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Centro de Investigación en Matemáticas, A.C.

THE CONVEX BODIES THAT ARE CLOSED UNIT BALLS OF TENSOR NORMED SPACES

T E S I S

Que para obtener el grado de

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Luisa Fernanda Higueras Montaño

Director de Tesis:

Dra. Maite Fernández Unzueta

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Contents

Introduction

The theory of convexity, as we know it today, had its origins in the works of H. Minkowski and H. Brunn at the end of the XIX century and the beginning of the XX century. In the seminals works *Geometrie der Zahlen* [27] *and Allgemeine Lehrsätze über die konvexen Polyeder* [28], H. Minkowski introduces the concept of convex body in $\mathbb{R}^d$ and starts the systematic study of convex bodies and its applications. One of the most important properties of Minkowski's work reveals that there exists a one to one correspondence between centrally symmetric convex bodies and norms on $\mathbb{R}^d$. This result appeared for the first time in *Geometrie der Zahlen*. Nowadays it is formulated as follows: *if P is a centrally symmetric convex body in $\mathbb{R}^d$ then for every $x \in \mathbb{R}^d$, $\|x\|_P := \inf\{\lambda > 0 : \lambda^{-1}x \in P\}$ defines a norm $\|\cdot\|_P$ on $\mathbb{R}^d$ for which P is the closed unit ball.*

On the other hand, the theory of norms on tensor products of Banach spaces had its origins in the works of R. Schatten and A. Grothendieck in the middle of the XX century. In his monograph *A theory of Cross-Spaces* [35], R. Schatten develops the first systematic study of classes of norms on the tensor product of Banach spaces. But the influential work of A. Grothendieck, *Résumé de la théorie des produits tensoriels topologiques* [15] demonstrated the enormous possibilities for the using of tensor products in Banach spaces (see the survey [32]).

Recently, there has been a continued interest in the algebraic and topological properties of the tensor product $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ and some of its subsets, as one can see in [11, 4, 22]. This has motivated the study on the interplay between the theory of tensor norms on finite dimensional normed spaces and the theory of convex sets.

This work lies in the intersection of the theory of tensor norms and the theory of convex sets. Here we have two principal goals: the first one is motivated by the bijection between centrally symmetric convex bodies and norms on $\mathbb{R}^d$. It consists in characterizing centrally symmetric convex bodies Q in $\otimes_{i=1}^{l}\mathbb{R}^{d_i} \simeq \mathbb{R}^{d_1\cdot\ldots\cdot d_l}$ which have the property that there exist norms $\|\cdot\|_i$ on $\mathbb{R}^{d_i}$ such that Q is the unitary ball of a reasonable crossnorm on $\otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$. We call this convex bodies, *tensorial 0-symmetric convex bodies.* The second goal is to define tensor products of centrally symmetric convex bodies in Euclidean spaces as an analogue to the construction of tensor norms on Banach spaces.

In the literature there appear different definitions of tensor products of special classes of convex sets. For example: I. Namioka and R. Phelps in [29] present the projective tensor product $K_1 \otimes K_2$ (defined by Z. Semadeni, [37]) of compact convex sets in locally convex Hausdorff topological vector spaces, and the tensor products $K_1 \triangle K_2$ and $K_1 \square K_2$. In [39], M. Velasco presents the projective tensor product of closed convex cones and the projective tensor product of convex bodies containing the origin in their interior. Both of them are in real vector spaces of finite dimension. In [5], G. Aubrun and S. Szarek introduce the projective tensor product of closed convex sets and the injective tensor product of centrally symmetric convex bodies, both of them in $\mathbb{R}^d$. Other references are [10, 17, 24].

To our knowledge the definition of tensor product of centrally symmetric convex bodies that we propose here is new. It is motivated by the theory of tensor norms on Banach spaces. In fact, we prove in Section 4.5 that there exists a bijection between our tensor products of centrally symmetric convex bodies and tensor norms on finite dimensonal normed spaces. We also define the injective and the projective tensor product of centrally symmetric convex bodies in Euclidean spaces. Even though both the projective and injective tensor product of centrally symmetric convex bodies are very natural definitions, we would like to notice they appeared, recently, in the book *Alice and Bob Meet Banach* [5], published in September 2017.

This work is divided in four parts. In Chapter 1 we introduce the notation, and present the basic results that we will use throughout the work. This chapter is about two topics: the theory of tensor norms in Banach spaces and the theory of convex sets.

In Chapter 2 we develop the mathematical tools needed to characterize a centrally symmetric convex body in $\otimes_{i=1}^{l} \mathbb{R}^{d_i} \simeq \mathbb{R}^{d_1 \cdots \cdot d_l}$ which has the property we described before (mentioned in the previous paragraph relating to unit balls of reasonable crossnorms). To this end, we first choose the inner product on $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$ given by the Hilbert tensor product $\otimes_{H,i=1}^{l} \left(\mathbb{R}^{d_i}, \|\cdot\|_2\right)$. Subsequently, we introduce the injective and the projective tensor product of centrally symmetric convex bodies in $\mathbb{R}^d$. See Definition 2.7. We characterize the property of being the unit ball of a reasonable crossnorm on $\otimes_{i=1}^{l} \left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$ for fixed norms $\|\cdot\|_i$ on $\mathbb{R}^{d_i}$. See Proposition 2.13. Finally, we present the characterization of a centrally symmetric convex Q in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$ for which there exist norms $\|\cdot\|_i$ on $\mathbb{R}^{d_i}$ such that Q is the unit ball of a reasonable crossnorm on $\otimes_{i=1}^{l} \left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$. See Theorem 2.20 . We algo give a characterization of the ellipsoids in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$ that are tensorial 0-symmetric convex bodies. See Theorem 2.30.

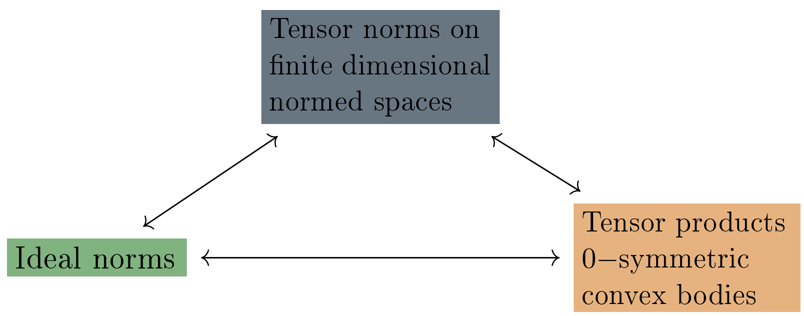
In Chapter 3 we study the topological structure of $\mathcal{B}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$ which is the set of tensorial 0-symmetric convex bodies in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$. To this end, we follow the ideas of [3, 2, 1]. We prove that $Gl_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$[1] acts properly on $\mathcal{B}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$. This lets us to show two fundamental properties of the space of tensorial0-symmetric convex bodies: the first one, the set of ellipsoids that also are tensorial 0-symmetric convex bodies $\mathscr{E}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$ is homeomorphic to the quotient space $Gl_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right) / O_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$[2]. The second one, $\mathcal{B}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$ is homeomorphic to $\mathscr{L}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right) \times \mathscr{E}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$ where $\mathscr{L}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$ is a compact $O_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$-global slice of $\mathcal{B}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$. See Theorem

[1] $Gl_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$ is the set of bijective linear maps on $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$ preserving decomposable tensors.

[2] $O_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$ is the set of linear isometries of $\otimes_{H,i=1}^{l} \mathbb{R}^{d_i}$ preserving decomposable tensors

3.32). At the end of this chapter we introduce the Σ-Banach-Mazur distance δ_{Σ}^{BM}. With this distance we define the analogue to the Banach-Mazur compactum in the context of tensorial 0-symmetric convex bodies. We wish to notice that the proofs and results of Section 3.3 and 3.4 are analogous to those of [3].

In Chapter 4 we introduce the definition of tensor product $\otimes_{\alpha}$ of order l of centrally symmetric (0-symmetric) convex bodies in Euclidean spaces (Section 4.3). We define the injective $\otimes_{\epsilon}$ and the projective $\otimes_{\pi}$ tensor product of 0-symmetric convex bodies. We prove that the projective tensor product $\otimes_{\pi}$ is defined by a universal property (Theorem 4.10). As analogue to the dual of a tensor norm, in Section 4.4 we define the dual of a tensor product $\otimes_{\alpha}$. Finally, in Section 4.5 we construct a bijection between tensor products of order l of 0-symmetric convex bodies in Euclidean spaces and tensor norms on finite dimensional normed spaces. This lets us to prove there exists a bijection between tensor products of order 2 of 0-symmetric convex bodies and ideal norms (See Definition 1.21). The following diagram describes the relation between tensor norms of order 2, ideal norms and tensor products of order 2 of 0-symmetric convex bodies. The symbol $\longleftrightarrow$ denotes that there exists a bijection between the sets.



Chapter 1

Preliminaries

In this chapter we introduce the notation and results that we will use throughout the work. The results that appear in Sections 1.1, 1.2 and 1.3 can be consulted in [34, 12, 14]. The details about the construction and properties of the Hilbert tensor product can be consulted in [20]. The results in Section 1.4 can be consulted in [16, 36]. The properties of the Banach-Mazur distance can be consulted in [38].

Notation

Here we introduce the notation and basic concepts that we will use through all the work. As usual the letters $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote the set of real numbers, complex numbers and natural numbers, respectively.

For a **normed vector space** we understand a pair $(X, \|\cdot\|)$ where X is a vector space over $\mathbb{R}$ or $\mathbb{C}$ and $\|\cdot\|$ is a norm defined on X. To shorten notation, we usually write X instead of $(X, \|\cdot\|)$ and we say that X is a normed (vector) space. The subset $\{x \in X : \|x\| \leq 1\}$ of a normed space X is called the **closed unit ball** of X. It is denoted by B_X.

Every normed space $(X, \|\cdot\|)$ is a metric space with the metric induced by the norm $\|\cdot\|$. Because of this, for a given subset $A \subseteq X$ of a normed space the symbols ∂A, $\overline{A}$ and $\text{int}A$ will denote the boundary of A, the closure of A and interior of A relative to the metric induced by the norm on X.

For a **Banach space** we understand a normed vector space $(X, \|\cdot\|)$ which is a complete metric space with the metric induced by the norm $\|\cdot\|$. The **dual of a Banach space** X, denoted by X^*, is the set of continuous linear transformations defined on X that take scalar values. The space X^* is a Banach space with the norm:

$$\|x^*\| := \sup\{|x^*(x)| : x \in B_X\} \text{ for } x^* \in X^*.$$

The set of continuous linear transformations (or bounded linear maps) $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ between two Banach spaces will be denoted by $\mathcal{L}(X, Y)$. The set $\mathcal{L}(X, Y)$ is a Banach space with the norm:

$$\|T\| := \sup \left\{ \|Tx\|_Y : \|x\|_X \leq 1 \right\}.$$

For a **pre-Hilbert space** we understand a pair $(H, \langle \cdot, \cdot \rangle)$ where H is a vector space over $\mathbb{R}$ or $\mathbb{C}$ and $\langle \cdot, \cdot \rangle$ is an inner product defined on H. A pre-Hilbert space $(H, \langle \cdot, \cdot \rangle)$ will be called a **Hilbert space** if it is a Banach space with the norm induced by the inner product $\langle \cdot, \cdot \rangle$. If $T : H_1 \to H_2$ is a continuous linear map between Hilbert spaces H_1, H_2 then T^t denotes the **transpose** map of T. That is, $T^t : H_2 \to H_1$ is a continuous linear map such that:

$$\left\langle x, T^t y \right\rangle_{H_1} = \langle Tx, y \rangle_{H_2} \text{ for every } x \in H_1, y \in H_2.$$

Unless otherwise stated, we will assume that the vector space $\mathbb{R}^d$ is a Hilbert space with the standard inner product denoted by $\langle \cdot, \cdot \rangle_2$ and closed unit ball B_2^d. As usual by $\mathbb{S}^{d-1}$ we denote the boundary of B_2^d. The vectors of the canonial basis of $\mathbb{R}^d$ are denoted by $e_j^d = (0, \ldots, 1, \ldots, 0)$ for $j = 1, \ldots, d$.

On the other hand, if $(M, d_M(\cdot))$ is a metric space we will use the symbols $x_n \underset{n\to\infty}{\longrightarrow} x$ and $x_n \to x$ to indicate that the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq M$ converges to the point $x \in M$.

Throughout the work we will use the well known fact that for every finite dimensional vector space E over $\mathbb{R}$ or $\mathbb{C}$, every pair of norms $||\cdot||, |||\cdot|||$ defined on E determine the same topology. This can be found in Theorem 1.21 [33] or Theorem 3.1 [8].

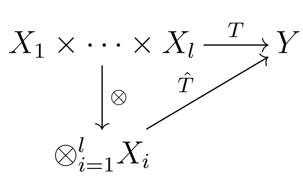
We will introduce new notation further in the work.

1.1 Tensor products of Banach spaces

Let $X_1, \ldots, X_l, Y$ be vector spaces. We say that a function $T : X_1 \times \cdots \times X_l \to Y$ is a **mutlilinear transformation** (or map) if T is a linear map on each variable. That is, for every scalar λ and vectors $x^i, z^i \in X_i$ for $i = 1, \ldots, l$ we have:

$$T\left(x^1, \ldots, \lambda x^i + z^i, \ldots, x^l\right) = \lambda T\left(x^1, \ldots, x^i, \ldots, x^l\right) + T\left(x^1, \ldots, z^i, \ldots, x^l\right).$$

The **tensor product** of the vector spaces $X_1, ..., X_l$, denoted by $\otimes_{i=1}^l X_i$ (or equivalently by $X_1 \otimes \cdots \otimes X_l$), is a vector space together with a multilinear map, $\otimes : X_1 \times \cdots \times X_l \to \otimes_{i=1}^l X_i$, with the property that for every vector space Y and every multilinear map $T : X_1 \times \cdots \times X_l \to Y$, there exists a unique linear transformation $\hat{T} : \otimes_{i=1}^l X_i \to Y$ such that $T = \hat{T} \circ \otimes$.



The image $\otimes\left(x^1,\ldots,x^l\right)$ of a tuple $\left(x^1,\ldots,x^l\right)$ will be denoted by $x^1\otimes\cdots\otimes x^l$. A vector $x^1\otimes\cdots\otimes x^l\in\otimes_{i=1}^l X_i$ is called a **decomposable tensor**. By $x^1\otimes\cdots x^{i-1}\otimes X_i\otimes x^{i+1}\otimes\cdots\otimes x^l$ we denote the subspace $\left\{x^1\otimes\cdots x^{i-1}\otimes a^i\otimes x^{i+1}\otimes\cdots\otimes x^l : a^i\in X_i\right\}\subseteq\otimes_{i=1}^l X_i$.

If $T_i : X_i\to Y_i$ for $i=1,\ldots,l$ are linear maps between vector spaces. Then,

$$\begin{aligned} X_1\times\cdots\times X_l &\to \otimes_{i=1}^l Y_i\\ \left(x^1,\ldots,x^l\right) &\to T_1x^1\otimes\cdots\otimes T_lx^l\end{aligned}$$

is a multilinear map. Its linearization is the linear map $T_1\otimes\cdots\otimes T_l$. The linear map $T_1\otimes\cdots\otimes T_l$ acts on decomposable tensors as follows:

$$\begin{aligned} T_1\otimes\cdots\otimes T_l : \otimes_{i=1}^l X_i &\to \otimes_{i=1}^l Y_i\\ x^1\otimes\cdots\otimes x^l &\to T_1x^1\otimes\cdots\otimes T_lx^l.\end{aligned}$$

In addition, if the spaces $X_1,\ldots,X_l$ are normed spaces then one can define norms on their tensor product $\otimes_{i=1}^l X_i$ that are compatible, in the sense of the following definition, with the norms on each space X_i, $i=1,\ldots,l$.

For every $x_i^*\in X_i^*$ with $i=1,\ldots,l$ the symbol $x_1^*\otimes\cdots\otimes x_l^*$ will denote both the linear map that sends the vector $x^1\otimes\cdots\otimes x^l\in\otimes_{i=1}^l X_i$ to $x_1^*\left(x^1\right)\cdots x_l^*\left(x^l\right)$ and the vector $x_1^*\otimes\cdots\otimes x_l^*\in\otimes_{i=1}^l X_i^*$.

Definition 1.1. Let $X_1,\ldots,X_l$ be normed spaces. We say that a norm α on $\otimes_{i=1}^l X_i$ is a reasonable crossnorm if the following conditions hold:

1. $\alpha\left(x^1\otimes\cdots\otimes x^l\right)\le\|x^1\|\cdots\|x^l\|$ for every $x^i\in X_i$, $i=1,\ldots,l$.
2. If $x_i^*\in X_i^*$, $i=1,\ldots,l$ then $x_1^*\otimes\cdots\otimes x_l^*\in\left(\otimes_{i=1}^l X_i,\alpha\right)^*$ and $\|x_1^*\otimes\cdots\otimes x_l^*\|\le\|x_1^*\|\cdots\|x_l^*\|$.

1.1.1 The projective tensor product of Banach spaces

Let $X_1,\ldots,X_l$ be Banach spaces. For each $u\in\otimes_{i=1}^l X_i$ **the projective norm** of u is defined as follows:

$$\pi(u) := \inf\left\{\sum_{i=1}^n\|x_i^1\|\cdots\|x_i^l\| : u=\sum_{i=1}^n x_i^1\otimes\cdots\otimes x_i^l\right\}.$$

Proposition 1.2. *Let $X_1,\ldots,X_l$ be Banach spaces. Then $\pi(\cdot)$ is a reasonable crossnorm on $\otimes_{i=1}^l X_i$ such that:*

1. *$\pi\left(x^1\otimes\cdots\otimes x^l\right)=\|x^1\|\cdots\|x^l\|$ for every $x^i\in X_i$ with $i=1,\ldots,l$.*
2. *If $x_i^*\in X_i^*$ for $i=1,...,l$ then $x_1^*\otimes\cdots\otimes x_l^*\in\left(\otimes_{i=1}^l X_i,\pi\right)^*$ and $\|x_1^*\otimes\cdots\otimes x_l^*\|=\|x_1^*\|\cdots\|x_l^*\|$.*

The completion of the normed space $\left(\otimes_{i=1}^{l} X_i, \pi\right)$ is called **the projective tensor product** of the Banach spaces $X_1, \ldots, X_l$, and is denoted by $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_l$.

The following proposition presents a characterization of the closed unit ball of the projective tensor product of Banach spaces which is fundamental for this work.

Proposition 1.3. *Let $X_1, \ldots, X_l$ be Banach spaces. The unit ball of $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_l$ is the closed convex hull:*

$$\overline{conv \left\{ x^1 \otimes \cdots \otimes x^l : x^1 \in B_{X_1}, \ldots, x^l \in B_{X_l} \right\}}.$$

Proposition 1.4. *Let X_i, Y_i for $i = 1, \ldots, l$ be Banach spaces. If $T_i \in \mathcal{L}(X_i; Y_i)$ for $i = 1, \ldots, l$, then $T_1 \otimes \cdots \otimes T_l : \left(\otimes_{i=1}^{l} X_i, \pi\right) \to \left(\otimes_{i=1}^{l} Y_i, \pi\right)$ is continuous and*

$$\|T_1 \otimes \cdots \otimes T_l\| = \|T_1\| \cdots \|T_l\|.$$

Proposition 1.5. *Let $i \in \{1, \ldots, l\}$ and let ψ be a quotient operator from the Banach space X_i to the Banach space W_i (i.e. ψ is a surjective linear map sending B_{X_i} onto B_{W_i}). Then, the operator $id_{X_1} \otimes \ldots \otimes \psi \otimes \cdots \otimes id_{X_l}$ is a quotient operator between spaces $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_l$ and $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi W_i \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_l$.*

The following theorem shows that in the same way that the tensor product of vector spaces linearizes multilinear maps, the projective tensor product of Banach spaces linearizes continuous multilinear maps.

Before we present the theorem, we will introduce some notation. If $X_1, \ldots, X_l, Y$ are Banach spaces, the set of continuous multilinear maps $T : X_1 \times \cdots \times X_l \to Y$ is denoted by $\mathcal{L}(X_1, \ldots, X_l; Y)$. The set $\mathcal{L}(X_1, \ldots, X_l; Y)$ is a Banach space with the norm,

$$\|T\| := \sup \left\{ \left\| T\left(x^1, \ldots, x^l\right) \right\|_Y : x^1 \in B_{X_1}, \ldots, x^l \in B_{X_l} \right\}.$$

Theorem 1.6. *(Universal property) Let $X_1, \ldots, X_l, Y$ be Banach spaces. For every multilinear operator $T \in \mathcal{L}(X_1, \ldots, X_l; Y)$ there exists a unique linear operator,*

$$\hat{T} \in \mathcal{L}\left(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_l; Y\right)$$

such that $T = \hat{T} \circ \otimes$. Reciprocally for each $\hat{S} \in \mathcal{L}\left(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_l; Y\right)$ and each $\left(x^1, \ldots, x^l\right) \in X_1 \times \cdots \times X_l$, the expresion $s\left(x^1, \ldots, x^l\right) := \hat{S}\left(x^1 \otimes \cdots \otimes x^l\right)$ determines a unique element $s \in \mathcal{L}(X_1, \ldots, X_l; Y)$. Furthermore, the correspondence $T \to \hat{T}$ establishes an linear isometry from $\mathcal{L}(X_1, ..., X_l; Y)$ onto $\mathcal{L}\left(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_l; Y\right)$.

1.1.2 The injective tensor product of Banach spaces

Let $X_1, \ldots, X_l$ be Banach spaces. For each $u \in \otimes_{i=1}^{l} X_i$ **the injective norm** of u is defined as follows:

$$\epsilon(u) := \sup \left\{ \left| x_1^* \otimes \cdots \otimes x_l^*(u) \right| : x_1^* \in B_{X_1^*}, \ldots, x_l^* \in B_{X_l^*} \right\}.$$

Proposition 1.7. *Let $X_1, \ldots, X_l$ be Banach spaces. Then $\epsilon(\cdot)$ is a reasonable cross-norm on $\otimes_{i=1}^{l} X_i$ such that:*

1. $\epsilon(u) \leq \pi(u)$ for every $u \in \otimes_{i=1}^{l} X_i$.

2. $\epsilon\left(x^1 \otimes \cdots \otimes x^l\right) = \|x^1\| \cdots \left\|x^l\right\|$ for every $x^i \in X_i$ with $i = 1, \ldots, l$.

3. If $x_i^* \in X_i^*$ for $i = 1, ..., l$ then $x_1^* \otimes \cdots \otimes x_l^* \in \left(\otimes_{i=1}^{l} X_i, \epsilon\right)^*$ and $\|x_1^* \otimes \cdots \otimes x_l^*\| = \|x_1^*\| \cdots \|x_l^*\|$.

The completion of the normed space $\left(\otimes_{i=1}^{l} X_i, \epsilon\right)$ is called **the injective tensor product** of the Banach spaces $X_1, \ldots, X_l$ and is denoted by $X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_l$.

Proposition 1.8. *Let X_i, Y_i $i = 1, \ldots, l$ be Banach spaces. If $T_i \in \mathcal{L}(X_i; Y_i)$ for $i = 1, \ldots, l$, then $T_1 \otimes \cdots \otimes T_l : \left(\otimes_{i=1}^{l} X_i, \epsilon\right) \to \left(\otimes_{i=1}^{l} Y_i, \epsilon\right)$ is continuous and*

$$\|T_1 \otimes \cdots \otimes T_l\| = \|T_1\| \cdots \|T_l\|.$$

Proposition 1.9. *Let $i \in \{1, \ldots, l\}$ and let $W_i \subseteq X_i$ be a closed subspace of the Banach space X_i, then $X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon W_i \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_l$ is a closed subspace of $X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_l$.*

1.1.3 Tensor product of Hilbert spaces

Let $\mathbb{E}_1, \ldots, \mathbb{E}_l$ be Hilbert spaces. Then for every $u, v \in \otimes_{i=1}^{l} \mathbb{E}_i$ with $u = \sum_{i=1}^{m} x_i^1 \otimes \cdots \otimes x_i^l$ and $v = \sum_{j=1}^{n} y_j^1 \otimes \cdots \otimes y_j^l$ the function:

$$\langle u, v \rangle_H := \sum_{i,j=1}^{m,n} \left\langle x_i^1, y_j^1 \right\rangle_{\mathbb{E}_1} \cdots \left\langle x_i^l, y_j^l \right\rangle_{\mathbb{E}_l}$$

is an inner product on $\otimes_{i=1}^{l} \mathbb{E}_i$. The norm determined by $\langle \cdot, \cdot \rangle_H$ will be denoted by $\|\cdot\|_H$. This norm is usually called the Frobenius norm [11].

Proposition 1.10. *If $\mathbb{E}_1, \ldots, \mathbb{E}_l$ are Hilbert spaces, then $\|\cdot\|_H$ is a reasonable cross-norm on $\otimes_{i=1}^{l} \mathbb{E}_i$.*

By $\otimes_{H,i=1}^{l} \mathbb{E}_i$ we denote the pre-Hilbert space $\left(\otimes_{i=1}^{l} \mathbb{E}_i, \|\cdot\|_H\right)$. The Hilbert space obtained as the completion of the pre-Hilbert space $\otimes_{H,i=1}^{l} \mathbb{E}_i$ is called **the Hilbert tensor product** of the Hilbert spaces $\mathbb{E}_1, \ldots, \mathbb{E}_l$ and is denoted by $\mathbb{E}_1 \hat{\otimes}_H \cdots \hat{\otimes}_H \mathbb{E}_l$. To shorten notation we will write $\otimes_{H,i=1}^{l} \mathbb{E}_i$ instead of $\left(\otimes_{i=1}^{l} \mathbb{E}_i, \|\cdot\|_H\right)$.

Proposition 1.11. *Let $\mathbb{E}_i, \mathbb{F}_i$ for $i = 1, \ldots, l$ be Hilbert spaces. If $T_i \in \mathcal{L}(\mathbb{E}_i; \mathbb{F}_i)$ then $T_1 \otimes \cdots \otimes T_l : \otimes_{H,i=1}^{l} \mathbb{E}_i \to \otimes_{H,i=1}^{l} \mathbb{F}_i$ is continuous and*

$$\|T_1 \otimes \cdots \otimes T_l\| \leq \|T_1\| \cdots \|T_l\|.$$

Proposition 1.12. *Let $\mathbb{E}_1, \ldots, \mathbb{E}_l$ be Hilbert spaces. If $\left\{x_j^i\right\}_{j \in J_i}$ is an orthonormal basis for $\mathbb{E}_i, i = 1, \ldots, l$ then $\left(x_{j_1}^1 \otimes \cdots \otimes x_{j_l}^l\right)_{j_1 \in J_1, \ldots, j_l \in J_l}$ is an orthonormal basis for $\mathbb{E}_1 \hat{\otimes}_H \cdots \hat{\otimes}_H \mathbb{E}_l$.*

1.2 Tensor norms on Banach Spaces

In the previous sections we introduced the injective norm and the projective norm on the tensor product of Banach spaces. Here we will see that this norms are members of a large class of norms that one can define on the tensor product of Banach spaces. The results that appear below can be consulted in [12] and [34].

The following proposition characterizes the reasonable crossnorms on the tensor product of the Banach spaces $X_1, \ldots, X_l$.

Proposition 1.13. *Let $X_1, \ldots, X_l$ be Banach spaces.*

1. A norm $\alpha(\cdot)$ on $\otimes_{i=1}^l X_i$ is a reasonable crossnorm if and only if

$$\epsilon(u) \leq \alpha(u) \leq \pi(u)$$

for every $u \in \otimes_{i=1}^l X_i$.

2. If α is a reasonable crossnorm on $\otimes_{i=1}^l X_i$, then $\alpha\left(x^1 \otimes \cdots \otimes x^l\right) = \|x^1\| \cdots \|x^l\|$ for every *$x^i \in X_i$ with $i = 1, \ldots, l$. Furthermore, for every $x_i^* \in X_i^*$ $i = 1, \ldots, l$, the norm of the linear functional $x_1^* \otimes \cdots \otimes x_l^* \in \left(\otimes_{i=1}^l X_i, \alpha\right)^*$ satisfies $\|x_1^* \otimes \cdots \otimes x_l^*\| = \|x_1^*\| \cdots \|x_l^*\|$.*

Definition 1.14. A **tensor norm** α of order l on the class of Banach spaces (resp. on the class of finite dimensional normed spaces) assigns to each l-tuple $(X_1, \ldots, X_l)$ of Banach spaces (resp. finite dimensional normed spaces) a norm $\alpha_{X_1,\ldots,X_l}(\cdot)$ on the tensor product $\otimes_{i=1}^l X_i$ such that the two following conditions are satisfied:

1. α is a **reasonable crossnorm**, i.e. for every $u \in \otimes_{i=1}^l X_i$

$$\epsilon(u) \leq \alpha_{X_1,\ldots,X_l}(u) \leq \pi(u).$$

2. α satisfies the **uniform property**: if $T_i \in \mathcal{L}(X_i; Y_i)$ for $i = 1, \ldots, l$, then

$$\left\|T_1 \otimes \cdots \otimes T_l : \left(\otimes_{i=1}^l X_i, \alpha\right) \to \left(\otimes_{i=1}^l Y_i, \alpha\right)\right\| \leq \|T_1\| \cdots \|T_l\|.$$

By $\otimes_{\alpha,i=1}^l X_i$ we denote the normed space $\left(\otimes_{i=1}^l X_i, \alpha\right)$, and by $X_1 \hat{\otimes}_\alpha \cdots \hat{\otimes}_\alpha X_l$ its completion.

From Propositions 1.4 and 1.8 it follows that the projective and the injective norm are tensor norms.

Remark 1.15. We would like to point out that there exist reasonable crossnorms that do not come from tensor norms in the class of Banach spaces (resp. finite dimensional normed spaces). An example of such norms is the norm $\|\cdot\|_H$ defined on the tensor product of Hilbert spaces. Other examples appear in [12].

1.2.1 Properties of tensor norms

In this section we state the basic properties of tensor norms that we will use in Chapter 3.

Definition 1.16. Let α be a tensor norm of order l. Then α is called **injective** if for every closed subspace E_i of the Banach space X_i with $i = 1, \ldots, l$, and every $u \in \otimes_{i=1}^{l} E_i$, we have

$$\alpha_{E_1,\ldots,E_l}(u) = \alpha_{X_1,\ldots,X_l}(u).$$

Definition 1.17. Let α be a tensor norm of order l. Then α is called **projective** if for every quotient operator $T_i : X_i \to Y_i$ between Banach spaces X_i, Y_i with $i = 1, \ldots, l$, we have that the extension of the map $T_1 \otimes \cdots \otimes T_l$ to the completion of $X_1 \hat{\otimes}_\alpha \cdots \hat{\otimes}_\alpha X_l$ and $Y_1 \hat{\otimes}_\alpha \cdots \hat{\otimes}_\alpha Y_l$ is a quotient operator

Proposition 1.18. *Let α be a tensor norm of order l. If $M_1, \ldots, M_l$ are finite dimensional normed spaces then the norm $\alpha'_{M_1,\ldots,M_l}$ defined as follows*

$$\alpha'_{M_1,\ldots,M_l}(u) := \sup\left\{|\varphi(u)| : \alpha_{M_1^*,\ldots,M_l^*}(\varphi) \le 1\right\} \text{ for } u \in \otimes_{i=1}^{l} M_i,$$

is a uniform reasonable crossnorm.

Corollary 1.19. *Let α be a tensor norm of order l, then α' is a tensor norm on the class of finite dimensional normed spaces.*

The norm α' is called the **dual norm** of α.

Proposition 1.20. *If $M_1, \ldots, M_l$ are finite dimensional normed spaces we have,*

$$\begin{aligned} \pi'_{M_1,\ldots,M_l} &= \epsilon_{M_1,\ldots,M_l} \\ \epsilon'_{M_1,\ldots,M_l} &= \pi_{M_1,\ldots,M_l}. \end{aligned}$$

1.3 Banach operator ideals

Here, we state the basic results about Banach operator ideals that we will use in Section 4.5. We present the correspondence between Banach operator ideals and tensor norms of finite dimensional normed spaces, see Theorems 1.22 and 1.23. For further details we refer to [34].

Below we present the definition of a Banach operator ideal that appears in Chapter 8, R. Ryan, [34]. Another equivalent formulation appears in [12].

Definition 1.21. A **Banach operator ideal** consists of an assignment to each pair of Banach spaces X, Y of a vector space $\mathcal{A}(X, Y)$ of bounded linear operators from X to Y, together with a norm, A, on this space, with the following properties:

1. $\mathcal{A}(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$ which contains the finite rank operators. Furthermore, for every $\varphi \in X^*$ and $y \in Y$, $A(\varphi(\cdot)y) = \|\varphi\| \|y\|$.

2. The ideal property: if $S \in \mathcal{A}(X_0, Y_0), T \in \mathcal{L}(X, X_0)$ and $R \in \mathcal{L}(Y_0, Y)$ then $RST \in \mathcal{A}(X,Y)$ and $A(RST) \leq \|R\| A(S) \|T\|$.

3. $(\mathcal{A}(X,Y), A)$ is a Banach space.

By $(\mathcal{A}, A)$ we denote a Banach operator ideal. Any Banach operator ideal restricted to the class of finite dimensional normed spaces is of the form $(\mathcal{L}, A)$, i.e. $\mathcal{A}(M,N) = \mathcal{L}(M,N)$ if M and N are finite dimensional normed spaces. For this reason we say that the norm A is an **ideal norm**.

The following theorems are consequences of the representation theorem for maximal operator ideals. See [12]. Before presenting the theorems 1.22 and 1.23, we recall there exists a linear isomorphism between the tensor product $M \otimes N$ of finite dimensional normed spaces, and the space of linear maps $T : M^* \to N$. The isomorphism is defined as follows:

Let $u = \sum_{i=1}^n x_i \otimes y_i$ be an element of the tensor product $M \otimes N$ of finite dimensional normed spaces. By T_u we denote the linear map:

$$\begin{aligned} T_u : M^* &\to N \\ x^* &\to \sum_{i=1}^n x^*(x_i)\, y_i. \end{aligned}$$

Conversely, let $T = \sum_{i=1}^n x_i^*(\cdot)\, y_i$ be a linear map from M to N. By u_T we denote the vector $\sum_{i=1}^n x_i^* \otimes y_i$ that belongs to $M^* \otimes N$.

Theorem 1.22. *Let $(\mathcal{A}, A)$ be a Banach operator ideal. For every pair M, N of finite dimensional normed spaces*

$$\alpha^{\mathcal{A}}_{M,N}(u) := A(T_u : M^* \to N),$$

is a reasonable uniform crossnorm. The norm $\alpha^{\mathcal{A}}$ is a tensor norm on the class of finite dimensional normed spaces.

Theorem 1.23. *Let α be a tensor norm of order two on the class of finite dimensional normed spaces. For every pair M, N of finite dimensional normed spaces define*

$$A_\alpha(T : M \to N) := \alpha_{M^*,N}(u_T).$$

Then A_α is an ideal norm.

Remark 1.24. As a consequence of the previous theorems there exists a bijection between tensor norms and ideal norms on the class of finite dimensional normed spaces.

1.4 Basic convexity

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A subset $C \subseteq V$ is **convex** if for every $x, y \in C$ and every real number $t \in [0,1]$, $tx + (1-t)\,y \in C$.

From the definition of convex set we have that intersection of convex sets are convex sets, the sum of convex sets is a convex set.

If A is a subset of a vector space V the **convex hull** of A is defined as:

$$conv\,(A) := \bigcap \{C \subseteq V : A \subseteq C \text{ and } C \text{ is convex}\}.$$

1.4.1 Convexity on Euclidean spaces

In this section we present the basic results about convex sets in real Euclidean spaces that we will use through the work. All of them are well known results. For a proof of this results see [36].

By a Euclidean space we mean a finite dimensional Hilbert space. The set of nonempty compact sets of a Euclidean space $\mathbb{E}$ is denoted by $\mathcal{C}(\mathbb{E})$.

Proposition 1.25. *Let A, B be subsets of $\mathbb{E}$ then:*

1. $conv\,(A) = \left\{\sum_{i=1}^{n} \lambda_i x_i : 0 \leq \lambda_i \leq 1, x_i \in A, \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N}\right\}.$
2. $conv\,(A+B) = conv\,(A) + conv\,(B).$

Proposition 1.26. *Let $\mathbb{E}$ be a Euclidean space. If $A \subseteq \mathbb{E}$ is compact then $conv\,(A)$ is compact.*

A subset A of a Euclidean space $\mathbb{E}$ is called a **convex body** if A is a convex compact set with nonempty interior. The set of convex bodies contained in $\mathbb{E}$ is denoted by $\mathcal{K}(\mathbb{E})$. If a convex body $A \subseteq \mathbb{E}$ is such that $A = -A$ then we say that A is a 0-**symmetric** (or centrally symmetric) convex body. The set of 0-symmetric convex bodies contained in $\mathbb{E}$ is denoted by $\mathcal{B}(\mathbb{E})$. If $\mathbb{E} = \mathbb{R}^d$ then $\mathcal{B}(\mathbb{E})$ is denoted by $\mathcal{B}(d)$.

Let $\mathbb{E}$ be a Euclidean space. For every $A \in \mathcal{K}(\mathbb{E})$, the **support function** of A, $h(A,\cdot) = h_A(\cdot)$, is defined by

$$h(A,x) := \sup\{\langle a, x\rangle_{\mathbb{E}} : a \in A\} \text{ for } x \in \mathbb{E}.$$

If $A \in \mathcal{B}(\mathbb{E})$, then the **Minkowski functional** (or **gauge function**) of A, $g(A,\cdot) = g_A(\cdot)$, is defined by

$$g(A,x) := \inf\left\{\lambda > 0 : \frac{1}{\lambda}x \in A\right\} \text{ for } x \in \mathbb{E}.$$

For every nonempty set $C \subseteq \mathbb{E}$ of a Euclidean space $\mathbb{E}$, the polar set of C is defined as follows:

$$C^{\circ} := \left\{x \in \mathbb{E} : \sup_{c \in C} \langle c, x\rangle_{\mathbb{E}} \leq 1\right\}.$$

The following proposition establishes a fundamental relation between the Minkowski functional of a 0-symmetric convex body A and the support function of its polar set A°.

Proposition 1.27. *Let $\mathbb{E}$ be a Euclidean space. If $A \in \mathcal{B}(\mathbb{E})$, then $g(A,\cdot) = h(A^\circ,\cdot)$.*

The theorem that appears below was proved by H. Minkowki in [27]. It exhibits the bijection, given by the Minkowski functional, between norms on a Euclidean space $\mathbb{E}$ and 0-symmetric convex bodies contained in $\mathbb{E}$.

Recall that for every $A \in \mathcal{B}(\mathbb{E})$ the polar set $A^\circ \in \mathcal{B}(\mathbb{E})$.

Theorem 1.28. *(H. Minkowski) Let $\mathbb{E}$ be a Euclidean space. If $A \in \mathcal{B}(\mathbb{E})$, then*

$$\|x\|_A := g(A,x) \text{ for } x \in \mathbb{E}$$

defines a norm $\|\cdot\|_A$ on $\mathbb{E}$ for which A is the closed unit ball. Furthermore, for every $x \in \mathbb{E}$ we have

$$\|x\|_{A^\circ} = \|\langle\cdot,x\rangle_{\mathbb{E}} : (\mathbb{E},\|\cdot\|_A) \to \mathbb{R}\|.$$

The following proposition is the analogue to Theorem 1.28 in complex Euclidean spaces. Let $\mathbb{E}$ be a complex Euclidean space. A convex body $A \subseteq \mathbb{E}$ is called **circled convex body** if $e^{i\theta}A = A$ for all $\theta \in \mathbb{R}$.

Proposition 1.29. *Let $\mathbb{E}$ be a complex Euclidean space. If A is a circled convex body in $\mathbb{E}$, then*

$$\|x\|_A := \inf\left\{\lambda > 0 : \lambda^{-1}x \in A\right\} \text{ for } x \in \mathbb{E}$$

defines a norm $\|\cdot\|_A$ on $\mathbb{E}$ for which A is the closed unit ball. Furthermore, for every $x \in \mathbb{E}$ we have

$$\|x\|_{A^\circ} = \|\langle\cdot,x\rangle : (\mathbb{E},\|\cdot\|_A) \to \mathbb{C}\|.$$

By abuse of notation, if $\mathbb{E}$ is a complex Euclidean space we use the same symbols $\mathcal{B}(\mathbb{E})$ to denote the set of circled convex bodies contained in $\mathbb{E}$ and $g(A,\cdot) = g_A(\cdot)$ to denote the Minkowski functional of a circled convex bodie A.

1.4.2 The Hausdorff metric

Let $\mathbb{E}$ be a Euclidean space. For every pair $C, D \in \mathcal{C}(\mathbb{E})$ the **Hausdorff metric** δ^H is defined by

$$\delta^H(C,D) := \max\left\{\sup_{x\in C}\inf_{y\in D}\|x-y\|, \sup_{y\in D}\inf_{x\in C}\|y-x\|\right\}$$

or, equivalently, by

$$\delta^H(C,D) = \min\{\lambda \geq 0 : C \subseteq D + \lambda B_{\mathbb{E}}, D \subseteq C + \lambda B_{\mathbb{E}}\}$$

Remark 1.30. It is not difficult to see that for $C, D \in \mathcal{C}(\mathbb{E})$, we have

$$\delta^H(conv(C), conv(D)) \leq \delta^H(C, D).$$

Theorem 1.31. *Let $\mathbb{E}$ be a Euclidean space. Then $\left(\mathcal{C}(\mathbb{E}), \delta^H\right)$ is a complete metric space.*

Theorem 1.32. *(Blaschke selection theorem) Let $\mathbb{E}$ be a Euclidean space. From every bounded sequence of nonempty compact convex sets in $\mathbb{E}$ one can select a subsequence converging to a nonempty compact convex set.*

The following characterization of the Hausdorff metric between convex bodies will be used throughout the work:

Proposition 1.33. *Let $\mathbb{E}$ be a Euclidean space. For every pair C, D of elements in $\mathcal{K}(\mathbb{E})$, we have*

$$\delta^H(C, D) = \sup_{x \in \partial B_{\mathbb{E}}} |h(C, x) - h(D, x)|.$$

Remark 1.34. The above proposition can be reformulated as follows: If $\bar{h}_C$ is the restriction of $h(C, \cdot)$ to $\partial B_{\mathbb{E}}$, then we have $\delta^H(C, D) = \left\|\bar{h}_C - \bar{h}_D\right\|_\infty$.

The following proposition shows that if a sequence of 0-symmetric convex bodies converges (in the Hausdorff metric) to a 0-symmetric convex body then so does the sequence of the polar sets. This property will be used repeatedly in this work. We present the proof because we did not find a proper reference.

Proposition 1.35. *Let $\mathbb{E}$ be a Euclidean space. For every sequence $\{C_n\}$ in $\mathcal{B}(\mathbb{E})$ and $C \in \mathcal{B}(\mathbb{E})$, we have:*

1. If $g_{C_n}(\cdot)$ converges uniformly on $\partial B_{\mathbb{E}}$ to $g_C(\cdot)$, then $g_{C_n^\circ}(\cdot)$ converges uniformly on $\partial B_{\mathbb{E}}$ to $g_{C^\circ}(\cdot)$.

2. If $C_n \to C$ in the Hausdorff metric, then $g_{C_n}(\cdot)$ converges uniformly on $\partial B_{\mathbb{E}}$ to $g_C(\cdot)$. Particularly, $C_n^\circ \to C^\circ$ in the Hausdorff metric.

Proof. 1. Let C_n, $n \in \mathbb{N}$ and C be as in the statement of the proposition. Since the pointwise convergence of norms on a finite dimensional space implies the uniform convergence on compact sets, we only need to prove that $g_{C_n^\circ}$ converges to g_{C° pointwise.

Assume that $g_{C_n}(\cdot)$ converges uniformly on $\partial B_{\mathbb{E}}$ to $g_C(\cdot)$. Let $\varepsilon > 0$. By the uniform convergence of $g_{C_n}(\cdot)$, we know there exists $N \in \mathbb{N}$ such that

$$\sup_{x \in \partial B_{\mathbb{E}}} |g_{C_n}(x) - g_C(x)| \leq \varepsilon \text{ for } n \geq N.$$

From the compactness of C we know there exists $r > 0$ such that $C \subseteq rB_{\mathbb{E}}$. Therefore, for every $x \in \partial C$ and $n \geq N$ we have

$$\left|g_{C_n}\left(\frac{1}{r}x\right) - \frac{1}{r}\right| \leq \varepsilon.$$

Which for $x \in C$ implies,

$$(1 - r\varepsilon)\, g_C(x) \leq g_{C_n}(x) \leq (1 + r\varepsilon)\, g_C(x) \text{ for } n \geq N.$$

From this, we obtain

$$\sup_{g_C(x) \leq \frac{1}{1+r\varepsilon}} |\langle x, y\rangle| \leq \sup_{g_{C_n}(x) \leq 1} |\langle x, y\rangle| \leq \sup_{g_C(x) \leq \frac{1}{1-r\varepsilon}} |\langle x, y\rangle|,$$

for any $y \in \mathbb{E}$. Thus,

$$\frac{1}{1 + r\varepsilon} g_{C^\circ}(y) \leq g_{C_n^\circ}(y) \leq \frac{1}{1 - r\varepsilon} g_{C^\circ}(y),$$

and $g_{C_n^\circ}$ converges to g_{C° pointwise. Since the pointwise convergence of norms implies the uniform convergence on compact sets, we have the desired result.

2. Assume that $C_n \to C$ in the Hausdorff metric. Since $C_n, C \in \mathcal{B}(\mathbb{E})$, Proposition 1.33 implies that $g_{C_n^\circ}(\cdot)$ converges uniformly on $\partial B_{\mathbb{E}}$ to $g_{C^\circ}(\cdot)$. Then by 1. we conclude that $g_{C_n^{\circ\circ}}(\cdot) = g_{C_n}(\cdot)$ converges uniformly on $\partial B_{\mathbb{E}}$ to $g_{C^{\circ\circ}}(\cdot) = g_C(\cdot)$. Applying Proposition 1.33 again, we get that $C_n^\circ \to C^\circ$ in the Hausdorff metric. □

1.4.3 The Banach-Mazur distance

Here, we introduce the results about the Banach-Mazur distance [6] that we will use in Chapter 3. The proofs of the results and further details can be consulted in [38].

If X and Y are isomorphic Banach spaces, then the **Banach-Mazur distance** between X and Y is defined as:

$$\delta^{BM}(X, Y) := \inf\left\{\|T\| \left\|T^{-1}\right\| : T \in \mathcal{L}(X, Y) \text{ and } T^{-1} \in \mathcal{L}(Y, X)\right\}.$$

The Banach-Mazur distance between 0-symmetric convex bodies of a Euclidean space $\mathbb{E}$ is defined as follows:

$$\delta^{BM}(P, Q) := \inf\{\lambda \geq 1 : T : \mathbb{E} \to \mathbb{E} \text{ is a bijective linear map and } Q \subseteq TP \subseteq \lambda Q\}.$$

By $\mathcal{BM}(d)$ we denote the set of equivalence classes of 0-symmetric convex bodies in $\mathbb{R}^d$ determined by the following equivalence relation: for every $P, Q \in \mathcal{B}(\mathbb{E})$, $P \sim Q$ if and only if there exists a bijective linear map such that $TP = Q$.

The Banach-Mazur distance determines a metric, $\log \delta^{BM}$, on the set $\mathcal{BM}(d)$. The space $\left(\mathcal{BM}(d), \log \delta^{BM}\right)$ is called the **Banach-Mazur compactum**.

Theorem 1.36. *The space* $\left(\mathcal{BM}(d), \log \delta^{BM}\right)$ *is a compact metric space.*

Chapter 2

Tensorial 0-symmetric convex bodies in $\mathbb{R}^d$

The fundamental theorem of H. Minkowski, Theorem 1.28, exhibits the bijective correspondence between 0-symmetric convex bodies and norms in $\mathbb{R}^d$. Motivated by this correspondence we start the study of 0-symmetric convex bodies Q in $\otimes_{i=1}^{l}\mathbb{R}^{d_i} \simeq \mathbb{R}^{d_1\cdot\dots\cdot d_l}$ with the property that there exist norms $\|\cdot\|_i$ on $\mathbb{R}^{d_i}$ for $i=1,\dots,l$ such that Q is the unit ball of a reasonable crossnorm on $\otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$.

Our main results are the characterizations of 0-symmetric convex bodies and ellipsoids with the above property, Theorem 2.20 and Theorem 2.30.

2.1 The set of decomposable tensors

Here we present some results about the set of decomposable tensors that we will use throughout the work.

Recall that a vector $x^1 \otimes \cdots \otimes x^l$ on the tensor product $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ is called a decomposable tensor. The set of decomposable tensors of the vector space $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ is denoted by $\Sigma_{\mathbb{R}^{d_1},\dots,\mathbb{R}^{d_l}}$. Below we prove that the set of decomposable tensors is closed (Corollary 2.2). This is a well known result (see [11]), we include its proof for the sake of completeness.

Recall that we always assume $\mathbb{R}^d$ is a Euclidean space with the standard inner product $\langle\cdot,\cdot\rangle_2$, norm$\|\cdot\|_2$ and closed unit ball B_2^d.

Proposition 2.1. *If $A_i \subseteq \mathbb{R}^{d_i}$ $i=1,\dots l$ are compact sets, then*

$$\Sigma_{A_1,\dots,A_l} := \left\{x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^{l}\mathbb{R}^{d_i} : x^i \in A_i\right\}$$

is a compact subset of $\otimes_{H,i=1}^{l}\mathbb{R}^{d_i}$.

Proof. Recall that the product topology on $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_l}$ coincides with the topology given by the norm:

$$\left\| \left(x^1, ..., x^l\right) \right\|_2 := \left(\sum_{i=1}^{l} \left\| x^i \right\|_2^2 \right)^{\frac{1}{2}}.$$

Thus, from the continuity of the multilinear map $\otimes : \mathbb{R}^{d_i} \times \cdots \times \mathbb{R}^{d_l} \to \otimes_{H,i=1}^{l} \mathbb{R}^{d_i}$ and the compactness of each A_i for $i = 1, ..., l$, we know that $\otimes (A_1, \ldots, A_l)$ is compact. Hence, $\Sigma_{A_1,...,A_l}$ is compact. □

Corollary 2.2. *The set of decomposable tensors is a closed subset of* $\otimes_{H,i=1}^{l} \mathbb{R}^{d_i}$.

Proof. Let $\left(x_n^1 \otimes \cdots \otimes x_n^l\right)_n$ be a sequence in $\Sigma_{\mathbb{R}^{d_1},...,\mathbb{R}^{d_l}}$ converging to some $z \in \otimes_{i=1}^{l} \mathbb{R}^{d_i}$. Notice that if $z = 0$ then $z \in \Sigma_{\mathbb{R}^{d_1},...,\mathbb{R}^{d_l}}$. Therefore we can assume that $z \neq 0$. Without loss of generality, suppose that every $x_n^1 \otimes \cdots \otimes x_n^1 \neq 0$. Let

$$y_n^1 = \left\| x_n^2 \right\|_2 \cdots \left\| x_n^l \right\|_2 x_n^1,$$
$$y_n^i = \frac{x_n^i}{\|x_n^i\|_2} \text{ for } i = 2, ..., l.$$

Then $y_n^1 \otimes \cdots \otimes y_n^l \to z$ and $\|y_n^1\|_2 = \left\| y_n^1 \otimes \cdots \otimes y_n^l \right\|_H \to \|z\|_H$. Thus, for some $r > 0$ we have $(y_n^1)_n \subseteq rB_2^{d_1}$. Hence, $y_n^1 \otimes \cdots \otimes y_n^l \in \Sigma_{rB_2^{d_1},...,B_2^{d_l}}$. From the compactness of $\Sigma_{rB_2^{d_1},...,B_2^{d_l}}$ we have $z \in \Sigma_{rB_2^{d_1},...,B_2^{d_l}}$. Which implies $z \in \Sigma_{\mathbb{R}^{d_1},...,\mathbb{R}^{d_l}}$. □

Let $T : \otimes_{i=1}^{l} \mathbb{R}^{d_i} \to \otimes_{j=1}^{k} \mathbb{R}^{c_j}$ be a linear map, we say that T *preserves decomposable tensors* if

$$T\left(\Sigma_{\mathbb{R}^{d_1},...,\mathbb{R}^{d_l}}\right) \subseteq \Sigma_{\mathbb{R}^{c_1},...,\mathbb{R}^{c_k}}.$$

The set of linear isomorphisms $T : \otimes_{i=1}^{l} \mathbb{R}^{d_i} \to \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ that preserve decomposable tensors is denoted by $GL_\Sigma\left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$. Similarly the set of linear maps $U : \otimes_{i=1}^{l} \mathbb{R}^{d_i} \to \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ that preserves the inner product $\langle \cdot, \cdot \rangle_H$ is denoted by $O_\Sigma\left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$. To shorten notation we usually write GL_Σ and O_Σ

The following theorem gives us a characterization of the elements of GL_Σ.

Theorem 2.3. *(Corollary 2.14, [25]) Let $l \geq 2$ and $d_i \geq 2$ be natural numbers for $i = 1, \ldots, l$. If T is an element of $GL_\Sigma\left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$, then there exist a permutation σ on $\{1, ..., l\}$ and bijective linear transformations $T_i : \mathbb{R}^{d_{\sigma(i)}} \to \mathbb{R}^{d_i}$ for $i = 1, \ldots, l$ such that*

$$T\left(x^1 \otimes \cdots \otimes x^l\right) = T_1\left(x^{\sigma(1)}\right) \otimes \cdots \otimes T_l\left(x^{\sigma(l)}\right),$$

for every $x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^{l} \mathbb{R}^{d_i}$.

Other properties of linear mappings that preserve decomposable tensors have been studied in [40, 41, 25, 26].

2.2 Tensor structure on $\mathbb{R}^d$

In this section we define a tensor structure on the Euclidean space $\mathbb{R}^d$. We will prove that the classical ℓ_p^d-balls correspond to reasonable crossnorms when we have a tensor structure on $\mathbb{R}^d$, see Proposition 2.6.

By a **tensor structure** on the Euclidean space $\mathbb{R}^d$ we mean a factorization $d = d_1 d_2 \cdots d_l$ with $d_i \in \mathbb{N}$ for $i = 1, ..., l$, and a linear bijective map $\Phi : \otimes_{i=1}^l \mathbb{R}^{d_i} \to \mathbb{R}^d$ preserving the inner products $\langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle_2$.

Notice that for any $m \in \mathbb{N}$ the space $\underbrace{\mathbb{R} \otimes \cdots \otimes \mathbb{R}}_{m} \otimes \mathbb{R}^d$ always gives us a tensor structure on $\mathbb{R}^d$:

$$\begin{aligned} \Phi : \mathbb{R} \otimes \cdots \otimes \mathbb{R} \otimes \mathbb{R}^d &\longrightarrow \mathbb{R}^d \\ \lambda_1 \otimes \cdots \otimes \lambda_m \otimes x &\to \lambda_1 \lambda_2 \cdots \lambda_m x. \end{aligned}$$

Since $\Sigma_{\mathbb{R},...,\mathbb{R},\mathbb{R}^d} = \mathbb{R} \otimes \cdots \otimes \mathbb{R} \otimes \mathbb{R}^d$, we call this tensor structure a **trivial tensor structure**.

On the other hand, if $d = d_1 d_2 \cdots d_l > 1$ then the scalar multiplication gives us a natural isomorphism between $\otimes_{i=1}^l \mathbb{R}^{d_i}$ and $\otimes_{d_i \neq 1} \mathbb{R}^{d_i}$. For example, when $l = 3$ and $d_2 = 1$ one has:

$$\begin{aligned} \mathbb{R}^{d_1} \otimes \mathbb{R} \otimes \mathbb{R}^{d_3} &\longrightarrow \mathbb{R}^{d_1} \otimes \mathbb{R}^{d_3} \\ x \otimes \lambda \otimes y &\to \lambda x \otimes y. \end{aligned}$$

Thus, the tensor structures given by the factorizations $d = d_1 d_3$ and $d = d_1 1 d_3$ are "essentially" the same.

The previous observation motivates the following definition: let $d = d_1 d_2 \cdots d_l$ and $d = c_1 c_2 \cdots c_k$ be two factorizations of $d > 1$. We say that the tensor structures given by $(d_1, d_2, ..., d_l)$ and $(c_1, c_2, ..., c_k)$ are *equivalent* if there exists a linear isomorphism $T : \otimes_{i=1}^l \mathbb{R}^{d_i} \to \otimes_{j=1}^k \mathbb{R}^{c_j}$ such that

$$T\left(\Sigma_{\mathbb{R}^{d_1},...,\mathbb{R}^{d_l}}\right) = \Sigma_{\mathbb{R}^{c_1},...,\mathbb{R}^{c_l}},$$

and T preserves the inner product $\langle \cdot, \cdot \rangle_H$. The following theorem tells us that for every $d > 1$ and every factorization $d = d_1 d_2 \cdots d_l$ there exists "essentially" one tensor structure on $\mathbb{R}^d$ given by $(d_1, d_2, ..., d_l)$.

Proposition 2.4. *Let $d > 1$ be a natural number and let $d = d_1 d_2 \cdots d_l$ and $d = c_1 c_2 \cdots c_k$ be two factorizations of d such that $c_j > 1$ and $d_i > 1$ for $j = 1, ..., k$, $i = 1, ..., l$. If the tensor structures given by $(d_1, d_2, ..., d_l)$ and $(c_1, c_2, ..., c_k)$ are equivalent, then $l = k$ and there exists a permutation $\sigma : \{1, ..., l\} \to \{1, ..., l\}$ such that $d_{\sigma(i)} = c_i$ for $i = 1, ..., l$.*

Proof. Suppose that $d = d_1 d_2 \cdots d_l$ and $d = c_1 c_2 \cdots c_k$ are two factorizations of d whithout ones. Since they induce equivalent tensor structures, we know there exists a linear isomorphism $T : \otimes_{i=1}^{l} \mathbb{R}^{d_i} \to \otimes_{j=1}^{k} \mathbb{R}^{c_j}$ such that $T\left(\Sigma_{\mathbb{R}^{d_1},\ldots,\mathbb{R}^{d_l}}\right) = \Sigma_{\mathbb{R}^{c_1},\ldots,\mathbb{R}^{c_k}}$. Therefore by Theorem 2.18 in [25] we have $l = k$ and there exist a permutation $\sigma : \{1, \ldots, l\} \to \{1, \ldots, l\}$ and linear isomorphisms $T_i : \mathbb{R}^{d_{\sigma(i)}} \to \mathbb{R}^{c_i}$ such that

$$T\left(x^1 \otimes \cdots \otimes x^l\right) = T_1\left(x^{\sigma(1)}\right) \otimes \cdots \otimes T_l\left(x^{\sigma(l)}\right) \text{ for } x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^{l} \mathbb{R}^{d_i}.$$

Hence, $d_{\sigma(i)} = c_i$. □

The following proposition shows that for trivial tensor structures, every 0-symmetric convex body is the closed unit ball of some reasonable crossnorm.

Proposition 2.5. *For every 0-symmetric convex body Q in $\mathbb{R} \otimes \mathbb{R}^d$, there exists a norm $\|\cdot\|$ on $\mathbb{R}^d$ such that Q is the unit ball of a reasonable crossnorm on the space $\mathbb{R} \otimes \left(\mathbb{R}^d, \|\cdot\|\right)$.*

Proof. Let g_Q be the Minkowski functional of Q. Since Q is a 0-symmetric convex body then,

$$Q_1 := \left\{x \in \mathbb{R}^d : 1 \otimes x \in Q\right\}$$

is a 0-symmetric convex body in $\mathbb{R}^d$ and $g_{Q_1}(x) = g_Q(1 \otimes x)$. If $\lambda \otimes x \in \mathbb{R} \otimes \mathbb{R}^d$ one has:

$$g_Q(\lambda \otimes x) = |\lambda|\, g_Q(1 \otimes x) = |\lambda|\, g_{Q_1}(x).$$

Now observe that for every $z = \sum_{j=1}^{n} \lambda_j \otimes x_j \in \mathbb{R} \otimes \mathbb{R}^d$, we have $g_Q(z) = g_{Q_1}\left(\sum_{j=1}^{n} \lambda_j x_j\right)$. Hence for every $a \in \mathbb{R}$ and $y^* \in \left(\mathbb{R}^d, g_Q\right)^*$ we have:

$$\begin{aligned}
\left|a \otimes y^*\left(\sum_{j=1}^{n} \lambda_j \otimes x_j\right)\right| &= \left|a \otimes y^*\left(1 \otimes \left(\sum_{j=1}^{n} \lambda_j x_j\right)\right)\right| \\
&= \left|a y^*\left(\sum_{j=1}^{n} \lambda_j x_j\right)\right| \\
&\leq |a|\, \|y^*\|\, g_{Q_1}\left(\sum_{j=1}^{n} \lambda_j x_j\right) \\
&= |a|\, \|y^*\|\, g_Q(z).
\end{aligned}$$

Therefore $\|a \otimes y^*\| \leq |a|\, \|y^*\|$.This proves that g_Q is a reasonable crossnorm on $\mathbb{R} \otimes \left(\mathbb{R}^d, \|\cdot\|\right)$. □

In the next proposition we will prove that for each tensor structure on $\mathbb{R}^d$ the classical ℓ_p^d-balls are closed unit balls of a reasonable crossnorm. To that end we introduce some notation: by ℓ_p^d we denote the space $\left(\mathbb{R}^d, \|\cdot\|_p\right)$, where $\|x\|_p := \left(\Sigma_{i=1}^d |x_i|^p\right)^{1/p}$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.
Let $d_i \geq 1$ for $i = 1, ..., l$ be natural numbers and let $d = d_1 \cdots d_l$. By $B_p^{d_1,...,d_l}$ for $1 \leq p \leq \infty$ we denote the sets:

$$B_p^{d_1,...,d_l} := \left\{ z \in \otimes_{i=1}^l \mathbb{R}^{d_i} : \sum_{j_1,...,j_l} \left| \left\langle z, e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l} \right\rangle_H \right|^p \leq 1 \right\} \text{ for } p \neq \infty$$

and

$$B_\infty^{d_1,...,d_l} := \left\{ z \in \otimes_{i=1}^l \mathbb{R}^{d_i} : \max_{j_1,...,j_l} \left| \left\langle z, e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l} \right\rangle_H \right| \leq 1 \right\}.$$

Since every linear isomorphism $T : \otimes_{i=1}^l \mathbb{R}^{d_i} \to \mathbb{R}^d$ that sends $\left\{ e_{j_i}^{d_1} \otimes \cdots \otimes e_{j_i}^{d_l} \right\}_{j_i=1,...,d_i}$, $i = 1, \ldots, l$ into the canonical basis of $\mathbb{R}^d$ is such that $T\left(B_p^{d_1,...,d_l}\right) = B_p^d$, the set $B_p^{d_1,...,d_l}$ is called ℓ_p^d-ball. In the following proposition we prove that the ℓ_p^d-balls are closed unit balls of reasonable crossnorms.

Proposition 2.6. *For every $1 \leq p \leq \infty$, the ℓ_p-ball $B_p^{d_1,...,d_l}$ is the closed unit ball of a reasonable crossnorm on $\otimes_{i=1}^l \ell_p^{d_i}$.*

Proof. We prove the result for $1 < p < \infty$, the cases $p = 1, \infty$ are analogous.
For every $x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^l \mathbb{R}^{d_i}$ we have,

$$\left\| x^1 \otimes \cdots \otimes x^l \right\|_p = \left(\sum_{j_1,...,j_l} \left| \left\langle x^1 \otimes \cdots \otimes x^l, e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l} \right\rangle_H \right|^p \right)^{\frac{1}{p}} = \left\| x^1 \right\|_p \cdots \left\| x^l \right\|_p.$$

On the other hand from Hölder inequality, and the previous equality for the conjugate index p^*, we have:

$$\left| \left\langle z, x^1 \otimes \cdots \otimes x^l \right\rangle_H \right| \leq \sum_{j_1,...,j_l} \left| \left\langle z, e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l} \right\rangle_H \left\langle e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l}, x^1 \otimes \cdots \otimes x^l \right\rangle_H \right|$$

$$\leq \left(\sum_{j_1,...,j_l} \left| \left\langle z, e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l} \right\rangle_H \right|^p \right)^{\frac{1}{p}} \left(\sum_{j_1,...,j_l} \left| \left\langle e_{j_1}^{d_1} \otimes \cdots \otimes e_{j_l}^{d_l}, x^1 \otimes \cdots \otimes x^l \right\rangle_H \right|^{p^*} \right)^{\frac{1}{p^*}}$$

$$\leq \|z\|_p \left\| x^1 \right\|_{p^*} \cdots \left\| x^l \right\|_{p^*}$$

Since $\left\| x^i \right\|_{p^*} = \left\| \langle \cdot, x^i \rangle_2 \right\|$ for $i = 1, \ldots, l$, then

$$\left\| \left\langle \cdot, x^1 \otimes \cdots \otimes x^l \right\rangle_H \right\| \leq \left\| \left\langle \cdot, x^1 \right\rangle_2 \right\| \cdots \left\| \left\langle \cdot, x^l \right\rangle_2 \right\|.$$

Therefore, $\epsilon(u) \leq \|u\|_p \leq \pi(u)$ for $u \in \otimes_{i=1}^l \ell_p^{d_i}$. □

2.3 The injective and the projective tensor product of 0-symmetric convex bodies

In this section we introduce the injective and the projective tensor product of 0-symmetric convex bodies $Q_1, ..., Q_l$. They are the natural analogues, in the context of 0-symmetric convex bodies, to the injective and the projective norm on the theory of tensor norms on Banach spaces.

Recall that we always assume $\mathbb{R}^d$, $d \in \mathbb{N}$ is a Euclidean space with the usual norm $\|\cdot\|_2$. See Chapter 1.

From now on, the inner product on the space $\otimes_{i=1}^l \mathbb{R}^{d_i}$ will be the one associated with the Hilbert tensor product $\otimes_{H,i=1}^l \mathbb{R}^{d_i}$ (see Section 1.1.3). In this way, if $Q \in \mathcal{B}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ then the polar body of Q is

$$Q^\circ = \left\{ z \in \otimes_{i=1}^l \mathbb{R}^{d_i} : \sup_{u \in Q} |\langle u, z \rangle_H| \leq 1 \right\}.$$

The closed unit ball of $\otimes_{H,i=1}^l \mathbb{R}^{d_i}$ will be denoted by $B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}$ or by $B_2^{d_1,\dots,d_l}$.

Definition 2.7. Let Q_i be a 0-symmetric convex body in $\mathbb{R}^{d_i}$ for $i = 1, \dots, l$. We define **the projective tensor product of** $Q_1, \dots, Q_l$ as

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l := conv\left(\Sigma_{Q_1,\dots,Q_l}\right),$$

and **the injective tensor product** as

$$Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l := \left(\Sigma_{Q_1^\circ,\dots,Q_l^\circ}\right)^\circ.$$

This definition of projective tensor tensor product of 0-symmetric bodies coincides with the definition of [4] when the latter is restricted to 0-symmetric convex bodies. Also, for every 0-symmetric convex body, Q, the tensor power $Q^{\otimes n}$ (Definition 2.24, [39]) and $Q \otimes_\pi \cdots \otimes_\pi Q$ are equal. Recently, in [5] appeared a definition of injective tensor product of 0-symmetric convex bodies that coincides with the one that we present here. One can directly see that $Q_1 \otimes_\pi \cdots \otimes_\pi Q_l$ and $Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l$ are 0-symmetric convex bodies in $\otimes_{H,i=1}^l \mathbb{R}^{d_i}$. Nevertheless, that is a consequence of the following proposition.

Proposition 2.8. *Let $Q_i \in \mathcal{B}(d_i)$ and let $g_{Q_i}(\cdot)$ be the Minkowski functional associated to Q_i for $i = 1, \dots, l$. Then:*

1. *$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l$ is the closed unit ball of the projective norm on $\otimes_{i=1}^l \left(\mathbb{R}^{d_i}, g_{Q_i}(\cdot)\right)$.*
2. *$Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l$ is the closed unit ball of the injective norm on $\otimes_{i=1}^l \left(\mathbb{R}^{d_i}, g_{Q_i}(\cdot)\right)$.*
3. *$(Q_1 \otimes_\pi \cdots \otimes_\pi Q_l)^\circ = Q_1^\circ \otimes_\epsilon \cdots \otimes_\epsilon Q_l^\circ$.*

Proof. 1. From the definition of the projective norm we have,

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \subseteq B_{\otimes_{\pi,i=1}^l \left(\mathbb{R}^{d_i}, g_{Q_i}\right)}.$$

To prove the reverse inclusion, suppose that z is in the open unit ball of $\otimes_{\pi,i=1}^{l}\left(\mathbb{R}^{d_i}, g_{Q_i}\right)$. Then $z=\sum_{j=1}^{N} x_j^1\otimes\cdots\otimes x_j^l$ where each x_j^i is non-zero and $\sum_{j=1}^{N} g_{Q_1}\left(x_j^1\right)\cdots g_{Q_l}\left(x_j^l\right)<1$. Let $y_j^i=\frac{x_j^i}{g_{Q_i}\left(x_j^i\right)}$ and $\lambda_j=g_{Q_1}\left(x_j^1\right)\cdots g_{Q_l}\left(x_j^l\right)$. Then $z=\sum_{j=1}^{N}\lambda_j y_j^1\otimes\cdots\otimes y_j^l\in Q_1\otimes_\pi\cdots\otimes_\pi Q_l$. It follows that the closed unit ball of $\otimes_{\pi,i=1}^{l}\left(\mathbb{R}^{d_i}, g_{Q_i}\right)$ is contained in $\overline{Q_1\otimes_\pi\cdots\otimes_\pi Q_l}$. Since $\Sigma_{Q_1,\ldots,Q_l}$ is compact (Proposition 2.1) then $Q_1\otimes_\pi\cdots\otimes_\pi Q_l$ is closed, this proves *1.*

To prove 2., take $z\in\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ then,

$$\epsilon(z)=\sup\left\{\left|\varphi_1\otimes\cdots\otimes\varphi_l(z)\right| : \varphi_i\in\left(\mathbb{R}^{d_i}, g_{Q_i}\right)^* \text{ with } \|\varphi_i\|\leq 1\, i=1,\ldots,l\right\}.$$

Since for every $\varphi_i\in\left(\mathbb{R}^{d_i}, g_{Q_i}\right)^*$ there exists $x^i\in\mathbb{R}^{d_i}$ such that $\varphi_i(x)=\langle x, x^i\rangle_2$ for all $x\in\mathbb{R}^{d_i}$. We have

$$\varphi_1\otimes\cdots\otimes\varphi_l(z)=\left\langle z, x^1\otimes\cdots\otimes x^l\right\rangle_H.$$

Hence,

$$\epsilon(z)=\sup\left\{\left|\left\langle z, x^1\otimes\cdots\otimes x^l\right\rangle_H\right| : x^i\in Q_i^\circ\right\}.$$

The proof of *3.* is straightforward:

$$Q_1^\circ\otimes_\epsilon\cdots\otimes_\epsilon Q_l^\circ=\left(\Sigma_{Q_1^{\circ\circ},\ldots,Q_l^{\circ\circ}}\right)^\circ=\left(\Sigma_{Q_1,\ldots,Q_l}\right)^\circ=\left(Q_1\otimes_\pi\cdots\otimes_\pi Q_l\right)^\circ,$$

this finishes the proof. □

Corollary 2.9. *The following equalities hold:* $B_1^{d_1,\ldots,d_l}=B_1^{d_1}\otimes_\pi\cdots\otimes_\pi B_1^{d_l}$ *and* $B_\infty^{d_1,\ldots,d_l}=B_\infty^{d_1}\otimes_\epsilon\cdots\otimes_\epsilon B_\infty^{d_l}$.

Proof. For every $x^1\otimes\cdots\otimes x^l\in\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ we know that

$$\begin{aligned} g_{B_1^{d_1}\otimes_\pi\cdots\otimes_\pi B_1^{d_l}}\left(x^1\otimes\cdots\otimes x^l\right) &= \left\|x^1\right\|_1\cdots\left\|x^l\right\|_1 \\ &= \left\|x^1\otimes\cdots\otimes x^l\right\|_1. \end{aligned}$$

Hence, $B_1^{d_1,\ldots,d_l}=B_1^{d_1}\otimes_\pi\cdots\otimes_\pi B_1^{d_l}$. From Proposition 2.8 we have that

$$\begin{aligned} B_\infty^{d_1}\otimes_\epsilon\cdots\otimes_\epsilon B_\infty^{d_l} &= \left(B_1^{d_1}\otimes_\pi\cdots\otimes_\pi B_1^{d_l}\right)^\circ \\ &= \left(B_1^{d_1,\ldots,d_l}\right)^\circ \\ &= B_\infty^{d_1,\ldots,d_l}. \end{aligned}$$

□

Now, we will see that the injective and the projective tensor product of 0-symmetric convex sets are continuous functions with respect to the Hausdorff metric.

Lemma 2.10. *Let $i \in \{1, \dots, l\}$. The function,*

$$r_i : \left(\mathcal{B}\left(d_i\right), \delta^H\right) \to \mathbb{R}$$
$$Q \mapsto r_i(Q) := \sup_{x \in Q} \|x\|_2$$

is uniformly continuous.

Proof. Let $P_i, Q_i \in \mathcal{B}\left(d_i\right)$ and take $\lambda > 0$ such that $P_i \subseteq Q_i + \lambda B_2^{d_i}$ and $Q_i \subseteq P_i + \lambda B_2^{d_i}$. Then, for every $x \in P_i$ we have $\|x\|_2 \leq r_i(Q) + \lambda$. Thus, $r_i(P_i) \leq r_i(Q_i) + \lambda$. In the same way we have $r_i(Q_i) \leq r_i(P_i) + \lambda$. Now, from the definition of Hausdorff metric we obtain

$$r_i(P_i) \leq r_i(Q_i) + \delta^H\left(P_i, Q_i\right)$$
$$r_i(Q_i) \leq r_i(P_i) + \delta^H\left(P_i, Q_i\right).$$

So, $|r_i(P_i) - r_i(Q_i)| \leq \delta^H\left(P_i, Q_i\right)$. This proves that r_i is uniformly continuous. □

Lemma 2.11. *Let P_i, Q_i be 0-symmetric convex bodies in $\mathbb{R}^{d_i}$ for $i = 1, \dots, l$. Then for each $i \in \{1, \dots, l\}$ we have*

$$\delta^H\left(Q_1 \otimes_\pi \cdots \otimes_\pi Q_i \otimes_\pi \cdots \otimes_\pi Q_l, Q_1 \otimes_\pi \cdots \otimes_\pi P_i \otimes_\pi \cdots \otimes_\pi Q_l\right) \leq \delta^H\left(Q_i, P_i\right) \prod_{j \neq i} r_j(Q_j).$$

Proof. First, lets us fix $i \in \{1, \dots, l\}$. From the definition of $\otimes_\pi$ we obtain:

$$\delta^H\left(Q_1 \otimes_\pi \cdots \otimes_\pi Q_i \otimes_\pi \cdots \otimes_\pi Q_l, Q_1 \otimes_\pi \cdots \otimes_\pi P_i \otimes_\pi \cdots \otimes_\pi Q_l\right) \leq$$
$$\delta^H\left(\Sigma_{Q_1, \dots, Q_i, \dots, Q_l}, \Sigma_{Q_1, \dots, P_i, \dots, Q_l}\right)$$

Now, take $\lambda > 0$ such that $P_i \subseteq Q_i + \lambda B_2^{d_i}$ and $Q_i \subseteq P_i + \lambda B_2^{d_i}$, then for every $x^i \in Q_i$, there exists $y^i \in P_i$ such that $x^i = y^i + \lambda u_i$ for some $u_i \in B_2^{d_i}$. Now if $x^j \in Q_j$ for $j = 1, \dots, i, \dots, l$ we have:

$$\begin{aligned} x^1 \otimes \cdots \otimes x^i \otimes \cdots \otimes x^l &= x^1 \otimes \cdots \otimes \left(y^i + \lambda u_i\right) \otimes \cdots \otimes x^l \\ &= x^1 \otimes \cdots \otimes y^i \otimes \cdots \otimes x^l + x^1 \otimes \cdots \otimes \lambda u_i \otimes \cdots \otimes x^l, \end{aligned}$$

Since $x^1 \otimes \cdots \otimes y^i \otimes \cdots \otimes x^l \in \Sigma_{Q_1, \dots, P_i, \dots, Q_l}$, and for $j \neq i$ $x^j \in r_j(Q_j) B_2^{d_j}$ we obtain

$$\Sigma_{Q_1, \dots, Q_i, \dots, Q_l} \subseteq \Sigma_{Q_1, \dots, P_i, \dots, Q_l} + \left(\lambda \prod_{j \neq i} r_j(Q_j)\right) B_2^{d_1, \dots, d_l}$$

In a similar way, we can prove that

$$\Sigma_{Q_1, \dots, P_i, \dots, Q_l} \subseteq \Sigma_{Q_1, \dots, Q_i, \dots, Q_l} + \left(\lambda \prod_{j \neq i} r_j(Q_j)\right) B_2^{d_1, \dots, d_l}.$$

Thus,

$$\delta^H\left(\Sigma_{Q_1,\ldots,Q_i,\ldots,Q_l},\Sigma_{Q_1,\ldots,P_i,\ldots,Q_l}\right)\leq\lambda\prod_{j\neq i}r_j(Q_j).$$

Since λ must be bigger than $\delta^H(Q_i,P_i)$ we obtain the desired inequality. □

Proposition 2.12. *The functions*

$$\begin{aligned}\otimes_\pi:\left(\mathcal{B}(d_1),\delta^H\right)\times\cdots\times\left(\mathcal{B}(d_l),\delta^H\right)&\rightarrow\left(\mathcal{B}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right),\delta^H\right)\\(Q_1,\ldots,Q_l)&\rightarrow Q_1\otimes_\pi\cdots\otimes_\pi Q_l,\end{aligned}$$

and

$$\begin{aligned}\otimes_\epsilon:\left(\mathcal{B}(d_1),\delta^H\right)\times\cdots\times\left(\mathcal{B}(d_l),\delta^H\right)&\rightarrow\left(\mathcal{B}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right),\delta^H\right)\\(Q_1,\ldots,Q_l)&\rightarrow Q_1\otimes_\epsilon\cdots\otimes_\epsilon Q_l,\end{aligned}$$

are continuous.

Proof. We are now in position to show that $\otimes_\pi$ is continuous. To that end for each $i=1,\ldots,l$ let $\{Q_i^n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{B}(d_i)$ converging to $Q_i\in\mathcal{B}(d_i)$ in the Hausdorff metric. Then by the triangle inequality and Lemma 2.11 we have:

$$\begin{gathered}\delta^H\left(Q_1\otimes_\pi\cdots\otimes_\pi Q_l,Q_1^n\otimes_\pi\cdots\otimes_\pi Q_l^n\right)\leq\\ \delta^H\left(Q_1\otimes_\pi Q_2\otimes_\pi\cdots\otimes_\pi Q_l,Q_1^n\otimes_\pi Q_2\otimes_\pi\cdots\otimes_\pi Q_l\right)+\\ +\delta^H\left(Q_1^n\otimes_\pi Q_2\otimes_\pi\cdots\otimes_\pi Q_l,Q_1^n\otimes_\pi Q_2^n\otimes_\pi Q_3\otimes_\pi\cdots\otimes_\pi Q_l\right)\\ +\delta^H\left(Q_1^n\otimes_\pi Q_2^n\otimes_\pi Q_3\otimes_\pi\cdots\otimes_\pi Q_l,Q_1^n\otimes_\pi Q_2^n\otimes_\pi Q_3^n\otimes_\pi Q_4\cdots\otimes_\pi Q_l\right)\\ +\cdots+\delta^H\left(Q_1^n\otimes_\pi Q_2^n\otimes_\pi\cdots\otimes_\pi Q_{l-1}^n\otimes_\pi Q_l,Q_1^n\otimes_\pi Q_2^n\otimes_\pi\cdots\otimes_\pi Q_{l-1}^n,Q_l^n\right)\\ \leq\delta^H\left(Q_1,Q_1^n\right)\prod_{j\neq1}r_j(Q_j)+\delta^H\left(Q_2,Q_2^n\right)r_1(Q_1^n)\prod_{j\neq1,2}r_j(Q_j)\\ +\delta^H\left(Q_3,Q_3^n\right)r_1(Q_1^n)r_2(Q_2^n)\prod_{j\neq1,2,3}r_j(Q_j)+\cdots+\delta^H\left(Q_l,Q_l^n\right)\prod_{j\neq l}r_j(Q_j^n).\end{gathered}$$

Thus, by the continuity of r_i (Lemma 2.10) and the fact $Q_i^n\rightarrow Q_i$ in the Hausdorff metric, we have $\delta^H\left(Q_1\otimes_\pi\cdots\otimes_\pi Q_l,Q_1^n\otimes_\pi\cdots\otimes_\pi Q_l^n\right)\rightarrow0$. This proves that $\otimes_\pi$ is continuous.

In order to prove that $\otimes_\epsilon$ is continuous. Observe that from Proposition 1.35, $Q_i^n\rightarrow Q_i$ implies that $(Q_i^n)^\circ\rightarrow Q_i^\circ$ on the Hausdorff metric. Thus, by the continuity of $\otimes_\pi$ we have

$$\begin{gathered}(Q_1^n)^\circ\otimes_\pi\cdots\otimes_\pi(Q_l^n)^\circ\rightarrow Q_1^\circ\otimes_\pi\cdots\otimes_\pi Q_l^\circ.\\ \left((Q_1^n)^\circ\otimes_\pi\cdots\otimes_\pi(Q_l^n)^\circ\right)^\circ\rightarrow\left(Q_1^\circ\otimes_\pi\cdots\otimes_\pi Q_l^\circ\right)^\circ\end{gathered}$$

Finally, applying Proposition 2.8 we obtain $\delta^H\left(Q_1\otimes_\epsilon\cdots\otimes_\epsilon Q_l,Q_1^n\otimes_\epsilon\cdots\otimes_\epsilon Q_l^n\right)\rightarrow0$. This completes the proof. □

2.4 Tensorial 0-symmetric convex bodies

In this section we give a characterization of the 0-symmetric convex bodies Q in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ with the property that there exist norms $\|\cdot\|_i$ on $\mathbb{R}^{d_i}$ for $i = 1, \dots, l$ such that Q is the unit ball of a reasonable crossnorm on $\otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$. To this end, in the following proposition we characterize the 0-symmetric convex bodies in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ that are closed unit balls of reasonable crossnorms for fixed norms in each $\mathbb{R}^{d_i}$ for $i = 1, \dots, l$.

Proposition 2.13. *Let $Q \in \mathcal{B}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$, $Q_i \in \mathcal{B}(d_i)$ and let $g_{Q_i}(\cdot)$ be the Minkowski functional associated to Q_i for $i = 1, \dots, l$. Then, Q is the closed unit ball of a reasonable crossnorm on $\otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, g_{Q_i}\right)$ if and only if*

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \subseteq Q \subseteq Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l. \tag{2.4.1}$$

Proof. In Proposition 2.8 we showed that $Q_1 \otimes_\pi \cdots \otimes_\pi Q_l$ and $Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l$ are the closed unit balls of the projective norm and the injective norm on the space $\otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, g_{Q_i}\right)$. Therefore, from our hipothesis the Minkowski functional g_Q has the following property

$$\epsilon(z) \leq g_Q(z) \leq \pi(z) \text{ for } z \in \otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, g_{Q_i}\right).$$

By Proposition 1.13 this is equivalent to be a reasonable crossnorm. This completes the proof. □

Remark 2.14. Let $Q \in \mathcal{B}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$, $Q_i \in \mathcal{B}(d_i)$ and let g_{Q_i}, g_Q be the Minkowski functionals associated to Q_i, Q. If Q is the closed unit ball of a reasonable crossnorm on $\otimes_{i=1}^{l}\left(\mathbb{R}^{d_i}, g_{Q_i}\right)$, then the duality between points and functionals given by the inner products on each $\mathbb{R}^{d_i}$ and $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ allows us to write the conditions of being a reasonable crossnorm as follows:

$$g_Q\left(x^1 \otimes \cdots \otimes x^l\right) \leq g_{Q_1}\left(x^1\right) \cdots g_{Q_l}\left(x^l\right)$$
$$g_{Q^\circ}\left(x^1 \otimes \cdots \otimes x^l\right) \leq g_{Q_1^\circ}\left(x^1\right) \cdots g_{Q_l^\circ}\left(x^l\right).$$

Now, we start the study of the class of convex bodies Q that satify inclusions (2.4.1). We call this convex bodies: *tensorial 0-symmetric convex bodies*. We will prove that they define reasonable crossnorms on $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ in the sense of Theorem 2.20.

Definition 2.15. We say that a 0-symmetric convex body $Q \subseteq \otimes_{i=1}^{l}\mathbb{R}^{d_i}$ is a **tensorial 0-symmetric convex body** with respect to $Q_i \in \mathcal{B}(d_i)$, $i = 1, ..., l$, if

$$Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \subseteq Q \subseteq Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l.$$

The subset of tensorial 0-symmetric convex bodies in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$, will be denoted by $\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$. The subset of tensorial 0-symmetric convex bodies with respect to $Q_1, \dots, Q_l$ will be denoted by $\mathcal{B}_{\Sigma_{Q_1,\dots,Q_l}}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$.

Remark 2.16. Notice that in Example 2.5 we proved that $\mathcal{B}_\Sigma\left(\mathbb{R}\otimes\mathbb{R}^d\right)=\mathcal{B}\left(\mathbb{R}\otimes\mathbb{R}^d\right)$.

Example 2.17. From Corollary 2.9 and Proposition 2.6 we have,

$$B_p^{d_1}\otimes_\pi\cdots\otimes_\pi B_p^{d_l}\subseteq B_p^{d_1\cdots d_l}\subseteq B_p^{d_1}\otimes_\epsilon\cdots\otimes_\epsilon B_p^{d_l} \text{ for } 1\leq p\leq\infty.$$

Therefore the ℓ_p-balls $B_p^{d_1,\dots,d_l}$ are tensorial 0-symmetric convex bodies.

Proposition 2.18. *Let $Q\subseteq\otimes_{i=1}^l\mathbb{R}^{d_i}$ be a tensorial 0-symmetric convex body with respect to $Q_i\in\mathcal{B}(d_i)$ $i=1,\dots,l$. Then*

1. $Q^\circ\in\mathcal{B}_{\Sigma_{Q_1^\circ,\dots,Q_l^\circ}}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$.
2. *For every real number $\lambda>0$, $\lambda Q\in\mathcal{B}_{\Sigma_{\lambda Q_1,\dots,Q_l}}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$.*

Proof. For 1. If $Q\in\mathcal{B}_{\Sigma_{Q_1,\dots,Q_l}}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ then

$$Q_1\otimes_\pi\cdots\otimes_\pi Q_l\subseteq Q\subseteq Q_1\otimes_\epsilon\cdots\otimes_\epsilon Q_l.$$

Thus $(Q_1\otimes_\epsilon\cdots\otimes_\epsilon Q_l)^\circ\subseteq Q^\circ\subseteq(Q_1\otimes_\pi\cdots\otimes_\pi Q_l)^\circ$.
By Proposition 2.8 we have $Q_1^\circ\otimes_\pi\cdots\otimes_\pi Q_l^\circ\subseteq Q^\circ\subseteq Q_1^\circ\otimes_\epsilon\cdots\otimes_\epsilon Q_l^\circ$.
For *2.* observe that if $\lambda>0$, then

$$\begin{aligned}\lambda Q_1\otimes_\pi\cdots\otimes_\pi Q_l&=\lambda conv\left(\Sigma_{Q_1,\dots,Q_l}\right)\\&=conv\left(\lambda\Sigma_{Q_1,\dots,Q_l}\right)\\&=conv\left(\Sigma_{\lambda Q_1,\dots,Q_l}\right)\\&=(\lambda Q_1)\otimes_\pi\cdots\otimes_\pi Q_l\end{aligned}$$

and

$$\begin{aligned}\lambda Q_1\otimes_\epsilon\cdots\otimes_\epsilon Q_l&=\lambda\left(\Sigma_{Q_1^\circ,\dots,Q_l^\circ}\right)^\circ\\&=\left(\lambda^{-1}\Sigma_{Q_1^\circ,\dots,Q_l^\circ}\right)^\circ\\&=\left(\Sigma_{\lambda^{-1}Q_1^\circ,\dots,Q_l^\circ}\right)^\circ\\&=\left(\Sigma_{(\lambda Q_1)^\circ,\dots,Q_l^\circ}\right)\\&=(\lambda Q_1)\otimes_\epsilon\cdots\otimes_\epsilon Q_l.\end{aligned}$$

□

The following proposition tells us that for every $Q\in\mathcal{B}_{\Sigma_{\lambda Q_1,\dots,Q_l}}$ the sections $Q\cap x^1\otimes\cdots x^{i-1}\otimes\mathbb{R}^{d_i}\otimes x^{i+1}\otimes\cdots\otimes x^l$ are "essentially" unique.

Proposition 2.19. *Let $Q\in\mathcal{B}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$, $P_i,\,Q_i\in\mathcal{B}(d_i)$, $i=1,\dots,l$. If $Q_1\otimes_\pi\cdots\otimes_\pi Q_l\subseteq Q\subseteq Q_1\otimes_\epsilon\cdots\otimes_\epsilon Q_l$ and $P_1\otimes_\pi\cdots\otimes_\pi P_l\subseteq Q\subseteq P_1\otimes_\epsilon\cdots\otimes_\epsilon P_l$ then there exist real numbers $\lambda_i>0$ such that $\lambda_1\cdots\lambda_l=1$ and $P_i=\lambda_iQ_i$, $i=1,\dots,l$.*

Proof. Let g_Q, g_{Q_i} and g_{P_i} be the Minkowski functionals associated to Q, Q_i and P_i respectively. From our hypothesis and Proposition 2.13 we have:

$$g_Q\left(x^1 \otimes \cdots \otimes x^l\right) = g_{Q_1}\left(x^1\right) \cdots g_{Q_l}\left(x^l\right),$$
$$g_Q\left(x^1 \otimes \cdots \otimes x^l\right) = g_{P_1}\left(x^1\right) \cdots g_{P_l}\left(x^l\right).$$

Now, fix an element $a^1 \otimes \cdots \otimes a^l \in \partial Q$. Then

$$g_{Q_1}\left(a^1\right) \cdots g_{Q_l}\left(a^l\right) = g_{P_1}\left(a^1\right) \cdots g_{P_l}\left(a^l\right) = 1,$$

and for each $i = 1, ..., l$ we have:

$$g_{Q_1}\left(a^1\right) \cdots g_{Q_{i-1}}\left(a^{i-1}\right) g_{Q_i}\left(x^i\right) g_{Q_{i+1}}\left(a^{i+1}\right) \cdots g_{Q_l}\left(a^l\right) =$$
$$g_{P_1}\left(a^1\right) \cdots g_{P_{i-1}}\left(a^{i-1}\right) g_{P_i}\left(x^i\right) g_{P_{i+1}}\left(a^{i+1}\right) \cdots g_{P_l}\left(a^l\right).$$

Mutiplying both sides of the above equation by $g_{Q_i}\left(a^i\right) g_{P_i}\left(a^i\right)$, we obtain

$$g_{P_i}\left(a^i\right) g_{Q_i}\left(x^i\right) = g_{Q_i}\left(a^i\right) g_{P_i}\left(x^i\right).$$

Thus $g_{P_i}\left(x^i\right) = \frac{g_{P_i}\left(a^i\right)}{g_{Q_i}\left(a^i\right)} g_{Q_i}\left(x^i\right)$.

Finally, if $\lambda_i := \frac{g_{Q_i}\left(a^i\right)}{g_{P_i}\left(a^i\right)}$, $i = 1, ..., l$ then $\lambda_1 \cdots \lambda_l = 1$ and $P_i = \lambda_i Q_i$. □

For every 0-symmetric convex body $Q \subseteq \otimes_{i=1}^l \mathbb{R}^{d_i}$. Each non-zero decomposable tensor $\mathbf{a} = a^1 \otimes \cdots \otimes a^l$ determines 0-symmetric compact convex sets in $\otimes_{i=1}^l \mathbb{R}^{d_i}$. They are the sections: $Q \cap a^1 \otimes \cdots a^{i-1} \otimes \mathbb{R}^{d_i} \otimes a^{i+1} \otimes \cdots \otimes a^l$ for $i = 1, ..., l$.

Using these sections we define 0-symmetric convex bodies on $\mathbb{R}^{d_i}$ for $i = 1, ..., l$ as follows:

$$Q_i\left(a^1 \otimes \cdots \otimes a^l\right) := \left\{x^i \in \mathbb{R}^{d_i} : a^1 \otimes \cdots a^{i-1} \otimes x^i \otimes a^{i+1} \otimes \cdots \otimes a^l \in Q\right\}.$$

Theorem 2.20. *Let Q be a 0-symmetric convex body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$. The following statements are equivalent:*

1. *There exist norms $\|\cdot\|_i$ on $\mathbb{R}^{d_i}$ for $i = 1, ..., l$ such that Q is the unitary ball of a reasonable crossnorm on $\otimes_{i=1}^l \left(\mathbb{R}^{d_i}, \|\cdot\|_i\right)$.*

2. *There exists a decomposable vector, $a^1 \otimes \cdots \otimes a^l \in \partial Q$, such that*

$$Q_1\left(a^1 \otimes \cdots \otimes a^l\right) \otimes_\pi \cdots \otimes_\pi Q_l\left(a^1 \otimes \cdots \otimes a^l\right) \subseteq Q$$
$$\subseteq Q_1\left(a^1 \otimes \cdots \otimes a^l\right) \otimes_\epsilon \cdots \otimes_\epsilon Q_l\left(a^1 \otimes \cdots \otimes a^l\right), \quad (2.4.2)$$

i.e. Q is a tensorial 0-symmetric convex body with respect to $Q_i\left(a^1 \otimes \cdots \otimes a^l\right)$ for $i = 1, ..., l$.

Proof. Let $g_{Q_i\left(a^1\otimes\cdots\otimes a^l\right)}$, g_Q, $g_{\left(Q_i\left(a^1\otimes\cdots\otimes a^l\right)\right)^\circ}$ and g_{Q° be the Minkowski functionals associated to $Q_i\left(a^1\otimes\cdots\otimes a^l\right)$, Q, $\left(Q_i\left(a^1\otimes\cdots\otimes a^l\right)\right)^\circ$ and Q° respectively. From Proposition 2.13, if Q satisfies equation (2.4.2) then g_Q is a reasonable crossnorm on

$$\otimes_{i=1}^l\left(\mathbb{R}^{d_i}, g_{Q_i\left(a^1\otimes\cdots\otimes a^l\right)}\right).$$

To prove the other implication, suppose Q is the unitary ball of a reasonable crossnorm $\|\cdot\|_\alpha$ on $\otimes_{i=1}^l\left(\mathbb{R}^{d_i},\|\cdot\|_i\right)$. Then $g_Q=\|\cdot\|_\alpha$ and for each $x^1\otimes\cdots\otimes x^l\in\Sigma_{\mathbb{R}^{d_1},\ldots,\mathbb{R}^{d_l}}$ we have

$$\begin{aligned}\left\|x^1\otimes\cdots\otimes x^l\right\|_\alpha &= \left\|x^1\right\|_1\cdots\left\|x^l\right\|_l \\ \left\|\left\langle\cdot,x^1\otimes\cdots\otimes x^l\right\rangle_H\right\| &= \left\|\left\langle\cdot,x^1\right\rangle\right\|\cdots\left\|\left\langle\cdot,x^l\right\rangle\right\|.\end{aligned}\tag{2.4.3}$$

Fix an element $a^1\otimes\cdots\otimes a^l\in\partial Q$. Then, $g_Q\left(a^1\otimes\cdots\otimes a^l\right)=\left\|a^1\right\|_1\cdots\left\|a^l\right\|_l=1$ and

$$\begin{aligned}\left\|a^1\otimes\cdots a^{i-1}\otimes x^i\otimes a^{i+1}\otimes\cdots\otimes a^l\right\|_\alpha &= \left\|a^1\right\|_1\cdots\left\|a^{i-1}\right\|_{i-1}\left\|x^i\right\|_i\left\|a^{i+1}\right\|_{i+1}\cdots\left\|a^l\right\|_l \\ &= \frac{1}{\left\|a^i\right\|_i}\left\|x^i\right\|_i.\end{aligned}$$

Now, by definition of $Q_i\left(a^1\otimes\cdots\otimes a^l\right)$ we obtain $g_{Q_i\left(a^1\otimes\cdots\otimes a^l\right)}\left(x^i\right)=\frac{1}{\left\|a^i\right\|_i}\left\|x^i\right\|_i$ which implies $g_{\left(Q_i\left(a^1\otimes\cdots\otimes a^l\right)\right)^\circ}\left(x^i\right)=\left\|a^i\right\|_i\left\|\left\langle\cdot,x^i\right\rangle\right\|$. Thus,

$$g_Q\left(x^1\otimes\cdots\otimes x^l\right)=g_{Q_1\left(a^1\otimes\cdots\otimes a^l\right)}\left(x^1\right)\cdots g_{Q_l\left(a^1\otimes\cdots\otimes a^l\right)}\left(x^l\right).$$

and

$$\begin{aligned}g_{Q^\circ}\left(x^1\otimes\cdots\otimes x^l\right) &= \left\|\left\langle\cdot,x^1\otimes\cdots\otimes x^l\right\rangle_H\right\| \\ &= \left\|\left\langle\cdot,x^1\right\rangle\right\|\cdots\left\|\left\langle\cdot,x^l\right\rangle\right\| \\ &= \left\|a^1\right\|_1\cdots\left\|a^l\right\|_l\left\|\left\langle\cdot,x^1\right\rangle\right\|\cdots\left\|\left\langle\cdot,x^l\right\rangle\right\| \\ &= g_{\left(Q_1\left(a^1\otimes\cdots\otimes a^l\right)\right)^\circ}\left(x^1\right)\cdots g_{\left(Q_l\left(a^1\otimes\cdots\otimes a^l\right)\right)^\circ}\left(x^l\right)\end{aligned}$$

Finally by Proposition 2.13 and remark 2.14 we have the desire result. □

In the previous theorem we proved that a 0-symmetric convex body $Q\subseteq\otimes_{i=1}^l\mathbb{R}^{d_i}$ determines a reasonable crossnorm if and only if it belongs to $\mathcal{B}_{\Sigma_{Q_1(a^1\otimes\cdots\otimes a^l),\ldots,Q_l(a^1\otimes\cdots\otimes a^l)}}$ for some $a^1\otimes\cdots\otimes a^l\in\partial Q$. In the following corollary we prove that the latter happens for every decomposable tensor.

Corollary 2.21. *Let Q be a 0-symmetric convex body Q in $\otimes_{i=1}^l\mathbb{R}^{d_i}$. The following propositions are equivalent:*

1. $Q\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$.
2. $Q\in\mathcal{B}_{\Sigma_{Q_1(a^1\otimes\cdots\otimes a^l),\ldots,Q_l(a^1\otimes\cdots\otimes a^l)}}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ *for some* $a^1\otimes\cdots\otimes a^l\in\partial Q$.
3. $Q\in\mathcal{B}_{\Sigma_{Q_1(a^1\otimes\cdots\otimes a^l),\ldots,Q_l(a^1\otimes\cdots\otimes a^l)}}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ *for every* $a^1\otimes\cdots\otimes a^l\in\partial Q$.

Corollary 2.21 is a consequence of Proposition 2.19, Proposition 2.13 and the previous theorem.
As a consequence of Proposition 2.19, the sections of a tensorial 0-symmetric convex body Q determined by subspaces

$$a^1 \otimes \cdots a^{i-1} \otimes \mathbb{R}^{d_i} \otimes a^{i+1} \otimes \cdots \otimes a^l,$$

are proportional. This allows us to choose the convex bodies $Q_1, ..., Q_l$ such that $Q \in \mathcal{B}_{\Sigma_{Q_1,...,Q_l}}$.
Remark 2.22. If Q is a 0-symmetric convex body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ and g_Q is the Minkowski functional associated to Q, then by Q^i we denote the convex bodies generated by the sections corresponding to $\frac{1}{g_Q\left(e_1^{d_1}\otimes\cdots\otimes e_1^{d_l}\right)} e_1^{d_1} \otimes \cdots \otimes e_1^{d_l}$ as follows:

$$Q^i := \left\{ x^i \in \mathbb{R}^{d_i} : e_1^{d_1} \otimes \cdots \otimes e_1^{d_{i-1}} \otimes x^i \otimes e_1^{d_{i+1}} \otimes \cdots \otimes \left(\frac{1}{g_Q\left(e_1^{d_1} \otimes \cdots \otimes e_1^{d_l}\right)} e_1^{d_l} \right) \in Q \right\}$$

for $i = 1, ..., l-1$, and

$$Q^l := \left\{ x^l \in \mathbb{R}^{d_l} : e_1^{d_1} \otimes \cdots \otimes e_1^{d_{l-1}} \otimes x^l \in Q \right\}.$$

The following example demonstrates that there exist 0-symmetric convex bodies that do not belong to the class of tensorial 0-symmetric convex bodies.

Example 2.23. The ellipsoid,

$$\mathcal{E} = \left\{ z = \sum_{i,j=1}^{2} z_{ij} e_i^2 \otimes e_j^2 : \frac{(z_{11})^2}{3} + (z_{12})^2 + (z_{21})^2 + \frac{(z_{22})^2}{2} = 1 \right\} \subseteq \mathbb{R}^2 \otimes \mathbb{R}^2$$

is not a tensorial symmetric convex body. We will proceed by contradiction.

Suppose that $\mathcal{E} \in B_\Sigma(\mathbb{R}^2 \otimes \mathbb{R}^2)$. Let $g_\mathcal{E}$ be the Minkowski functional associated to $\mathcal{E}$. Consider the convex bodies generated by $e_1^2 \otimes \sqrt{3} e_1^2$ and $e_2^2 \otimes \sqrt{2} e_2^2$,

$$\begin{array}{ll} \mathcal{E}^1 = \left\{ x \in \mathbb{R}^2 : x \otimes \sqrt{3} e_1^2 \in \mathcal{E} \right\} & \mathcal{E}_1(e_2^2 \otimes e_2^2) = \left\{ x \in \mathbb{R}^2 : x \otimes \sqrt{2} e_2^2 \in \mathcal{E} \right\}, \\ \mathcal{E}^2 = \left\{ y \in \mathbb{R}^2 : e_1^2 \otimes y \in \mathcal{E} \right\} & \mathcal{E}_2(e_2^2 \otimes e_2^2) = \left\{ y \in \mathbb{R}^2 : e_2^2 \otimes y \in \mathcal{E} \right\}. \end{array}$$

We denote by $g_{\mathcal{E}^1}, g_{\mathcal{E}_1(e_2^2 \otimes e_2^2)}$ the Minkowski functionals associated to $\mathcal{E}^1, \mathcal{E}_1(e_2^2 \otimes e_2^2)$ respectively. From Proposition 2.19 and Corollary 2.21, there exists $\lambda > 0$ such that $\mathcal{E}^1 = \lambda \mathcal{E}_1(e_2^2 \otimes e_2^2)$. However,

$$g_{\mathcal{E}^1} \begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{x^2 + 3y^2}$$

$$g_{\mathcal{E}_1(e_2^2 \otimes e_2^2)} \begin{pmatrix} x \\ y \end{pmatrix} = \sqrt{2x^2 + y^2},$$

which is a contradiction. Thus $\mathcal{E} \notin B_\Sigma(\mathbb{R}^2 \otimes \mathbb{R}^2)$.

2.4.1 The set of tensorial 0-symmetric convex bodies B_Σ

Here we show that the set of tensorial 0-symmetric convex bodies is a path-connected and closed subset of the metric space $\left(\mathcal{B}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right), \delta^H\right)$.

Proposition 2.24. *$\mathcal{B}_{\Sigma_{Q_1,\ldots,Q_l}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is a closed convex subset of $\left(\mathcal{B}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right), \delta^H\right)$.*

Proof. To prove that $\mathcal{B}_{\Sigma_{Q_1,\ldots,Q_l}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is convex, observe that for $0 \leq t \leq 1$ we have:

$$\begin{aligned} tQ_1 \otimes_\pi \cdots \otimes_\pi Q_l + (1-t)\, Q_1 \otimes_\pi \cdots \otimes_\pi Q_l &= Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \\ tQ_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l + (1-t)\, Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l &= Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l. \end{aligned}$$

Thus, for every $P, Q \in \mathcal{B}_{\Sigma_{Q_1,\ldots,Q_l}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ we have $tQ+(1-t)\,P \in \mathcal{B}_{\Sigma_{Q_1,\ldots,Q_l}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. Now, we will prove that $\mathcal{B}_{\Sigma_{Q_1,\ldots,Q_l}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is closed. Let $\{P_n\}$ be a sequence in $\mathcal{B}_{\Sigma_{Q_1,\ldots,Q_l}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ converging in the Hausdorff metric to some $P \in \mathcal{B}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. From Proposition 1.35 we know that $P_n^\circ \to P^\circ$ in the Hausdorff metric. Let $g_{P_n}, g_P, g_{P_n^\circ}, g_{P^\circ}, g_{Q_i}, g_{(Q_i)^\circ}$ be the Minkowski functionals associated to $P_n, P, P_n^\circ, P^\circ, Q_i, (Q_i)^\circ$ respectively.

Since $P_n^\circ \to P^\circ$ in the Hausdorff metric then $g_{P_n} \to g_P$ pointwise. Therefore:

$$\begin{aligned} g_P\left(x^1 \otimes \cdots \otimes x^l\right) &= \lim_{n\to\infty} g_{P_n}\left(x^1 \otimes \cdots \otimes x^l\right) \\ &= \lim_{n\to\infty} g_{Q_1}\left(x^1\right) \cdots g_{Q_l}\left(x^l\right) \\ &= g_{Q_1}\left(x^1\right) \cdots g_{Q_l}\left(x^l\right). \end{aligned}$$

Similarly, from the pointwise convergence of $g_{(P_n)^\circ} \to g_{P^\circ}$ we have

$$g_{P^\circ}\left(x^1 \otimes \cdots \otimes x^l\right) = g_{(Q_1)^\circ}\left(x^1\right) \cdots g_{(Q_l)^\circ}\left(x^l\right),$$

thus by Proposition 2.13, $P \in \mathcal{B}_{\Sigma_{Q_1,\ldots,Q_l}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. □

Proposition 2.25. *$\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is a path-connected closed subset of $\left(\mathcal{B}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right), \delta^H\right)$.*

Proof. First, we prove that $\mathcal{B}_\Sigma$ is closed. Take a sequence $\{P_n\}$ in $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ converging to $P \in \mathcal{B}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. Without lost of generality, we assume that $e_1^{d_1} \otimes \cdots \otimes e_1^{d_l} \in \partial P$. Let $g_{P_n}, g_P, g_{P_n^\circ}, g_{P^\circ}, g_{P_n^i}, g_{(P_n^i)^\circ}$ be the Minkowski functionals associated to $P_n, P, P_n^\circ, P^\circ, P_n^i, (P_n^i)^\circ$ respectively.

Since $P_n^\circ \to P^\circ$ in the Hausdorff metric then $g_{P_n} \to g_P$ pointwise therefore:

$$\begin{aligned} &\lim_{n\to\infty} g_{P_n}\left(e_1^{d_1} \otimes \cdots \otimes e_1^{d_{i-1}} \otimes e_1^{d_i} \otimes e_1^{d_{i+1}} \otimes \cdots \otimes e_1^{d_l}\right) \\ &= g_P\left(e_1^{d_1} \otimes \cdots \otimes e_1^{d_{i-1}} \otimes e_1^{d_i} \otimes e_1^{d_{i+1}} \otimes \cdots \otimes e_1^{d_l}\right) \\ &= 1 \end{aligned}$$

Hence, for $i \in \{1, ..., l-1\}$ we obtain

$$\begin{aligned}
& g_P \left(e_1^{d_1} \otimes \cdots \otimes e_1^{d_{i-1}} \otimes x^i \otimes e_1^{d_{i+1}} \otimes \cdots \otimes e_1^{d_l}\right) \\
&= \lim_{n\to\infty} g_{P_n} \left(e_1^{d_1} \otimes \cdots \otimes e_1^{d_{i-1}} \otimes x^i \otimes e_1^{d_{i+1}} \otimes \cdots \otimes e_1^{d_l}\right) \\
&= \lim_{n\to\infty} g_{P_n^1} \left(e_1^{d_1}\right) \cdots g_{P_n^{i-1}} \left(e_1^{d_{i-1}}\right) g_{P_n^i} \left(x^i\right) g_{P_n^{i+1}} \left(e_1^{d_{i+1}}\right) \cdots g_{P_n^l} \left(e_1^{d_l}\right) \\
&= \lim_{n\to\infty} g_{P_n^i} \left(x^i\right) g_{P_n^l} \left(e_1^{d_l}\right) \\
&= g_P \left(e_1^{d_1} \otimes \cdots \otimes e_1^{d_{i-1}} \otimes e_1^{d_i} \otimes e_1^{d_{i+1}} \otimes \cdots \otimes e_1^{d_l}\right) \lim_{n\to\infty} g_{P_n^i} \left(x^i\right) \\
&= \lim_{n\to\infty} g_{P_n^i} \left(x^i\right).
\end{aligned}$$

For $i = l$ we have

$$\begin{aligned}
g_{P^l} \left(x^l\right) &= g_P \left(e_1^{d_1} \otimes e_1^{d_2} \otimes \cdots \otimes x^l\right) \\
&= \lim_{n\to\infty} g_{P_n} \left(e_1^{d_1} \otimes e_1^{d_2} \otimes \cdots \otimes x^l\right) \\
&= \lim_{n\to\infty} g_{P_n^1} \left(e_1^{d_1}\right) g_{P_n^2} \left(e_1^{d_2}\right) \cdots g_{P_n^l} \left(x^l\right) \\
&= \lim_{n\to\infty} g_{P_n^l} \left(x^l\right).
\end{aligned}$$

So for each $i = 1, ..., l$, $g_{P_n^i} \to g_{P^i}$ pointwise. It implies:

$$\begin{aligned}
g_P \left(x^1 \otimes \cdots \otimes x^l\right) &= \lim_{n\to\infty} g_{P_n} \left(x^1 \otimes \cdots \otimes x^l\right) \\
&= \lim_{n\to\infty} g_{P_n^1} \left(x^1\right) \cdots g_{P_n^l} \left(x^l\right) \\
&= \lim_{n\to\infty} g_{P_n^1} \left(x^1\right) \cdots \lim_{n\to\infty} g_{P_n^l} \left(x^l\right) \\
&= g_{P^1} \left(x^1\right) \cdots g_{P^l} \left(x^l\right).
\end{aligned}$$

Now, the pointwise convergence $g_{P_n^i} \to g_{P^i}$ implies the uniform convergence of $g_{P_n^i} \to g_{P^i}$ on $\partial B_2^{d_i}$ thus we have $g_{(P_n^i)^\circ} \to g_{(P^i)^\circ}$ uniformly on $\partial B_2^{d_i}$. Since $g_{(P_n)^\circ} \to g_{P^\circ}$pointwise then,

$$\begin{aligned}
g_{P^\circ} \left(x^1 \otimes \cdots \otimes x^l\right) &= \lim_{n\to\infty} g_{P_n^\circ} \left(x^1 \otimes \cdots \otimes x^l\right) \\
&= \lim_{n\to\infty} g_{(P_n^1)^\circ} \left(x^1\right) \cdots g_{\left(P_n^l\right)^\circ} \left(x^l\right) \\
&= g_{(P^1)^\circ} \left(x^1\right) \cdots g_{\left(P^l\right)^\circ} \left(x^l\right).
\end{aligned}$$

By Proposition 2.13, $P \in \mathcal{B}_{\Sigma_{P^1,\dots,P^l}} \left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$.

To prove that $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is path-connected, for every $P \in \mathcal{B}_{\Sigma_{P_1,\dots,P_l}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ and $Q \in \mathcal{B}_{\Sigma_{Q_1,\dots,Q_l}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ define:

$$\alpha_1(t) = (1-3t)\,Q + 3tQ_1 \otimes_\pi \cdots \otimes_\pi Q_l \text{ for } 0 \le t \le \frac{1}{3}$$
$$\alpha_2(t) = ((2-3t)\,Q_1 + (3t-1)\,P_1) \otimes_\pi \cdots \otimes_\pi ((2-3t)\,Q_l + (3t-1)\,P_l) \text{ for } \frac{1}{3} \le t \le \frac{2}{3}$$
$$\alpha_3(t) = (3-3t)P_1 \otimes_\pi \cdots \otimes_\pi P_l + (3t-2)P \text{ for } \frac{2}{3} \le t \le 1.$$

The continuity of α_2 is a consequence of that of $t \to (2-3t)\,Q_i + (3t-1)\,P_i$ $i = 1, ..., l$ and $\otimes_\pi$.

We now turn to prove that α_1 is continuous. Let $r_1, r_2 > 0$ such that $Q \subseteq r_1 B_2^{d_1 \cdots d_l}$ and $Q_1 \otimes_\pi \cdots \otimes_\pi Q_l \subseteq r_2 B_2^{d_1 \cdots d_l}$. If $s, t \in \left[0, \frac{1}{2}\right]$ then

$$\begin{aligned}
&\delta^H\left((1-3t)\,Q + 3tQ_1 \otimes_\pi \cdots \otimes_\pi Q_l, (1-3s)\,Q + 3sQ_1 \otimes_\pi \cdots \otimes_\pi Q_l\right)\\
&\le \delta^H\left((1-3t)\,Q, (1-3s)\,Q\right)\\
&+ \delta^H\left(3tQ_1 \otimes_\pi \cdots \otimes_\pi Q_l, 3sQ_1 \otimes_\pi \cdots \otimes_\pi Q_l\right)\\
&\le |3t-3s|\,r_1 + |3t-3s|\,r_2.
\end{aligned}$$

So α_1 is continuous. The proof of the continuity of α_3 is analogous to the proof for α_1. We have proved that $\alpha(t) = \begin{cases} \alpha_1(t) & 0 \le t \le \frac{1}{3} \\ \alpha_2(t) & \frac{1}{3} \le t \le \frac{2}{3} \\ \alpha_3(t) & \frac{2}{3} \le t \le 1 \end{cases}$ is a continuous path connecting Q, P. This completes the proof. □

2.4.2 Characterization of tensorial ellipsoids

The aim of this section is to characterize the ellipsoids in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ which also are tensorial 0-symmetric convex bodies. We prove in Corollary 2.31 that this ellipsoids are the image of $B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}$ by elements of $GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$.

The usual definition of an ellipsoid is the following: a subset $\mathcal{E}$ contained in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ is an ellipsoid if there exists a linear isomorphism $T : \mathbb{R}^d \to \otimes_{i=1}^l \mathbb{R}^{d_i}$ with $d = d_1 \cdots d_l$ such that $\mathcal{E} = T\left(B_2^d\right)$. Nevertheless, observe that for every linear map $U : \mathbb{R}^d \to \otimes_{i=1}^l \mathbb{R}^{d_i}$ preserving the inner products $\langle \cdot, \cdot \rangle_2$ and $\langle \cdot, \cdot \rangle_H$, and every ellipsoid $\mathcal{E} = T\left(B_2^d\right)$ contained in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ one has:

$$\mathcal{E} = T\left(B_2^d\right) = TU\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right).$$

For this reason, in this work, we say that a subset $\mathcal{E}$ in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ is an **ellipsoid** if there exists a linear isomorphism $T : \otimes_{i=1}^l \mathbb{R}^{d_i} \to \otimes_{i=1}^l \mathbb{R}^{d_i}$ such that $\mathcal{E} = T\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right)$.

We say that an ellipsoid $\mathcal{E} \subseteq \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ is a **tensorial ellipsoid** if $\mathcal{E} \in \mathcal{B}_\Sigma$. The subset of tensorial ellipsoids will be denoted by $\mathscr{E}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i} \right)$.

Let $\mathcal{E}_i \subseteq \mathbb{R}^{d_i}$ for $i = 1, ..., l$ be ellipsoids. We define **the Hilbert tensor product of** $\mathcal{E}_1, ..., \mathcal{E}_l$ as follows;

$$\mathcal{E}_1 \otimes_H \cdots \otimes_H \mathcal{E}_l := \left\{ z \in \otimes_{i=1}^{l} \mathbb{R}^{d_i} : z \in B_{\underset{i=1}{\overset{l}{\otimes_H}} \left(\mathbb{R}^{d_i}, g_{\mathcal{E}_i} \right)} \right\}$$

Clearly, $\mathcal{E}_1 \otimes_H \cdots \otimes_H \mathcal{E}_l \in \mathscr{E}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i} \right)$. We denote by $\langle \cdot, \cdot \rangle_{\mathcal{E}_1 \otimes_H \cdots \otimes_H \mathcal{E}_l}$ the inner product determined by $\mathcal{E}_1 \otimes_H \cdots \otimes_H \mathcal{E}_l$.

Lemma 2.26. *Let m, n be natural numbers and let $d = mn$. If $S \in M_{dxd}(\mathbb{R})$ is a positive definite matrix and there exist antisymmetric matrices $A_{ij}, B_{ij} \in M_{n,n}(\mathbb{R})$, such that*

$$S = \begin{bmatrix} I_n & A_{11} & A_{12} & \dots & A_{1,m-1} \\ -A_{11} & I_n & A_{21} & \dots & A_{2,m-2} \\ -A_{12} & -A_{21} & I_n & \dots & A_{3,m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_{1,m-1} & -A_{2,m-2} & -A_{3,m-3} & \dots & I_n \end{bmatrix}$$

and

$$S^{-1} = \begin{bmatrix} I_n & B_{11} & B_{12} & \dots & B_{1,m-1} \\ -B_{11} & I_n & B_{21} & \dots & B_{2,m-2} \\ -B_{12} & -B_{21} & I_n & \dots & B_{3,m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -B_{1,m-1} & -B_{2,m-2} & -B_{3,m-3} & \dots & I_n \end{bmatrix},$$

where I_n is the identity matrix of dimension n, then $S = I_d$ is the identity matrix on $M_{dxd}(\mathbb{R})$.

Proof. Let $n \geq 1$ be an arbitrary, but fixed, natural number. We will prove the result by induction on m.

Step 1. If $m = 1$ the result is true by definition of S. We prove the result for $m = 2$.

By our hypothesis we have:

$$S = \begin{bmatrix} I_n & A_{11} \\ -A_{11} & I_n \end{bmatrix} \text{ and } S^{-1} = \begin{bmatrix} I_n & B_{11} \\ -B_{11} & I_n \end{bmatrix}$$

then

$$SS^{-1} = \begin{bmatrix} I_n - A_{11}B_{11} & A_{11} + B_{11} \\ -A_{11} - B_{11} & I_n - A_{11}B_{11} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}.$$

Thus, $B_{11} = -A_{11}$ and $-A_{11}^2 = 0$. Since A_{11} is antisymmetric, the latter equality implies that $A_{11}^t A_{11} = 0$ so $A_{11} = 0$ and the proof is complete.

Step 2. Assume the result is valid for $m-1$ and denote :

$$E = \begin{bmatrix} I_n & A_{11} & \dots & A_{1,m-2} \\ -A_{11} & I_n & \cdots & A_{2,m-3} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{1,m-2} & -A_{2,m-3} & \cdots & I_n \end{bmatrix}_{n(m-1),n(m-1)} , F = \begin{bmatrix} A_{1,m-1} \\ A_{2,m-2} \\ A_{3,m-3} \\ \vdots \\ A_{m-1,1} \end{bmatrix}_{n(m-1),n} ,$$

$$G = \begin{bmatrix} I_n & B_{11} & \dots & B_{1,m-2} \\ -B_{11} & I_n & \cdots & B_{2,m-3} \\ \vdots & \vdots & \ddots & \vdots \\ -B_{1,m-2} & -B_{2,m-3} & \cdots & I_n \end{bmatrix}_{n(m-1),n(m-1)} , H = \begin{bmatrix} B_{1,m-1} \\ B_{2,m-2} \\ B_{3,m-3} \\ \vdots \\ B_{m-1,1} \end{bmatrix}_{n(m-1),n} ,$$

then

$$S = \begin{bmatrix} E & F \\ F^t & I_n \end{bmatrix} , S^{-1} = \begin{bmatrix} G & H \\ H^t & I_n \end{bmatrix}$$

and

$$SS^{-1} = \begin{bmatrix} EG + FH^t & EH + FI_n \\ F^tG + I_nH^t & F^tH + I_n \end{bmatrix} = \begin{bmatrix} I_{n(m-1)} & 0_{n(m-1),n} \\ 0_{n,n(m-1)} & I_n \end{bmatrix} .$$

Therefore $F^tH + I_n = I_n$ and $F^tH = 0_{n,n}$. Since we also have $F^tG + I_nH^t = 0_{n,n(m-1)}$ then $H^t = -F^tG$. This implies:

$$H = -GF. \tag{2.4.4}$$

From the previous equations we get $0 = F^tH = -F^tGF$ and

$$F^tGF = 0_{n,n} \tag{2.4.5}$$

Now, if we write F_i, $i = 1, 2, \dots, n$ for the columns of F then from equation (2.4.5) we have

$$F_i^tGF_i = 0. \tag{2.4.6}$$

On the other hand, since S is positive definite, then S^{-1}, E, G also are positive definite matrices. So, by equation (2.4.6) $F_i = 0$ for $i = 1, 2, \dots, n$ and $F = 0$. Consequently $H = 0$, by equation (2.4.4).

Finally, we can apply our inductive hypothesis to E and $E^{-1} = G$ and we have $E = I_{n(m-1)}$. Therefore $S = I_d$. □

Every ellipsoid $\mathcal{E} = T\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right) \subset \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ is the closed unit ball associated to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{E}} := \langle T^{-1}(\cdot), T^{-1}(\cdot) \rangle_H$. In view of this, the following proposition describes the relation between $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ and $\langle \cdot, \cdot \rangle_H$ on decomposable vectors, when $\mathcal{E}$ is a tensorial ellipsoid in $\mathbb{R}^m \otimes \mathbb{R}^n$.

Proposition 2.27. *Let $m, n \geq 1$ be natural numbers. If $T \in GL(\mathbb{R}^m \otimes \mathbb{R}^n)$ and $\mathcal{E} = T(B_2^m \otimes_H B_2^n)$ then,*

$$B_2^m \otimes_\pi B_2^n \subseteq \mathcal{E} \subseteq B_2^m \otimes_\epsilon B_2^n$$

if and only if the following relations hold :

$$\langle x \otimes y, z \otimes w \rangle_H = \frac{\langle T^{-1}(x \otimes y), T^{-1}(z \otimes w) \rangle_H + \langle T^{-1}(x \otimes w), T^{-1}(z \otimes y) \rangle_H}{2}$$

$$\langle x \otimes y, z \otimes w \rangle_H = \frac{\langle T^{t}(x \otimes y), T^{t}(z \otimes w) \rangle_H + \langle T^{t}(x \otimes w), T^{t}(z \otimes y) \rangle_H}{2},$$

for each $x, z \in \mathbb{R}^m$ and $y, w \in \mathbb{R}^n$.

Proof. Recall that if $\mathcal{E} = T(B_2^m \otimes_H B_2^n) \subset \mathbb{R}^m \otimes \mathbb{R}^n$ is an ellipsoid then $\mathcal{E}^\circ = (T^t)^{-1}(B_2^m \otimes_H B_2^n.)$. Since the "if" part is inmediate we give the proof for the "only if" part.

Let $x, z \in \mathbb{R}^m$ and $y, w \in \mathbb{R}^n$. Since $B_2^m \otimes_\pi B_2^n \subseteq \mathcal{E} \subseteq B_2^m \otimes_\epsilon B_2^n$ then

$$\langle x \otimes y, x \otimes y \rangle_H = \left\langle T^{-1}(x \otimes y), T^{-1}(x \otimes y) \right\rangle_H .$$

Now, by the polarization formula we have:

$$\begin{aligned}
\langle x \otimes y, z \otimes w \rangle_H &= \langle x, z \rangle_2 \langle y, w \rangle_2 \\
&= \frac{1}{16}\left(\|x+z\|^2 - \|x-z\|^2\right)\left(\|y+w\|^2 - \|y-w\|^2\right) \\
&= \frac{1}{16}\left(\|x+z\|^2\|y+w\|^2 - \|x+z\|^2\|y-w\|^2 - \|x-z\|^2\|y+w\|^2 + \|x-z\|^2\|y-w\|^2\right) \\
&= \frac{1}{16}\left\langle T^{-1}(x+z)\otimes(y+w), T^{-1}(x+z)\otimes(y+w)\right\rangle_H \\
&\quad -\frac{1}{16}\left\langle T^{-1}(x+z)\otimes(y-w), T^{-1}(x+z)\otimes(y-w)\right\rangle_H \\
&\quad -\frac{1}{16}\left\langle T^{-1}(x-z)\otimes(y+w), T^{-1}(x-z)\otimes(y+w)\right\rangle_H \\
&\quad +\frac{1}{16}\left\langle T^{-1}(x-z)\otimes(y-w), T^{-1}(x-z)\otimes(y-w)\right\rangle_H \\
&= \frac{\langle T^{-1}(x \otimes y), T^{-1}(z \otimes w) \rangle_H + \langle T^{-1}(x \otimes w), T^{-1}(z \otimes y) \rangle_H}{2}.
\end{aligned}$$

Since $\mathcal{E}^\circ$ is also a tensorial ellipsoid, the result holds analogously for T^t. □

Theorem 2.28. *Let $\mathcal{E} \in \mathcal{B}_\Sigma(\mathbb{R}^m \otimes \mathbb{R}^n)$ be an ellipsoid. If*

$$B_2^m \otimes_\pi B_2^n \subseteq \mathcal{E} \subseteq B_2^m \otimes_\epsilon B_2^n$$

then $\mathcal{E} = B_2^m \otimes_H B_2^n$.

Proof. Suppose $\mathcal{E} \subseteq \mathbb{R}^m \otimes \mathbb{R}^n$ is an ellipsoid such that $B_2^m \otimes_\pi B_2^n \subseteq \mathcal{E} \subseteq B_2^m \otimes_\epsilon B_2^n$.

If $\mathcal{E} = T(B_2^m \otimes_H B_2^n)$, for some $T \in GL(\mathbb{R}^m \otimes \mathbb{R}^n)$, then by Proposition 2.27 for each $x, z \in \mathbb{R}^m$ and $y, w \in \mathbb{R}^n$ we have:

$$\begin{aligned}
\langle x \otimes y, z \otimes w \rangle_H &= \frac{\langle T^{-1}(x \otimes y), T^{-1}(z \otimes w)\rangle_H + \langle T^{-1}(x \otimes w), T^{-1}(z \otimes y)\rangle_H}{2} \\
\langle x \otimes y, z \otimes w \rangle_H &= \frac{\langle T^{t}(x \otimes y), T^{t}(z \otimes w)\rangle_H + \langle T^{t}(x \otimes w), T^{t}(z \otimes y)\rangle_H}{2},
\end{aligned}$$

Which, for $S = TT^t$ is equivalent to:

$$\begin{aligned}
\langle x \otimes y, z \otimes w \rangle_H &= \frac{\langle S^{-1}(x \otimes y), z \otimes w\rangle_H + \langle S^{-1}(x \otimes w), z \otimes y\rangle_H}{2} \\
\langle x \otimes y, z \otimes w \rangle_H &= \frac{\langle S(x \otimes y), z \otimes w\rangle_H + \langle S(x \otimes w), z \otimes y\rangle_H}{2}.
\end{aligned}$$

On the other hand, for the canonical basis $\{e_\sigma\}_{\sigma=1,\ldots,d} \subseteq \mathbb{R}^d$, $\{e_i\}_{i=1,\ldots,d} \subseteq \mathbb{R}^m$ and $\{e_j\}_{j=1,\ldots,d} \subseteq \mathbb{R}^n$. Let $\Phi_{(m,n)} : \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}^d$ be the bijective map such that

$$\Phi_{(m,n)}(e_i \otimes e_j) = e_{(i-1)n+j}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. Clearly, $\Phi_{(m,n)}$ preserves the inner products $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_2$.

Hence, we have:

$$\begin{aligned}
\left\langle e_{(i-1)n+j}, e_{(k-1)n+l} \right\rangle_2 &= \frac{\left\langle S^{-1}e_{(i-1)n+j}, e_{(k-1)n+l} \right\rangle_2 + \left\langle S^{-1}e_{(i-1)n+l}, e_{(k-1)n+j} \right\rangle_2}{2} \\
\left\langle e_{(i-1)n+j}, e_{(k-1)n+l} \right\rangle_2 &= \frac{\left\langle Se_{(i-1)n+j}, e_{(k-1)n+l} \right\rangle_2 + \left\langle Se_{(i-1)n+l}, e_{(k-1)n+j} \right\rangle_2}{2},
\end{aligned}$$

for $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$. Therefore:

$$\left\langle Se_{(i-1)n+j}, e_{(k-1)n+l} \right\rangle_2 = \begin{cases} 1 & \text{if } k = i, l = j \\ 0 & \text{if } k = i, l \neq j \\ 0 & \text{if } k \neq i, l = j \\ -\left\langle Se_{(i-1)n+l}, e_{(k-1)n+j} \right\rangle_2 & \text{if } k \neq i, l \neq j \end{cases}$$

$$\left\langle S^{-1}e_{(i-1)n+j}, e_{(k-1)n+l} \right\rangle_2 = \begin{cases} 1 & \text{if } k = i, l = j \\ 0 & \text{if } k = i, l \neq j \\ 0 & \text{if } k \neq i, l = j \\ -\left\langle S^{-1}e_{(i-1)n+l}, e_{(k-1)n+j} \right\rangle_2 & \text{if } k \neq i, l \neq j \end{cases}$$

So the positive definite matrices associated with S, S^{-1} can be written as follows:

$$S = \begin{bmatrix} I_n & A_{11} & A_{12} & \dots & A_{1,m-1} \\ -A_{11} & I_n & A_{21} & \dots & A_{2,m-2} \\ -A_{12} & -A_{21} & I_n & \dots & A_{3,m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_{1,m-1} & -A_{2,m-2} & -A_{3,m-3} & \dots & I_n \end{bmatrix},$$

$$S^{-1} = \begin{bmatrix} I_n & B_{11} & B_{12} & \dots & B_{1,m-1} \\ -B_{11} & I_n & B_{21} & \dots & B_{2,m-2} \\ -B_{12} & -B_{21} & I_n & \dots & B_{3,M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -B_{1,m-1} & -B_{2,m-2} & -B_{3,m-3} & \dots & I_n \end{bmatrix}.$$

Where $A_{ij}, B_{ij} \in M_{n,n}(\mathbb{R})$ are antisymmetric and I_n is the identity matrix of dimension n.

Finally, Lemma 2.26 implies that $S = I_d$ hence $T \in O\left(\mathbb{R}^d\right)$ this is equivalent to $\mathcal{E} = B_2^m \otimes_H B_2^n$. □

Lemma 2.29. *Let $\mathcal{E}$ be an ellipsoid in $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. For every $z^l \in \partial B_2^{d_l}$, let*

$$i_{z^l} : \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}} \to \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}} \otimes \mathbb{R}^{d_l}$$
$$x^1 \otimes \cdots \otimes x^{l-1} \to x^1 \otimes \cdots \otimes x^{l-1} \otimes z^l,$$

and $\mathcal{E}_{z^l} := i_{z^l}^{-1}(\mathcal{E})$. Then, if

$$B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_l} \subseteq \mathcal{E} \subseteq B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_l},$$

one has

$$B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_{l-1}} \subseteq \mathcal{E}_{z^l} \subseteq B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_{l-1}}.$$

Proof. For $z^l \in \partial B_2^{d_l}$ we denote by $\langle \cdot, \cdot \rangle_{z^l}, \|\cdot\|_{z^l}$ the inner product and the norm induced on $\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}$ by $i_{z^l}^{-1}(\mathcal{E})$.

Let $x^1 \otimes \cdots \otimes x^{l-1} \in \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}$, then we have:

$$\left\langle x^1 \otimes \cdots \otimes x^{l-1}, y^1 \otimes \cdots \otimes y^{l-1} \right\rangle_{z^l} := \left\langle x^1 \otimes \cdots \otimes x^{l-1} \otimes z^l, x^1 \otimes \cdots \otimes x^{l-1} \otimes z^l \right\rangle_{\mathcal{E}}.$$

Thus, from our hypothesis we obtain $\left\|x^1 \otimes \cdots \otimes x^{l-1}\right\|_{z^l} = \|x^1\|_2 \cdots \left\|x^{l-1}\right\|_2$. We also have:

$$\begin{aligned} \left\|x^1 \otimes \cdots \otimes x^{l-1}\right\|_{(\mathcal{E}_{z^l})^\circ} &= \sup_{\|a\|_{z^l} \le 1} \left|\left\langle a, x^1 \otimes \cdots \otimes x^{l-1}\right\rangle_H\right| \\ &= \sup_{\left\|i_{z^l}(a)\right\|_{\mathcal{E}} \le 1} \left|\left\langle i_{z^l}(a), x^1 \otimes \cdots \otimes x^{l-1} \otimes z^l\right\rangle_H\right| \\ &\le \left\|x^1 \otimes \cdots \otimes x^{l-1} \otimes z^l\right\|_{\mathcal{E}^\circ} \\ &= \left\|x^1\right\|_2 \cdots \left\|x^{l-1}\right\|_2. \end{aligned}$$

Which is equivalent to $B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_{l-1}} \subseteq \mathcal{E}_{z^l} \subseteq B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_{l-1}}$. □

Theorem 2.30. *Let $\mathcal{E} \subseteq \otimes_{i=1}^{l} \mathbb{R}^{d_i}$ be an ellipsoid such that*

$$B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_l} \subseteq \mathcal{E} \subseteq B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_l}, \tag{2.4.7}$$

then $\mathcal{E} = B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}$.

Proof. We will prove the result using induction on the number l of factors on the tensor product.

The case $l = 2$ is Theorem 2.28. Now we assume that the result holds for $l-1$. This means that for every ellipsoid $\mathcal{E} \in \mathcal{B}_\Sigma \left(\otimes_{i=1}^{l-1} \mathbb{R}^{d_i}\right)$ such that

$$B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_{l-1}} \subseteq \mathcal{E} \subseteq B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_{l-1}},$$

we have $\mathcal{E} = B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_{l-1}}$.

Let $\mathcal{E}$ be an ellipsoid in $\mathcal{B}_\Sigma \left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$ satisfying equation (2.4.7), and $\|\cdot\|_{\mathcal{E}}$ its induced norm. By Lemma 2.29, for every $z^l \in \partial B_2^{d_l}$ we have

$$B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_{l-1}} \subseteq \mathcal{E}_{z^l} \subseteq B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_{l-1}}.$$

Applying the induction hypothesis we obtain $\mathcal{E}_{z^l} = B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_{l-1}}$. Therefore for every $\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \in \otimes_{i=1}^{l-1} \mathbb{R}^{d_i}$ we have

$$\begin{aligned} \left\| \sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes z^l \right\|_{\mathcal{E}} &= \left\| i_{z^l} \left(\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \right) \right\|_{\mathcal{E}} \\ &= \left\| \sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \right\|_{z^l} \\ &= \left\| \sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \right\|_2 . \end{aligned}$$

Since $\mathcal{E}^\circ$ also satisfies equation (2.4.7), we have that $\left\|\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes z^l\right\|_{\mathcal{E}^\circ} = \left\|\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1}\right\|_2$. Here, $\|\cdot\|_{\mathcal{E}^\circ}$ is the norm induced by $\mathcal{E}^\circ$.

Now consider canonical isomorphism

$$\begin{aligned} \psi : \left(\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}\right) \otimes \mathbb{R}^{d_l} &\to \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}} \otimes \mathbb{R}^{d_l} \\ \left(x_i^1 \otimes \cdots \otimes x_i^{l-1}\right) \otimes x^l &\to x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes x^l, \end{aligned}$$

and denote by $\tilde{\mathcal{E}}, \|\cdot\|_{\tilde{\mathcal{E}}}$ the ellipsoid and the norm on $\left(\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}\right) \otimes \mathbb{R}^{d_l}$ determined by this isomorphism and $\mathcal{E}$.

For each non-zero $x^l \in \mathbb{R}^{d_l}$, and $u = \sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \in \mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}$ we have

$$\begin{aligned}
\left\|u \otimes x^l\right\|_{\tilde{\mathcal{E}}} &= \left\|\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes x^l\right\|_{\mathcal{E}} \\
&= \left\|\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes \frac{x^l}{\|x^l\|_2}\right\|_{\mathcal{E}} \left\|x^l\right\|_2 \\
&= \left\|i_{\frac{x^l}{\|x^l\|_2}} \left(\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1}\right)\right\|_{\mathcal{E}} \left\|x^l\right\|_2 \\
&= \left\|\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1}\right\|_{x^l} \left\|x^l\right\|_2 \\
&= \left\|\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1}\right\|_{2} \left\|x^l\right\|_2 \\
&= \|u\|_2 \left\|x^l\right\|_2 .
\end{aligned}$$

And,

$$\begin{aligned}
\left\|u \otimes x^l\right\|_{\tilde{\mathcal{E}}^\circ} &= \sup_{a \in \tilde{\mathcal{E}}} \left|\left\langle a, u \otimes x^l \right\rangle_H\right| \\
&= \sup_{\|\psi(a)\|_{\mathcal{E}} \leq 1} \left|\left\langle \psi(a), \sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes x^l \right\rangle_H\right| \\
&\leq \left\|\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes x^l\right\|_{\mathcal{E}^\circ} \\
&\leq \left\|\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1} \otimes \frac{x^l}{\|x^l\|_2}\right\|_{\mathcal{E}^\circ} \left\|x^l\right\|_2 \\
&= \left\|\sum_{i=1}^{N} x_i^1 \otimes \cdots \otimes x_i^{l-1}\right\|_{2} \left\|x^l\right\|_2 \\
&= \|u\|_2 \left\|x^l\right\|_2 .
\end{aligned}$$

We have proved that $\tilde{\mathcal{E}} \in \mathcal{B}_\Sigma\left(\left(\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_{l-1}}\right) \otimes_H \mathbb{R}^{d_l}\right)$.

Now, let $d = d_1 \cdots d_{l-1}$ and $\Phi_{(d_1, d_2, \ldots, d_{l-1})} : \otimes_{i=1}^{l-1} \mathbb{R}^{d_i} \to \mathbb{R}^d$ be a bijective linear map defined as,

$$\begin{aligned}
\Phi_{(d_1, d_2, \ldots, d_{l-1})} &: \otimes_{i=1}^{l-1} \mathbb{R}^{d_i} \to \mathbb{R}^d \\
e_{j_1}^{d_1} \otimes e_{j_2}^{d_2} \otimes \cdots \otimes e_{l-1}^{d_{l-1}} &\to e^d_{(j_1-1)d_2 \cdots d_{l-1} + (j_2-1)d_3 \cdots d_{l-1} + \cdots + (j_{l-2}-1)d_{l-1} + j_{l-1}} .
\end{aligned}$$

Since $\Phi_{(d_1,d_2,\dots,d_{l-1})}$ preserves the inner products $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_2$, we have,

$$\tilde{\mathcal{E}} \in \mathcal{B}_\Sigma\left(\mathbb{R}^{d_1\cdots d_{l-1}} \otimes_H \mathbb{R}^{d_l}\right).$$

By Theorem 2.28, we obtain $\tilde{\mathcal{E}} = B_2^{d_1\cdots d_{l-1}} \otimes_H B_2^{d_l}$. Since $\mathcal{E} = \psi\left(\tilde{\mathcal{E}}\right)$ and ψ is an orthogonal transformation, we obtain $\mathcal{E} = B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}$. □

Corollary 2.31. *Let $T \in GL\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ such that $T\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right)$ belongs to $\mathscr{E}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. Then there exist $U \in O\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ and $T_i \in GL\left(d_i\right)$ for $i = 1, ..., l$ such that*

$$T = T_1 \otimes \cdots \otimes T_l U.$$

Proof. Assume that $T\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right)$ belongs to $\mathscr{E}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$, then there exists $A_i \in \mathcal{B}(\mathbb{R}^{d_i})$ for $i = 1, ..., l$ such that

$$A_1 \otimes_\pi \cdots \otimes_\pi A_l \subseteq T\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right) \subseteq A_1 \otimes_\epsilon \cdots \otimes_\epsilon A_l.$$

Since $T\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right)$ is an ellipsoid we must have that all A_i are ellipsoids. Thus there exist $T_i \in GL\left(\mathbb{R}^{d_i}\right)$ for $i = 1, ..., l$ with $A_i = T_i\left(B_2^{d_i}\right)$.

From this, we deduce:

$$T_1\left(B_2^{d_1}\right) \otimes_\pi \cdots \otimes_\pi T_l\left(B_2^{d_l}\right) \subseteq T\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right) \subseteq T_1\left(B_2^{d_1}\right) \otimes_\epsilon \cdots \otimes_\epsilon T_l\left(B_2^{d_l}\right)$$

$$B_2^{d_1} \otimes_\pi \cdots \otimes_\pi B_2^{d_l} \subseteq \left(T_1^{-1} \otimes \cdots \otimes T_l^{-1}\right) T\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right) \subseteq B_2^{d_1} \otimes_\epsilon \cdots \otimes_\epsilon B_2^{d_l}.$$

Therefore Theorem 2.30 implies that,

$$\left(T_1^{-1} \otimes \cdots \otimes T_l^{-1}\right) T\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right) = B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}.$$

Finally $U := \left(T_1^{-1} \otimes \cdots \otimes T_l^{-1}\right) T \in O\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. This completes the proof. □

Corollary 2.32. *If $\mathcal{E}$ is an ellipsoid in $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ then there exists*

$$T \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right),$$

such that $\mathcal{E} = T\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right)$.

Chapter 3

Topological structure of $\mathcal{B}_{\Sigma}$

In [2, 1], S. Antonyan described the topological structure of the set of 0-symmetric convex bodies in $\mathbb{R}^d$, $\mathcal{B}(d)$. S. Antonyan's idea consists in studying the properties of the natural action of the general linear group $GL(d)$ in $\mathcal{B}(d)$. Following this idea, we study the topological structure of $\mathcal{B}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ through the properties of the action determined on it by the set of linear bijective maps on $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ preserving decomposable tensors, $GL_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$. To this end, we use the Löwner ellipsoid (See Section 3.4.1) to define the set $\mathscr{L}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ which is the analogue to the set of 0-symmmetric convex bodies in $\mathbb{R}^d$ for which the Euclidean ball B_2^d is the Löwner ellipsoid, $L(d)$ [3, 19]. In fact, we prove that $\mathscr{L}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ is a compact global $O_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$-slice. Recall that $O_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ is the group of linear isometries of $\otimes_{H,i=1}^{l}\mathbb{R}^{d_i}$ preserving decomposable tensors. This allows us to prove that $\mathcal{B}_{\Sigma}$ is homeomophic to $\mathscr{E}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) \times \mathscr{L}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$, where $\mathscr{E}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ is the set of tensorial 0-symmetric ellipsoids. The results and proof of Sections 3.3 and 3.4 are natural adaptations of analogous results in [3].

At the end of the chapter we introduce the Σ-Banach-Mazur distance δ_{Σ}^{BM} between tensorial 0-symmetric convex bodies,

$$\delta_{\Sigma}^{BM}(P,Q) := \inf\left\{\lambda \geq 1 : Q \subseteq TP \subseteq \lambda Q \text{ for some } T \in Gl_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)\right\}.$$

We prove that $\log\delta_{\Sigma}^{BM}$ is a metric on the set of equivalence classes determined by Gl_{Σ} in $\mathcal{B}_{\Sigma}$, $\mathcal{BM}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$. In Theorem 3.36 we prove that $\left(\mathcal{BM}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right), \log\delta_{\Sigma}^{BM}\right)$ is a compact metric space.

3.1 Topological groups

In this section we introduce the background about the actions of topological groups needed to follow each part of this chapter. The notions and results that we present here can be consulted in [7] and [30].

A **topological group** G is a group with a topology such that G is a Hausdorff space, the product

$$\begin{aligned} G \times G &\to G \\ (g,h) &\to gh, \end{aligned}$$

and the function

$$\begin{aligned} G &\to G \\ g &\to g^{-1} \end{aligned}$$

are continuous.

Example 3.1. The set of linear bijective maps $GL\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ is a topological group with the topology that inherites from $\mathcal{L}\left(\otimes_{H,i=1}^{l}\mathbb{R}^{d_i}\right)$.

Example 3.2. The set of orthogonal transformations $O\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ is a topological group with the topology that inherites from $\mathcal{L}\left(\otimes_{H,i=1}^{l}\mathbb{R}^{d_i}\right)$.

3.2 Actions of topological groups

Let G be a topological group and let X be a Hausdorff topological space. A **continuous action** of G on X is a continuous function $\theta : G \times X \to X$ such that:

1. $\theta(e,x) = x$ for all $x \in X$.
2. $\theta(h, \theta(g,x)) = \theta(hg, x)$ for all $g, h \in G$ and $x \in X$.

To shorten notation for every $g \in G$ and $x \in X$ we will write gx instead of $\theta(g,x)$.

A G-**space** is a pair (X, θ) where G is a topological group and θ is a continuous action of G on X. If X is a G-space and $x \in X$ then the set

$$G(x) := \{gx : g \in G\},$$

is called the **orbit** of x. The set

$$G_x := \{g \in G : gx = x\},$$

is called the **isotropy group** (or **stability group**) of G at x.

Let $H \subseteq G$ be a subgroup. The family of all subgroups of G that are conjugate to H is denoted by $[H]$. That is $[H] := \{gHg^{-1} : g \in G\}$. If H_1 and H_2 are subgroups of G, then one says that $[H_1] \preceq [H_2]$ if and only if $H_1 \subseteq gH_2g^{-1}$ for some $g \in G$. The relation $\preceq$ is a partial order on the set $\{[H] : H \subseteq G \text{ is a subgroup}\}$.

For a subset $S \subseteq X$ and a subgroup $H \subseteq G$,

$$H(S) := \{hs : h \in H, s \in S\}$$

denotes the H-saturation of S. A subset $S \subseteq X$ is called H-**invariant** if $H(S) = S$.

A continuous map $f : X \to Y$ between two $G-$spaces is called **equivariant** or a G-**map** if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$.

3.2.1 Orbit space

The action of a group G on a set X determines a partition of X for which the orbits $G(x)$, $x \in X$ are the equivalence classes.

Proposition 3.3. *Let X be a G-space. If $x_1, x_2 \in X$ then $G(x_1) = G(x_2)$ or $G(x_1) \cap G(x_2) = \emptyset$.*

Let X/G denote the set whose elements are the orbits of the elements of a G-space X. Let $\pi_X : X \to X/G$ denote the orbit map taking x into its orbit $G(x)$. Then X/G endowed with the quotient topology is called the **orbit space** of X (with respect to G).

Proposition 3.4. *Let X be a G-space. Then $\pi_X : X \to X/G$ is a continuous open map.*

Theorem 3.5. *Let X be a G-space. If G is compact then:*

1. *X/G is Hausdorff.*
2. *$\pi_X : X \to X/G$ is closed.*
3. *$\pi_X : X \to X/G$ is proper (i.e. pre-images of compact sets are compact sets).*
4. *X is compact if and only if X/G is compact.*
5. *X is locally compact if and only if X/G is locally compact.*

Let X be a G-space and $\theta : G \times X \to X$ be the action of G on X. For each $x \in X$, we can define the function:

$$\begin{aligned} \theta_x : G &\to X \\ g &\to gx. \end{aligned}$$

Observe that the continuity of θ implies that of θ_x. Also, $\theta_x(G) = G(x)$ and $\theta_x^{-1}(\{x\}) = G_x$. The following proposition follows directly from the previous observation.

Proposition 3.6. *Let X be a G-space. If G is compact then:*

1. *For every $x \in X$, $G(x)$ is compact.*
2. *For every $x \in X$, G_x is closed.*

Let x be a point in a G-space X. Then G_x acts on G as follows:

$$\begin{aligned} G_x \times G &\to G \\ (h, g) &\to gh^{-1}. \end{aligned}$$

The continuity of the product on G implies the continuity of the previous action. Thus, G is a G_x-space.

Let G/G_x be the orbit space of G with respect to G_x. Observe that,

$$G/G_x = \{gG_x : g \in G\}.$$

Therefore, we can define an action of G on G/G_x as follows:

$$\begin{aligned} G \times G/G_x &\to G/G_x \\ (g, hG_x) &\to ghG_x. \end{aligned}$$

This action is continuous. So, G/G_x is a G-space.

On the other hand, observe that the function $\theta_x : G \to G(x)$ satisfies that

$$\theta_x(gh) = (gh)x = g(hx) = gx = \theta_x(g),$$

for every $g \in G$ and $h \in G_x$.

Thus by the universal property of quotients, there exists a continuous function

$$\overline{\theta_x} : G/G_x \to G(x),$$

such that $\overline{\theta_x} \circ \pi_G = \theta_x$.

Proposition 3.7. *Let X be a G-space. If G is compact then for every $x \in X$,*

$$\overline{\theta_x} : G/G_x \to G(x)$$

is a G-equivariant homeomorphism.

3.2.2 Proper actions

Let G be a G-space. For every $S \subseteq X$ and $T \subseteq X$, the set

$$[S, T] := \{g \in G : gS \cap T \neq \emptyset\},$$

is called **the transporter** from S to T.

We will say that a subset $S \subseteq X$ of a G-space X is **small**, if any $x \in X$ has a neighborhood V such that the set $[S, T]$ has compact closure in G.

Definition 3.8. (R. Palais, [31]) Let G be a locally compact group and X a Tychonoff G-space. We say that the action of G on X is **proper** (in the sense of Palais) if any $x \in X$ has a small neighborhood V.

Proposition 3.9. *Let G be a locally compact group acting properly on a Tychonoff space X. Then,*

1. *The orbit space X/G is a Tychonoff space.*
2. *For every $x \in X$, $G(x)$ is a closed subset of X. Also, $\theta_x : G \to G(x)$ is open.*
3. *For evey $x \in X$, the isotropy group G_x is a compact subgroup of G. The function $\overline{\theta_x} : G/G_x \to G(x)$ is a G-equivariant homeomorphism.*

3.2.3 Slices

Definition 3.10. Let X be a G-space and H a closed subgroup of G. An H-invariant subset $S \subseteq X$ is called an H-**slice** in X, if $G(S)$ is open in X and there exists a G-equivariant map $f : G(S) \to G/H$ such that $S = f^{-1}(eH)$. The saturation $G(S)$ is called a *tubular* set. If $G(S) = X$, then we say that S is a *global* H-slice of X.

Theorem 3.11. *(G. Bredon, [7]) Let G be a compact group and H a closed subgroup. A subset $S \subseteq X$ of a G-space X is an H-slice if and only if it satisfies the following conditions:*

1. *S is H-invariant.*
2. *$G(S)$ is open in X.*
3. *S is closed in $G(S)$.*
4. *If $g \in G \setminus H$ then $gS \cap S = \emptyset$.*

A Lie group is a group G which is also a smooth manifold, and whose group operations are smooth functions on G.

Theorem 3.12. *(Slice Theorem) Let G be a compact Lie group, X a Tychonoff G-space and $x \in X$. Then:*

1. *There exists a G_x-slice $S \subseteq X$ such that $x \in S$.*
2. *$[G_y] \preceq [G_x]$ for each point $y \in G(S)$.*

3.3 The action of GL_Σ on $\mathcal{B}_\Sigma$

In this section we prove that GL_Σ acts continuously on $\mathcal{B}_\Sigma$. To this end, we need the following result about actions of topological groups on compact subsets of metric spaces. Throughout this chapter we will assume that $d_i \geq 2$ for $i = 1, \ldots, l$ and $l \geq 2$ in $\otimes_{i=1}^{l} \mathbb{R}^{d_i}$.

Let (X, d) be a metric space and let G be a topological group acting continuously on X. Consider the set 2^X consisting of all nonempty compact subsets of X equipped with the Hausdorff metric topology. Define an action of G on 2^X by the rule:

$$\begin{aligned} G \times 2^X &\to 2^X \\ (g, Q) &\to gQ := \{gx : x \in Q\}. \end{aligned}$$

Proposition 3.13. *(Proposition 3.1.1, [19]) The action defined above is continuous.*

As consequence of Proposition 3.13 the action of $GL\left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$ on $\otimes_{H,i=1}^{l} \mathbb{R}^{d_i}$ defines a continuous action on the space of nonempty compact subsets of $\otimes_{H,i=1}^{l} \mathbb{R}^{d_i}$.

Proposition 3.14. $GL_\Sigma\left(\otimes_{i=1}^{l} \mathbb{R}^{d_i}\right)$ *is a closed subgroup of* $GL\left(\otimes_{H,i=1}^{l} \mathbb{R}^{d_i}\right)$.

Proof. Clearly $S \circ T \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ whenever $S, T \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$.
If $T \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ then by Theorem 2.3, $T\left(\Sigma_{\mathbb{R}^{d_1},\ldots,\mathbb{R}^{d_l}}\right) = \Sigma_{\mathbb{R}^{d_1},\ldots,\mathbb{R}^{d_l}}$. So T^{-1} also preserves decomposable tensors.

To see that $GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is a closed subgroup of $GL\left(\otimes_{H,i=1}^l \mathbb{R}^{d_i}\right)$, let $\{T_k\}$ be a sequence in $GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$, and let $T \in GL\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ be such that $\|T_k - T\| \to 0$. Then for every $x^1 \otimes \cdots \otimes x^l \in \Sigma_{\mathbb{R}^{d_1},\ldots,\mathbb{R}^{d_l}}$, we know that $\left\{T_k\left(x^1 \otimes \cdots \otimes x^l\right)\right\}$ is a sequence contained in $\Sigma_{\mathbb{R}^{d_1},\ldots,\mathbb{R}^{d_l}} \subseteq \otimes_{H,i=1}^l \mathbb{R}^{d_i}$ such that

$$\lim_{k\to\infty} T_k\left(x^1 \otimes \cdots \otimes x^l\right) = T\left(x^1 \otimes \cdots \otimes x^l\right).$$

By Corollary 2.2 $\Sigma_{\mathbb{R}^{d_1},\ldots,\mathbb{R}^{d_l}}$ is closed in $\otimes_{H,i=1}^l \mathbb{R}^{d_i}$. Therefore $T\left(x^1 \otimes \cdots \otimes x^l\right) \in \Sigma_{\mathbb{R}^{d_1},\ldots,\mathbb{R}^{d_l}}$ and $T \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. Hence GL_Σ is a closed subgroup of $GL\left(\otimes_{H,i=1}^l \mathbb{R}^{d_i}\right)$. □

Corollary 3.15. *$O_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is compact.*

Remark 3.16. Since $GL\left(\otimes_{H,i=1}^l \mathbb{R}^{d_i}\right)$ is a locally compact space and $GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is closed, we know that $GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is locally compact.

Proposition 3.17. *(GL_Σ preserves tensorial 0-symmetric convex bodies). Assume $d_i \geq 2$ for $i = 1, \ldots, l$. Let $T \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ and $Q \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$, then $TQ \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$.*

Proof. Let T be an element in $GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$, from Theorem 2.3 there exist a permutation σ of the set $\{1, ..., l\}$, and bijective linear maps $T_i : \mathbb{R}^{d_{\sigma(i)}} \to \mathbb{R}^{d_i}$ such that:

$$T\left(x^1 \otimes \cdots \otimes x^l\right) = T_1\left(x^{\sigma(1)}\right) \otimes \cdots \otimes T_l\left(x^{\sigma(l)}\right) \text{ for all } x^1 \otimes \cdots \otimes x^l \in \otimes_{i=1}^l \mathbb{R}^{d_i}.$$

Now if $Q_i \in \mathcal{B}(d_i)$ for $i = 1, \ldots, l$ then,

$$\begin{aligned} T\left(Q_1 \otimes_\pi \cdots \otimes_\pi Q_l\right) &= T\left(conv\left(\Sigma_{Q_1,\ldots,Q_l}\right)\right) \\ &= conv\left(T\left(\Sigma_{Q_1,\ldots,Q_l}\right)\right) \\ &= conv\left(\Sigma_{T_1 Q_{\sigma(1)},\ldots,T_l Q_{\sigma(l)}}\right) \\ &= T_1 Q_{\sigma(1)} \otimes_\pi \cdots \otimes_\pi T_l Q_{\sigma(l)}. \end{aligned} \tag{3.3.1}$$

On the other hand,

$$\begin{aligned} T\left(Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l\right) &= T\left(\left(\Sigma_{Q_1^\circ,\ldots,Q_l^\circ}\right)^\circ\right) \\ &= \left(T^{-t}\left(\Sigma_{Q_1^\circ,\ldots,Q_l^\circ}\right)\right)^\circ \\ &= \left(\Sigma_{T_1^{-t} Q_{\sigma(1)}^\circ,\ldots,T_l^{-t} Q_{\sigma(l)}^\circ}\right)^\circ \\ &= \left(\Sigma_{\left(T_1 Q_{\sigma(1)}\right)^\circ,\ldots,\left(T_l Q_{\sigma(1)}\right)^\circ}\right)^\circ \\ &= T_1 Q_{\sigma(1)} \otimes_\epsilon \cdots \otimes_\epsilon T_l Q_{\sigma(l)}. \end{aligned}$$

Therefore, if $Q \in \mathcal{B}_{\Sigma_{Q_1,\ldots,Q_l}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ then $TQ \in \mathcal{B}_{\Sigma_{T_1 Q_{\sigma(1)},\ldots,T_l Q_{\sigma(l)}}}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. □

Theorem 3.18. *(GL_Σ acts continuosly on $\mathcal{B}_\Sigma$). The function*

$$GL_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)\times\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)\to\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) \quad (3.3.2)$$
$$(T,Q)\to TQ:=\{Tx : x\in Q\}$$

is a continuous action.

Proof. By Proposition 3.17 the action is well defined. The continuity of the action of the linear isomorphisms on the space of nonempty compact convex sets of $\otimes_{H,i=1}^{l}\mathbb{R}^{d_i}$ implies the continuity of 3.3.2 (Proposition 3.13). □

3.4 GL_Σ acts properly on $\mathcal{B}_\Sigma$

In this section we prove that GL_Σ acts properly (Definition 3.8) on $\mathcal{B}_\Sigma$.

The following lemma is a direct consequence of Lemma 3.1 of [3]. For the sake of completeness, we have decided to include the proof here.

Lemma 3.19. *Let $\varepsilon>0$ and $P\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ be such that $2\varepsilon B_2^{d_1,\ldots,d_l}\subseteq P$. If $\delta^H(P,Q)<\varepsilon$ for some $Q\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$, then $\varepsilon B_2^{d_1,\ldots,d_l}\subseteq Q$.*

Proof. Let $\varepsilon>0$ and $P\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ be such that $2\varepsilon B_2^{d_1,\ldots,d_l}\subseteq P$. Assume that there exists $Q\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ with $\delta^H(P,Q)<\varepsilon$ and $\varepsilon B_2^{d_1,\ldots,d_l}\nsubseteq Q$. Let $x_0\in\varepsilon B_2^{d_1,\ldots,d_l}\setminus Q$. Since Q is compact, there exists $z\in Q$ such that

$$dist\,(x_0,Q):=\inf_{x\in Q}\|x_0-x\|_H=\|x_0-z\|_H\,.$$

Let M be the hyperplane through z in $\otimes_{i=1}^{l}\mathbb{R}^{d_i}$ orthogonal to the ray $\overrightarrow{x_0z}$. Since Q is convex, it lies in the halfspace determined by M which does not contain the point x_0. Let a be the intersection point of the ray $\overrightarrow{x_0z}$ with the boundary $2\varepsilon\partial B_2^{d_1,\ldots,d_l}\subseteq P$. Clearly, $\|a\|_H=2\varepsilon$ and

$$\|a-z\|_H=\inf_{x\in M}\|a-x\|_H\le dist\,(a,Q)\le\delta^H(P,Q)<\varepsilon.$$

Since, $\|x_0\|_H\le\varepsilon$, the triangle inequality implies that

$$\varepsilon>\|a-z\|_H>\|a-x_0\|_H\ge\|a\|_H-\|x_0\|_H\ge\varepsilon.$$

This contradiction proves the lemma. □

The following lemma is analogous to Lemma 3.2 of [3].

Lemma 3.20. *Let $\varepsilon>0$ and $P\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ be such that $2\varepsilon B_2^{d_1,\ldots,d_l}\subseteq P$. Then the set,*

$$V_P(\varepsilon):=\left\{Q\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right):\delta^H(P,Q)<\varepsilon\right\}$$

is a relatively compact set in $\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$.

Proof. Assume that $2\varepsilon B_2^{d_1,\ldots,d_l} \subseteq P$ and let $\{Q_k\}$ be a sequence contained in $V_P(\varepsilon)$. Clearly $\{Q_k\}$ is a bounded sequence in $\mathcal{K}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. By the Blaschke selection theorem (Theorem 1.32), there exists a nonempty compact convex set $Q \subseteq \bigotimes_{i=1}^{l} \mathbb{R}^{d_i}$ such that $Q_{k_i} \to Q$ (in the Hausdorff metric) for some subsequence of $\{Q_k\}$. By Lemma 3.19, $\varepsilon B_2^{d_1,\ldots,d_l} \subseteq Q_{k_i}$ for all $i \in \mathbb{N}$. Thus we must have that $Q \in \mathcal{B}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. Since $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is closed in $\mathcal{B}\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ (Proposition 2.25) we have that $Q \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. This completes the proof. □

Theorem 3.21. *Let $l \geq 2$ and $d_i \geq 2$ for $i = 1, \ldots, l$. The action of $GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ on $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is proper.*

Proof. Let $P \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ and $\varepsilon > 0$ such that $2\varepsilon B_2^{d_1,\ldots,d_l} \subseteq P$. We claim that $V_P(\varepsilon)$ is a small neighborhood of P.

Indeed, Let C be a 0-symmetric convex body in $\otimes_{i=1}^l \mathbb{R}^{d_i}$. Then there exists $\lambda > 0$ such that $\lambda B_2^{d_1,\ldots,d_l} \subseteq C$. We will prove that the transporter,

$$\Gamma = \left\{T \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) : TV_P(\varepsilon) \cap V_C(\lambda) \neq \emptyset\right\}$$

has compact closure in $GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$.

Let $\{T_k\}$ be a sequence contained in Γ. Then, for every $k \in \mathbb{N}$ there exists $Q_k \in V_P(\varepsilon)$ such that $T_k(Q_k) \in V_C(\lambda)$. From Lemma 3.19, we have that $\varepsilon B_2^{d_1,\ldots,d_l} \subseteq Q_k$ for all k. Since $\delta^H(T_k(Q_k), C) < \lambda$, we also have that $T_k(Q_k) \subseteq C + \lambda B_2^{d_1,\ldots,d_l}$. Therefore,

$$T_k\left(\varepsilon B_2^{d_1,\ldots,d_l}\right) \subseteq T_k(Q_k) \subseteq C + \lambda B_2^{d_1,\ldots,d_l}.$$

This implies that there exists $r > 0$ with the property that $\|T_k\| \leq r$ for all k (recall that $\|T_k\|$ is the operator norm of $T_k \in \otimes_{H,i=1}^l \mathbb{R}^{d_i}$). Since each T_k is an operator between finite dimensional normed spaces, then $\{T_k\}$ has a convergent subsequence in $\mathcal{L}\left(\otimes_{H,i=1}^l \mathbb{R}^{d_i}\right)$.

Let $T \in \mathcal{L}\left(\otimes_{H,i=1}^l \mathbb{R}^{d_i}\right)$ be such that $T_{k_i} \to T$. By Proposition 3.14 it is enough to prove that T is a linear isomorphism. To do this, observe that $\{Q_{k_i}\} \subseteq V_P(\varepsilon)$ and $\{T_{k_i}(Q_{k_i})\} \subseteq V_C(\lambda)$. Thus from Lemma 3.20, there exists a sub-subsequence $\left\{Q_{k_{i_j}}\right\}$ such that $Q_{k_{i_j}} \to Q$ and $T_{k_i}(Q_{k_i}) \to D$ for some $Q, D \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. Therefore $TQ = D$ and then T is a linear isomorphism. This completes the proof. □

The following corollary is a consequence of the properness of the action of GL_Σ on $\mathcal{B}_\Sigma$, and the characterization of the set of tensorial ellipsoids (Corollary 2.31).

Corollary 3.22. *Let $l \geq 2$ and $d_i \geq 2$ for $i = 1, \ldots, l$. Then $GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)/O_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is homeomorphic to $\mathcal{E}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$.*

3.4.1 A global slice for the action of Gl_Σ on $\mathcal{B}_\Sigma$

In this section we prove that:

$$\mathscr{L}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) := \left\{P \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) : L\ddot{o}w\left(P^1 \otimes_\pi \cdots \otimes_\pi P^l\right) = B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right\},$$

is a compact global slice for $\mathcal{B}_\Sigma$. In Theorem 3.32, we prove that $\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ is homeomorphic to $\mathscr{L}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) \times \mathscr{E}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$.

Let Q be a 0-symmetric convex body. By $L\ddot{o}w(Q)$ we denote the minimal-volume ellipsoid $L\ddot{o}w(Q)$ containing Q (respectively, the maximal-volume ellipsoid $John\,(Q)$ contained in Q) [18, 9]. Below we present the well known characterization of the Löwner ellipsoid of the closed unit ball of a finite dimensional normed space. This characterization will be used to prove Proposition 3.24.

Theorem 3.23. *(Theorem 15.4, [38]) Let $(E, \|\cdot\|)$ be an n-dimensional Banach space. Then there exist an inner product $\langle\cdot,\cdot\rangle$ on E, inducing the Euclidean norm $|||\cdot|||_2$, and vectors $x_1, \ldots, x_N$ in E and positive numbers $c_1, \ldots, c_N$ such that*

1. *$|||x|||_2 \leq \|x\|$ for every $x \in E$.*
2. *$|||x_i|||_2 = \|x_i\| = 1$ for $i = 1, \ldots, N$, $\Sigma_{i=1}^{N} c_i = n$.*
3. *$x = \Sigma_{i=1}^{N} c_i \langle x, x_i\rangle x_i$ for $x \in E$.*
4. *$N \leq n(n+1)/2$ in the real case and $N \leq n^2$ in the complex case.*

Moreover, $\langle\cdot,\cdot\rangle$ is induced by the ellipsoid of minimal-volume containing the unit ball B_E of E.

The following proposition proves that the Löwner ellipsoid of the the projective tensor product $P_1 \otimes_\pi \cdots \otimes_\pi P_l$ of 0-symmetric convex bodies is the Hilbert tensor product of the Löwner ellipsoids $L\ddot{o}w(P_1) \otimes_H \cdots \otimes_H L\ddot{o}w(P_l)$. This result appears in [4].

Proposition 3.24. *(Lemma 1,[4]) Let $P_i \in \mathcal{B}(d_i)$ for $i = 1, \ldots, l$. Then*

$$L\ddot{o}w\left(P_1 \otimes_\pi \cdots \otimes_\pi P_l\right) = L\ddot{o}w\left(P_1\right) \otimes_H \cdots \otimes_H L\ddot{o}w\left(P_l\right).$$

Proof. For $i = 1, ..., l$, let $\langle\cdot,\cdot\rangle_{L\ddot{o}w(P_i)}$ be the inner product on $\mathbb{R}^{d_i}$ determined by $L\ddot{o}w(P_i)$ and let $\langle\cdot,\cdot\rangle_{L\ddot{o}w(P_1)\otimes_H\cdots\otimes_H L\ddot{o}w(P_l)}$ be the one determined by $L\ddot{o}w(P_1) \otimes_H \cdots \otimes_H L\ddot{o}w(P_l)$. Then by Theorem 3.23, there exist $\left\{x_{j_i}^i\right\}_{j_i=1,\ldots,m_i} \subseteq \mathbb{R}^{d_i}$ and positive scalars $c_{j_i}^i$ such that

$$I_{\mathbb{R}^{d_i}} = \sum_{j_i=1}^{m_i} c_{j_i}^i \left\langle\cdot, x_{j_i}^i\right\rangle_{L\ddot{o}w(P_i)} x_{j_i}^i,$$

where $g_{P_i}\left(x_{j_i}^i\right) = \left\langle x_{j_i}^i, x_{j_i}^i\right\rangle_{L\ddot{o}w(P_i)} = 1$ for $j_i = 1, \ldots, m_i$ and $\sum_{j_i=1}^{m_i} c_{j_i}^i = d_i$.

It is not difficult to see that $P_i \subseteq L\ddot{o}w(P_i)$ for $i = 1, \ldots, l$ implies $P_1 \otimes_\pi \cdots \otimes_\pi P_l \subseteq L\ddot{o}w(P_1) \otimes_H \cdots \otimes_H L\ddot{o}w(P_l)$.

Since, $I_{\otimes_{i=1}^{l}\mathbb{R}^{d_i}} = I_{\mathbb{R}^{d_1}} \otimes \cdots \otimes I_{\mathbb{R}^{d_l}}$ we have

$$I_{\otimes_{i=1}^{l}\mathbb{R}^{d_i}} = \sum_{j_1,\dots,j_l} c_{j_1}^1 \cdots c_{j_l}^l \left\langle \cdot, x_{j_1}^1 \right\rangle_{Löw(P_1)} \cdots \left\langle \cdot, x_{j_l}^l \right\rangle_{Löw(P_1)} x_{j_1}^1 \otimes \cdots \otimes x_{j_l}^l,$$

with

$$\begin{aligned} g_{P_1\otimes_\pi\cdots\otimes_\pi P_l}\left(x_{j_1}^1 \otimes \cdots \otimes x_{j_l}^l\right) &= \left\langle x_{j_1}^1 \otimes \cdots \otimes x_{j_l}^l, x_{j_1}^1 \otimes \cdots \otimes x_{j_l}^l \right\rangle_{Löw(P_1)\otimes_H\cdots\otimes_H Löw(P_l)} \\ &= 1, \end{aligned}$$

and

$$\sum_{j_1,\dots,j_l} c_{j_1}^1 \cdots c_{j_l}^l = d_1 \cdots d_l.$$

Therefore from Theorem 3.23, we conclude

$$Löw\left(P_1 \otimes_\pi \cdots \otimes_\pi P_l\right) = Löw\left(P_1\right) \otimes_H \cdots \otimes_H Löw\left(P_l\right).$$

□

From this proposition and the duality between the Löwner ellipsoid and the John ellipsoid (See §15, [38]) we have:

Corollary 3.25. *Let $P_i \in \mathcal{B}(d_i)$ for $i = 1, \dots, l$. Then,*

$$John\left(P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l\right) = John\left(P_1\right) \otimes_H \cdots \otimes_H John\left(P_l\right).$$

Proof. Let $P_i \in \mathcal{B}(d_i)$ for $i = 1, \dots, l$. Then,

$$\begin{aligned} John\left(P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l\right) &= John\left(\left(P^\circ{}_1 \otimes_\pi \cdots \otimes_\pi P^\circ{}_l\right)^\circ\right) \\ &= \left(Löw\left(P^\circ{}_1 \otimes_\pi \cdots \otimes_\pi P^\circ{}_l\right)\right)^\circ \\ &= \left(Löw\left(P_1^\circ\right) \otimes_H \cdots \otimes_H Löw\left(P_l^\circ\right)\right)^\circ \\ &= \left(Löw\left(P_1^\circ\right)\right)^\circ \otimes_H \cdots \otimes_H \left(Löw\left(P_l^\circ\right)\right)^\circ \\ &= John\left(P_1^{\circ\circ}\right) \otimes_H \cdots \otimes_H John\left(P_l^{\circ\circ}\right) \\ &= John\left(P_1\right) \otimes_H \cdots \otimes_H John\left(P_l\right). \end{aligned}$$

□

Let $Löw$ be the **Löwner map** defined in [1] as follows:

$$\begin{aligned} Löw : \mathcal{B}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) &\to \mathscr{E}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) \\ C &\to Löw\left(C\right). \end{aligned}$$

And,

$$\mathscr{L}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) := \left\{C \in \mathcal{B}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) : Löw\left(C\right) = B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right\}.$$

Now we will define the analogue to the Löwner map, *Löw*, and $\mathscr{L}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$, in the context of tensorial 0-symmetric convex bodies. To this end, we define $conv_\Sigma$ as the map:

$$\begin{aligned} conv_\Sigma : \mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) &\to \mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) \\ P &\to conv\left(P \cap \Sigma :_{\mathbb{R}^{d_1},\ldots,\mathbb{R}^{d_l}}\right). \end{aligned}$$

Remark 3.26. Let $P \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$. If P^i for $i = 1, \ldots, l$ are the convex bodies, associated to P, defined in Remark 2.22, then

$$conv_\Sigma(P) = P^1 \otimes_\pi \cdots \otimes_\pi P^l.$$

Using the previous remark and Proposition 3.24, for every $P \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ we have

$$Löw\left(conv\left(P \cap \Sigma\right)\right) \in \mathscr{E}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right).$$

This lets us define l_Σ as the composition, $l_\Sigma := Löw \circ conv_\Sigma$. Thus,

$$\begin{aligned} l_\Sigma : \mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) &\to \mathscr{E}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) \\ P &\to Löw\left(P^1 \otimes_\pi \cdots \otimes_\pi P^l\right). \end{aligned}$$

Finally, we define $\mathscr{L}_\Sigma$ as:

$$\mathscr{L}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) := \left\{P \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right) : l_\Sigma(P) = B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right\}.$$

Proposition 3.27. *The following statements hold:*

1. *The function $conv_\Sigma$ is continuous and GL_Σ*-equivariant.
2. *l_Σ is an GL_Σ-equivariant retraction from $\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ onto $\mathscr{E}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$.*

Proof. The equivariance of $conv_\Sigma$ follows directly from the definition. The continuity of $conv_\Sigma$ is a consequence of Remark 3.26 and the continuity of $\otimes_\pi$ (Proposition 2.12). Indeed, observe that if $\{P_k\}$ is a sequence in $\mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ converging to some $P \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$, then for each $i \in \{1, \ldots, l\}$ we have $P_k^i \to P^i$ (this was proved in Proposition 2.25).

The Löwner map is continuous and $GL\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$-equivariant (Theorem 3.6, [3]). Therefore, l_Σ, which is the composition of $conv_\Sigma$ and *Löw* is continuous and GL_Σ-equivariant.

Now if $\mathcal{E} \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ then Corollary 2.31 implies that there exist $T_i \in GL(d_i)$, $i = 1, .., l$ such that $\mathcal{E} = T_1 \otimes \cdots \otimes T_l\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right)$. Thus,

$$\begin{aligned} l_\Sigma(\mathcal{E}) &= l_\Sigma\left(T_1 \otimes \cdots \otimes T_l\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right)\right) \\ &= (T_1 \otimes \cdots \otimes T_l)\, l_\Sigma\left(\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right)\right) \\ &= T_1 \otimes \cdots \otimes T_l\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right) \\ &= \mathcal{E}. \end{aligned}$$

This proves that l_Σ is an GL_Σ-equivariant retraction to $\mathscr{E}_\Sigma\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$. □

Proposition 3.28. *$\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ satisfies the following properties:*

1. *$\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ is O_Σ-invariant.*
2. *The saturation $GL_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\left(\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\right)$ coincides with $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$.*
3. *Let $T\in GL_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$. If $T\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\cap\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\neq\emptyset$, then*
$$T\in O_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right).$$
4. *$\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ is compact.*

Proof. 1. Let $P\in\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ and $U\in O_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$. Then
$$\begin{aligned} l_\Sigma(UP) &= L\ddot{o}w\left(conv_\Sigma(UP)\right)\\ &= UL\ddot{o}w\left(conv_\Sigma(P)\right)\\ &= U\left(B_2^{d_1}\otimes_H\cdots\otimes_H B_2^{d_l}\right)\\ &= B_2^{d_1}\otimes_H\cdots\otimes_H B_2^{d_l}.\end{aligned}$$

Therefore $UP\in\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$.

2. Let $Q\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$. For every $i\in\{1,...,l\}$ there exists a linear isomorphism $T_i:\mathbb{R}^{d_i}\to\mathbb{R}^{d_i}$ such that $T_i\left(B_2^{d_i}\right)=L\ddot{o}w\left(Q^i\right)$. Set $P=T_1^{-1}\otimes\cdots\otimes T_l^{-1}(Q)$, then
$$\begin{aligned} l_\Sigma(P) &= l_\Sigma\left(T_1^{-1}\otimes\cdots\otimes T_l^{-1}(Q)\right)\\ &= T_1^{-1}\otimes\cdots\otimes T_l^{-1}\left(l_\Sigma(Q)\right)\\ &= T_1^{-1}\otimes\cdots\otimes T_l^{-1}\left(L\ddot{o}w\left(Q^1\right)\otimes_H\cdots\otimes_H L\ddot{o}w\left(Q^l\right)\right)\\ &= T_1^{-1}\otimes\cdots\otimes T_l^{-1}\left(T_1\left(B_2^{d_1}\right)\otimes_H\cdots\otimes_H T_l\left(B_2^{d_l}\right)\right)\\ &= B_2^{d_1}\otimes_H\cdots\otimes_H B_2^{d_l}.\end{aligned}$$

Thus $Q\in GL_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\left(\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\right)$.

3. Assume that $T\in Gl_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$. If there exists $P\in\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ such that $TP\in\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$, then
$$B_2^{d_1}\otimes_H\cdots\otimes_H B_2^{d_l}=l_\Sigma(TP)=T\left(l_\Sigma(P)\right)=T\left(B_2^{d_1}\otimes_H\cdots\otimes_H B_2^{d_l}\right).$$

Therefore, $T\in O\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\cap GL_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)=O_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$.

4. Let $\{P_k\}_{k\in\mathbb{N}}$ be a sequence contained in $\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$. For each k we have:
$$L\ddot{o}w\left(P_k^1\otimes_\pi\cdots\otimes_\pi P_k^l\right)=B_2^{d_1}\otimes_H\cdots\otimes_H B_2^{d_l}.$$

Thus, $P_k^1\otimes_\pi\cdots\otimes_\pi P_k^l\in\mathscr{L}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$. Since $\mathscr{L}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ is a compact set (Theorem 4 and Remark 1, [1]), there exists a subsequence $P_{k_j}^1\otimes_\pi\cdots\otimes_\pi P_{k_j}^l$ converging to some $D\in\mathcal{B}\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ such that $L\ddot{o}w(D)=B_2^{d_1}\otimes_H\cdots\otimes_H B_2^{d_l}$. Hence, from the convergence $P_{k_j}^1\otimes_\pi\cdots\otimes_\pi P_{k_j}^l\to D$ we have $P_{k_j}^i\to D^i$ for $i=1,\ldots,l$.

On the other hand, from Proposition 2.12 we have that $P_{k_j}^1 \otimes_\epsilon \cdots \otimes_\epsilon P_{k_j}^l \to D^1 \otimes_\epsilon \cdots \otimes_\epsilon D^l$. So, there exists $r > 0$ such that

$$P_{k_j} \subseteq P_{k_j}^1 \otimes_\epsilon \cdots \otimes_\epsilon P_{k_j}^l \subseteq r B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}.$$

By the Blaschke selection theorem (Theorem 1.32) we can assume that the sequence P_{k_j} converges to some nonempty compact convex set $P \subseteq \otimes_{i=1}^l \mathbb{R}^{d_i}$. Since

$$P_{k_j}^1 \otimes_\pi \cdots \otimes_\pi P_{k_j}^l \subseteq P_{k_j} \text{ for } j \in \mathbb{N},$$

we have that $D \subseteq P$. Therefore, $P \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ and

$$l_\Sigma(P) = \lim_{j\to\infty} l_\Sigma\left(P_{k_j}\right) = B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}.$$

This completes the proof. □

Theorem 3.29. *$\mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is a compact global O_Σ-slice for the proper GL_Σ-space $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$.*

Proof. We first observe that,

$$\mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) = l_\Sigma^{-1}\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right).$$

The compactness of $\mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ was proved in Proposition 3.28. By Corollary 2.31 $\mathscr{E}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is the GL_Σ-orbit of $B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l} \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. It is clear that $O_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is the stabilizer of $B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}$. From this, we have a homeomorphism between $\mathscr{E}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ and GL_Σ/O_Σ (see Proposition 3.7). This, together with Proposition 3.27, yields a GL_Σ-equivariant map $f : \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) \to GL_\Sigma/O_\Sigma$ such that $\mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) = f^{-1}\left(O_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)\right)$. This proves the theorem. □

Corollary 3.30. *The following statements hold:*

1. *The $Gl_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$-orbit space $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)/GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is compact.*
2. *The orbit spaces $\mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)/O_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ and $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)/GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ are homeomorphic.*

Proof. 1. From Proposition 3.28 we know that $\mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is compact and $GL_\Sigma(\mathscr{L}_\Sigma) = \mathcal{B}_\Sigma$. Since the orbit map

$$\pi : \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) \to \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)/GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right),$$

is continuous, we have that

$$\pi(\mathscr{L}_\Sigma) = \mathcal{B}_\Sigma/GL_\Sigma$$

is compact.

2. Denote by $\pi_|$ the restriction of the orbit map to $\mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. We already know that $\pi_|$ is a continuous surjective map from $\mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ onto $\mathcal{B}_\Sigma/GL_\Sigma$. Notice that from Proposition 3.28, for every $P, Q \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ we have $\pi_|(P) = \pi_|(Q)$ if and only if P and Q have the same O_Σ-orbit. Hence $\pi_|$ induces a continuous bijective map

$$\rho : \mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)/O_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) \to \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)/GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right).$$

Since $\mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)/O_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ is compact (Theorem 3.5), and

$$\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)/GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$$

is Hausdorff (Proposition 3.9), we conclude that ρ is a homeomorphism. □

For a strictly positive operator we mean a linear operator $T : \mathbb{R}^d \to \mathbb{R}^d$ (resp. $T : \otimes_{i=1}^l \mathbb{R}^{d_i} \to \otimes_{i=1}^l \mathbb{R}^{d_i}$) such that $\langle x, Tx\rangle > 0$ (resp. $\langle x, Tx\rangle_H > 0$) for every non-zero $x \in \mathbb{R}^d$.

Lemma 3.31. *Let $d_i \geq 2$ for $i = 1, \ldots, l$, and let*

$$\mathcal{A} := \left\{S_1 \otimes \cdots \otimes S_l : S_i \in Gl\left(\mathbb{R}^{d_i}\right), S_i \text{ is self-adjoint and strictly positive, } i = 1, \ldots, l\right\}.$$

Then, $\mathcal{A}$ is closed in $GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ and

$$\begin{aligned} \Psi : \mathcal{A} \times O_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) &\to GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) \\ (S, U) &\to SU \end{aligned}$$

is a homeomorphism.

Proof. Notice that if $S_i \in GL\left(\mathbb{R}^{d_i}\right)$ is self-adjoint and strictly positive for $i = 1, \ldots, l$, then $S_1 \otimes \cdots \otimes S_l$ is a self-adjoint linear isomorphism. The spectral theorem (Theorem 5, [23]) implies that it is also positive definite.

Let $\{S_{1,n} \otimes \cdots \otimes S_{l,n}\}$ be a sequence in $\mathcal{A}$ converging to some $S \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. It follows directly that S is self-adjoint and strictly positive. From Theorem 2.3, there exist a permutation σ of the set $\{1, \ldots, l\}$ and linear isomorphisms $T_i \in GL(d_i)$ such that,

$$S\left(x^1 \otimes \cdots \otimes x^l\right) = T_1\left(x^{\sigma(1)}\right) \otimes \cdots \otimes T_l\left(x^{\sigma(l)}\right).$$

This is equivalent to

$$S\left(x^1 \otimes \cdots \otimes x^l\right) = \left(T_1 \otimes \cdots \otimes T_l\right) \circ U_\sigma\left(x^1 \otimes \cdots \otimes x^l\right),$$

where U_σ is an orthogonal map defined as $U_\sigma\left(x^1 \otimes \cdots \otimes x^l\right) := x^{\sigma(1)} \otimes \cdots \otimes x^{\sigma(l)}$.

Now, let $T_i = S_i W_i$ be the polar decomposition of each T_i (i.e. S_i is a self-adjoint positive linear isomorphism and W_i is an orthogonal map) then

$$\begin{aligned} S\left(x^1 \otimes \cdots \otimes x^l\right) &= \left(T_1 \otimes \cdots \otimes T_l\right) \circ U_\sigma\left(x^1 \otimes \cdots \otimes x^l\right) \\ &= \left(S_1 W_1 \otimes \cdots \otimes S_l W_l\right) U_\sigma\left(x^1 \otimes \cdots \otimes x^l\right) \\ &= \left(S_1 \otimes \cdots \otimes S_l\right)\left(W_1 \otimes \cdots \otimes W_l\right) U_\sigma\left(x^1 \otimes \cdots \otimes x^l\right). \end{aligned}$$

Hence $S=(S_1\otimes\cdots\otimes S_l)(W_1\otimes\cdots\otimes W_l)U_\sigma$. Since, the polar decomposition (Theorem 60, [21]) of a linear isomorphism is unique we have $(W_1\otimes\cdots\otimes W_l)U_\sigma = Id_{\otimes_{i=1}^l\mathbb{R}^{d_i}}$. Thus

$$S=(S_1\otimes\cdots\otimes S_l)\in\mathcal{A}.$$

We proceed to show that Ψ is a homeomorphism. Observe that, the previous argument proved that if $T\in GL_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ then it can be written as $T=SW$ where $S\in\mathcal{A}$ and $W\in O_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$, therefore Ψ is surjective. The polar decomposition theorem implies that Ψ is injective.

Clearly Ψ is continuous, so we only need to check that if $T_n,T\in GL_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ and $T_n\to T$ then $\Psi^{-1}(T_n)\to\Psi^{-1}(T)$. Let $T_n=\Psi(S_n,U_n)$ and $T=\Psi(S,U)$.

Observe that $O_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ is closed in $O\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$, thefore it is compact. The compactness of $O_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$, and the polar decomposition theorem guarantee that every subsequence $\{U_{n_k}\}$ has a convergent sub-subsquence $U_{n_{k_j}}$ converging to U. Thus the sequence U_n converges to U, and $S_n\to S$. From this we have Ψ is a homeomorphism, which completes the proof. □

Theorem 3.32. *Let $l\geq 2$ and $d_i\geq 2$ for $i=1,\ldots,l$. The following statements hold:*

1. *There exists an $O_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$-equivariant retraction*

$$r:\mathcal{B}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\to\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$$

such that $r(P)$ belongs to the $GL_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$-orbit of P.

2. *$\mathcal{B}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ is homeomorphic to $\mathscr{L}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\times\mathscr{E}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$.*

Proof. 1. Let $f:GL_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\to\mathscr{E}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ be defined by

$$f(T):=T\left(B_2^{d_1}\otimes_H\cdots\otimes_H B_2^{d_l}\right).$$

Then, by Corollary 2.31 and Proposition 3.9, f induces a GL_Σ-equivariant homeomorphism

$$\tilde{f}:GL_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)/O_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\to\mathscr{E}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right),$$

and f is the composition of the following two maps:

$$GL_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\xrightarrow{\pi}Gl_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)/O_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\xrightarrow{\tilde{f}}\mathscr{E}_\Sigma\left(\bigotimes_{i=1}^l\mathbb{R}^{d_i}\right)$$

where π is the natural quotient map. Since $O_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)=O\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)\cap GL_\Sigma$ is compact, π is closed (Theorem 3.5). From this we have f is closed.

This yields that the restriction $f_{|\mathcal{A}}:\mathcal{A}\to\mathscr{E}_\Sigma$ is a homeomorphism. If we let O_Σ act on $\mathcal{A}$ as follows:

$$\begin{aligned}O_\Sigma\times\mathcal{A}&\to\mathcal{A}\\(g,S)&\to gSg^{-1},\end{aligned}$$

and on $\mathscr{E}_\Sigma$ by the action induced from $\mathcal{B}_\Sigma$. Then $f_{|\mathcal{A}}$ is O_Σ-equivariant.

Denote by $\xi : \mathscr{E}_\Sigma \to \mathcal{A}$ the inverse map of $f_{|\mathcal{A}}$. Then we have the following property of ξ :

$$[\xi(\mathcal{E})]^{-1}\mathcal{E} = B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l} \text{ for all } \mathcal{E} \in \mathscr{E}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right). \tag{3.4.1}$$

Now, we define

$$r(P) := [\xi(l_\Sigma(P))]^{-1} P \text{ for every } P \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right).$$

It follows directly that r is continuous.

From Equation (3.4.1) and the equivariance of l_Σ we have:

$$\begin{aligned} l_\Sigma(r(P)) &= l_\Sigma\left([\xi(l_\Sigma(P))]^{-1} P\right) \\ &= [\xi(l_\Sigma(P))]^{-1} l_\Sigma(P) \\ &= B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}, \end{aligned}$$

therefore $r(P) \in \mathscr{L}_\Sigma$. If $P \in \mathscr{L}_\Sigma$, then

$$\begin{aligned} r(P) &= [\xi(l_\Sigma(P))]^{-1} P \\ &= \left[\xi\left(B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}\right)\right]^{-1} P \\ &= I_{\bigotimes_{i=1}^l \mathbb{R}^{d_i}} P \\ &= P. \end{aligned}$$

Thus r is a retraction on $\mathscr{L}_\Sigma$.

Now, we prove that r is O_Σ-equivariant. Let $g \in O_\Sigma$ and $P \in \mathcal{B}_\Sigma$. Then

$$r(gP) = [\xi(l_\Sigma(gP))]^{-1} gP = [\xi(gl_\Sigma(P))]^{-1} gP.$$

From the equivariance of ξ we have $\xi(gl_\Sigma(P)) = g\xi(l_\Sigma(P))g^{-1}$, and hence

$$[\xi(gl_\Sigma(P))]^{-1} = g[\xi(l_\Sigma(P))]^{-1} g^{-1}.$$

Consequently,

$$r(gP) = \left(g[\xi(l_\Sigma(P))]^{-1} g^{-1}\right) gP = g\left([\xi(l_\Sigma(P))]^{-1} P\right) = gr(P),$$

as required. We have proved that r is an O_Σ-retraction. From its definition we know $r(P)$ belongs to the GL_Σ-orbit of P.

2. We define

$$\begin{aligned} \varphi : \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) &\to \mathscr{L}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) \times \mathscr{E}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) \\ P &\to (r(P), l_\Sigma(P)). \end{aligned}$$

Then φ is an O_Σ-equivariant homeomorphism with inverse map given by $\varphi^{-1}(Q,\mathcal{E}) = \xi(\mathcal{E})Q$. □

3.5 The space $\left(\mathcal{BM}_\Sigma, \delta_\Sigma^{BM}\right)$

We denote by $\mathcal{BM}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ the set of equivalence classes of tensorial 0-symmetric convex bodies in $\otimes_{i=1}^l \mathbb{R}^{d_i}$ determined by the following equivalence relation: for every $P, Q \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$, $P \sim Q$ if and only if there exists a bijective linear map $T \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ such that $TP = Q$.

Let P, Q be tensorial 0-symmetric convex bodies in $\otimes_{i=1}^l \mathbb{R}^{d_i}$. We define the Σ-**Banach-Mazur distance** $\delta_\Sigma^{BM}(P,Q)$ as follows:

$$\delta_\Sigma^{BM}(P,Q) := \inf\left\{\lambda \geq 1 : Q \subseteq TP \subseteq \lambda Q \text{ for some } T \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)\right\}.$$

Since, for every $P, Q \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ there exist real numbers $r_1, r_2 > 0$ such that $Q \subseteq r_1 P \subseteq r_2 Q$. We conclude that $\delta_\Sigma^{BM}(P,Q)$ is well defined.

Also notice that for every $P, Q \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ we have:

$$\delta^{BM}(P,Q) \leq \delta_\Sigma^{BM}(P,Q). \tag{3.5.1}$$

Proposition 3.33. *For every $P, Q \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ there exists $T \in GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$, such that*

$$Q \subseteq TP \subseteq \left(\delta_\Sigma^{BM}(P,Q)\right) Q.$$

Proof. Let $\lambda = \delta_\Sigma^{BM}(P,Q)$ and $\{T_n\} \subseteq Gl_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ such that $Q \subseteq T_n P \subseteq \lambda_n Q$ for some sequence $\{\lambda_n\}$ converging to λ.
Considering each T_n as a bounded linear map,

$$T_n : \left(\otimes_{i=1}^l \mathbb{R}^{d_i}, g_P(\cdot)\right) \to \left(\otimes_{i=1}^l \mathbb{R}^{d_i}, g_Q(\cdot)\right),$$

we have that $\|T_n\| \leq \lambda_n$ and $\|T_n^{-1}\| \leq 1$.

Since the sequence $\{\lambda_n\}$ is bounded, then $\{T_n\}$ is bounded. So, there exists a convergent subsequence

$$T_{n_k} \to T \in \mathcal{L}\left(\left(\otimes_{i=1}^l \mathbb{R}^{d_i}, g_P(\cdot)\right); \left(\otimes_{i=1}^l \mathbb{R}^{d_i}, g_Q(\cdot)\right)\right). \tag{3.5.2}$$

From this, we have $TP \subseteq \lambda Q$.

Observe that if $w \in Q$ then for every $k \in \mathbb{N}$, $T_{n_k}^{-1}(w) \in P$. Due to the compactness of P there exists $z \in P$ such that $T_{n_{k_j}}^{-1}(w) \to z$ for some sub-subsequence. This implies,

$$g_Q\left(T_{n_{k_j}}(z) - w\right) = g_Q\left(T_{n_{k_j}}(z) - T_{n_{k_j}} T_{n_k}^{-1}(w)\right) \leq \left\|T_{n_{k_j}}\right\| g_P\left(z - T_{n_k}^{-1}(w)\right).$$

From this we can conclude that $w = T(z)$. Therefore $Q \subseteq TP$ and T is a linear isomorphism.

Recall that $\mathcal{L}\left(\left(\otimes_{i=1}^l \mathbb{R}^{d_i}, g_P\right); \left(\otimes_{i=1}^l \mathbb{R}^{d_i}, g_Q\right)\right)$ is a finite dimensional normed space so the convergence of Equation (3.5.2) is equivalent to the convergence on $\mathcal{L}\left(\otimes_{H,i=1}^l \mathbb{R}^{d_i}\right)$. Thus $T \in Gl_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$. Clearly,

$$\lambda = \left\|T : \left(\otimes_{i=1}^l \mathbb{R}^{d_i}, g_P(\cdot)\right) \to \left(\otimes_{i=1}^l \mathbb{R}^{d_i}, g_Q(\cdot)\right)\right\|$$

which completes the proof. □

Corollary 3.34. *If $\delta_\Sigma^{BM}(P,Q)=1$ for some $P,Q\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$, then there exists $T\in Gl_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ such that $TP=Q$.*

Proposition 3.35. *The function $\log\delta_\Sigma^{BM}$ is a metric on the set $\mathcal{BM}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$.*

Proof. Clearly $\log\delta_\Sigma^{BM}$ is positive. By Corollary 3.34, we have $\log\delta_\Sigma^{BM}(P,Q)=0$ for $P,Q\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ if and only if there exists $T\in Gl_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$ such that $TP=Q$. Thus the Gl_Σ-orbit of P and Q coincide. The proof of the triangle inequality and the symmetry are straightforward. □

Theorem 3.36. *$l\geq 2$. The metric space $\left(\mathcal{BM}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right),\log\delta_\Sigma^{BM}\right)$ is compact.*

Proof. For $P\in B_\Sigma$ we denote by $[P]$ the GL_Σ-orbit of P. Also, let us denote by d the product $d_1\cdot\ldots\cdot d_l$.
We will prove that every sequence $\{[P_n]\}$ in B_Σ/GL_Σ has a convergent subsequence. From Lemma 3.38 and Proposition 2.8 we have,

$$P_n^1\otimes_\pi\cdots\otimes_\pi P_n^l\subseteq P\subseteq\frac{d}{d_l}P_n^1\otimes_\pi\cdots\otimes_\pi P_n^l \text{ for every } n\in\mathbb{N}.$$

On the other hand, it is a well known fact that for every P_n^i $i=1,\ldots,l$ there exists $T_{i,n}\in GL_\Sigma\left(\mathbb{R}^{d_i}\right)$, such that

$$B_1^{d_i}\subseteq T_{i,n}P_n^i\subseteq d_iB_1^{d_i}.$$

This, in combination with Equation (3.3.1) and Example 2.17 implies:

$$B_1^d\subseteq T_{1,n}P_n^1\otimes_\pi\cdots\otimes_\pi T_{l,n}P_n^l\subseteq\left(T_{1,n}\otimes\cdots\otimes T_{l,n}\right)P\subseteq\frac{d}{d_l}T_{1,n}P_n^1\otimes_\pi\cdots\otimes_\pi T_{l,n}P_n^l\subseteq\frac{d^2}{d_l}B_1^d.$$

Now for every $n\in\mathbb{N}$, set $Q_n=\left(T_{1,n}\otimes\cdots\otimes T_{l,n}\right)P$. Then, the above equation can be written as

$$B_1^d\subseteq Q_n\subseteq\frac{d^2}{d_l}B_1^d,\ \text{for every } n\in\mathbb{N}. \tag{3.5.3}$$

Since the sequence $\{Q_n\}$ is bounded, the Blaschke selection theorem (Theorem 1.32) implies the existence of a subsequence $\{Q_{n_k}\}$ converging in the Hausdorff metric to some compact convex set Q. It follows from Equation (3.5.3) that $B_1^d\subseteq Q$. Hence $Q\in B_\Sigma$ (recall that B_Σ is closed by Proposition 2.25).

From Proposition 1.35 we know that $g_{Q_{n_k}}\to g_Q$ uniformly on $B_2^{d_1}\otimes_H\cdots\otimes_H B_2^{d_l}$. From this, it follows that the indentity map:

$$I_k:\left(\otimes_{i=1}^l\mathbb{R}^{d_i},g_{Q_{n_k}}\right)\to\left(\otimes_{i=1}^l\mathbb{R}^{d_i},g_{Q_{n_k}}\right)$$
$$z\to z$$

satisfies $\|I_k\|\left\|I_k^{-1}\right\|\underset{k\to\infty}{\longrightarrow}1$. This implies

$$\delta_\Sigma^{BM}\left(Q_{n_k},Q\right)\underset{k\to\infty}{\longrightarrow}1.$$

Since $\delta_\Sigma^{BM}\left(P_{n_k},Q\right)=\delta_\Sigma^{BM}\left(Q_{n_k},Q\right)$, we have $[P_{n_k}]$ converges to Q, as we required. □

Theorem 3.37. *The space* $\left(\mathcal{BM}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right), \log\delta_\Sigma^{BM}\right)$ *is homeomorphic to the orbit space* $\mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)/GL_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$.

Proof. As we did on the proof of Theorem 3.36, if $P \in \mathcal{B}_\Sigma$ then $[P]$ denotes the GL_Σ-orbit of P.

Define Ψ as follows:

$$\begin{aligned} \Psi : \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) &\to \mathcal{BM}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right) \\ P &\to [P]. \end{aligned}$$

We will prove that Ψ induces a homeomorphism between $\mathcal{B}_\Sigma/GL_\Sigma$ and $\mathcal{BM}_\Sigma$ Clearly Ψ is surjective. From Corollary 3.34 we have,

$$\Psi(P) = \Psi(Q) \text{ if and only if } [P] = [Q]. \tag{3.5.4}$$

We claim that Ψ is continuous. To see this, take a sequence $\{P_n\}$ in $\mathcal{B}_\Sigma$ converging to some $P \in \mathcal{B}_\Sigma$. From Proposition 1.35 we have that g_{P_n} converges uniformly on $B_2^{d_1} \otimes_H \cdots \otimes_H B_2^{d_l}$ to g_P. As we saw in the proof of Theorem 3.36, this implies that $\delta_\Sigma^{BM}(P_n, P)$ converges to 1. Therefore $[P_n]$ converges to $[P]$ on $\mathcal{BM}_\Sigma$. This proves the continuity of Ψ.

Since Ψ is continuous and surjective, Equation (3.5.4) implies that Ψ induces a continuous bijective map $\tilde{\Psi} : \mathcal{B}_\Sigma/Gl_\Sigma \to \mathcal{BM}_\Sigma$, such that Ψ is the composition of the orbit map $\pi : \mathcal{B}_\Sigma \to \mathcal{B}_\Sigma/Gl_\Sigma$ and $\tilde{\Psi}$. Finally, due to the compactness of $\mathcal{B}_\Sigma/Gl_\Sigma$ and the fact that $\mathcal{BM}_\Sigma$ is Hausdorff, we conclude that $\tilde{\Psi}$ is a homeomorphism. □

Now we calculate upper bounds for the Σ-Banach-Mazur distance between tensorial 0-symmetric convex bodies. The bounds that we present below are consequences of the following results of [13] and Corollary 4.15.

Lemma 3.38. *(Proposition 2.4, [13]) Let* $F_1, \ldots, F_l$ *be normed spaces of finite dimension* $d_1, \ldots, d_l$ *respectively. Then for every* $z \in F_1 \otimes \cdots \otimes F_l$,

$$\pi(z) \leq d_1 \cdots d_{l-1} \epsilon(z).$$

Corollary 3.39. *(Corollary 2.5, [13]) Let* $P_i, \in \mathcal{B}(d_i)$ *for* $i = 1, \ldots, l$. *Then,*

$$\delta_\Sigma^{BM}\left(P^1 \otimes_\pi \cdots \otimes_\pi P^l, P^1 \otimes_\epsilon \cdots \otimes_\epsilon P^l\right) \leq d_1 \cdots d_{l-1}.$$

Proposition 3.40. *For every* $P, Q \in \mathcal{B}_\Sigma\left(\otimes_{i=1}^l \mathbb{R}^{d_i}\right)$ *we have,*

$$\delta_\Sigma^{BM}(P, Q) \leq (d_1 \cdots d_{l-1})^2 \left(\prod_{i=1}^l \delta_\Sigma^{BM}\left(P^i, Q^i\right)\right).$$

Proof. By Corollary 4.15 and Lemma 3.38 we have,

$$\begin{aligned}
&\delta_\Sigma^{BM}(P,Q)\\
&\le \delta_\Sigma^{BM}\left(P, P^1\otimes_\pi\cdots\otimes_\pi P^l\right)\delta_\Sigma^{BM}\left(P^1\otimes_\pi\cdots\otimes_\pi P^l, Q^1\otimes_\pi\cdots\otimes_\pi Q^l\right)\delta_\Sigma^{BM}\left(Q^1\otimes_\pi\cdots\otimes_\pi Q^l, Q\right)\\
&\le \delta_\Sigma^{BM}\left(P, P^1\otimes_\pi\cdots\otimes_\pi P^l\right)\left(\prod_{i=1}^{l}\delta_\Sigma^{BM}\left(P^i,Q^i\right)\right)\delta_\Sigma^{BM}\left(Q^1\otimes_\pi\cdots\otimes_\pi Q^l, Q\right)\\
&\le (d_1\cdots d_{l-1})^2\left(\prod_{i=1}^{l}\delta_\Sigma^{BM}\left(P^i,Q^i\right)\right).
\end{aligned}$$

This completes the proof. □

The following upper bound for the diameter of $\mathcal{BM}_\Sigma$ is a direct consequence of the previous proposition and John's result (see [38],Chapter 9):

$$\sup\left\{\delta^{BM}(P,Q) : P,Q\in\mathcal{B}(d)\right\}\le d.$$

Corollary 3.41. *Let $P,Q\in\mathcal{B}_\Sigma\left(\otimes_{i=1}^l\mathbb{R}^{d_i}\right)$. Then,*

$$\delta_\Sigma^{BM}(P,Q)\le (d_1\cdots d_{l-1})^2(d_1\cdots d_l).$$

Chapter 4

Tensor products of centrally symmetric convex bodies

The aim of this chapter is to introduce the definition of tensor products of 0-symmetric convex bodies in Euclidean spaces and to present its basic properties. In Section 4.1, we define the category of centrally symmetric convex bodies in Euclidean spaces. In Section 4.2, we define the injective and the projective tensor product of 0-symmetric convex bodies in Euclidean spaces. We prove that the projective tensor product is defined by a universal property, Theorem 4.10. In Section 4.3, we define tensor products of 0-symmetic convex bodies. We define the concept of projective and injective tensor product of 0-symmetric convex bodies. In Section 4.4, we introduce the dual of a tensor product of 0-symmetric convex bodies. Finally, in Section 4.5 we prove that there exists a bijection between tensor products of 0-symmetric convex bodies and tensor norms on finite dimensional normed spaces.

4.1 The category of centrally symmetric convex bodies

In this section we introduce the category of centrally symmetric convex bodies. In Section 4.3, we prove that every tensor product of 0-symmetric convex bodies (definition 4.11) determines a functor in this category.

Let $\mathcal{SCB}$ denote the category of 0-symmetric convex bodies in real Euclidean spaces. Its objects are pairs $(P, \mathbb{E})$ where $\mathbb{E}$ is a Euclidean space, and P is a 0-symmetric convex body in $\mathbb{E}$ (i.e. $P \in \mathcal{B}(\mathbb{E})$). The morphisms between objects $(P_1, \mathbb{E}_1)$ and $(P_2, \mathbb{E}_2)$ denoted by $Hom\left((P_1, \mathbb{E}_1), (P_2, \mathbb{E}_2)\right)$, are linear transformations $T : \mathbb{E}_1 \to \mathbb{E}_2$ such that $T(P_1) \subseteq P_2$.

Proposition 4.1. *$\mathcal{SCB}$ is a category.*

Proof. The proof is straightforward. Notice that for every object $(P, \mathbb{E})$ the identity map $I_{\mathbb{E}}$ always belongs to $Hom\left((P, \mathbb{E}), (P, \mathbb{E})\right).$ Also, if $T \in Hom\left((P_1, \mathbb{E}_1), (P_2, \mathbb{E}_2)\right)$

and $S \in Hom\left(\left(P_2, \mathbb{E}_2\right), \left(P_3, \mathbb{E}_3\right)\right)$ then,

$$ST(P_1) \subseteq S\left(P_2\right) \subseteq P_3.$$

Thus, the composition $S \circ T \in Hom\left(\left(P_1, \mathbb{E}_1\right), \left(P_3, \mathbb{E}_3\right)\right).$ □

Remark 4.2. Notice that for circled convex bodies (see Section 1.4.1). It is possible to define an analogue category.

4.2 The injective and the projective tensor product of 0-symmetric convex bodies

In what follows, the letters $\mathbb{E}$ and $\mathbb{F}$ denote real or complex Euclidean spaces. The letter l will denote a positive natural number. On the tensor product $\otimes_{i=1}^{l}\mathbb{E}_i$ of finite dimensional Euclidean spaces, we use the inner product given by the Hilbert tensor product $\otimes_{H,i=1}^{l}\mathbb{E}_i$. In this way the polarity will be determined by the inner product $\langle\cdot,\cdot\rangle_H$.

Definition 4.3. Let $\mathbb{E}_1, ..., \mathbb{E}_l$ be real (resp. complex) Euclidean spaces. If P_i is a 0-symmetric (resp. circled) convex body contained in $\mathbb{E}_i$ for $i = 1, ..., l$, then we define *the projective tensor product* as:

$$P_1 \otimes_\pi \cdots \otimes_\pi P_l := conv\left(\Sigma_{P_1,...,P_l}\right).$$

In the same way, we define *the injective tensor product* as:

$$P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l := \left(\Sigma_{P_1^\circ,...,P_l^\circ}\right)^\circ.$$

Remark 4.4. We would like to notice that the previous definition of the injective and the projective tensor product of 0-symmetric convex bodies coincides with the definition of G. Aubrun and S. Szarek [5].

From now on, given a Euclidean space $\mathbb{E}$ and a 0-symmetric (resp. circled) convex body $P \subseteq \mathbb{E}$ by $(\mathbb{E}, g_P)$ we denote the vector space $\mathbb{E}$ with the norm determined by the Minkowski functional $g_P(\cdot)$. That is $(\mathbb{E}, g_P)$ is a normed space with closed unit ball P.

Theorem 4.5. *Let $P_i \subseteq \mathbb{E}_i$ be a 0-symmetric (resp. circled) convex body and let g_{P_i} be the Minkowski functional of P_i for $i = 1, ..., l$. Then,*

1. $P_1 \otimes_\pi \cdots \otimes_\pi P_l$ is the closed unit ball of the projective tensor product $\otimes_{\pi,i=1}^{l} E_i$ of the normed spaces $E_i := \left(\mathbb{E}_i, g_{P_i}\right)$.

2. $P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l$ is the closed unit ball of the injective tensor product $\otimes_{\epsilon,i=1}^{l} E_i$ of the normed spaces $E_i := \left(\mathbb{E}_i, g_{P_i}\right)$.

Proof. From the definition of the projective norm we have $P_1 \otimes_\pi \cdots \otimes_\pi P_l \subseteq B_{\otimes^l_{\pi,i=1} E_i}$. To prove the reverse inclusion, suppose that z is in the open unit ball of $\otimes^l_{\pi,i=1} E_i$. Then $z = \sum_{j=1}^N x^1_j \otimes \cdots \otimes x^l_j$ where each x^i_j is non-zero and $\sum_{j=1}^N g_1\left(x^1_j\right) \cdots g_l\left(x^l_j\right) < 1$. Let $y^i_j = \frac{x^i_j}{g_i\left(x^i_j\right)}$ and $\lambda_j = g_1\left(x^1_j\right) \cdots g_l\left(x^l_j\right)$. Then $z = \sum_{j=1}^N \lambda_j y^1_j \otimes \cdots \otimes y^l_j \in P_1 \otimes_\pi \cdots \otimes_\pi P_l$. It follows that the closed unit ball of $\otimes^l_{\pi,i=1} E_i$ is contained in $\overline{P_1 \otimes_\pi \cdots \otimes_\pi P_l}$. Since $\Sigma_{P_1,\ldots,P_l}$ is compact then $P_1 \otimes_\pi \cdots \otimes_\pi P_l$ is closed, therefore $P_1 \otimes_\pi \cdots \otimes_\pi P_l = B_{\otimes^l_{\pi,i=1} E_i}$. For the second assertion. Observe $z \in P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l$ if and only if $\left|\left\langle z, x^1 \otimes \cdots \otimes x^l \right\rangle_H\right| \leq 1$ for every $x^1 \otimes \cdots \otimes x^l \in \Sigma_{P^\circ_1,\ldots,P^\circ_l}$. This is equivalent to,

$$\sup\left\{\left|\left\langle z, x^1 \otimes \cdots \otimes x^l \right\rangle_H\right| x^i \in P^\circ_i \text{ for } i = 1, \ldots, l\right\} \leq 1.$$

By definition of the injective norm we have $z \in B_{\otimes^l_{\epsilon,i=1} E_i}$. This completes the proof. □

Remark 4.6. For every tuple $(P_i, \mathbb{E}_i)$ $i = 1, \ldots, l$ of objects in $\mathcal{SCB}$, we have

$$\left(P_1 \otimes_\pi \cdots \otimes_\pi P_l, \otimes^l_{H,i=1} \mathbb{E}_i\right)$$

and $\left(P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l, \otimes^l_{H,i=1} \mathbb{E}_i\right)$ are objects in $\mathcal{SCB}$.

Proposition 4.7. *(Uniform property) Let $T_i : \mathbb{E}_i \to \mathbb{F}_i$ be a linear map for $i = 1, \ldots, l$. If $T_i(P_i) \subseteq Q_i$ and $P_i \subseteq \mathbb{E}_i$, $Q_i \subseteq \mathbb{F}_i$ are 0-symmetric (resp. circled) convex bodies for $i = 1, \ldots, l$, then*

$$T_1 \otimes \cdots \otimes T_l\left(P_1 \otimes_\pi \cdots \otimes_\pi P_l\right) \subseteq Q_1 \otimes_\pi \cdots \otimes_\pi Q_l,$$

and $T_1 \otimes \cdots \otimes T_l\left(P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l\right) \subseteq Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l$.

Proof. To prove the first inclusion, let us observe that

$$\left(T_1 \otimes \cdots \otimes T_l\right)\left(\Sigma_{P_l,\ldots,P_l}\right) = \Sigma_{T_l(P_l),\ldots,T_l(P_l)} \subseteq \Sigma_{Q_l,\ldots,Q_l}.$$

The linearity of $T_1 \otimes \cdots \otimes T_l$ implies,

$$conv\left(\left(T_1 \otimes \cdots \otimes T_l\right)\left(\Sigma_{P_l,\ldots,P_l}\right)\right) = T_1 \otimes \cdots \otimes T_l\left(conv\left(\Sigma_{P_l,\ldots,P_l}\right)\right).$$

Hence,

$$T_1 \otimes \cdots \otimes T_l\left(P_1 \otimes_\pi \cdots \otimes_\pi P_l\right) \subseteq Q_1 \otimes_\pi \cdots \otimes_\pi Q_l.$$

We will prove the second inclusion. Recall that $T_i(P_i) \subseteq Q_i$ for $i = 1, \ldots, l$ implies $T^t_i\left(Q^\circ_i\right) \subseteq P^\circ_i$ for $i = 1, .., l$. Thus, for every $z \in P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l$ and $y^i \in Q^\circ_i$ we have

$$\sup\left\{\left|\left\langle z, T^t_1 y^1 \otimes \cdots \otimes T^t_l y^l \right\rangle_H\right| : y^i \in Q^\circ_i \text{ for } i = 1, \ldots, l\right\} \leq 1,$$

or equivalently

$$\sup\left\{\left|\left\langle \left(T_1 \otimes \cdots \otimes T_l\right) z, y^1 \otimes \cdots \otimes y^l \right\rangle_H\right| : y^i \in Q^\circ_i \text{ for } i = 1, \ldots, l\right\} \leq 1.$$

So, $\left(T_1 \otimes \cdots \otimes T_l\right) z \in Q_1 \otimes_\epsilon \cdots \otimes_\epsilon Q_l$. This completes the proof □

Remark 4.8. In Section 4.3 we prove that $\otimes_\pi$ and $\otimes_\epsilon$ define functors in $\mathcal{SCB}$.

Assume that P_i is a 0-symmetric (resp. circled) convex body contained in a Euclidean space $\mathbb{E}_i$ for $i = 1, ..., l$. We say that a 0-symmetric (resp. circled) convex body P contained in a Euclidean space $\mathbb{E}$ and a multilinear map $\varphi : \mathbb{E}_1 \times \cdots \times \mathbb{E}_l \to \mathbb{E}$ such that $\varphi(P_1, ..., P_l) \subseteq P$ have the property (*) if:

(*) For every Euclidean space $\mathbb{F}$, every 0-symmetric (resp. circled) convex body $Q \subseteq \mathbb{F}$ and every multilinear function $T : \mathbb{E}_1 \times \cdots \times \mathbb{E}_l \to \mathbb{F}$. If $T(P_1, ..., P_l) \subseteq Q$, then there exists a unique linear function $T_\varphi : \mathbb{E} \to \mathbb{F}$ such that $T_\varphi(P) \subseteq Q$ and $T_\varphi \circ \varphi = T$.

Proposition 4.9. *Suppose that P_i is a 0-symmetric (resp. circled) convex body contained in the Euclidean space $\mathbb{E}_i$ for $i = 1, ..., l$. Let P, P' be 0-symmetric (resp. circled) convex bodies contained in Euclidean spaces $\mathbb{E}, \mathbb{E}'$, and let $\varphi : \mathbb{E}_1 \times \cdots \times \mathbb{E}_l \to \mathbb{E}$ and $\varphi' : \mathbb{E}_1 \times \cdots \times \mathbb{E}_l \to \mathbb{E}'$ be multilinear maps such that $\varphi(P_1, ..., P_l) \subseteq P$, $\varphi'(P_1, ..., P_l) \subseteq P'$. If the pairs (P, φ) and (P', φ') have the property (*) then there exists a linear isomorphism $\Psi : \mathbb{E} \to \mathbb{E}'$ such that $\Psi(P) = P'$ and $\Psi\varphi = \varphi'$.*

Proof. Assume that the pairs (P, φ) and (P', φ') have the property (*).
Since $\varphi(P_1, ..., P_l) \subseteq P$ and $\varphi'(P_1, ..., P_l) \subseteq P'$, then there exist linear maps $S : \mathbb{E} \to \mathbb{E}'$ and $T : \mathbb{E}' \to \mathbb{E}$ such that $S \circ \varphi = \varphi'$, $T \circ \varphi' = \varphi$, $S(P) \subseteq P'$ and $T(P') \subseteq P$.

Clearly, the linear map $TS : \mathbb{E} \to \mathbb{E}$ verifies:

$$TS\varphi\left(x^1, ..., x^l\right) = \varphi\left(x^1, ..., x^l\right) \text{ for every } \left(x^1, ..., x^l\right) \in \mathbb{E}_1 \times \cdots \times \mathbb{E}_l.$$

Since (P, φ) has the property (*). Then TS must be equal to the identity map $I_{\mathbb{E}}$ of $\mathbb{E}$ (this is a consequence of the uniqueness of the linear extension). Therefore,

$$P' = ST(P') \subseteq S(P) \subseteq P'.$$

Thus, $S(P) = P'$ and $S \circ \varphi = \varphi'$. This completes the proof. □

Theorem 4.10. *Let P_i be a 0-symmetric (resp. circled) convex body contained in the Euclidean space $\mathbb{E}_i$ for $i = 1, ..., l$. Then $P_1 \otimes_\pi \cdots \otimes_\pi P_l$ has the property (*).*

Proof. From the definition of $\otimes_\pi$ we know that $\otimes(P_1, ..., P_l) \subseteq P_1 \otimes_\pi \cdots \otimes_\pi P_l$. The universal property of the tensor product of vector spaces implies the existence of a unique linear function $\hat{T} : \otimes_{i=1}^l \mathbb{E}_i \to \mathbb{E}$ such that $\hat{T} \circ \otimes = T$. Therefore, we only need to check that $\hat{T}(P_1 \otimes_\pi \cdots \otimes_\pi P_l) \subseteq Q$. However, this is a direct consequence of $\otimes(P_1, ..., P_l) = \Sigma_{P_1, ..., P_l}$, the linearity of $\hat{T}$ and the property $\hat{T} \circ \otimes = T$. □

4.3 Tensor products of 0-symmetric convex bodies

In this section we introduce the definition of tensor products of 0-symmetric convex bodies, and we begin the study of its basic properties. We introduce the concepts of injective and projective tensor products of 0-symmetric convex bodies. They are the analogue to the concepts of injective and projective tensor norms (see Chapter 1). In Proposition 4.18 we prove that $\otimes_\epsilon$ is an injective tensor product of 0-symmetric convex bodies. In Proposition 4.19 we prove that $\otimes_\pi$ is a projective tensor product of 0-symmetric convex bodies.

Definition 4.11. A ***tensor product*** $\otimes_\alpha$ ***of*** 0***-symmetric convex bodies of order*** l assigns to each tuple $P_i \subseteq \mathbb{E}_i, i = 1, ..., l$ of 0-symmetric (resp. circled) convex bodies in real (resp. complex) Euclidean spaces $\mathbb{E}_i$, a 0-symmetric (resp. circled) convex body $P_1 \otimes_\alpha \cdots \otimes_\alpha P_l$ in $\otimes^l_{H,i=1}\mathbb{E}_i$ such that the following conditions are satisfied:

1. $P_1 \otimes_\pi \cdots \otimes_\pi P_l \subseteq P_1 \otimes_\alpha \cdots \otimes_\alpha P_l \subseteq P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l$.
2. (*Uniform property*) For every linear map $T_i : \mathbb{E}_i \to \mathbb{F}_i$ for $i = 1, .., l$. If $T_i(P_i) \subseteq Q_i$ and $P_i \subseteq \mathbb{E}_i$, $Q_i \subseteq \mathbb{F}_i$ are 0-symmetric (resp. circled) convex bodies for $i = 1, ..., l$. Then,

$$T_1 \otimes \cdots \otimes T_l (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l) \subseteq Q_1 \otimes_\alpha \cdots \otimes_\alpha Q_l.$$

For abbreviation, we write $\otimes_\alpha$ is a tensor product of 0-symmetric convex bodies instead of $\otimes_\alpha$ is a tensor product of 0-symmetric convex bodies of order l.
Let $\otimes_\alpha$ be a tensor product of 0-symmetric convex bodies. We define $F_{\otimes_\alpha} : \mathcal{SCB} \times \cdots \times \mathcal{SCB} \to \mathcal{SCB}$ as follows:
For every tuple of objects $(P_i, \mathbb{E}_i)$ $i = 1, ..., l$,

$$F_{\otimes_\alpha}((P_1, \mathbb{E}_1), \dots, (P_l, \mathbb{E}_l)) = \left(P_1 \otimes_\alpha \cdots \otimes_\alpha P_l, \otimes^l_{H,i=1}\mathbb{E}_i\right).$$

For $T_i \in Hom((P_i, \mathbb{E}_i), (Q_i, \mathbb{F}_i))$ with $i = 1, ..., l$,

$$F_{\otimes_\alpha}(T_1, ..., T_l) = T_1 \otimes \cdots \otimes T_l.$$

Proposition 4.12. *Let $\otimes_\alpha$ be a tensor product of* 0*-symmetric convex bodies, then $F_{\otimes_\alpha}$ is a functor.*

Proof. The proof is a direct consequence of the uniform property, and the fact that for linear maps $T_i : \mathbb{E}_i \to \mathbb{F}_i$ and $S : \mathbb{E}_i \to \mathbb{F}_i$ with $i = 1, ..., l$ we have,

$$S_1 \otimes \cdots \otimes S_l \circ T_1 \otimes \cdots \otimes T_l = S_1T_1 \otimes \cdots \otimes S_lT_l.$$

□

4.3.1 Properties of tensor products of 0-symmetric convex bodies

The following propositions of tensor products of 0-symmetric (resp. circled) convex bodies are consequences of the uniform property.

Proposition 4.13. *Let $P_i \subseteq \mathbb{E}_i$ be a* 0*-symmetric (resp. circled) convex body and let $\mathbb{F}_i$ be a Euclidean space for $i = 1, ..., l$. Then for each bijective linear map $T_i : \mathbb{E}_i \to \mathbb{F}_i$ with $i = 1, ..., l$ we have:*

$$T_1 \otimes \cdots \otimes T_l (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l) = T_1P \otimes_\alpha \cdots \otimes_\alpha T_lP.$$

Proof. The proof is straightforward. Since $T_i : \mathbb{E}_i \to \mathbb{F}_i$ is bijective, we know $Q_i = T_i(P_i) \in B(\mathbb{F}_i)$ and $T_i^{-1}(Q_i) \in \mathcal{B}(\mathbb{E}_i)$ for $i = 1, ..., l$. From this and the uniform property of $\otimes_\alpha$ we have

$$T_1 \otimes \cdots \otimes T_l (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l) \subseteq T_1 P \otimes_\alpha \cdots \otimes_\alpha T_l P,$$

and

$$T_1^{-1} \otimes \cdots \otimes T_l^{-1} (Q_1 \otimes_\alpha \cdots \otimes_\alpha Q_l) \subseteq T_1^{-1} Q_1 \otimes_\alpha \cdots \otimes_\alpha T_l^{-1} Q_l.$$

Therefore, $T_1 \otimes \cdots \otimes T_l (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l) = T_1 P \otimes_\alpha \cdots \otimes_\alpha T_l P$. □

Proposition 4.14. *Let $P_i, Q_i \in \mathcal{B}(\mathbb{E}_i)$ for $i = 1, ..., l$. If $\otimes_\alpha$ is a tensor product of 0-symmetric convex bodies then,*

$$\delta^{BM}(P_1 \otimes_\alpha \cdots \otimes_\alpha P_l, Q_1 \otimes_\alpha \cdots \otimes_\alpha Q_l) \le \delta^{BM}(P_1, Q_1) \cdots \delta^{BM}(P_l, Q_l).$$

Proof. First, for $i \in \{1, ..., l\}$ we prove the following inequality:

$$\delta^{BM}(P_1 \otimes_\alpha \cdots \otimes_\alpha P_i \otimes_\alpha \cdots \otimes_\alpha P_l, P_1 \otimes_\alpha \cdots \otimes_\alpha Q_i \otimes_\alpha \cdots \otimes_\alpha P_l) \le \delta^{BM}(P_i, Q_i). \tag{4.3.1}$$

Let $T_i : \mathbb{E}_i \to \mathbb{E}_i$ be a linear isomorphism and $\lambda \ge 1$ such that $Q_i \subseteq T_i(P_i) \subseteq \lambda Q_i$. From the uniform property and Proposition 4.13 we have,

$$\begin{aligned} P_1 \otimes_\alpha \cdots \otimes_\alpha Q_i \otimes_\alpha \cdots \otimes_\alpha P_l &\subseteq I_{\mathbb{E}_1} \otimes \cdots \otimes T_i \otimes \cdots \otimes I_{\mathbb{E}_l} (P_1 \otimes_\alpha \cdots \otimes_\alpha P_i \otimes_\alpha \cdots \otimes_\alpha P_l) \\ &\subseteq P_1 \otimes_\alpha \cdots \otimes_\alpha \lambda Q_i \otimes_\alpha \cdots \otimes_\alpha P_l \\ &= I_{\mathbb{E}_1} \otimes \cdots \otimes \lambda I_{\mathbb{E}_i} \otimes \cdots \otimes I_{\mathbb{E}_l} (P_1 \otimes_\alpha \cdots \otimes_\alpha Q_i \otimes_\alpha \cdots \otimes_\alpha P_l) \\ &= \lambda P_1 \otimes_\alpha \cdots \otimes_\alpha Q_i \otimes_\alpha \cdots \otimes_\alpha P_l. \end{aligned}$$

Since $\lambda \ge \delta^{BM}(P_i, Q_i)$ we have proved the inequality 4.3.1.

To prove the result, observe that the multiplicative triangle inequalty of δ^{BM} implies:

$$\begin{gathered} \delta^{BM}(P_1 \otimes_\alpha \cdots \otimes_\alpha P_l, Q_1 \otimes_\alpha \cdots \otimes_\alpha Q_l) \le \\ \delta^{BM}(P_1 \otimes_\alpha P_2 \otimes_\alpha \cdots \otimes_\alpha P_l, Q_1 \otimes_\alpha P_2 \otimes_\alpha \cdots \otimes_\alpha P_l) \\ \cdot \delta^{BM}(Q_1 \otimes_\alpha P_2 \otimes_\alpha \cdots \otimes_\alpha P_l, Q_1 \otimes_\alpha Q_2 \otimes_\alpha P_3 \otimes_\alpha \cdots \otimes_\alpha P_l) \\ \cdot \delta^{BM}(Q_1 \otimes_\alpha Q_2 \otimes_\alpha P_3 \otimes_\alpha \cdots \otimes_\alpha P_l, Q_1 \otimes_\alpha Q_2 \otimes_\alpha Q_3 \otimes_\alpha P_4 \cdots \otimes_\alpha P_l) \cdots \\ \cdots \delta^{BM}(Q_1 \otimes_\alpha Q_2 \otimes_\alpha \cdots \otimes_\alpha Q_{l-1} \otimes_\alpha P_l, Q_1 \otimes_\alpha Q_2 \otimes_\alpha \cdots \otimes_\alpha Q_l). \end{gathered}$$

From Inequality 4.3.1 we have

$$\delta^{BM}(P_1 \otimes_\alpha \cdots \otimes_\alpha P_l, Q_1 \otimes_\alpha \cdots \otimes_\alpha Q_l) \le \delta^{BM}(P_1, Q_1) \cdots \delta^{BM}(P_l, Q_l).$$

□

Corollary 4.15. *Let $P_i, Q_i \in \mathcal{B}(d_i)$ for $i = 1, ..., l$. If $\otimes_\alpha$ is a tensor product 0-symmetric convex bodies, then*

$$\delta_\Sigma^{BM}(P_1 \otimes_\alpha \cdots \otimes_\alpha P_l, Q_1 \otimes_\alpha \cdots \otimes_\alpha Q_l) \leq \delta^{BM}(P_1, Q_1) \cdots \delta^{BM}(P_l, Q_l) \leq d_1 \cdots d_l.$$

Proof. In Proposition 4.15 we proved that for each $i \in \{1, ..., l\}$,

$$P_1 \otimes_\alpha \cdots \otimes_\alpha Q_i \otimes_\alpha \cdots \otimes_\alpha P_l \subseteq I_{\mathbb{E}_1} \otimes \cdots \otimes T_i \otimes \cdots \otimes I_{\mathbb{E}_l}(P_1 \otimes_\alpha \cdots \otimes_\alpha P_i \otimes_\alpha \cdots \otimes_\alpha P_l) \subseteq P_1 \otimes_\alpha \cdots \otimes_\alpha Q_i \otimes_\alpha \cdots \otimes_\alpha P_l.$$

Thus,

$$\delta_\Sigma^{BM}(P_1 \otimes_\alpha \cdots \otimes_\alpha P_i \otimes_\alpha \cdots \otimes_\alpha P_l, P_1 \otimes_\alpha \cdots \otimes_\alpha Q_i \otimes_\alpha \cdots \otimes_\alpha P_l) \leq \delta^{BM}(P_i, Q_i).$$

Hence from the multiplicative triangle inequality of δ_Σ^{BM}, and John's theorem $\delta^{BM}(P_i, Q_i) \leq d_i$ (See [18, 38]). We obtain the desired result. □

Definition 4.16. We say that a tensor product $\otimes_\alpha$ of 0-symmetric (resp. circled) convex bodies is ***injective***, if for each 0-symmetric (resp. circled) convex body $P_i \subseteq \mathbb{E}_i$ and subspaces $M_i \subseteq \mathbb{E}_i$ for $i = 1, ..., l$ we have that:

$$(P_1 \cap M_1) \otimes_\alpha \cdots \otimes_\alpha (P_l \cap M_l) = (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l) \cap M_1 \otimes \cdots \otimes M_l.$$

Here, the inner product $\langle \cdot, \cdot \rangle_{M_i}$ on the space M_i is the restricition of the inner product $\langle \cdot, \cdot \rangle_{\mathbb{E}_i}$ to M_i. In this way each $P_i \cap M_i$ is a 0-symmetric convex body in M_i.

Definition 4.17. We say that a tensor product $\otimes_\alpha$ of 0-symmetric convex bodies is ***projective***, if for each 0-symmetric (resp. circled) convex body $P_i \subseteq \mathbb{E}_i$ and every surjective linear map $T_i : \mathbb{E}_i \to \mathbb{F}_i$ with $i = 1, ..., l$ we have that:

$$T_1 \otimes \cdots \otimes T_l (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l) = T_1 P \otimes_\alpha \cdots \otimes_\alpha T_l P.$$

Proposition 4.18. *The injective tensor product $\otimes_\epsilon$ is an injective tensor product of 0-symmetric (resp. circled) convex bodies.*

Proof. Let M_i be a subspaces of $\mathbb{E}_i$ $i = 1, ...l$. From the uniform property of $\otimes_\epsilon$ we have

$$(P_1 \cap M_1) \otimes_\epsilon \cdots \otimes_\epsilon (P_l \cap M_l) \subseteq (P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l)$$
$$(P_1 \cap M_1) \otimes_\epsilon \cdots \otimes_\epsilon (P_l \cap M_l) \subseteq (P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l) \cap M_1 \otimes \cdots \otimes M_l.$$

If $z \in (P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l) \cap M_1 \otimes \cdots \otimes M_l$, then $z = \sum_{j=1}^N z_j^1 \otimes \cdots z_j^l$ with $z_j^i \in M_i$, and

$$\sup \left\{ \left| \left\langle z, x^1 \otimes \cdots \otimes x^l \right\rangle_H \right| x^i \in P_i^\circ \text{ for } i = 1, ..., l \right\} \leq 1.$$

Take $y^i \in (P_i \cap M_i)^\circ \subseteq M_i$. From the Hanh-Banach theorem we know there exists $x^i \in P_i^\circ$ such that $\langle m^i, x^i \rangle_{\mathbb{E}_i} = \langle m^i, y^i \rangle_{M_i}$ for every $m^i \in M_i$. Hence,

$$\begin{aligned}
\left| \left\langle z, y^1 \otimes \cdots \otimes y^l \right\rangle_{\otimes_{H,i=1}^l M_i} \right| &= \left| \left\langle \sum_{j=1}^N z_j^1 \otimes \cdots z_j^l, y^1 \otimes \cdots \otimes y^l \right\rangle_{\otimes_{H,i=1}^l M_i} \right| \\
&= \left| \sum_{j=1}^N \left\langle z_j^1, y^1 \right\rangle_{M_1} \cdots \left\langle z_j^l, y^l \right\rangle_{M_l} \right| \\
&= \left| \sum_{j=1}^N \left\langle z_j^1, x^1 \right\rangle_{\mathbb{E}_1} \cdots \left\langle z_j^l, x^l \right\rangle_{\mathbb{E}_l} \right| \\
&= \left| \left\langle z, x^1 \otimes \cdots \otimes x^l \right\rangle_H \right| \leq 1.
\end{aligned}$$

Therefore, $z \in (P_1 \cap M_1) \otimes_\epsilon \cdots \otimes_\epsilon (P_l \cap M_l)$. This completes the proof. □

Proposition 4.19. *The projective tensor product $\otimes_\pi$ is a projective tensor product of 0-symmetric (resp. circled) convex bodies.*

Proof. For every $1 \leq i \leq l$, let $T_i : \mathbb{E}_i \to \mathbb{F}_i$ be a surjective linear map and $P_i \in \mathcal{B}(\mathbb{E})$. Then $T_i(P_i) \in \mathcal{B}(\mathbb{F}_i)$ and

$$\begin{aligned}
T_1 \otimes \cdots \otimes T_l \left(\Sigma_{P_l,\ldots,P_l}\right) &= \Sigma_{T_1 P_l,\ldots,T_l P_l} \\
T_1 \otimes \cdots \otimes T_l \left(conv\left(\Sigma_{P_l,\ldots,P_l}\right)\right) &= conv\left(\Sigma_{T_1 P_l,\ldots,T_l P_l}\right) \\
T_1 \otimes \cdots \otimes T_l \left(P_1 \otimes_\pi \cdots \otimes_\pi P_l\right) &= T_1 P \otimes_\pi \cdots \otimes_\pi T_l P.
\end{aligned}$$

□

4.4 The dual of a tensor product of 0-symmetric convex bodies

In this section we introduce the dual $\otimes_{\alpha'}$ of a tensor product of 0-symmetric convex bodies $\otimes_\alpha$. In Theorem 4.24 we prove that a tensor product 0-symmetric convex bodies $\otimes_\alpha$ is projective if and only if $\otimes_{\alpha'}$ is injective.

Theorem 4.20. *(Dual of a tensor product) Let $\otimes_\alpha$ be a tensor product of 0-symmetric (resp. circled) convex bodies. Then the dual of $\otimes_\alpha$, $\otimes_{\alpha'}$, defined for each $P_i \in \mathcal{B}(\mathbb{E}_i)$ for $i = 1..., l$, by*

$$P_1 \otimes_{\alpha'} \cdots \otimes_{\alpha'} P_l := (P_1^\circ \otimes_\alpha \cdots \otimes_\alpha P_l^\circ)^\circ$$

is a tensor product of 0-symmetric (resp. circled) convex bodies.

Proof. First, we prove that for every $P_i \in \mathcal{B}(\mathbb{E})$ $i = 1, ..., l$:

$$P_1 \otimes_\pi \cdots \otimes_\pi P_l \subseteq P_1 \otimes_{\alpha'} \cdots \otimes_{\alpha'} P_l \subseteq P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l.$$

Since, $\otimes_\alpha$ is a tensor product of 0-symmetric convex bodies (resp. circled) we have

$$P_1^\circ \otimes_\pi \cdots \otimes_\pi P_l^\circ \subseteq P_1^\circ \otimes_\alpha \cdots \otimes_\alpha P_l^\circ \subseteq P_1^\circ \otimes_\epsilon \cdots \otimes_\epsilon P_l^\circ$$

which implies

$$\begin{aligned}
(P_1^\circ \otimes_\epsilon \cdots \otimes_\epsilon P_l^\circ)^\circ \subseteq (P_1^\circ \otimes_\alpha \cdots \otimes_\alpha P_l^\circ)^\circ \subseteq (P_1^\circ \otimes_\pi \cdots \otimes_\pi P_l^\circ)^\circ \\
\left(\Sigma_{P_l^{\circ\circ},...,P_l^{\circ\circ}}\right)^{\circ\circ} \subseteq (P_1^\circ \otimes_\alpha \cdots \otimes_\alpha P_l^\circ)^\circ \subseteq \left(conv\left(\Sigma_{P_l^\circ,...,P_l^\circ}\right)\right)^\circ \\
conv\left(\Sigma_{P_l,...,P_l}\right) \subseteq (P_1^\circ \otimes_\alpha \cdots \otimes_\alpha P_l^\circ)^\circ \subseteq \left(conv\left(\Sigma_{P_l^\circ,...,P_l^\circ}\right)\right)^\circ \\
P_1 \otimes_\pi \cdots \otimes_\pi P_l \subseteq P_1 \otimes_{\alpha'} \cdots \otimes_{\alpha'} P_l \subseteq P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l.
\end{aligned}$$

To prove the uniform property, recall that for every linear map $T_i : \mathbb{E}_i \to \mathbb{F}_i$ such that $T_i(P_i) \subseteq Q_i$ $i = 1, ..., l$, one has $T_i^t(Q_i^\circ) \subseteq P_i^\circ$.

From the uniform property of $\otimes_\alpha$ we have,

$$T_1^t \otimes \cdots \otimes T_l^t (Q_1^\circ \otimes_\alpha \cdots \otimes_\alpha Q_l^\circ) \subseteq P_1^\circ \otimes_\alpha \cdots \otimes_\alpha P_l^\circ.$$

Therefore,

$$\begin{aligned}
\left(T_1^t \otimes \cdots \otimes T_l^t\right)^t (P_1^\circ \otimes_\alpha \cdots \otimes_\alpha P_l^\circ)^\circ \subseteq (Q_1^\circ \otimes_\alpha \cdots \otimes_\alpha Q_l^\circ)^\circ \\
T_1 \otimes \cdots \otimes T_l (P_1 \otimes_{\alpha'} \cdots \otimes_{\alpha'} P_l) \subseteq Q_1 \otimes_{\alpha'} \cdots \otimes_{\alpha'} Q_l.
\end{aligned}$$

□

Hence $\otimes_{\alpha'}$ is a tensor product of 0-symmetric (resp. circled) convex bodies.

Proposition 4.21. *The dual of $\otimes_\pi$ (resp. $\otimes_\epsilon$) is $\otimes_\epsilon$ (resp.$\otimes_\pi$).*

Proof. Let $P_i \in \mathcal{B}(\mathbb{E})$ $i = 1, ..., l$. Then,

$$\begin{aligned}
P_1 \otimes_{\pi'} \cdots \otimes_{\pi'} P_l = (P_1^\circ \otimes_\pi \cdots \otimes_\pi P_l^\circ)^\circ &= \left(conv\left(\Sigma_{P_l^\circ,...,P_l^\circ}\right)\right)^\circ \\
&= \left(\Sigma_{P_l^\circ,...,P_l^\circ}\right)^\circ \\
&= P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l.
\end{aligned}$$

And,

$$\begin{aligned}
P_1 \otimes_{\epsilon'} \cdots \otimes_{\epsilon'} P_l = (P_1^\circ \otimes_\epsilon \cdots \otimes_\epsilon P_l^\circ)^\circ &= \left(\left(\Sigma_{P_l^{\circ\circ},...,P_l^{\circ\circ}}\right)^\circ\right)^\circ \\
&= conv\left(\Sigma_{P_l,...,P_l}\right) \\
&= P_1 \otimes_\pi \cdots \otimes_\pi P_l.
\end{aligned}$$

□

Proposition 4.22. *Let $\otimes_\alpha$ be a tensor product of* 0*-symmetric (resp. circled) convex bodies, then $\otimes_{\alpha''} = \otimes_\alpha$.*

Proof. Let $P_j \in \mathcal{B}(\mathbb{E})$ $j = 1, ..., l$. Then,

$$\begin{aligned} P_1 \otimes_{\alpha''} \cdots \otimes_{\alpha''} P_l &= (P_1^\circ \otimes_{\alpha'} \cdots \otimes_{\alpha'} P_l^\circ)^\circ \\ &= ((P_1^{\circ\circ} \otimes_\alpha \cdots \otimes_\alpha P_l^{\circ\circ})^\circ)^\circ \\ &= conv\,(P_1 \otimes_\alpha \cdots \otimes_\alpha P_l) \\ &= P_1 \otimes_\alpha \cdots \otimes_\alpha P_l. \end{aligned}$$

□

Lemma 4.23. *Let $\mathbb{E}$ be a Euclidean space, $M \subseteq \mathbb{E}$ be a subspace and $i_M : M \to \mathbb{E}$ be the inclusion map. If $P \in \mathcal{B}(\mathbb{E})$ then,*

$$i^t(P^\circ) = (P \cap M)^\diamond.$$

Where, $(P \cap M)^\diamond$ is the polar body of $P \cap M \subseteq M$ determined by the restriction of the inner product $\langle \cdot, \cdot \rangle_{\mathbb{E}}$ to the subspace M.

Proof. Let $\langle \cdot, \cdot \rangle_M$ be the inner product of $\mathbb{E}$ restricted to the subspace M and $P \in \mathcal{B}(\mathbb{E})$. Clearly $P \cap M$ is a 0-symmetric (resp. circled) convex body on $(M, \langle \cdot, \cdot \rangle_M)$.

If $y \in P^\circ$, then $\sup\{|\langle x, y \rangle_{\mathbb{E}}| : x \in P\} \leq 1$. This implies that for every $m \in P \cap M$,

$$\left|\left\langle m, i_M^t(y) \right\rangle_M\right| = |\langle i_M(m), y \rangle_{\mathbb{E}}| = |\langle m, y \rangle_{\mathbb{E}}| \leq 1.$$

Therefore $y \in (P \cap M)^\diamond$ so $i^t(P^\circ) \subseteq (P \cap M)^\diamond$.

On the other hand, take $y \in (P \cap M)^\diamond$. By the Hanh-Banach theorem, there exists $z \in \mathbb{E}$ such that $z \in P^\circ$ and

$$\langle m, z \rangle_{\mathbb{E}} = \langle m, y \rangle_M \text{ for every } m \in M.$$

From this, we have $i_M^t(z) = y$. Therefore $(P \cap M)^\diamond \subseteq i^t(P^\circ)$ which completes the proof. □

Theorem 4.24. *Let $\otimes_\alpha$ be a tensor product of* 0*-symmetric (resp. circled) convex bodies, then $\otimes_\alpha$ is projective if and only if $\otimes_{\alpha'}$ is injective.*

Proof. Assume that $\otimes_\alpha$ is projective. Let M_j $j = 1, ..., l$ be a subspace of $\mathbb{E}_j$. As we did on the previous lemma. For every $P_j \in \mathcal{B}(\mathbb{E}_j)$ we denote by $(P_j \cap M_j)^\diamond$ the polar body of $P_j \cap M_j \subseteq M_j$ determined by the inner product on M_j induced by the one on $\mathbb{E}_i$.

Let $i_{M_j} : M_j \to \mathbb{E}_j$ be the inclusion map, then $i_{M_j}^t$ is a surjective linear map. Since $\otimes_\alpha$ is projective we know

$$i_{M_1}^t \otimes \cdots \otimes i_{M_l}^t (P_1^\circ \otimes_\alpha \cdots \otimes_\alpha P_l^\circ) = i_{M_1}^t P_1^\circ \otimes_\alpha \cdots \otimes_\alpha i_{M_l}^t P_l^\circ.$$

Applying Lemma 4.23 twice we have:

$$\begin{aligned}(i_{M_1}\otimes\cdots\otimes i_{M_l})^t\,(P_1^\circ\otimes_\alpha\cdots\otimes_\alpha P_l^\circ) &= (P_1\cap M_1)^\diamond\otimes_\alpha\cdots\otimes_\alpha(P_l\cap M_l)^\diamond\\ \left((P_1^\circ\otimes_\alpha\cdots\otimes_\alpha P_l^\circ)^\circ\cap\otimes_{j=1}^l M_j\right)^\diamond &= (P_1\cap M_1)^\diamond\otimes_\alpha\cdots\otimes_\alpha(P_l\cap M_l)^\diamond\\ \left((P_1^\circ\otimes_\alpha\cdots\otimes_\alpha P_l^\circ)^\circ\cap\otimes_{j=1}^l M_j\right)^{\diamond\diamond} &= ((P_1\cap M_1)^\diamond\otimes_\alpha\cdots\otimes_\alpha(P_l\cap M_l)^\diamond)^\diamond\end{aligned}$$

Thus,

$$\begin{aligned}(P_1\cap M_1)\otimes_{\alpha'}\cdots\otimes_{\alpha'}(P_l\cap M_l) &= (P_1^\circ\otimes_\alpha\cdots\otimes_\alpha P_l^\circ)^\circ\cap\otimes_{j=1}^l M_j\\ &= P_1\otimes_{\alpha'}\cdots\otimes_{\alpha'}P_l\cap\otimes_{j=1}^l M_j.\end{aligned}$$

and $\otimes_{\alpha'}$ is injective.
Conversely suppose that $\otimes_{\alpha'}$ is injective. Let $T_j:\mathbb{E}_j\to\mathbb{F}_j$ be a surjective linear map and $P_j\in\mathcal{B}(\mathbb{E}_j)$ for $j=1,...,l$.
Since each T_j is surjective we know that: $T_jP_j\in\mathcal{B}(\mathbb{F})$, T_j^t is injective and

$$T_j^t\left((T_jP_j)^\circ\right)=P_j^\circ\cap T_j^t(\mathbb{F}_j)\,.$$

From this and the injectivity of $\otimes_{\alpha'}$ we have:

$$\begin{aligned}T_1^t\left((T_1P_l)^\circ\right)\otimes_{\alpha'}\cdots\otimes_{\alpha'}T_l^t\left((T_1P_l)^\circ\right) &= P_1^\circ\cap T_1^t(\mathbb{F}_1)\otimes_{\alpha'}\cdots\otimes_{\alpha'}P_l^\circ\cap T_l^t(\mathbb{F}_l)\\ &= (P_1^\circ\otimes_{\alpha'}\cdots\otimes_{\alpha'}P_l^\circ)\cap\otimes_{j=1}^l T_j^t(\mathbb{F}_j)\\ &= (P_1\otimes_\alpha\cdots\otimes_\alpha P_l)^\circ\cap\otimes_{j=1}^l T_j^t(\mathbb{F}_j)\,.\end{aligned}$$

Since $T_1\otimes\cdots\otimes T_l$ is surjective we know that $T_1^t\otimes\cdots\otimes T_l^t$ is injective, and

$$T_1^t\otimes\cdots\otimes T_l^t\left((T_1\otimes\cdots\otimes T_l\,(P_1\otimes_\alpha\cdots\otimes_\alpha P_l))^\circ\right)=(P_1\otimes_\alpha\cdots\otimes_\alpha P_l)^\circ\cap\otimes_{j=1}^l T_j^t(\mathbb{F}_j)\,.$$

Hence,

$$T_1^t\left((T_1P_1)^\circ\right)\otimes_{\alpha'}\cdots\otimes_{\alpha'}T_l^t\left((T_lP_l)^\circ\right)=T_1^t\otimes\cdots\otimes T_l^t\left((T_1\otimes\cdots\otimes T_l\,(P_1\otimes_\alpha\cdots\otimes_\alpha P_l))^\circ\right)\tag{4.4.1}$$

Now, for each $j=1,...,l$ we define

$$\begin{aligned}S_j:\mathbb{F}_j&\to T_j^t(\mathbb{F}_j)\\ y^j&\to T_j^t\left(y^j\right).\end{aligned}$$

Then S_j is a bijective linear map. From Proposition 4.13 it follows

$$S_1\otimes\cdots\otimes S_l\left((T_1P_1)^\circ\otimes_{\alpha'}\cdots\otimes_{\alpha'}(T_1P_l)^\circ\right)=S_1\left((T_1P_1)^\circ\right)\otimes_{\alpha'}\cdots\otimes_{\alpha'}S_l\left((T_lP_l)^\circ\right),$$

which is equivalent to

$$T_1^t\otimes\cdots\otimes T_l^t\left((T_1P_1)^\circ\otimes_{\alpha'}\cdots\otimes_{\alpha'}(T_1P_l)^\circ\right)=T_1^t\left((T_1P_1)^\circ\right)\otimes_{\alpha'}\cdots\otimes_{\alpha'}T_l^t\left((T_lP_l)^\circ\right).$$

From this and Equation (4.4.1),

$$T_1^t \otimes \cdots \otimes T_l^t \left((T_1 P_1)^\circ \otimes_{\alpha'} \cdots \otimes_{\alpha'} (T_1 P_l)^\circ \right) = T_1^t \otimes \cdots \otimes T_l^t \left((T_1 \otimes \cdots \otimes T_l (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l))^\circ \right).$$

Therefore,

$$\begin{aligned}
(T_1 P_1)^\circ \otimes_{\alpha'} \cdots \otimes_{\alpha'} (T_1 P_l)^\circ &= (T_1 \otimes \cdots \otimes T_l (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l))^\circ \\
(T_1 P_1 \otimes_\alpha \cdots \otimes_\alpha T_1 P_l)^\circ &= (T_1 \otimes \cdots \otimes T_l (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l))^\circ \\
(T_1 P_1 \otimes_\alpha \cdots \otimes_\alpha T_1 P_l)^{\circ\circ} &= (T_1 \otimes \cdots \otimes T_l (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l))^{\circ\circ} \\
T_1 P_1 \otimes_\alpha \cdots \otimes_\alpha T_1 P_l &= T_1 \otimes \cdots \otimes T_l (P_1 \otimes_\alpha \cdots \otimes_\alpha P_l).
\end{aligned}$$

This proves that $\otimes_\alpha$ is projective. □

4.5 A bijection between tensor products of 0-symmetric convex bodies and tensor norms

Proposition 4.25. *Let $\otimes_\alpha$ be a tensor product of 0-symmetric (resp. circled) convex bodies, let M_i be a finite dimensional vector space and suppose that $\langle \cdot, \cdot \rangle_i$ is an inner product on M_i for $i = 1, ..., l$. If for each $i \in \{1, ..., l\}$, $\|\cdot\|_i$ is a norm on M_i with closed unit ball B_i. Then $B_1 \otimes_\alpha \cdots \otimes_\alpha B_l$ is a 0-symmetric (resp. circled) convex body in $\otimes_{H,i=1}^l (M_i, \langle \cdot, \cdot \rangle_i)$, and*

$$\|z\|_{\otimes_\alpha} := g_{B_1 \otimes_\alpha \cdots \otimes_\alpha B_l}(z) \text{ for every } z \in \otimes_{i=1}^l M_i.$$

is a uniform reasonable crossnorm on $\otimes_{i=1}^l (M_i, \|\cdot\|_i)$.

Proof. Let M_i and $\|\cdot\|_i$ as in the stament of the theorem. We denote by $\mathbb{M}_i = (M_i, \langle \cdot, \cdot \rangle_i)$. Clearly each closed unit ball B_i belongs to $\mathcal{B}(\mathbb{M}_i)$ for $i = 1, ..., l$.

Since $\otimes_\alpha$ is a tensor product of 0-symmetric (resp. circled) convex bodies $B_1 \otimes_\alpha \cdots \otimes_\alpha B_l$ is well defined. We will prove that

$$\|\cdot\|_{\otimes_\alpha} := g_{B_1 \otimes_\alpha \cdots \otimes_\alpha B_l}(\cdot)$$

is a uniform reasonable crossnorm.

From the fact that $\otimes_\alpha$ is a tensor product of 0-symmetric (resp. circled) convex bodies we know:

$$B_1 \otimes_\pi \cdots \otimes_\pi B_l \subseteq B_1 \otimes_\alpha \cdots \otimes_\alpha B_l \subseteq B_1 \otimes_\epsilon \cdots \otimes_\epsilon B_l,$$

Hence, if $z \in \otimes_{i=1}^l M_i$ we have

$$g_{B_1 \otimes_\epsilon \cdots \otimes_\epsilon B_l}(z) \leq g_{B_1 \otimes_\alpha \cdots \otimes_\alpha B_l}(z) \leq g_{B_1 \otimes_\pi \cdots \otimes_\pi B_l}(z).$$

From Theorem 4.5 we know $g_{B_1\otimes_\pi\cdots\otimes_\pi B_l}(\cdot)$ and $g_{B_1\otimes_\epsilon\cdots\otimes_\epsilon B_l}(\cdot)$ are the closed unit ball of the spaces $\otimes_{\pi,i=1}^{l}(M_i,\|\cdot\|_i)$ and $\otimes_{\epsilon,i=1}^{l}(M_i,\|\cdot\|_i)$, respectively.

Therefore,

$$\epsilon_{M_1,\dots,M_l}(z) \le g_{B_1\otimes_\alpha\cdots\otimes_\alpha B_l}(z) \le \pi_{M_1,\dots,M_l}(z),$$

and so $\|\cdot\|_{\otimes_\alpha}$ is a reasonable crossnorm.

To see that $\|\cdot\|_{\otimes_\alpha}$ is uniform. Take $T_i \in \mathcal{L}((M_i,\|\cdot\|_i), N_i)$ such that $\|T_i\| \le 1$. Then $T_i(B_i) \subseteq B_{N_i}$, and by the uniform property of $\otimes_\alpha$ we have

$$T_1\otimes\cdots\otimes T_l(B_1\otimes_\alpha\cdots\otimes_\alpha B_l) \subseteq B_{N_1}\otimes_\alpha\cdots\otimes_\alpha B_{N_l}.$$

Which implies that,

$$T_1\otimes\cdots\otimes T_l : \left(\otimes_{i=1}^{l}(M_i,\|\cdot\|_i), \|\cdot\|_{\otimes_\alpha}\right) \to \left(\otimes_{i=1}^{l}N_i, \|\cdot\|_{\otimes_\alpha}\right)$$

has norm less than or equal to one. This proof that $\|\cdot\|_{\otimes_\alpha}$ is uniform. □

Lemma 4.26. *Let $\otimes_\alpha$ be a tensor product of 0-symmetric (resp. circled) convex bodies. For $i = 1,\dots,l$ let M_i be a finite dimensional vector space and $\|\cdot\|_i$ be a norm on it. Then $\|\cdot\|_{\otimes_\alpha}$ does not depend on the election of the inner products on the spaces M_i.*

Proof. This is a consequence of Proposition 4.13. To see this let M_i $i = 1,\dots,l$ be normed spaces of finite dimension and $[\cdot,\cdot]_i$, $\langle\cdot,\cdot\rangle_i$ be inner products on M_i. Denote by $T_i : (M_i,[\cdot,\cdot]_i) \to (M_i,\langle\cdot,\cdot\rangle_i)$ the identity map. Let $Q_i = T_iB_{M_i}$. Hence $B_{M_1}\otimes_\alpha\cdots\otimes_\alpha B_{M_l} \in \mathcal{B}\left(\otimes_{H,i=1}^{l}(M_i,[\cdot,\cdot]_i)\right)$, $Q_1\otimes_\alpha\cdots\otimes_\alpha Q_l \in \mathcal{B}\left(\otimes_{H,i=1}^{l}(M_i,\langle\cdot,\cdot\rangle_i)\right)$ and

$$\begin{aligned} B_{M_1}\otimes_\alpha\cdots\otimes_\alpha B_{M_l} &= T_1\otimes\cdots\otimes T_l(B_{M_1}\otimes_\alpha\cdots\otimes_\alpha B_{M_l}) \\ &= T_1B_{M_1}\otimes_\alpha\cdots\otimes_\alpha T_lB_{M_l} \\ &= Q_1\otimes_\alpha\cdots\otimes_\alpha Q_l. \end{aligned}$$

Therefore $g_{B_{M_1}\otimes_\alpha\cdots\otimes_\alpha B_{M_l}} = g_{Q_1\otimes_\alpha\cdots\otimes_\alpha Q_l}$. This implies that $\|\cdot\|_{\otimes_\alpha}$ does not depend on the election of the inner products on the spaces M_i. □

Proposition 4.27. *Let $\alpha(\cdot)$ be a tensor norm of order l on finite dimensional normed spaces. Define $\otimes_{\|\cdot\|_\alpha}$ as follows: for every 0-symmetric (resp. circled) convex body $P_i \subseteq \mathbb{E}_i$ for $i = 1,\dots,l$.*

$$P_1\otimes_{\|\cdot\|_\alpha}\cdots\otimes_{\|\cdot\|_\alpha} P_l := B_{\otimes_{\alpha,i=1}^{l}(\mathbb{E}_i,g_{P_i})}.$$

Then, $\otimes_{\|\cdot\|_\alpha}$ is a tensor product of order l of finite dimensional 0-symmetric (resp. circled) convex bodies.

Proof. Let $P_i \in \mathcal{B}(\mathbb{E}_i)$ and g_{P_i} be the Minkowski functional of P_i for $i = 1,\dots,l$. Then $E_i = (\mathbb{E}_i, g_{P_i})$ is a finite dimensional normed space.

Since $\alpha(\cdot)$ is a tensor norm on finite dimensional normed spaces. We have $B_{\otimes^l_{\alpha,i=1}E_i} \in \mathcal{B}\left(\otimes^l_{H,i=1}\mathbb{E}_i\right)$ and so

$$P_1 \otimes_{\|\cdot\|_\alpha} \cdots \otimes_{\|\cdot\|_\alpha} P_l \in \mathcal{B}\left(\otimes^l_{H,i=1}\mathbb{E}_i\right).$$

Since $\alpha(\cdot)$ is a reasonable crossnorm we know that:

$$\epsilon(z) \leq \alpha(z) \leq \pi(z) \text{ for } z \in \otimes^l_{i=1}E_i.$$

From Theorem 4.5 the above inequality is equivalent to

$$P_1 \otimes_\pi \cdots \otimes_\pi P_l \subseteq P_1 \otimes_{\|\cdot\|_\alpha} \cdots \otimes_{\|\cdot\|_\alpha} P_l \subseteq P_1 \otimes_\epsilon \cdots \otimes_\epsilon P_l.$$

To see that $\otimes_{\|\cdot\|_\alpha}$ has the uniform property. Let $P_i \in \mathcal{B}(\mathbb{E}_i)$ and $Q_i \in \mathcal{B}(\mathbb{F}_i)$ with $i = 1, ..., l$.

If $T_i : \mathbb{E}_i \to \mathbb{F}_i\, i = 1, ..., l$ is a linear map such that $T_i(P_i) \subseteq Q_i$. Then $T_i \in \mathcal{L}((\mathbb{E}_i, g_{P_i}), (\mathbb{F}_i, g_{Q_i}))$ and $\|T_i\| \leq 1$ for $i = 1, ..., l$.

From this and the uniform property of $\alpha(\cdot)$ we have $\|T_1 \otimes \cdots \otimes T_l\| \leq 1$, which is equivalent to

$$T_1 \otimes \cdots \otimes T_l\left(P_1 \otimes_{\|\cdot\|_\alpha} \cdots \otimes_{\|\cdot\|_\alpha} P_l\right) \subseteq Q_1 \otimes_{\|\cdot\|_\alpha} \cdots \otimes_{\|\cdot\|_\alpha} Q_l.$$

□

Theorem 4.28. *There exists a bijection between tensor products of order l of 0-symmetric (resp. circled) convex bodies and tensor norms of order l on finite dimensional normed spaces.*

Proof. From Proposition 4.25 we know that the map $\otimes_\alpha \to \|\cdot\|_{\otimes_\alpha}$ sends tensor products of 0-symmetric (resp. circled) convex bodies into tensor norms on finite dimensional normed spaces. From Proposition 4.27 the map $\|\cdot\|_\alpha \to \otimes_{\|\cdot\|_\alpha}$ sends tensor norms on finite dimensional normed spaces into tensor products of 0-symmetric (resp. circled) convex bodies.

Let $\otimes_\alpha$ be a tensor product of 0-symmetric (resp. circled) convex bodies. We will prove that if $\beta(\cdot) = \|\cdot\|_{\otimes_\alpha}$, then $\otimes_\alpha = \otimes_\beta$.

Let $P_i \subseteq \mathbb{E}_i$ be a 0-symmetric (resp. circled) convex body for $i = 1, ..., l$, then

$$\begin{aligned} P_1 \otimes_\beta \cdots \otimes_\beta P_l &= \left\{z \in \otimes^l_{i=1}(\mathbb{E}_i, g_{P_i}) : \beta(z) \leq 1\right\} \\ &= \left\{z \in \otimes^l_{i=1}(\mathbb{E}_i, g_{P_i}) : g_{P_1 \otimes_\alpha \cdots \otimes_\alpha P_l}(z) \leq 1\right\} \\ &= P_1 \otimes_\alpha \cdots \otimes_\alpha P_l. \end{aligned}$$

On the other hand, let $\|\cdot\|_\alpha$ be a tensor norm on finite dimensional normed spaces. We will prove that if $\otimes_\beta = \otimes_{\|\cdot\|_\alpha}$ then $\|\cdot\|_{\otimes_\beta} = \|\cdot\|_\alpha$.

Let M_i be a finite dimensional normed space for $i = 1, ..., l$. Then

$$\|z\|_{\otimes_\beta} = g_{B_{M_1} \otimes_\beta \cdots \otimes_\beta B_{M_l}}(z).$$

Since,

$$B_{M_1} \otimes_\beta \cdots \otimes_\beta B_{M_l} = \left\{ z \in \otimes_{i=1}^{l} M_i : \|z\|_\alpha \leq 1 \right\}.$$

We have $\|z\|_{\otimes_\beta} = \|z\|_\alpha$ for every $z \in \otimes_{i=1}^{l} M_i$. This completes the proof. $\square$

From the previous theorem and Theorem 1.22 we obtain the following corollary. Recall that for every linear map $T = \sum_{i=1}^{n} x_i^* (\cdot) y_i$ from M to N we denote by u_T the vector $\sum_{i=1}^{n} x_i^* \otimes y_i$ that belongs to $M^* \otimes N$.

Corollary 4.29. *If $\otimes_\alpha$ is a tensor product of order 2 of 0-symmetric (resp. circled) convex bodies, then there exists a Banach operator ideal such that: for every pair M, N of finite dimensional normed spaces one has:*

$$A(T : M \to N) := \|u_T\|_{\otimes_\alpha}$$

i.e. A is an ideal norm.

Proposition 4.30. *Let $\otimes_\alpha$ be a tensor product of 0-symmetric (resp. circled) convex bodies, then $\otimes_\alpha$ is injective (resp. projective) if and only if $\|\cdot\|_{\otimes_\alpha}$is an injective tensor norm (resp. projective).*

Proof. Assume that $\otimes_\alpha$ is injective. Let M_i $i = 1, ..., l$ be finite dimensional normed spaces and $\langle \cdot, \cdot \rangle_i$ be an inner product on M_i. On every subspace $N_i \subseteq M_i$ consider the inner product dermined by the restriction of $\langle \cdot, \cdot \rangle_i$ to M_i.

Now $\otimes_\alpha$ is injective if and only if

$$(B_{M_1} \cap N_1) \otimes_\alpha \cdots \otimes_\alpha (B_{M_l} \cap N_l) = B_{M_1} \otimes_\alpha \cdots \otimes_\alpha B_{M_l} \cap \otimes_{i=1}^{l} N_i.$$

But, $B_{M_i} \cap N_i = B_{N_i}$. Therefore, the above equation is equivalent to,

$$g_{B_{N_1} \otimes_\alpha \cdots \otimes_\alpha B_{N_l}}(z) = g_{B_{M_1} \otimes_\alpha \cdots \otimes_\alpha B_{M_l}}(z) \text{ for } z \in \otimes_{i=1}^{l} N_i.$$

And $\|\cdot\|_{\otimes_\alpha}$is injective. We have proved that $\otimes_\alpha$ is injective if and only if $\|\cdot\|_{\otimes_\alpha}$is injective.

Assume that $\otimes_\alpha$ is projective. Let E_i, M_i be finite dimensional normed spaces. Recall that a linear map $T_i : M_i \to E_i$ is a quotient operator if and only if T_i is surjective and $T_i(B_{M_i}) = B_{E_i}$.

Since $\otimes_\alpha$ is projective if and only if

$$\begin{aligned} T_1 \otimes \cdots \otimes T_l (B_{M_1} \otimes_\alpha \cdots \otimes_\alpha B_{M_l}) &= T_1 B_{M_1} \otimes_\alpha \cdots \otimes_\alpha T_l B_{M_l} \\ &= B_{E_1} \otimes_\alpha \cdots \otimes_\alpha B_{E_l}. \end{aligned}$$

We conclude that $T_1 \otimes \cdots \otimes T_l : \left(\otimes_{i=1}^{l} M_i, \|\cdot\|_{\otimes_\alpha}\right) \to \left(\otimes_{i=1}^{l} E_i, \|\cdot\|_{\otimes_\alpha}\right)$ is a quotient operator. This proves that $\otimes_\alpha$ is projective if and only if $\|\cdot\|_{\otimes_\alpha}$is projective. $\square$

Proposition 4.31. *In real Euclidean spaces if* $\otimes_\alpha$ *is a tensor product of* 0*-symmetric convex bodies, then* $\|\cdot\|_{\otimes_{\alpha'}}$ *is the dual norm of* $\|\cdot\|_{\otimes_\alpha}$.

Proof. Let M_i $i = 1, ..., l$ be finite dimensional normed spaces, and $\langle \cdot, \cdot \rangle_i$ be an inner product on M_i. Set $\beta(\cdot) = \|\cdot\|_{\otimes_\alpha}$. We will prove that

$$\beta'(z) = g_{B_{M_1} \otimes_{\alpha'} \cdots \otimes_{\alpha'} B_{M_l}}(z) \text{ for every } z \in \otimes_{i=1}^l M_i.$$

Let $T_i : M_i \to M_i^*$ be the canonical map sending $x \in M_i$ to $\langle \cdot, x \rangle_i$. Since each M_i is finite dimensional we know that T_i is a linear isomorphism. From the definition of T_i it follows that $T_i\left(B_{M_i}^\circ\right) = B_{M_i^*}$. Observe that $T_1 \otimes \cdots \otimes T_l$ is the canonical map:

$$\begin{aligned} T_1 \otimes \cdots \otimes T_l : \otimes_{i=1}^l M_i &\to \otimes_{i=1}^l M_i^* \\ w &\to \langle \cdot, w \rangle_H . \end{aligned}$$

Now, take $z \in \otimes_{i=1}^l M_i$ such that

$$\beta'(z) = \sup\left\{ |\varphi(z)| : g_{B_{M_1^*} \otimes_\alpha \cdots \otimes_\alpha B_{M_l^*}}(\varphi) \le 1 \right\} \le 1.$$

From Proposition 4.13 we know,

$$\begin{aligned} T_1 \otimes \cdots \otimes T_l\left(B_{M_1}^\circ \otimes_\alpha \cdots \otimes_\alpha B_{M_l}^\circ\right) &= T_1\left(B_{M_1}^\circ\right) \otimes_\alpha \cdots \otimes_\alpha T_l\left(B_{M_l}^\circ\right) \\ &= B_{M_1^*} \otimes_\alpha \cdots \otimes_\alpha B_{M_l^*}. \end{aligned}$$

Therefore, if $w \in B_{M_1}^\circ \otimes_\alpha \cdots \otimes_\alpha B_{M_l}^\circ$ then $|\langle z, w \rangle_H| \le 1$. Which implies,

$$z \in \left(B_{M_1}^\circ \otimes_\alpha \cdots \otimes_\alpha B_{M_l}^\circ\right)^\circ = B_{M_1} \otimes_{\alpha'} \cdots \otimes_{\alpha'} B_{M_l}.$$

Conversely, if $z \in B_{M_1} \otimes_{\alpha'} \cdots \otimes_{\alpha'} B_{M_l}$. Let $\varphi \in \otimes_{i=1}^l M_i^*$ such that $g_{B_{M_1^*} \otimes_\alpha \cdots \otimes_\alpha B_{M_l^*}}(\varphi) \le 1$. Then

$$T_1^{-1} \otimes \cdots \otimes T_l^{-1}(\varphi) \in B_{M_1}^\circ \otimes_\alpha \cdots \otimes_\alpha B_{M_l}^\circ.$$

Since

$$\left\langle z, T_1^{-1} \otimes \cdots \otimes T_l^{-1}(\varphi) \right\rangle_H = \varphi(z),$$

we conclude that $|\varphi(z)| \le 1$ for every φ with $g_{B_{M_1^*} \otimes_\alpha \cdots \otimes_\alpha B_{M_l^*}}(\varphi) \le 1$. This proves that $\beta'(z) \le 1$.

We have proved that

$$g_{B_{M_1} \otimes_{\alpha'} \cdots \otimes_{\alpha'} B_{M_l}}(\cdot) = \beta'(\cdot).$$

This completes the proof. □

List of Symbols

$\mathbb{N}$ Set of natural numbers,1

$\mathbb{R}$ Set of real numbers,1

$\mathbb{S}^{d-1}$ $d-1$-sphere, 2

$\mathcal{A}(X,Y)$ 7

$\mathcal{BM}(d)$ The Banach Mazur compactum, 12

$\mathcal{BM}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ 54

$\mathcal{B}(\mathbb{E})$ The set of centrally symmetric convex bodies, 9

$\mathcal{B}(d)$ 9

$\mathcal{B}_{\Sigma_{Q_1,\dots,Q_l}}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ 22

$\mathcal{B}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ 22

$\mathcal{C}(\mathbb{E})$ 8

$\mathcal{E}_1\otimes_H\cdots\otimes_H\mathcal{E}_l$ 30

$\mathcal{K}(\mathbb{E})$ The set of convex bodies, 9

$\mathcal{L}(X,Y)$ Space of continuous linear maps, 1

$\mathcal{L}(X_1,...,X_l;Y)$ Space of continuous multilinear maps, 4

$\mathcal{SCB}$ 58

$\mathscr{E}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ Set of tensorial ellipsoids, 30

$\mathscr{L}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ 48

$\mathscr{L}_{\Sigma}\left(\otimes_{i=1}^{l}\mathbb{R}^{d_i}\right)$ 48

$\otimes_{\alpha,i=1}^{l}X_i$ 6

$\otimes_{i=1}^{l}X_i$, $X_1\otimes\cdots\otimes X_l$ 2

$\overline{\theta_x}$ 41

$\overline{A}$ Closure of a set,1

∂A Boundary of a set,1

$\pi(u)$ 3

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