V) is α-stable for all $\alpha > 0$. □

Corollary 6.3. *For any curve X and $d \geq 4g - 2$, $G(\alpha : 2, d, 4) \neq \emptyset$ for all $\alpha > 0$. Moreover, $U(2, d, 4) \neq \emptyset$ and for $d \geq 4(g-1), U(2, d, 2+s) = \emptyset$ if $s > d - 2g$.*

Proof. Since any stable bundle of degree $d \geq 2(2g-1)$ is generated, the first part follows from Proposition 6.2. The last part follows from the Riemann–Roch theorem. □

Remark 6.4. Recall from the Brill–Noether theory for vector bundles of rank $n \geq 2$ (see [6, 13, 7]) that if $0 < d < 2n$, there exists a semistable vector bundle E of rank n and degree d with k sections if and only if $n \leq d + (n-k)g$. Hence, if $0 < d < 2n$ and $k > n + \frac{d-n}{g}$, then $U^s(n, d, k) = \emptyset$. Moreover, if $d > n(2g-2)$, then by the Riemann–Roch theorem, every semistable bundle E has $h^0(E) = d+n(1-g)$; so, if $k > d + n(1 - g)$, $U^s(n, d, k) = \emptyset$.

We shall now consider the case $g = 2$. Any curve of genus $g = 2$ is a Petri curve. From Corollary 3.10, if $d < n + 2$, $G(\alpha : n, d, k) = \emptyset$ for all $\alpha > 0$ and $k > n$.

From Theorem 4.3, we have

Proposition 6.5. *For X of genus $g = 2$ and $d = n + 2, n \geq 3$,*

(1) *$G(\alpha : n, d, n + 1) \neq \emptyset$ for all $\alpha > 0$.*
(2) *$G(\alpha : n, d, n + 1) = G(\alpha' : n, d, n + 1)$ for all $\alpha, \alpha' > 0$ and $\alpha_L = 0$.*
(3) *$G(n, d, n + 1)$ is smooth and irreducible of dimension 2.*
(4) *$U(n, d, n + 1) = G(n, d, n + 1)$.*
(5) *If $k \geq n + 2$, $G(\alpha : n, d, k) = \emptyset$ for all $\alpha > 0$.*

Proof. Parts (1)–(4) follow from Theorem 4.3. Part (5) follows from Remark 6.4 and Proposition 3.7, since for the existence of a semistable bundle with at least k sections we need $k - n \leq \frac{d-n}{2}$. □

From Remark 6.4 and Proposition 2.6, we have

(1) if $n + 2 < d < 2n$ and $k > \frac{d+n}{2}$, $U^s(n, d, k) = \emptyset$;
(2) if $d > 2n$ and $k > d - n$, $U^s(n, d, k) = \emptyset$;
(3) if $d \geq 3n$, $G(\alpha : n, d, n + 1) \neq \emptyset$ for all $\alpha > 0$. Moreover, $U(n, d, n + 1) \neq \emptyset$.

In particular, for $n = 2$, from Propositions 2.5 and 6.3, Corollary 3.10 and the Riemann–Roch theorem, we have

(1) If $d < 4$ and $k \geq 3$, $G(\alpha : 2, d, k) = \emptyset$ for all $\alpha > 0$;
(2) if $d = 5$ and $k > 3$, $U(2, d, k) = \emptyset$ and $G_0(2, 5, k) = \emptyset$;
(3) if $d \geq 6$ and $k = 3, 4$, $G(\alpha : 2, d, k) \neq \emptyset$ for all $\alpha > 0$. Moreover, $U(2, d, k) \neq \emptyset$;
(4) if $d \geq 6$ and $k > d - 2$, $U(2, d, k) = \emptyset$ and $G_0(2, d, k) = \emptyset$.

For $d = 4$, we need the following lemmas.

Lemma 6.6. (1) *$B(2, 4, k) = \emptyset$ for $k \geq 3$.*
(2) *$\widetilde{B}(2, 4, k) = \emptyset$ for $k \geq 5$.*
(3) *$\widetilde{B}(2, 4, 3) \neq \emptyset$.*
(4) *$\widetilde{B}(2, 4, 4) = \{K \oplus K\}$.*

Proof. Let E be a semistable vector bundle of rank 2 and degree $d = 4 = 2(2g-2)$. From the Riemann–Rock theorem, $h^0(E) = 2 + h^1(E)$. If $h^1(E) = h^0(E^* \otimes K) \geq 1$, then E is an extension

$$\xi : 0 \to L \to E \to K \to 0 \tag{6.1}$$

of K by L, where L is a line bundle of degree 2. Thus, E cannot be stable, that is, $B(2, 4, k) = \emptyset$ for $k \geq 3$.

Since $h^1(L) \leq 1$ and $h^1(K) = 1$, from the cohomology sequence of (6.1), $h^1(E) \leq 2$. Hence, $\widetilde{B}(2,4,k) = \emptyset$ for $k \geq 5$.

If $L \ncong K$, $H^1(L) = 0$, $H^0(L) \cong \mathbb{C}$ and $h^1(K^* \otimes L) = 1$. Hence, there exist non-trivial extensions (6.1), and $h^0(E) = 3$, that is, $\widetilde{B}(2,4,3) \neq \emptyset$.

Let $L \cong K$. If ξ is non-trivial, from the cohomology sequence of

$$0 \to \mathcal{O} \to E^* \otimes K \to \mathcal{O} \to 0,$$

$H^0(E^* \otimes K) \cong H^0(\mathcal{O})$. Hence, $h^0(E) = 3$.

Therefore, $\widetilde{B}(2,4,4) = \{K \oplus K\}$. □

Note that if (L, W) is a coherent system of type $(1,2,2)$, then $(L, W) = (K, H^0(K))$.

Lemma 6.7. *If $(K, H^0(K))$ is a coherent subsystem of a coherent system (E, V) of type $(2,4,3)$, then (E, V) is not α-semistable for any $\alpha > 0$.*

Proof. For any $\alpha > 0$, $\mu_\alpha(K, H^0(K)) = 2 + 2\alpha > 2 + \alpha\frac{3}{2} = \mu_\alpha(E, V)$. □

Corollary 6.8. *The coherent systems*

(1) $(L \oplus K, H^0(L) \oplus H^0(K))$ *and*
(2) $(E, H^0(E))$ *with E a non-trivial extension of K by K*

are not α-semistable for any $\alpha > 0$.

Lemma 6.9. *Let (E, V) be a coherent system of type $(2,4,3)$. If E is a non-trivial extension ξ of K by L, with $L \ncong K$, (E, V) is generated. Moreover, (E, V) is α-stable for all $\alpha > 0$.*

Proof. If $n_{I_E} = 1$, then $I_E = K$, which is a contradiction since $\xi \neq 0$. If $n_{I_E} = 2$ and $d_{I_E} < 4$, from Lemma 3.1, we get a contradiction. Therefore, (E, V) is generated. From Proposition 2.5, (E, V) is α-stable for all $\alpha > 0$. □

Proposition 6.10. *If $(E, V) \in G_L(2,4,3)$, then E is semistable and (E, V) is α-stable for all $\alpha > 0$.*

Proof. The proposition follows at once from Propositions 3.7 and 3.8. □

Theorem 6.11. (1) *$U(2,4,k) = \emptyset$ for $k \geq 3$.*
(2) *$G_0(2,4,k) = \emptyset$ for $k \geq 5$.*
(3) *$G(\alpha : 2,4,3) \neq \emptyset$ for all $\alpha > 0$.*
(4) *$U^s(2,4,3) \cong G_L(2,4,3) \cong \mathrm{Pic}^4(X)$.*

Proof. (1)–(3) follow from Lemma 6.6. $U^s(2,4,3) \cong G_L(2,4,3)$ follows from Lemma 6.9 and Proposition 6.10.

To prove $G_L(2,4,3) \cong \mathrm{Pic}^4(X)$, suppose $(E,V) \in G_L(2,4,3) \cong U^s(2,4,3)$, so E is semistable, generically generated and $h^0(I_E^*) = 0$. If E is not generated, then $\deg I_E \leq 3$. Moreover, I_E must be stable, for otherwise it has a quotient line bundle Q of degree ≤ 1, hence with $h^0(Q) \leq 1$. The corresponding subbundle L has $\dim(V \cap H^0(L)) \geq 2$, contradicting the α-stability of (E,V). However $U(2,3,3) = \emptyset$, so I_E cannot exist. Thus E is generated and it follows that E arises from an extension

$$0 \to L^* \to V \otimes \mathcal{O} \to E \to 0$$

or dually

$$0 \to E^* \to V^* \otimes \mathcal{O} \to L \to 0, \tag{6.2}$$

where L is a line bundle of degree 4.

Conversely, any line bundle L of degree 4 is generated and $h^0(L) = 3$ by the Riemann–Roch theorem. So there is a unique extension (6.2) for each L. Certainly then E is generated with $h^0(E^*) = 0$, so $(E,V) \in G_L(2,4,3)$. □

Moreover,

Theorem 6.12. (1) $\widetilde{G}_0(2,4,4) = \{(K \oplus K, H^0(K \oplus K))\}$.
(2) $G_0(2,4,4) = \emptyset$.
(3) $\widetilde{G}(\alpha : 2,4,4) \neq \emptyset$ *for all* $\alpha > 0$.

Proof. Parts (1) and (2) follow from Lemma 6.6. Since $(K \oplus K, H^0(K \oplus K)) \cong (K, H^0(K)) \oplus (K, H^0(K))$, it is α-semistable for all $\alpha > 0$, so $\widetilde{G}(\alpha : 2,4,4) \neq \emptyset$ for all $\alpha > 0$. □

For $d = 5$ and $k = 3$, we have

Theorem 6.13. (1) $G_0(2,5,3) \neq \emptyset$.
(2) $U(2,5,3) \neq \emptyset$.
(3) $U(2,5,3) \neq G_0(2,5,3)$.

Proof. Let E be a non-trivial extension

$$\phi : 0 \to L \to E \to M \to 0 \tag{6.3}$$

of M by L, where L is a line bundle of degree 2 and M a general line bundle of degree 3 with $h^0(M) = 2$. Note that $h^1(M^* \otimes L) = 2$.

It is well known that E is stable and from the cohomology sequence of (6.3), $h^0(E) = 3$. Hence, $(E, H^0(E)) \in G_0(2,5,3)$.

Let (F, W) be any coherent subsystem of $(E, H^0(E))$, with F a line sub-bundle. Since E is stable, $d_F < \mu(E) = 2 + \frac{1}{2}$, so $\dim W \leq h^0(F) \leq 2$. If $\dim W = 2$, $F \cong K$.

Now, if in the extension (6.3), $L \ncong K$ and M is general and generated, $H^0(K^* \otimes M) = 0$ i.e. K cannot be a subbundle of E. Hence, for all coherent subsystems (F, W), $\dim W \leq 1$ and from Remark 2.3, $(E, H^0(E))$ is α-stable for all $\alpha > 0$. Therefore, $U(2, 5, 3) \neq \emptyset$.

However, if $L \cong K$, for any coherent subsystem (F, W) of $(E, H^0(E))$, with F a line subbundle, $\mu_\alpha(F, W) \leq \mu_\alpha(K, H^0(K))$. Thus, since $(K, H^0(K))$ is a coherent subsystem of $(E, H^0(E))$,

$$\mu_\alpha(K, H^0(K)) < \mu_\alpha(E, V)$$

if and only if $\alpha < 1$. For $\alpha = 1$, $\mu_\alpha(K, H^0(K)) = \mu_\alpha(E, V)$. Therefore, $(E, H^0(E)) \notin U(2, 5, 3)$. □

Acknowledgments

The author thanks Peter E. Newstead for useful conversations and for his suggestions and comments on previous versions of this work. Thanks are also due to the referee for valuable comments and suggestions towards improvement of the paper. She also thanks the International Center for Theoretical Physics, Trieste, where part the work was carried out, for their hospitality and acknowledges the support of CONACYT grant 48263-F. The author is a member of the research group VBAC (Vector Bundles on Algebraic Curves), which was partially supported by EAGER (EC FP5 Contract no. HPRN-CT-2000-00099) and by EDGE (EC FP5 Contract no. HPRN-CT-2000-00101).

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of Algebraic Curves*, Vol. 1 (Springer-Verlag, New York, 1985).

[2] S. B. Bradlow, G. Daskalopoulos, O. Garcia-Prada and R. Wentworth, Stable augmented bundles over Riemann surfaces, *Vector Bundles in Algebraic Geometry*, Durham 1993, eds. N. J. Hitchin, P. E. Newstead and W. M. Oxbury, London Mathematical Society Lecture Notes Series, Vol. 208 (Cambridge University Press, Cambridge, 1995), pp. 15–67.

[3] S. B. Bradlow and O. Garcia-Prada, An application of coherent systems to a Brill–Noether problem, *J. Reine Angew Math.* **551** (2002) 123–143.

[4] S. B. Bradlow, O. Garcia-Prada, V. Muñoz and P. E. Newstead, Coherent systems and Brill–Noether theory. *Int. J. Math.* **14** (2003) 683–733.

[5] S. B. Bradlow, O. Garcia-Prada, V. Mercat, V. Muñoz and P. E. Newstead, On the geometry of moduli spaces of coherent systems on algebraic curves, to appear in *Int. J. Math.*

[6] L. Brambila-Paz, I. Grzegorczyk and P. E. Newstead, Geography of Brill–Noether loci for small slopes, *J. Algebra Geom.* **6** (1997) 645–669.

[7] L. Brambila-Paz, V. Mercat, P. E. Newstead, and F. Ongay, Nonemptiness of Brill–Noether loci, *Int. J. Math.* **11** (2000) 737–760.

[8] D. C. Butler, Birational maps of moduli of Brill–Noether pairs, preprint, arXiv:math.AG/9705009.
[9] P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (Wiley-Interscience, New York, 1978).
[10] M. He, Espaces de modules de systèmes cohérents, *Int. J. Math.* **9** (1998) 545–598.
[11] A. King and P. E. Newstead, Moduli of Brill–Noether pairs on algebraic curves, *Int. J. Math.* **6** (1995) 733–748.
[12] J. Le Potier, Faisceaux semistables et systèmes cohérents, *Vector Bundles in Algebraic Geometry*, Durham 1993, eds. N. J. Hitchin, P. E. Newstead and W. M. Oxbury, London Mathematical Society Lecture Notes Series, Vol. 208 (Cambridge University Press, Cambridge, 1995), pp. 179–239.
[13] V. Mercat, Le problème de Brill–Noether pour des fibrés stables de petite pente, *J. Reine Angew. Math.* **506** (1999) 1–41.
[14] K. Paranjape and S. Ramanan, On the canonical ring of a curve, *Algebraic Geometry and Commutative Algebra*, in Honor of Masayoshi Nagata (Kinokuniya Publications Tokyo, 1987), pp. 503–516.
[15] N. Raghavendra and P. A. Vishwanath, Moduli of pairs and generalised theta divisors, *Tôhoku Math. J.* **46** (1994) 321–340.
[16] C. Yee, On the Brill–Noether map, preprint.