CIMAT

CENTRO DE INVESTIGACIÓN EN MATEMÁTICAS A.C.

The Duality between Tensor Norms and Ideals of Multilinear Operators, a Geometrical Approach

TESIS

que para obtener el grado de

Doctor en Ciencias

con orientación en

Matemáticas Básicas

PRESENTA

Samuel García Hernández

Director de Tesis

Maite Fernández Unzueta

Guanajuato, Gto.

Abstract

The purpose of this dissertation is to establish a theoretical framework for the study of bounded multilinear operators via ideals of Σ-operators and Σ-tensor norms. Throughout this work we motivate the definitions of these two concepts, explore their basic properties and exhibit the relations between them. In addition, several examples are developed.

The author of [54] shows that every multilinear operator T is associated with a Σ-operator f_T. This associated function and its domain allow to study T within a geometrical environment. As well as the theory of bounded linear operators, the theory of Σ-operators is closely related with tensor product of Banach spaces and norms on it, namely, Σ-tensor norms.

The most important results we have obtained are the representation theorem for maximal ideals of Σ-operators and the duality theorem for Σ-tensor norms. We strengthen this theory by exploring a wide range of ideal properties. To be specific, we study the classes of compact, weakly-compact, approximable, nuclear, (p, q)-dominated and (p, q)-factorable Σ-operators. In each case we show the implications for the associated multilinear operator, and the Σ-tensor norms involved, these varying from the projective and injective to the Lapresté Σ-tensor norms. It is worth pointing out that unlike the linear theory of tensor norms, the Σ-tensor norms are presented in two different types, these are, Σ-tensor norms on duals and Σ-tensor norms on spaces.

To describe our results recall that the representation theorem for maximal ideals asserts that

$$(E \otimes_{\alpha'} F)^* = \mathcal{A}(E, F^*)$$

where the maximal ideal $\mathcal{A}$ is related with the finitely generated tensor norm α, see [41, Sec. 17.5]. The representation theorem in this setting acquires the form

$$\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^{\beta}\right)^* = \mathcal{A}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y^*\right)$$

where the maximal ideal of Σ-operators $\mathcal{A}$ is related with the finitely generated Σ-tensor norm on spaces α, Theorem 3.13.

The duality theorem for tensor norms ensures that

$$E^* \otimes_{\overleftarrow{\alpha}} F^* \to (E \otimes_{\alpha'} F)^*$$

is an isometry where $\overleftarrow{\alpha}$ is the cofinite hull of α and α' is its dual tensor norm, see [41, Sec. 15.5]. In our case, the duality theorem relates the two types of Σ-tensor norms by showing that

$$\left(\mathcal{L}^{\beta}\left(X_1, \ldots, X_n\right) \otimes Y, \overleftarrow{\nu}_{\beta}\right) \to \left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \alpha^{\beta}\right)^*$$

is also an isometry. Here, $\overleftarrow{\nu}$ is the cofinite hull of the Σ-tensor norm on duals ν and the Σ-tensor norm on spaces α is in duality with ν, see Theorem 3.9.

In the case of a compact Σ-operator we prove that the associated multilinear bounded operator is compact in the classical sense. This property is closely related with compact Lipschitz functions, see Proposition 4.10. The weakly-compact case produces similar results.

We succesfully relate the projective Σ-tensor norm on duals with nuclear Σ-operators via an accurate approximation property, see Proposition 4.21.

We use the developed theory in this dissertation to establish the definition of the class of (p, q)-dominated Σ-operators. To be precise, this class is defined as the maximal ideal of Σ-operators associated to the Σ-tensor norm on duals defined by the Lapresté Σ-tensor norm on spaces, see Definitions 4.46, 4.44, Theorem 2.23 and Definitions 2.17, 2.20. In this case, the principal result is the factorization of multilinear operators through subsets of spaces $L_p(\mu)$ see Theorem 4.49.

We finish the dissertation with the case of (p, q)-factorable Σ-operators. This class is defined as the maximal ideal of Σ-operators associated to the Lapresté Σ-tensor norm on duals, see Definitions 4.52, 4.50 and 2.20. As we see in Theorem 4.57, every (p, q)-factorable Σ-operator f_T induces a factorization of the associated multilinear operator T through subsets of spaces $L_q(\mu)$ and $L_p(\mu)$.

Agradecimientos

A Maite Fernández por aceptar dirigir el trabajo de tesis. Por contagiarme su manera de disfrutar, compartir y hacer matemáticas. Por tener tanta paciencia en las largas y exhaustivas sesiones de trabajo. Quiero agradecer que semana a semana me haya motivado a continuar.

A Dr. Richard M. Aron, Dr. Alejandro Chávez Domínguez, Dr. Fernando Galaz Fontes, Dr. Francisco Eduardo Castillo Santos y al Dr. Ocatavio Arizmendi Echegaray por conformar el jurado, por la paciencia y dedicación el realizar la lectura.

A todo el personal de CIMAT por hacer que el proceso de estudiar un doctorado sea más llevadero y poco tormentoso. Quiero expresar mi gratitud al personal acadédmico por el excelente nivel que se aprecia en los cursos. Al personal de servicios escolares por su eficacia y amabilidad en todo momento.

A los compañeros de CIMAT con los que he compartido esta etapa fundamental en nuestras vidas. Por la complicidad y ser una fuente de desahogo.

A mi madre que me ha apoyado toda la vida. Por ser un gran ejemplo de humanidad, nobleza y sinceridad. A mis hermanos por apoyarme con todos los recursos a su alcance. A mi padre.

A Dany, esa persona tan especial que me ha apoyado desde la distancia y de manera incondicional por tanto tiempo. Por hacer de Guanajuato un lugar más especial.

Deseo expresar mi gratitud al CONACYT por la beca de doctorado No. 376073 y proyecto 182296 y a CIMAT por los numerosos apoyos económicos otorgados en cada congreso al que asistí a lo largo del doctorado y por al ayuda para finalizar el proceso de graduación.

Contents

Introduction

The theory of tensor products goes back to Schatten on his work about cross-spaces [97]. Despite this, it is well known that Grothendieck was the first mathematician who showed the usefulness of tensor products in functional analysis in the well known *Résumé de la théorie métric des produits tensoriels topologiques* [57]. In particular, he exhibited that many properties of Banach spaces have a local behavior and operators between Banach spaces can be studied in tensor terms. Due to the complexity of the "*Résumé*", as it is refered to these days, and the reach of the journal where it was published, the work of Grothendieck was totally understood many years later after its release. Nowadays, tensorial techniques are easier to handle and apply thanks to the effort of many other mathematicians who translated the work of Grothendieck and showed its usefulness. The first example of this translation and the utility of tensor products is the work of Lindentrauss and Pełczyński [66] about absolutely p-summing operators.

Technically speaking, among the results of the ideas of Grothendieck, the concept of tensor norms arose on the side of tensor products and that of operator ideals on the side of linear operators. The concepts of ideals of operators and tensor norms were developed separately though operator ideals was more popular since tensorial techniques seemed to be quite difficult to handle. The theory of operator ideals was explored by A. Piestch et al. in the monorgaph *Operator ideals* [85] making no reference of tensor products whatsoever. On the other hand, it took several years to have a general reference of the study of tensor norms until the book *Tensor norms and operator ideals* by A. Defant and K. Floret [41] where the authors have made a very exhaustive study of tensor norms and operator ideals in tandem. Nowadays there are more references, for example, *The metric theory of tensor products, Grothendieck Résumé revisited* by J. Diestel, J. H Fourie and J. Swart [45] and *Introduction to tensor products of Banach spaces* by R. Ryan [93].

A natural generalization of the theory of linear operators is the multilinear setting. As is well known, the theory of operator ideals for the multilinear setting was started by A. Pietsch in [86] where he gave a possible way to follow for the multilinear case by establishing ideals for the development of ideals of multilinear forms. In the reference [56], K. Floret and S. Hungfeld gave a generalization for the vector valued case currently known as multi-ideals. Also, tensor norms for the n-fold tensor product of Banach spaces were generalized and a representation theorem for maximal multi-ideals of operators by finitely generated tensor norms is given as well. Another proposal is given by G. Botelho and E.R. Torres in [22] where the authors develop the theory of hyper-ideals. This is a slightly different approach which considers more general compositions.

In recent years, concrete examples of ideal properties of operators have been studied by many authors in the context of multilinear operators. Among them we find compactness, nuclearity, p-summability, dominated operators. etc.. These collections fit in the theory of multi-ideals and many of them are represented by tensor norms in the sense of [56].

Although the theory of multilinear operators has plenty of examples of multi-ideals and tensor norms, there is no general reference for their relation in the style of [41]. The authors of [17] studied the representability of multi-ideals by smooth tensor norms and they show that representation of ideals by smooth tensor norms is not a common situation. Certain properties of multi-ideals can be found in [21] and [22].

In [54] M. Fernández-Unzueta shows that multilinear operators can be studied by their associated Σ-operators. In [7], p-summability for Σ-operators is developed by J. Angúlo and M. Fernández-Unzueta. Their results can be interpreted as a new approach to p-summability for multilinear operators. Any other ideal property can be studied in the context of Σ-operators and, as a consequence, enhance the knowledge of multilinear operators.

This dissertation is focused on the study of Σ-operators in terms of ideals. As well as in the linear case, Σ-operators admit a translation to the setting of tensor products. Thus, tensor norms appear naturally but in this context two notions of tensor norms have to be considered, namely, Σ-tensor norms on spaces and Σ-tensor norms on duals. This dissertation studies the duality between ideals of Σ-operators and Σ-tensor norms in the sense of [41].

Next, we briefly describe how this approach works. In [54] the author shows that the subset of simple tensors, denoted by $\Sigma_{X_1 \dots X_n}$, of a tensor product of Banach spaces $X_1 \otimes \dots \otimes X_n$ admits a unique metric topology induced by reasonable crossnorms β on the tensor product. This observation lets us associate a Σ-operator $f_T : \Sigma^{\beta}_{X_1 \dots X_n} \to Y$ to each bounded multilinear operator $T : X_1 \times \cdots \times X_n \to Y$. The important point to note here is that the topologies induced on $\Sigma_{X_1 \dots X_n}$ allows us to study T through its associated Σ-operator f_T within the geometrical environment provided by the normed space $(X_1 \otimes \dots \otimes X_n, \beta)$ on $\Sigma^{\beta}_{X_1 \dots X_n}$.

The first success of this approach is done in [7] where the authors develop the theory of p-summability for Σ-operators. They prove that a p-summing Σ-operator admits a factorization through a subset of $L_p(\mu)$. This result is a generalization of the factorization theorem of A. Pietsch for p-summing linear operators. As a consequence, the associated multilinear operator admits a factorization diagram. In particular, the authors provide a new approach to p-summability for the multilinear case. As we said above, this procedure can be done for other ideal properties. Even more, those cases fit in a duality theory between ideals of Σ-operators and Σ-tensor norms.

A component in an ideal $\mathcal{A}$, denoted by $\mathcal{A}\left(\Sigma^{\beta}_{X_1\ldots X_n}; Y\right)$, is defined for each election of Banach spaces $X_1, \ldots, X_n, Y$ and a reasonable crossnorm β on $X_1 \otimes \ldots \otimes X_n$. The definition of an ideal of Σ-operators is based on the consideration of finite rank Σ-operators. The translation to the tensorial context of this consideration leads us to define Σ-tensor norms on duals, Definition 2.20. On the other hand, every component in a maximal ideal is represented by functionals with the help of Σ-tensor norms on spaces, see Definition 2.17 and Theorem 3.13.

Summarizing, for giving a satisfactory duality between Σ-tensor norms and ideals of Σ-operators for the study of bounded multilinear operators, the following four points are highlighted:

- The subset $\Sigma_{X_1\ldots X_n} \subset X_1 \otimes \ldots \otimes X_n$ of simple tensors inherits a topology induced by a reasonable crossnorm β on $X_1 \otimes \ldots \otimes X_n$.
- Every bounded multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ has an associated Σ-operator $f_T : \Sigma^{\beta}_{X_1\ldots X_n} \to Y$.
- Every bounded finite rank operator is considered.
- There are two types of Σ-tensor norms, namely, Σ-tensor norms on duals and Σ-tensor norms on spaces.

In the next paragraphs we describe the contents chapter by chapter.

In Chapter 1 we collect some results about Σ-operators presented in [54]. We begin the chapter by recalling the definition of Σ-operators and presenting the tensorial representation by tensors and functionals on a tensor product. Next, we give a precise definition of bounded Σ-operators and the topological considerations for the domain of a typical bounded Σ-operator. In this chapter we present the algebraic behavior of the ideal property for Σ-operators and the uniform properties for Σ-tensor norms. We finish by establishing the relation of the projective tensor norm (in the sense of [56]) and the 2-fold injective tensor norm with Σ-operators.

In Chapter 2 we define the principal objects of our interest, namely, ideals of Σ-operators and Σ-tensor norms in the versions on duals and on spaces. In the first section of this chapter we study the classes of bounded and p-summing Σ-operators (these last studied in [7]). Inspired by the behavior of these collections we present the notion of ideals of Σ-operators. In the tensorial context, we define the projective and injective Σ-tensor norms as well as the generalization of the Chevet-Saphar norm d_p defined by J. Angúlo in his doctoral dissertation [6]. Next, we present the definition of Σ-tensor norms in the two versions. The last section of this chapter establishes the relation between these three concepts in the class of finite dimensional normed spaces.

In Chapter 3 we prove our most important results, these are the representation theorem for maximal ideals of Σ-operators and the duality theorem for Σ-tensor norms, in short RT and DT respectively. The RT describes, in tensor terms, any component of a maximal ideal of Σ-operators, Theorem 3.13. On the other hand, the DT describes, in the Banach space case, the relation between Σ-tensor norms on duals and Σ-tensor norms on spaces. To prove the RT and the DT we need to extend the definitions of Σ-tensor norms and ideals of Σ-operators from the class $\mathcal{FIN}$ to the class $\mathcal{BAN}$. This is developed at the beginning of the chapter.

Chapter 4 is devoted to exhibiting concrete examples of ideals of Σ-operators and Σ-tensor norms. We begin with the generalizations of injective and surjective ideals. On the tensorial side, we introduce the notion of an injective Σ-tensor norm on duals and a projective Σ-tensor norm on spaces. As an application of the RT, we relate these tensorial properties with injective and surjective ideals of Σ-operators, see Proposition 4.8. Next, we study the classes of compact, weakly-compact, and nuclear Σ-operators. We relate the collection of nuclear Σ-operators with a certain approximation property. We continue with generalizations of the collections of operators that admit a factorization through a Hilbert space and 2-dominated operators. In both cases we present the respective Σ-tensor norms on spaces which are in duality. We finish the dissertation with the generalization of the Lapresté tensor norms to the setting of Σ-tensor norms. These norms enable us to define the maximal ideals of (p,q)-dominated and (p,q)-factorable Σ-operators. As we will see, every bounded multilinear operator in these collections has a natural factorization.

Preliminaries and Notation

We will use the standard notation of the theory of Banach spaces. The letter $\mathbb{K}$ denotes the field of complex or real numbers. The capital letters X, Y, Z, W, E, F denote vector, normed or Banach spaces over the field $\mathbb{K}$. The unit ball of the normed space X will be denoted by B_X. The canonical inclusion of X into X^{**} is denoted by K_X even in the algebraic case. The set of finite dimensional subspaces of the vector space X is denoted by $\mathcal{F}(X)$, while the set of finite codimensional subspaces of X is denoted by $\mathcal{CF}(X)$.

For vector spaces X and Y we denote by $L(X,Y)$ the set of linear operators from X to Y. In the case of Banach spaces, the set of bounded operators from X to Y endowed with the uniform norm is denoted by $\mathcal{L}(X,Y)$. In the case $Y = \mathbb{K}$ we simply write $X^{\#}$ for vector spaces and X^* for normed spaces.

Recall that the function $T : X_1 \times \cdots \times X_n \to Y$ between vector spaces is said to be multilinear (or n-linear) if

$$T(x^1, \dots, (a + \lambda b), \dots, x^n) = T(x^1, \dots, a, \dots, x^n) + \lambda\, T(x^1, \dots, b, \dots, x^n)$$

holds for all $a, b \in X_i$, $\lambda \in \mathbb{K}$ and $1 \leq i \leq n$. The vector space of all multilinear operators from $X_1 \times \cdots \times X_n$ to Y is denoted by $L(X_1, \dots, X_n; Y)$. If $X_1, \dots, X_n$ and Y are normed spaces then T is said to be bounded if

$$\|T\| := \sup \left\{ \, \|T(x^1, \dots, x^n)\| \mid x^i \in B_{X_i} \, \right\} < \infty.$$

The symbol $\mathcal{L}\,(X_1, \dots, X_n; Y)$ denotes the Banach space of all bounded multilinear operators from $X_1 \times \cdots \times X_n$ into Y. In the case $Y = \mathbb{K}$ we simply write $L\,(X_1, \dots, X_n)$ and $\mathcal{L}\,(X_1, \dots, X_n)$ while their members are called n-forms and bounded n-forms, respectively.

The reader is supposed to know the construction and elementary properties of the tensor product of vector spaces, see [93, Ch. 1]. The universal property of the tensor product of vector spaces establishes that: For every multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ there exists a unique linear operator $\widetilde{T} : X_1 \otimes \ldots \otimes X_n \to Y$ such that

$$\begin{array}{ccc} X_1 \times \cdots \times X_n & \xrightarrow{T} & Y \\ {\scriptstyle\otimes}\downarrow & \nearrow_{\widetilde{T}} & \\ X_1 \otimes \ldots \otimes X_n & & \end{array}$$

commutes. If $X_1, \ldots, X_n$ and Y are Banach spaces, β is a norm on $X_1 \otimes \ldots \otimes X_n$ and $\widetilde{T} : (X_1 \otimes \ldots \otimes X_n, \beta) \to Y$ is bounded, then T is said to be β-bounded.

A tensor norm α on the class of Banach spaces assigns to each pair of Banach spaces X and Y a norm on the algebraic tensor product $X \otimes Y$ (the resulting normed space denoted by $X \otimes_\alpha Y$) such that:

i) α **is crossed:** $\alpha(x \otimes y) \leq \|x\| \, \|y\|$ for all x and y.

ii) α **is reasonable:** For every $x^* \in X^*$ and $y^* \in Y$ the functional defined by $x^* \otimes y^*$ is bounded on $X \otimes_\alpha Y$ and $\|x^* \otimes y^*\| \leq \|x^*\| \, \|y^*\|$.

iii) **Uniform property:** If $T : X \to Z$ and $S : Y \to W$ are bounded linear operators, then $T \otimes S : X \otimes_\alpha Y \to Z \otimes_\alpha W$ is bounded and $\|T \otimes S\| \leq \|T\| \, \|S\|$.

An operator ideal on the class of Banach spaces $[\mathcal{A}, A]$ assigns to each pair of Banach spaces X and Y a linear subspace $\mathcal{A}(X, Y)$ of $\mathcal{L}(X, Y)$ together with a norm A such that:

i) $(\mathcal{A}(X, Y), A)$ is a Banach space.

ii) Every rank-one operator $x^* \cdot y : X \to Y$ belongs to $\mathcal{A}(X, Y)$ and $A(x^* \cdot y) \leq \|x^*\| \, \|y\|$ for all $x^* \in X^*$ and $y \in Y$.

iii) **Ideal property:** If in the composition

$$Z \xrightarrow{R} X \xrightarrow{T} Y \xrightarrow{S} W$$

R and S denote bounded linear operators and T belongs to $\mathcal{A}(X, Y)$, then STR belongs to $\mathcal{A}(Z, W)$ and $A(STR) \leq \|R\| \, A(T) \, \|S\|$.

We will frequently use the following algebraic identification, it only consist in grouping the first n spaces

$$X_1 \otimes \ldots \otimes X_n \otimes Y = (X_1 \otimes \ldots \otimes X_n) \otimes Y.$$

For example, under this identification we have that if $x^1 \otimes \ldots \otimes x^n \otimes y$ and $z^1 \otimes \ldots \otimes z^n \otimes y$ are elements in $X_1 \otimes \ldots \otimes X_n \otimes Y$ then

$$x^1 \otimes \ldots \otimes x^n \otimes y - z^1 \otimes \ldots \otimes z^n \otimes y = (x^1 \otimes \ldots \otimes x^n - z^1 \otimes \ldots \otimes z^n) \otimes y.$$

If $f : A \to B$ is a function then we use the notation $\langle f, a \rangle$ for the value of f at the point $a \in A$, i.e. $f(a)$.

Classical references for the theory of Banach spaces and functional analysis in general are [39, 44, 47, 49, 50, 51, 52, 64, 67, 96, 99, 101, 103].

Chapter 1

Σ-Operators

This chapter is dedicated to establish the foundations of the theory of ideals of Σ-operators and Σ-tensor norms. For this end, we collect some results from [54]. We begin the chapter by giving a precise definition of a Σ-operator. We also show that Σ-operators can be represented in a tensorial context. Next, we present the adequate representation of the domain of a multilinear operator inside a normed tensor product. We continue by treating the case of bounded multilinear operators and indicating the corresponding bounded Σ-operators. The fundamental considerations for a consistent behavior of the theory are presented. We finish the chapter with the tensorial representation of bounded Σ-operators.

1.1 Σ-Operators and their Tensorial Representation

This dissertation is motivated by the study of multilinear operators under the assumption that the domain of a typical multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ is immersed within an accurate geometric environment, see (1.5). Thus, we may study T via an auxiliary function f_T, see Definition 1.4. This idea gives place to the notion of Σ-operators which, in particular, are functions whose domain is contained in a tensor product.

In this section, n is a natural number and $X_1, \ldots, X_n, Y$ denote vector spaces. The set of simple tensors of the algebraic tensor product $X_1 \otimes \ldots \otimes X_n$ is denoted by $\Sigma_{X_1 \ldots X_n}$. This is,

$$\Sigma_{X_1 \ldots X_n} := \left\{ x^1 \otimes \ldots \otimes x^n \mid x^i \in X_i \right\}.$$

If the context is clear we write Σ. The set Σ is well known as the Segre cone, and its projectivization as the Segre variety, see [70, Sec. 4.3.4]. In this dissertation we are interested in the Segre cone because it helps us to represent $X_1 \times \cdots \times X_n$ inside $X_1 \otimes \ldots \otimes X_n$.

Let $T : X_1 \times \cdots \times X_n \to Y$ be a multilinear operator and let $\widetilde{T} : X_1 \otimes \ldots \otimes X_n \to Y$ be its linearization. We define the function

$$\begin{aligned} f_T : \Sigma_{X_1 \ldots X_n} &\to Y \\ x^1 \otimes \ldots \otimes x^n &\mapsto T(x^1, \ldots, x^n). \end{aligned}$$

In other words, f_T is the restriction of $\widetilde{T}$ to the set $\Sigma_{X_1 \ldots X_n}$. We have the following commutative diagram

$$\begin{array}{ccc} X_1 \times \cdots \times X_n & & \\ \downarrow & \searrow^{T} & \\ \Sigma_{X_1 \ldots X_n} & \xrightarrow[f_T]{} & Y \\ \downarrow & \nearrow_{\widetilde{T}} & \\ X_1 \otimes \ldots \otimes X_n & & \end{array} \quad , \tag{1.1}$$

where the unlabeled arrows are the natural inclusion in the tensor product and inclusion as sets respectively. It is worth to notice that the function f_T cannot be a linear operator since $\Sigma_{X_1 \ldots X_n}$ is not a vector space; however, the cone property implies that f_T is homogeneous. In the concrete examples of Chapter 4 we will see that this property becomes useful.

A function $f : \Sigma_{X_1 \ldots X_n} \to Y$ is named Σ-operator if there exists a multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ such that $f = f_T$. The operator T, if it exists, is unique. Under these circumstances f and T are said to be associated. The collection of all Σ-operators from $\Sigma_{X_1 \ldots X_n}$ to Y, denoted by $L(\Sigma_{X_1 \ldots X_n}; Y)$, is a vector space with the sum and multiplication by scalars defined pointwise.

In analogy with linear operators, Σ-operators can be represented by functionals defined on a tensor product. To show this, let $f : \Sigma_{X_1 \ldots X_n} \to Y$ be a Σ-operator. Define

$$\begin{aligned} \varphi_f : X_1 \otimes \ldots \otimes X_n \otimes Y^{\#} &\to \mathbb{K} \\ x^1 \otimes \ldots \otimes x^n \otimes y^{\#} &\mapsto \left\langle y^{\#}, f(x^1 \otimes \ldots \otimes x^n) \right\rangle. \end{aligned}$$

It is clear that φ_f is well defined since $f(x^1 \otimes \ldots \otimes x^n) = T(x^1, \ldots, x^n)$. Here, T is the associated multilinear operator of f. The functional φ_f is called the functional associated to the Σ-operator f. It is a simple matter to prove that

$$\begin{aligned} L(\Sigma_{X_1 \ldots X_n}; Y) &\to \left(X_1 \otimes \ldots \otimes X_n \otimes Y^{\#} \right)^{\#} \\ f &\mapsto \varphi_f \end{aligned} \tag{1.2}$$

is a morphism between vector spaces.

In the opposite direction, every linear functional $\varphi : X_1 \otimes \ldots \otimes X_n \otimes Y \to \mathbb{K}$ gives rise to a Σ-operator defined by

$$\begin{array}{rcl} f_\varphi : \Sigma_{X_1 \ldots X_n} & \to & Y^\# \\ x^1 \otimes \ldots \otimes x^n & \mapsto & f_\varphi(x^1 \otimes \ldots \otimes x^n) : y \mapsto \varphi(x^1 \otimes \ldots \otimes x^n \otimes y), \end{array}$$

where the associated multilinear map of f_φ is

$$\begin{array}{rcl} T : X_1 \times \cdots \times X_n & \to & Y^\# \\ (x^1, \ldots, x^n) & \mapsto & T(x^1, \ldots, x^n) : y \mapsto \varphi(x^1 \otimes \ldots \otimes x^n \otimes y). \end{array}$$

In this situation, we say that f_φ is the associated Σ-operator to the functional φ. An easy computation shows that

$$\begin{array}{rcl} (X_1 \otimes \ldots \otimes X_n \otimes Y)^\# & \to & L\left(\Sigma_{X_1 \ldots X_n}; Y^\#\right) \\ \varphi & \mapsto & f_\varphi \end{array} \tag{1.3}$$

is a linear morphism.

In the linear case, the duality between operators and functionals on a tensor product leads us to consider extension to double duals. Nowadays, these extensions are called canonical extensions, see [41, 9]. Other extensions to double duals can be find in [8]. To clarify the representation of Σ-operators by functionals let us denote, for $\varphi : X_1 \otimes \ldots \otimes X_n \otimes Y \to \mathbb{K}$, its canonical extension by $\overline{\varphi}$. This functional is given by

$$\begin{array}{rcl} \overline{\varphi} : X_1 \otimes \ldots \otimes X_n \otimes Y^{\#\#} & \to & \mathbb{K} \\ x^1 \otimes \ldots \otimes x^n \otimes y^{\#\#} & \mapsto & \left\langle y^{\#\#}, f_\varphi(x^1 \otimes \ldots \otimes x^n) \right\rangle. \end{array}$$

If we consider $X_1 \otimes \ldots \otimes X_n \otimes Y$ as a linear subspace of $X_1 \otimes \ldots \otimes X_n \otimes Y^{\#\#}$, the functional $\overline{\varphi}$ is actually an extension of φ since for every $x^i \in X_i$ and $y \in Y$ we have

$$\begin{array}{rcl} \overline{\varphi}(x^1 \otimes \ldots \otimes x^n \otimes y) & = & \left\langle f_\varphi(x^1 \otimes \ldots \otimes x^n), y \right\rangle \\ & = & \varphi(x^1 \otimes \ldots \otimes x^n \otimes y). \end{array}$$

The next proposition clarifies the representation of Σ-operators by functionals via the canonical extension.

Proposition 1.1. *Let $X_1, \ldots, X_n$ and Y be vector spaces. If $\varphi : X_1 \otimes \ldots \otimes X_n \otimes Y \to \mathbb{K}$ is a linear functional and if $f : \Sigma_{X_1 \ldots X_n} \to Y$ is a Σ-operator, then*

i) $\varphi_{f_\varphi} = \overline{\varphi}$.

ii) $f_{\varphi_f} = K_Y f$.

iii) The spaces $(X_1 \otimes \ldots \otimes X_n \otimes Y)^{\#}$ *and* $L\left(\Sigma_{X_1 \ldots X_n}; Y^{\#}\right)$ *are linearly isomorphic via (1.3).*

Proof. Identity (i) is immediate from the definition of the canonical extension of φ. To prove (ii) it is enough to see that

$$\begin{aligned} \left\langle f_{\varphi_f}(p), y^{\#} \right\rangle &= \left\langle \varphi_f, p \otimes y^{\#} \right\rangle \\ &= \left\langle y^{\#}, f(p) \right\rangle \\ &= \left\langle K_Y f(p), y^{\#} \right\rangle \end{aligned}$$

holds for all $p \in \Sigma_{X_1 \ldots X_n}$ and $y^{\#} \in Y^{\#}$.

The morphism inverse of (1.3) is

$$\begin{aligned} L\left(\Sigma_{X_1 \ldots X_n}; Y^{\#}\right) &\to (X_1 \otimes \ldots \otimes X_n \otimes Y)^{\#} \\ f &\mapsto \varphi_f|_{X_1 \otimes \ldots \otimes X_n \otimes Y}, \end{aligned}$$

since (i) implies

$$\varphi_{f_\varphi}|_{X_1 \otimes \ldots \otimes X_n \otimes Y} = \overline{\varphi}|_{X_1 \otimes \ldots \otimes X_n \otimes Y} = \varphi.$$

Conversely,

$$\begin{aligned} \left\langle f_{\varphi_f|_{X_1 \otimes \ldots \otimes X_n \otimes Y}}(p)\, , \, y \right\rangle &= \langle \varphi_f|_{X_1 \otimes \ldots \otimes X_n \otimes Y}\, , \, p \otimes K_Y(y) \rangle \\ &= \langle \varphi_f\, , \, p \otimes K_Y(y) \rangle \\ &= \langle K_Y(y)\, , \, f(p) \rangle \\ &= \langle f(p)\, , \, y \rangle \end{aligned}$$

for all $p \in \Sigma$ and $y \in Y$. ■

A simple case of Σ-operators occurs when the range is contained in a finite dimensional vector space. We say that the Σ-operator $f : \Sigma_{X_1 \ldots X_n} \to Y$ has finite rank if $f(\Sigma_{X_1 \ldots X_n})$ is contained in a finite dimensional subspace of Y. This is equivalent to saying that the linearization of the associated multilinear operator $\widetilde{T}$ has finite rank as a linear operator between vector spaces. The collection of all Σ-operators of finite rank from $\Sigma_{X_1 \ldots X_n}$ to Y is denoted by $\mathcal{F}(\Sigma_{X_1 \ldots X_n}; Y)$.

For the case of finite rank Σ-operators, the space $\mathcal{F}\left(\Sigma_{X_1\ldots X_n};Y\right)$ has a simpler tensorial representation. In this case, we have the following linear isomorphism

$$L(X_1,\ldots,X_n)\otimes Y \cong \mathcal{F}\left(\Sigma_{X_1\ldots X_n};Y\right)$$

described by $\varphi\otimes y \sim \varphi\cdot y$. Here,

$$\begin{array}{rcl}\varphi\cdot y:\Sigma_{X_1\ldots X_n} & \to & Y\\ p & \mapsto & \varphi(p)\,y.\end{array}$$

If the tensor v in $L(X_1,\ldots,X_n)\otimes Y$ and the Σ-operator f are related under the above linear isomorphism, then they are called associated.

Since $X_1\otimes\ldots\otimes X_n\otimes Y$ is linearly isomorphic to $(X_1\otimes\ldots\otimes X_n)\otimes Y$ it is easy to see that $L(X_1,\ldots,X_n)\otimes Y^{\#}$ is algebraically embedded in $(X_1\otimes\ldots\otimes X_n\otimes Y)^{\#}$ under the morphism generated by

$$\begin{array}{rcl}\varphi\otimes y^{\#}: X_1\otimes\ldots\otimes X_n\otimes Y & \to & \mathbb{K}\\ x^1\otimes\ldots\otimes x^n\otimes y & \mapsto & \varphi(x^1\otimes\ldots\otimes x^n)\,y^{\#}(y).\end{array}$$

The translation of this algebraic embedding to the context of Σ-operators is the inclusion of $\mathcal{F}\left(\Sigma_{X_1\ldots X_n};Y^{\#}\right)$ in $L\left(\Sigma_{X_1\ldots X_n};Y^{\#}\right)$. A picture of this situation is presented by the commutative diagram

$$\begin{array}{ccc}(X_1\otimes\ldots\otimes X_n\otimes Y)^{\#} & \longrightarrow & L\left(\Sigma_{X_1\ldots X_n};Y^{\#}\right)\\ \uparrow & & \uparrow\\ L(X_1,\ldots,X_n)\otimes Y^{\#} & \longrightarrow & \mathcal{F}\left(\Sigma_{X_1\ldots X_n};Y^{\#}\right)\end{array} \tag{1.4}$$

where the horizontal arrows are linear isomorphisms and the vertical ones are linear embeddings.

The implications of Diagram (1.4) for multilinear operators are the well known facts that $(X_1\otimes\ldots\otimes X_n\otimes Y)^{\#}$ is isomorphic to $L\left(X_1,\ldots,X_n;Y^{\#}\right)$, and the vector space of all finite rank multilinear operators is linearly isomorphic to $L(X_1,\ldots,X_n)\otimes Y^{\#}$.

Remark 1.2. *Notice that the considered tensor products have the form $X_1\otimes\ldots\otimes X_n\otimes Y$, where the first n factors are those spaces which constitute the domain of a typical Σ-operator f (and so the domain of the associated multilinear operator of f).*

Throughout this dissertation we deal with tensor products of the form $X_1\otimes\ldots\otimes X_n\otimes Y$ i.e. tensor products of $n+1$ factors whose first n spaces are distinguished. This approach becomes fundamental in the definitions of Σ-tensor norms and ideals of Σ-operators, see definitions 2.17, 2.20 and 2.3.

1.2 Bounded Σ-Operators

In this section we push diagram (1.4) one step beyond by dealing with the case of bounded multilinear operators between Banach spaces. As a consequence, we will see how the notion of bounded Σ-operators arises.

Throughout this section n is a natural number, and now $X_1, \dots, X_n, Y$ are Banach spaces. This assumption is not essential since all the results remain valid for normed spaces.

First, let us recall the definition of a reasonable crossnorm in the sense of [56]. A reasonable crossnorm β on the n-fold tensor product $X_1 \otimes \dots \otimes X_n$ is, by definition, a norm with the following properties:

i) $\beta(x^1 \otimes \dots \otimes x^n) \leq \|x^1\| \cdots \|x^n\|$ for all $x^1 \otimes \dots \otimes x^n \in \Sigma_{X_1 \dots X_n}$.

ii) If $x_i^* \in X_i^*$, then the linear functional

$$\begin{array}{rcl} x_1^* \otimes \dots \otimes x_n^* : (X_1 \otimes \dots \otimes X_n, \beta) & \to & \mathbb{K} \\ x^1 \otimes \dots \otimes x^n & \mapsto & x_1^*(x_1) \cdots x_n^*(x_n) \end{array}$$

is bounded and $\|x_1^* \otimes \dots \otimes x_n^*\| \leq \|x_1^*\| \cdots \|x_n^*\|$.

These conditions say, by definition, that β is crossed and reasonable respectively. It is easy to see that if the two conditions are verified simultaneously, then both are indeed equalities. This type of norms was studied in [56] where the authors use them to represent maximal ideals of multilinear operators. In this dissertation we are interested only in the behavior of the topologies induced on Σ by the resulting normed space. The most important examples of reasonable crossnorms (tensor norms in the style of [56]) are the projective tensor norm

$$\pi(u) = \inf \left\{ \sum_i \|x_i^1\| \cdots \|x_i^n\| \;\middle|\; u = \sum_i x_i^1 \otimes \cdots \otimes x_i^n \right\}$$

and the injective tensor norm

$$\varepsilon(u) = \sup \left\{ \, |\langle x_1^* \otimes \dots \otimes x_n^* \, , \, u \rangle| \; | \; x_i^* \in B_{X_i^*} \, \right\}.$$

It is easy to see that β is reasonable and crossed if and only if $\varepsilon(u) \leq \beta(u) \leq \pi(u)$ holds for all $u \in X_1 \otimes \dots \otimes X_n$. As a consequence, all the reasonable crossnorms coincide on Σ. This is, the value of $\beta(x^1 \otimes \dots \otimes x^n)$ is $\|x^1\| \cdots \|x^n\|$ no matter the reasonable crossnorm β.

The algebraic tensor product $X_1 \otimes \dots \otimes X_n$ endowed with a reasonable crossnorm β is denoted by $(X_1 \otimes \dots \otimes X_n, \beta)$.

Remark 1.3. *We do not use the notation $X_1 \otimes_\beta \dots \otimes_\beta X_n$; commonly, it implies more properties on the norm β (uniformity for instance).*

From now on, β denotes a reasonable crossnorm on $X_1 \otimes \ldots \otimes X_n$.

The normed space $(X_1 \otimes \ldots \otimes X_n, \beta)$ helps us to give Σ a structure of topological space. For this end, we have two options: the topology given by the norm and the weak topology. For the case of the norm, it turns out that no matter how we chose the norm β, the topology on Σ is the same. On the other hand, the weak topology behaves differently for each reasonable crossnorm.

In the case of the norm, the lack of enough algebraic structure of Σ implies that the topology induced on it is not given by a norm but by a metric. This metric has the habitual definition

$$\begin{aligned} d_\beta : \Sigma \times \Sigma &\rightarrow \mathbb{R}^+ \cup \{0\} \\ (p,q) &\mapsto \beta(p-q). \end{aligned}$$

The values of d_β are determined by the values of β on the set Σ-Σ (or equivalently Σ+Σ), this is, the set of tensors of rank less than or equal to two. In this way, we have a metric on Σ for each reasonable crossnorm. Despite this, we have the same metric topology as we may read in Theorem 2.1 of [54] . It establishes that

Theorem. *Let $X_1 \ldots X_n$ be Banach spaces, $r \in \mathbb{N}$ and α and β be reasonable crossnorms on $X_1 \otimes \ldots \otimes X_n$. Then, the following metric spaces are Lipschitz equivalent:*

$$\left(\mathcal{S}^r_{X_1,\ldots,X_n}, d_\alpha\right) \cong \left(\mathcal{S}^r_{X_1,\ldots,X_n}, d_\beta\right).$$

In fact, for every $w, z \in \mathcal{S}^r_{X_1,\ldots,X_n}$ $d_\alpha(w,z) \leq (2r)^{n-1} d_\beta(w,z)$ and $\alpha(z) \leq r^{n-1}\beta(z)$.

In this theorem the symbol $\mathcal{S}^r_{X_1,\ldots,X_n}$ denotes the set of tensors of rank less than or equal to r and d_α denotes the induced metric by α. The case $r = 2$ asserts that

$$\pi(p-q) \leq 2^{n-1}\beta(p-q)$$

holds for all reasonable crossnorm β and $p, q \in \Sigma$. This inequality in addition to $\beta \leq \pi$ leads us to conclude that all the reasonable crossnorms on $X_1 \otimes \ldots \otimes X_n$ induce the same metric topology on Σ.

The topological structure induced on Σ by the weak topology of $(X_1 \otimes \ldots \otimes X_n, \beta)$ becomes indispensable since many ideal properties (p-summability for instance) are established in terms of weak summable sequences. Unlike the case of the norm, two different reasonable crossnorms may produce two different weak topological structures on Σ. More details of this affirmation can be found in Remark 3.5 of [54] where the author highlights that the sequence $\{e_i \otimes e_i\}_i$ converges weakly to zero in the prehilbert tensor product $(\ell_2 \otimes \ell_2, H)$ but it does not in the projective tensor product $(\ell_2 \otimes \ell_2, \pi)$.

When we deal with subspaces, a good behavior of weak topologies is essential for properties as maximality (of an ideal) and finite generation (of a tensor norm) since they depend on the behavior on $\Sigma_{E_1 \ldots E_n}$ where E_i is a finite dimensional subspace of X_i. Although the weak case does not work very well in general, for subspaces we have a satisfactory behavior.

Let E_i be a closed subspace of X_i. Let us denote by $\beta|_{E_1,\ldots,E_n}$ the norm induced on $E_1 \otimes \ldots \otimes E_n$ by $(X_1 \otimes \ldots \otimes X_n, \beta)$. If the context is clear we write $\beta|$. It is convenient to have the next diagram in mind

$$\begin{array}{ccc} E_1 \otimes \ldots \otimes E_n & \longrightarrow & X_1 \otimes \ldots \otimes X_n \\ \uparrow & & \uparrow \\ \Sigma_{E_1 \ldots E_n} & \longrightarrow & \Sigma_{X_1 \ldots X_n} \end{array} ,$$

where all the arrows are algebraic inclusions. It is easy to see that $\beta|$ is a reasonable crossnorm on $E_1 \otimes \ldots \otimes E_n$. Thus, we may consider the metric induced by $\beta|$ on $\Sigma_{E_1 \ldots E_n}$ but this is the same as that induced by $\Sigma_{X_1 \ldots X_n}$ and $(X_1 \otimes \ldots \otimes X_n, \beta)$. The weak topology induced on $\Sigma_{E_1 \ldots E_n}$ is also the same since the weak topology of $(E_1 \otimes \ldots \otimes E_n, \beta|)$ is precisely the one induced on $E_1 \otimes \ldots \otimes E_n$ by the weak topology of $(X_1 \otimes \ldots \otimes X_n, \beta)$. This observation becomes fundamental in Definitions 3.4, 3.1 and 3.7.

Let us fix some notation. The set $\Sigma_{X_1 \ldots X_n}$ endowed with the metric induced by the normed space $(X_1 \otimes \ldots \otimes X_n, \beta)$ is denoted by $\Sigma^{\beta}_{X_1 \ldots X_n}$. This notation, besides denoting a metric space, recalls the isometry

$$\Sigma^{\beta}_{X_1 \ldots X_n} \to (X_1 \otimes \ldots \otimes X_n, \beta) \, . \tag{1.5}$$

Thus, if we have to consider a weak topology on $\Sigma_{X_1 \ldots X_n}$ we use that induced by the normed space $(X_1 \otimes \ldots \otimes X_n, \beta)$.

Now, consider a multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ between Banach spaces. Due to the properties of the projective tensor norm (in the sense of [56]) it is easy to prove that the following statements are equivalent:

i) $T : X_1 \times \cdots \times X_n \to Y$ is bounded.

ii) $f_T : \Sigma^{\pi}_{X_1 \ldots X_n} \to Y$ is Lipschitz.

iii) $\widetilde{T} : (X_1 \otimes \ldots \otimes X_n, \pi) \to Y$ is bounded.

In this situation, $\|T\| = Lip^{\pi}(f_T) = \|\widetilde{T}\|$. This result is contained in [54] as Theorem 3.2. We may conclude that, just as the linear operator $\widetilde{T}$ captures the boundedness of the multilinear operator T, the Σ-operator f_T also does it but in terms of a metric space and a Lipschitz function. This observation is crucial since it says that we may study the bounded multilinear operator T via its associated Σ-operator f_T which is a Lipschitz function whose domain

$\Sigma^{\pi}_{X_1\dots X_n}$ is contained in the normed space $(X_1 \otimes \ldots \otimes X_n, \pi)$. Thus, we may use the richness that the topological environment $(X_1 \otimes \ldots \otimes X_n, \pi)$ offers, for example, the norm π and its weak topology.

The fact that all the metric spaces $\Sigma^{\beta}_{X_1\dots X_n}$ are Lipschitz equivalent implies that the Σ-operator $f_T : \Sigma^{\beta}_{X_1\dots X_n} \to Y$ is a Lipschitz function for all reasonable crossnorms β and, due to the comments of the last paragraph, this occurs exactly when T is bounded. Even more, the Lipschitz norms are related as follows

$$Lip^{\pi}(f_T) \leq Lip^{\beta}(f_T) \leq 2^{n-1} Lip^{\pi}(f_T).$$

Proposition 3.4 of [54] asserts that the equivalence of (ii) and (iii) above, only occurs (up to isomorphisms) for the reasonable crossnorm π. As a consequence, the Lipschitz property of $f_T : \Sigma^{\beta}_{X_1\dots X_n} \to Y$ implies no more than the boundedness of T.

Throughout this dissertation we are not interested in the boundedness of the linear operator $\widetilde{T} : (X_1 \otimes \ldots \otimes X_n, \beta) \to Y$ but in the (ideal) properties we may define for the associated Σ-operator $f_T : \Sigma^{\beta}_{X_1\dots X_n} \to Y$. As we can see in Chapter 4, the more interesting cases appear when we consider weak properties, and as we have already said, it is important the choice of the norm β.

In the sequel, almost all the definitions are stated for spaces $X_1, \ldots, X_n$ and Y and a reasonable crossnorm β. For a good manipulation in the notation and the language, we say that an arrangement $(X_1, \ldots, X_n, Y, \beta)$, where $X_1, \ldots, X_n, Y$ are Banach spaces and β is a reasonable crossnorm on $X_1 \otimes \ldots \otimes X_n$, will be named an election in $\mathcal{BAN}$. Elections in $\mathcal{FIN}$, for finite dimensional normed spaces, and $\mathcal{NORM}$ for normed spaces follow obvious definitions.

Definition 1.4. *The Σ-operator $f : \Sigma^{\beta}_{X_1\dots X_n} \to Y$, where $(X_1, \ldots, X_n, Y, \beta)$ is an election in $\mathcal{BAN}$, is said to be bounded if its associated multilinear operator is bounded.*

The collection of all bounded Σ-operators form $\Sigma^{\beta}_{X_1\dots X_n}$ into Y is denoted by $\mathcal{L}\left(\Sigma^{\beta}_{X_1\dots X_n}, Y\right)$ and it becomes a Banach space with the norm Lip^{β}. This is,

$$Lip^{\beta}(f) = \sup_{\beta(p-q)\leq 1} \|f(p) - f(q)\|$$

for every Σ-operator $f : \Sigma^{\beta}_{X_1\dots X_n} \to Y$.

As the reader knows, the theory of ideals of bounded linear operators is based on the ideal property. This establishes that ideal properties are preserved by composition with bounded linear operators. For the multilinear case there is a popular notion of this property as we can see in [56]. In this section, we present an accurate notion of the ideal property for bounded Σ-operators. For this end, we describe the compositions to be considered. Since the approach we are presenting is developed for the study of multilinear operator we translate this kind of compositions to multilinear language.

Consider Banach spaces $X_1, \ldots, X_n$, $Z_1, \ldots, Z_n$ and let β and θ be reasonable crossnorms on $X_1 \otimes \ldots \otimes X_n$ and $Z_1 \otimes \ldots \otimes Z_n$, respectively. Assume that the bounded multilinear operator $R : Z_1 \times \cdots \times Z_n \to (X_1 \otimes \ldots \otimes X_n, \beta)$ satisfies:

i) $\widetilde{R} : (Z_1 \otimes \ldots \otimes Z_n, \theta) \to (X_1 \otimes \ldots \otimes X_n, \beta)$ is bounded.

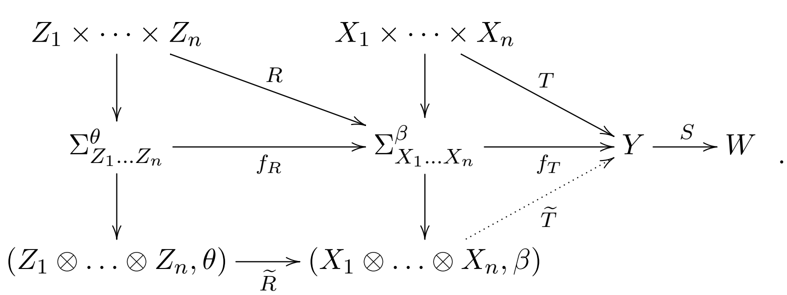
ii) $f_R\left(\Sigma_{Z_1 \ldots Z_n}\right) \subset \Sigma_{X_1 \ldots X_n}$.

Then, the bounded Σ-operator $f_R : \Sigma^{\theta}_{Z_1 \ldots Z_n} \to (X_1 \otimes \ldots \otimes X_n, \beta)$ associated to R is named Σ-θ-operator. The first condition lets us manipulate the induced topologies on $\Sigma^{\theta}_{Z_1 \ldots Z_n}$ and $\Sigma^{\beta}_{X_1 \ldots X_n}$ by the normed spaces $(Z_1 \otimes \ldots \otimes Z_n, \theta)$ and $(X_1 \otimes \ldots \otimes X_n, \beta)$, respectively. The second property lets us compose Σ-θ-operators, Σ-operators and linear operators as is shown in the diagram

$$\Sigma^{\theta}_{Z_1 \ldots Z_n} \xrightarrow{f_R} \Sigma^{\beta}_{X_1 \ldots X_n} \xrightarrow{f_T} Y \xrightarrow{S} W. \tag{1.6}$$

This type of composition is considered for the ideal property of an ideal of Σ-operators, see Definition 2.3. In [74] and [102] the authors exhibit the general form of the operators on tensor products that preserve Σ.

The next diagram exhibits the complete situation of a typical composition. In it, all the involved operators are present



All the labeled arrows mean boundedness or Lipschitz conditions respectively while the unlabeled arrows are the natural inclusions and isometries respectively. The dotted line means that $\widetilde{T}$ may not be bounded. The bounded multilinear operator $Sf_T R : Z_1 \times \cdots \times Z_n \to W$ has the composition $Sf_T f_R$ as its associated bounded Σ-operator.

Classical compositions for multilinear operators like

$$\begin{array}{ccccc} Z_1 \times \cdots \times Z_n & & & & \\ {\scriptstyle T_1}\downarrow \qquad \downarrow{\scriptstyle T_n} & & & & \\ X_1 \times \cdots \times X_n & \xrightarrow[T]{} & Y & \xrightarrow[S]{} & W \end{array}$$

are considered for the case $\beta = \pi(\,\cdot\,; X_1, \ldots, X_n)$ and $\theta = \pi(\,\cdot\,; Z_1, \ldots, Z_n)$, or any other tensor norm in the sense of [56] because they verify the uniform property.

The next proposition exhibits the associated functional of a composition like (1.6).

Proposition 1.5. *Consider the composition*

$$\Sigma^{\theta}_{Z_1 \ldots Z_n} \xrightarrow{f_R} \Sigma^{\beta}_{X_1 \ldots X_n} \xrightarrow{f} Y \xrightarrow{S} W$$

where S is a bounded linear operator, f is a bounded Σ-operator and f_R is a Σ-θ-operator with associated multilinear operator R. Then, the functional associated to the composition $Sff_R : \Sigma^{\theta}_{Z_1 \ldots Z_n} \to W$ is:

i) $\varphi_f \circ (\widetilde{R} \otimes S^*) : Z_1 \otimes \ldots \otimes Z_n \otimes W^* \to \mathbb{K}$.

ii) $v_{Sff_R} = \widetilde{R}^* \otimes S(v_f) \in \mathcal{L}(Z_1, \ldots, Z_n) \otimes W$ *if* $f = \sum_i \varphi_i \cdot y_i \in \mathcal{F}(\Sigma_{X_1 \ldots X_n}; Y)$.

Proof. By definition, we have that

$$\begin{aligned} \left\langle \varphi_{Sff_R}\,,\, z^1 \otimes \ldots \otimes z^n \otimes w^* \right\rangle &= \left\langle w^*\,,\, Sff_R(z^1 \otimes \ldots \otimes z^n) \right\rangle \\ &= \left\langle S^* w^*\,,\, ff_R(z^1 \otimes \ldots \otimes z^n) \right\rangle \\ &= \left\langle \varphi_f\,,\, f_R(z^1 \otimes \ldots \otimes z^n) \otimes S^* w^* \right\rangle \\ &= \left\langle \varphi_f \widetilde{R} \otimes S^*\,,\, z^1 \otimes \ldots \otimes z^n \otimes w^* \right\rangle \end{aligned}$$

for all $z^1 \otimes \ldots \otimes z^n \otimes w^* \in Z_1 \otimes \ldots \otimes Z_n \otimes W^*$. This proves (i).

On the other hand, for (ii) we have that

$$\begin{aligned} Sff_R(q) &= S\left(\sum_i \varphi_i(f_R(q))\, y_i\right) \\ &= \sum_i \varphi_i(f_R(q))\, Sy_i \\ &= \left\langle \sum_i (\varphi_i \circ f_R) \cdot Sy_i\,,\, q \right\rangle \\ &= \left\langle \sum_i (\widetilde{R}^* \varphi_i) \cdot Sy_i\,,\, q \right\rangle \end{aligned}$$

holds for all $q \in \Sigma_{Z_1 \ldots Z_n}$. Hence

$$\begin{aligned} v_{Sff_R} &= \sum_i (\widetilde{R}^* \varphi_i) \otimes S y_i \\ &= \sum_i \left\langle \widetilde{R}^* \otimes S \, , \, \varphi_i \otimes y_i \right\rangle \\ &= \widetilde{R}^* \otimes S(v_f). \end{aligned}$$

■

Σ-θ-operators are also useful to define uniformity for Σ-tensor norms on spaces. To explain this, let $f_R : \Sigma^{\theta}_{Z_1 \ldots Z_n} \to \Sigma^{\beta}_{X_1 \ldots X_n}$ be a Σ-θ-operator and let $S : W \to Y$ be a bounded linear operator. The assignment

$$\begin{aligned} f_R \otimes S : Z_1 \otimes \ldots \otimes Z_n \otimes W &\to X_1 \otimes \ldots \otimes X_n \otimes Y \\ z^1 \otimes \ldots \otimes z^n \otimes w &\mapsto f_R(z^1 \otimes \ldots \otimes z^n) \otimes Sw \end{aligned}$$

is a well defined function since $f_R = \widetilde{R}$ in $\Sigma_{Z_1 \ldots Z_n}$, see Definition 2.17.

For the case of Σ-tensor norms on duals we require an accurate uniform property. This is established in terms of another type of operator that we describe next. First, we fix some notation.

Any multilinear form $\varphi : X_1 \times \cdots \times X_n \to \mathbb{K}$ such that $\tilde{\varphi} : (X_1 \otimes \ldots \otimes X_n, \beta) \to \mathbb{K}$ is bounded is called β-form. We set

$$\mathcal{L}^{\beta}(X_1, \ldots, X_n) := \{ \, \varphi : X_1 \times \cdots \times X_n \to \mathbb{K} \mid \varphi \text{ is a } \beta\text{-form} \, \} \, .$$

Clearly, the Banach space $(X_1 \otimes \ldots \otimes X_n, \beta)^*$ induces a Banach structure on $\mathcal{L}^{\beta}(X_1, \ldots, X_n)$ under the identification $\varphi \sim \tilde{\varphi}$. If $\varphi \in \mathcal{L}^{\beta}(X_1, \ldots, X_n)$ we denote its norm by $\|\varphi\|_{\beta}$. Set δ as the evaluation map

$$\begin{aligned} \delta : (X_1 \otimes \ldots \otimes X_n, \beta) &\to \mathcal{L}^{\beta}(X_1, \ldots, X_n)^* \\ u &\mapsto \delta_u : \varphi \mapsto \langle \varphi, u \rangle \, . \end{aligned}$$

We say that the bounded linear operator $A : \mathcal{L}^{\beta}(X_1, \ldots, X_n) \to \mathcal{L}^{\theta}(Z_1, \ldots, Z_n)$ preserves Σ if for every simple tensor $q \in \Sigma_{Z_1 \ldots Z_n}$ there exists a simple tensor $p \in \Sigma_{X_1 \ldots X_n}$ such that

$$A^*(\delta_q) = \delta_p.$$

In other words, $A^* : \mathcal{L}^{\theta}(Z_1, \ldots, Z_n)^* \to \mathcal{L}^{\beta}(X_1, \ldots, X_n)^*$ maps evaluations on simple tensors into evaluations on simple tensors.

It is well known that many ideal properties are described in terms of weak summable sequences. In the case of bounded Σ-operators, this also occurs. Let us make the kind of sequences to be considered precise.

Let $p \in [1, \infty)$. For each pair of sequences $(p_i), (q_i)$ in $\Sigma^{\beta}_{X_1 \dots X_n}$ we write

$$\|(p_i - q_i)\|_p^{w\beta} := \sup \left(\sum_i |\varphi(p_i) - \varphi(q_i)|^p \right)^{\frac{1}{p}}$$

where the supremum is taken over all $\varphi \in \mathcal{L}^{\beta}(X_1, \dots, X_n)$ with $\|\varphi\|_{\beta} \leq 1$. If the last supremum is finite, we are in the case of a weakly p-summable sequence of the form $(p_i - q_i)$ in the normed space $(X_1 \otimes \dots \otimes X_n, \beta)$. We are not interested in all the weakly p-summable sequences in the normed space $(X_1 \otimes \dots \otimes X_n, \beta)$.

Analogously, for each sequence (φ_i) in $\mathcal{L}^{\beta}(X_1, \dots, X_n)$ we write

$$\|(\varphi_i)\|_p^{wd} := \sup \left(\sum_i |\varphi_i(p) - \varphi_i(q)|^p \right)^{\frac{1}{p}}$$

where the supremum is taken over all $p, q \in \Sigma$ such that $\beta(p - q) \leq 1$. In this case, we are not in the case of a weakly p-summable sequence in the normed space $\mathcal{L}^{\beta}(X_1, \dots, X_n)$ since the supremum just takes into account functionals of the form $\delta_p - \delta_q$. These considerations can be interpreted as a combination of the Lipschitz and linear theory.

The next proposition becomes useful in the particular examples of Σ-tensor norms and ideals of Σ-operators that we present in Chapter 4. It describes the behavior of weakly p-summable sequences under Σ-θ-operators and operators that preserve Σ respectively.

Proposition 1.6. *Let $(a_i), (b_i)$ be sequences in $\Sigma^{\theta}_{Z_1 \dots Z_n}$ and (ψ_i) be a sequence in $\mathcal{L}^{\theta}(Z_1, \dots, Z_n)$. If $f_R : \Sigma^{\theta}_{Z_1 \dots Z_n} \to \Sigma^{\beta}_{X_1 \dots X_n}$ is a Σ-θ-operator and $A : \mathcal{L}^{\theta}(Z_1, \dots, Z_n) \to \mathcal{L}^{\beta}(X_1, \dots, X_n)$ preserves Σ, then:*

i) $\|(f_R a_i - f_R b_i)\|_p^{w\beta} \leq \|\widetilde{R}\| \|(a_i - b_i)\|_p^{w\theta}$.

ii) $\|(A(\psi_i))\|_p^{wd} \leq \|A\| \|(\psi_i)\|_p^{wd}$.

Proof. To prove (i) let us estimate

$$
\begin{aligned}
\|(f_R a_i - f_R b_i)\|_p^{w\beta} &= \sup_{\|\varphi\|_\beta \le 1} \left(\sum_i |\varphi(f_R a_i) - \varphi(f_R b_i)|^p \right)^{\frac{1}{p}} \\
&= \sup_{\|\varphi\|_\beta \le 1} \left(\sum_i |\varphi(R a_i) - \varphi(R b_i)|^p \right)^{\frac{1}{p}} \\
&= \sup_{\|\varphi\|_\beta \le 1} \left(\sum_i |\varphi \widetilde{R}(a_i - b_i)|^p \right)^{\frac{1}{p}} \\
&\le \sup_{\|\psi\|_\theta \le 1} \left(\sum_i |\psi(a_i - b_i)|^p \right)^{\frac{1}{p}} \ \|\widetilde{R}\|.
\end{aligned}
$$

Analogously, for (ii) we have

$$
\begin{aligned}
\|(A(\psi_i))\|_p^{wd} &= \sup_{\beta(p-q) \le 1} \left(\sum_i |A(\psi_i)(p) - A(\psi_i)(q)|^p \right)^{\frac{1}{p}} \\
&= \sup_{\beta(p-q) \le 1} \left(\sum_i |A(\psi_i)(p-q)|^p \right)^{\frac{1}{p}} \\
&= \sup_{\beta(p-q) \le 1} \left(\sum_i |\psi_i(A^*\delta_p - A^*\delta_q)|^p \right)^{\frac{1}{p}} \\
&\le \sup_{\theta(a-b) \le 1} \left(\sum_i |\psi_i(a) - \psi_i(b)|^p \right)^{\frac{1}{p}} \ \|A\|
\end{aligned}
$$

■

The definition of a bounded Σ-operator implies that every bounded Σ-operator f is a Lipschitz function. The following operators help us to control the Lipschitz norm of bounded Σ-operators. Even more, in Chapter 3 we show that they are useful to establish that a Σ-tensor norm on duals is crossed. For each $p, q \in \Sigma_{X_1 \ldots X_n}$ and $y^* \in Y^*$ define the linear operator

$$
\begin{aligned}
L_{pqy^*} : Y &\to X_1 \otimes \ldots \otimes X_n \\
y &\mapsto y^*(y)(p-q).
\end{aligned}
$$

Let $T : X_1 \times \cdots \times X_n \to Y$ be a multilinear operator. The composition

$$\begin{array}{ccc} X_1 \otimes \ldots \otimes X_n & \xrightarrow{L_{pqy^*}\widetilde{T}} & X_1 \otimes \ldots \otimes X_n \\ & {\scriptstyle \widetilde{T}} \searrow \qquad \nearrow {\scriptstyle L_{pqy^*}} & \\ & Y & \end{array}$$

verifies $\left\langle L_{pqy^*}\widetilde{T}, u \right\rangle = \left\langle y^*\widetilde{T}, u \right\rangle (p-q)$ for all $u \in X_1 \otimes \ldots \otimes X_n$. The trace of the composition $L_{pqy^*}\widetilde{T}$ is well defined since it is a finite rank linear operator. Even more, it is given by

$$tr(L_{pqy^*}\widetilde{T}) = y^*\widetilde{T}(p-q) = y^*(f_T p - f_T q).$$

We conclude this chapter by giving the representation of bounded Σ-operators via the projective tensor norm (in the sense of [56]) defined on the tensor product $X_1 \otimes \ldots \otimes X_n \otimes Y$. Even more, we represent bounded Σ-operators of finite rank in terms of the injective norm of the 2-fold tensor product. As we will see in the next chapter, these norms are particular cases of the projective Σ-tensor norm on spaces and the injective Σ-tensor norm on duals.

If $(X_1, \ldots, X_n, Y, \beta)$ is an election in $\mathcal{BAN}$, we define

$$\mathcal{F}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right) := \left\{ f_T : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y \mid \widetilde{T} \text{ is } \beta\text{-bounded and has finite rank} \right\}.$$

In this case, notice that the norm β becomes relevant since there exist bounded forms $\varphi : X_1 \otimes \ldots \otimes X_n \to \mathbb{K}$ such that $\tilde{\varphi} : (X_1 \otimes \ldots \otimes X_n, \beta) \to \mathbb{K}$ is not bounded. In particular, $\mathcal{F}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ and $\mathcal{F}\left(\Sigma^{\theta}_{X_1 \ldots X_n}; Y\right)$ may not be isomorphic for different reasonable crossnorms β and θ.

Proposition 1.7. *Let $X_1, \ldots, X_n, Y$ be Banach spaces. Let π be the projective tensor norm on the $(n+1)$-fold tensor product $X_1 \otimes \ldots \otimes X_n \otimes Y$. Then the mapping*

$$\begin{array}{rcl} (X_1 \otimes \ldots \otimes X_n \otimes Y, \pi)^* & \to & \mathcal{L}\left(\Sigma^{\pi}_{X_1 \ldots X_n}; Y^*\right) \\ \varphi & \mapsto & f_{\varphi}. \end{array}$$

is a linear isometric isomorphism. If ε denotes the 2-fold injective tensor norm, then

$$\begin{array}{rcl} \left(\mathcal{L}^{\pi}\left(X_1, \ldots, X_n\right) \otimes Y^*, \varepsilon\right) & \to & \mathcal{F}\left(\Sigma^{\pi}_{X_1 \ldots X_n}, Y^*\right) \\ \varphi \otimes y^* & \mapsto & \varphi \cdot y^*. \end{array}$$

is a linear isometric isomorphism.

Proof. From the linear isometry $(X_1 \otimes \ldots \otimes X_n \otimes Y, \pi) = ((X_1 \otimes \ldots \otimes X_n, \pi) \otimes Y, \pi)$ we deduce that

$$\begin{aligned} (X_1 \otimes \ldots \otimes X_n \otimes Y, \pi)^* &= ((X_1 \otimes \ldots \otimes X_n, \pi) \otimes Y, \pi)^* \\ &= \mathcal{L}\left((X_1 \otimes \ldots \otimes X_n, \pi); Y^*\right) \\ &= \mathcal{L}\left(\Sigma^{\pi}_{X_1 \ldots X_n}; Y^*\right) \end{aligned}$$

are linear isometric isomorphisms. For the second case recall that the norm induced on $\mathcal{L}^{\pi}(X_1, \ldots, X_n) \otimes Y^*$ by the space $((X_1 \otimes \ldots \otimes X_n, \pi) \otimes Y, \pi)^*$ is the injective tensor norm for the case of two factors. ∎

In particular, Proposition 1.7 ensures that the diagram

$$\begin{array}{ccc} (X_1 \otimes \ldots \otimes X_n \otimes Y, \pi)^* & \longrightarrow & \mathcal{L}\left(\Sigma^{\pi}_{X_1 \ldots X_n}; Y^*\right) \\ \uparrow & & \uparrow \\ (\mathcal{L}^{\pi}(X_1, \ldots, X_n) \otimes Y^*, \varepsilon) & \longrightarrow & \mathcal{F}\left(\Sigma^{\pi}_{X_1 \ldots X_n}, Y^*\right) \end{array} \tag{1.7}$$

is commutative. The horizontal arrows are linear isometric isomorphisms and the vertical ones are linear isometries.

Chapter 2

Ideals of Σ-Operators and Σ-Tensor Norms

In this chapter we present the most important concepts of the dissertation, namely, ideals of Σ-operators and Σ-tensor norms. We begin by studying the collections of bounded and p-summing Σ-operators. Next, we establish the definition of an ideal of Σ-operators. We continue with the projective and injective Σ-tensor norms. These cases allow us to explain the necessity of two types of Σ-tensor norms. We conclude the chapter by showing the relation between ideals of Σ-operators, Σ-tensor norms on duals and Σ-tensor norms on spaces in the class $\mathcal{FIN}$.

2.1 Ideals of Σ-Operators

The notion of ideal of multilinear operators has its origin in [86]. The subsequent research following this approximation (and related topics) is huge and has plenty of references, see for instance [1, 15, 16, 17, 18, 20, 21, 29, 32, 43, 55, 56, 75, 77, 88, 90]. On the side of Lipschitz theory, a recent notion of ideals of Lipschitz mappings (from a metric space into a Banach space) has appeared. Some references for this approximation are [2, 24, 26, 36, 60].

Before presenting the definition of an ideal of Σ-operators let us make some remarks of the collections of bounded and p-summing Σ-operators presented in Chapter 1 and [7] respectively. The properties these collections enjoy will let us introduce the definition of an ideal of Σ-operators easier.

Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. Plainly, $\mathcal{L}\left(\Sigma^{\beta}_{X_1 \ldots X_n}, Y\right)$ is a vector space with the sum and multiplication by scalars defined pointwise. Moreover, it is a Banach space with the Lipschitz norm Lip^{β}.

Every $f_T \in \mathcal{F}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ verifies, by definition, that $\widetilde{T} : (X_1 \otimes \ldots \otimes X_n, \beta) \to Y$ is bounded. In particular T is bounded as a multilinear operator. Thus, $\mathcal{L}\left(\Sigma^{\beta}_{X_1 \ldots X_n}, Y\right)$ contains $\mathcal{F}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ as a linear subspace since the algebraic operations are the same. For the simple case of a rank-one Σ-operator $f = \varphi \cdot y$ with φ in $\mathcal{L}^{\beta}(X_1, \ldots, X_n)$ and $y \in Y$ we have that

$$\begin{aligned} \|f(p) - f(q)\| &= |\varphi(p) - \varphi(q)| \|y\| \\ &\leq \|\varphi\|_{\beta} \|y\| \beta(p - q) \end{aligned}$$

holds for all $p, q \in \Sigma_{X_1 \ldots X_n}$. This means that $Lip^{\beta}(\varphi \cdot y) \leq \|\varphi\|_{\beta} \|y\|$.

For arbitrary $p, q \in \Sigma_{X_1 \ldots X_n}$ and $y^* \in Y^*$, the dominations

$$\begin{aligned} \left|tr(\widetilde{T} L_{pqy^*})\right| &= |y^*(f_T(p) - f_T(q))| \\ &\leq \|y^*\| \|f_T(p) - f_T(q)\| \\ &\leq \beta(p - q) \|y^*\| Lip^{\beta}(f_T) \end{aligned}$$

assert that

$$\sup_{Lip^{\beta}(f_T) \leq 1} \left|tr(\widetilde{T} L_{pqy^*})\right| \leq \beta(p - q) \|y^*\|$$

holds for all $p, q \in \Sigma_{X_1 \ldots X_n}$ and $y^* \in Y^*$. This domination means that the linear functional

$$\begin{aligned} L^{pqy^*} : \mathcal{L}\left(\Sigma^{\beta}_{X_1 \ldots X_n}, Y\right) &\to \mathbb{K} \\ T &\mapsto tr\left(\widetilde{T} L_{pqy^*}\right) \end{aligned}$$

is bounded, and $\|L^{pqy^*}\| \leq \beta(p - q) \|y\|$.

The collection of bounded Σ-operators behaves well under compositions with Σ-θ-operators and linear operators. To see this, consider the composition

$$\Sigma^{\theta}_{Z_1 \ldots Z_n} \xrightarrow{f_R} \Sigma^{\beta}_{X_1 \ldots X_n} \xrightarrow{f} Y \xrightarrow{S} W$$

where f_R is a Σ-θ-operator, f is an element of $\mathcal{L}\left(\Sigma^{\beta}_{X_1 \ldots X_n}, Y\right)$ and $S : Y \to W$ is a bounded linear operator. Then, the inequalities

$$\begin{aligned} \|S f f_R(p) - S f f_R(q)\| &\leq \|S\| \, \|f f_R(p) - f f_R(q)\| \\ &\leq \|S\| \, Lip^{\beta}(f) \, \beta(f_R(p) - f_R(q)) \\ &= \|S\| \, Lip^{\beta}(f) \, \beta(\widetilde{R}(p - q)) \\ &\leq \|S\| \, Lip^{\beta}(f) \, \|\widetilde{R}\| \theta(p - q) \end{aligned}$$

imply $Lip^{\theta}(Sff_R) \leq \|\widetilde{R}\| Lip^{\beta}(f)\|S\|$.

As we will see in Chapter 4, these properties are enjoyed by other collections of Σ-operators. A non trivial example of the same phenomenon is the case of p-summability defined and developed in [7]. Let us recall the definition of a p-summing Σ-operator.

Definition 2.1. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$ and $p \in [1, \infty)$. The Σ-operator $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ is said to be p-summing if there exists a constant $C > 0$ such that for all finite sequences (p_i), (q_i) in $\Sigma^{\beta}_{X_1 \ldots X_n}$ it is verified*

$$\left(\sum_i \|f(p_i) - f(q_i)\|^p \right)^{\frac{1}{p}} \leq C \; \|(p_i - q_i)\|^{w\beta}_p .$$

The p-summing norm of f, denoted by $\pi_p(f)$, is defined by the infimum of the constants C as above.

Given an election $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$, the collection of all p-summing Σ-operators from $\Sigma^{\beta}_{X_1 \ldots X_n}$ into Y is denoted by $\Pi_p\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$. It becomes a Banach space with the norm π_p.

Proposition 2.2. *For any election $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$ the space $\Pi_p\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ verifies:*

i) $\Pi_p\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ *is a Banach space.*

ii) $\mathcal{F}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ *is contained as a linear space in* $\Pi_p\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$.

iii) $\pi_p(\varphi \cdot y) \leq \|\varphi\|_{\beta} \|y\|$ *for all* $\varphi \in \mathcal{L}^{\beta}(X_1, \ldots, X_n)$ *and* $y \in Y$.

iv) $\sup\limits_{\pi^{\beta}_p(f_T) \leq 1} \left| tr(\widetilde{T} L_{pqy^*}) \right| \leq \beta(p - q) \, \|y^*\|$ *for all* $p, q \in \Sigma_{X_1 \ldots X_n}$ *and* $y^* \in Y^*$.

v) If in the composition

$$\Sigma^{\theta}_{Z_1 \ldots Z_n} \xrightarrow{f_R} \Sigma^{\beta}_{X_1 \ldots X_n} \xrightarrow{f} Y \xrightarrow{S} W$$

f_R is a Σ-θ-operator, f is an element of $\Pi_p\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ and $S : Y \to W$ is a bounded linear operator, then Sff_R belongs to $\Pi_p\left(\Sigma^{\theta}_{Z_1 \ldots Z_n}; W\right)$ and $\pi_p(Sff_R) \leq \|\widetilde{R}\| \; \pi_p(f) \; \|S\|$.

Proof. The proof of (i) is contained in [7]. We first prove (iii). Given finite sequences (p_i) and (q_i) in Σ we have

$$\begin{aligned}\sum_i |\varphi(p_i)\, y - \varphi(q_i)\, y|^p &\leq \|\varphi\|_\beta^p \sum_i \left|\frac{\varphi}{\|\varphi\|_\beta}(p_i) - \frac{\varphi}{\|\varphi\|_\beta}(q_i)\right|^p \|y\|^p \\ &\leq \|\varphi\|_\beta^p\, \|y\|^p \sup_{\psi \in B_{(X_1\otimes\ldots\otimes X_n,\beta)^*}} \sum_i |\psi(p_i) - \psi(q_i)|^p.\end{aligned}$$

In other words, every rank-one Σ-operator is p-summing and $\pi_p^\beta(\varphi \cdot y) \leq \|\varphi\|\, \|y\|$. Plainly, (ii) is deduced from (iii).

The item (iv) is easy to check since

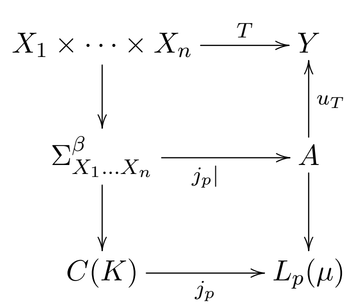
$$\begin{aligned}\left|tr(\widetilde{T} L_{pqy^*})\right| &= |y^*(f_T(p) - f_T(q))| \\ &\leq \|y^*\|\, \|f_T(p) - f_T(q)\| \\ &\leq \|y^*\|\, \pi_p^\beta(f_T) \sup_{\|\varphi\|_\beta \leq 1} |\varphi(p) - \varphi(q)| \\ &= \|y^*\|\, \pi_p^\beta(f_T)\, \beta(p-q).\end{aligned}$$

The property (v) follows from Proposition 1.6. Let (p_i) and (q_i) be finite sequences in $\Sigma_{Z_1\ldots Z_n}$, then

$$\begin{aligned}\sum_i |Sff_R(p_i) - Sff_R(q_i)|^p &\leq \|S\|^p \sum_i |ff_R(p_i) - ff_R(q_i)|^p \\ &\leq \|S\|\, \pi_p^\beta(f)^p \left(\|(f_R(p_i) - f_R(q_i))\|_p^{w\beta}\right)^p \\ &\leq \|S\|^p\, \pi_p^\beta(f)^p\, \|\widetilde{R}\|^p \left(\|((p_i) - (q_i))\|_p^{w\theta}\right)^p.\end{aligned}$$

This way $Sff_R : \Sigma^\theta_{Z_1\ldots Z_n} \to W$ is p-summing and $\pi_p^\theta(SFf_R) \leq \|S\|\, \pi_p^\beta(f)\, \|\widetilde{R}\|$. ∎

One of the most important results of [7] is the factorization theorem for p-summing Σ-operators. It implies that the associated multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ of a p-summing Σ-operator $f_T : \Sigma^\beta_{X_1\ldots X_n} \to Y$ factors as follows:



where K is the unit ball of $(X_1 \otimes \ldots \otimes X_n, \beta)^*$ considered with the w^*-topology (hence a compact subset), μ is a regular Borel probability measure on K, j_p is the natural inclusion, $A = j_p(\Sigma^\beta_{X_1 \ldots X_n}) \subset L_p(\mu)$, the unlabeled arrows are the natural inclusions and the evaluation map respectively and $u_T : A \to Y$ is a Lipschitz function such that $Lip(u_T) \leq \pi_p(f_T)$.

Bearing in mind the behavior of the collections of bounded and p-summing Σ-operators we define an ideal of Σ-operators as follows.

Definition 2.3. *An ideal of* Σ*-operators* $[\mathcal{A}, A]$ *defined on* $\mathcal{BAN}$ *is a subclass* $\mathcal{A}$ *of the class of all bounded* Σ*-operators. For each election* $(X_1, \ldots, X_n, Y, \beta)$ *in* $\mathcal{BAN}$ *a component is defined by*

$$\mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right) := \mathcal{A} \cap \mathcal{L}\left(\Sigma^\beta_{X_1 \ldots X_n}, Y\right)$$

and contains the space $\mathcal{F}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right)$*. The component* $\mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right)$ *is supplied with a complete norm* A *which verifies:*

I1 $A(\varphi \cdot y) \leq \|\varphi\|_\beta \, \|y\|$ *for all* $\varphi \in \mathcal{L}^\beta\left(X_1, \ldots, X_n\right)$ *and* $y \in Y$*.*

I2 $\sup\limits_{A^\beta(f_T) \leq 1} \left|tr(\widetilde{T} L_{pqy^*})\right| \leq \beta(p-q)\,\|y^*\|$ *for all* $p, q \in \Sigma_{X_1 \ldots X_n}$ *and* $y^* \in Y^*$*.*

I3 If in the composition

$$\Sigma^\theta_{Z_1 \ldots Z_n} \xrightarrow{f_R} \Sigma^\beta_{X_1 \ldots X_n} \xrightarrow{f} Y \xrightarrow{S} W$$

f_R *is a* Σ*-*θ*-operator,* f *is an element of* $\mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right)$ *and* $S : Y \to W$ *is a bounded linear operator, then* Sff_R *belongs to* $\mathcal{A}\left(\Sigma^\theta_{Z_1 \ldots Z_n}; W\right)$ *and* $A(Sff_R) \leq \|\widetilde{R}\|\, A(T)\, \|S\|$*.*

Notice that a component in the ideal $[\mathcal{A}, A]$ is defined for every election in $\mathcal{BAN}$, that is, it is not enough to specify the spaces X_i and Y. The property *I1* lets us control the ideal norm of a rank one Σ-operator. The condition *I2* tells us that the linear functional defined by the operator L_{pqy^*} is bounded and we have control of its norm. Clearly, the property $I3$ says that the ideal property is preserved by compositions as (1.6).

Definition 2.3 in the case $n = 1$ coincides with the definition of an ideal of bounded linear operators, see [46, p. 131]. This is clear since $\mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right)$ reduces to $\mathcal{A}\left(X, Y\right)$ with $\beta = \|\cdot\|_X$. The space $\mathcal{F}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right)$ becomes the space of linear operators of finite rank from X into Y. Every Σ-operator of the form $\varphi \cdot y$ is a linear rank-one operator with $\varphi \in X^*$. The properties *I2* and *I3* reduce to $\|T\| \leq A(T)$ and the linear ideal property respectively.

On the other hand, the principal difference between the classical theory of ideals of multilinear operators presented in [56] (well known as multi-ideals) and the approach we present is the consideration of the finite rank operators. In the classical approach, an ideal $[\mathcal{A}, A]$ of multilinear operators requires that for any Banach spaces $X_1, \ldots, X_n$ and Y, the component $\mathcal{A}(X_1, \ldots, X_n; Y)$ contains all the multilinear operators of finite type, that is, all operators $T : X_1 \times \cdots \times X_n \to Y$ which can be expressed as a finite sum of the form

$$T(x^1, \ldots, x^n) = \sum_i x^*_{1i}(x^1) \ldots x^*_{ni}(x^n)\, y_i$$

where $x^*_{ji} \in X^*_j$ and $y_i \in Y$. The proposal we present is based on the requirement that $\mathcal{F}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ is contained in $\mathcal{A}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$. Notice that any rank-one operator from the cartesian product $X_1 \times \cdots \times X_n$ to Y of finite type has the form

$$x^*_1 \otimes \ldots \otimes x^*_n \cdot y : (x^1, \ldots, x^n) \mapsto x^*_1(x^1) \ldots x^*_n(x^n)\, y$$

and since β is a reasonable crossnorm on $X_1 \otimes \ldots \otimes X_n$, the linearization of $x^*_1 \otimes \ldots \otimes x^*_n \cdot y$ is an element of $\mathcal{F}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$. This means that any operator of finite type is an element of $\mathcal{A}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$. Even more, the property *I1* in the case $\varphi = x^*_1 \otimes \ldots \otimes x^*_n$ asserts that $A(x^*_1 \otimes \ldots \otimes x^*_n \cdot y) \leq \|x^*_1\| \ldots \|x^*_n\| \|y\|$. Thus, the finite type operators are considered in the perspective of Σ-operators and we have the same control over the ideal norm.

There is a recent notion of ideals of Lipchitz operators as we may see, for instance, in [3, 26]. In this case we have a richer structure in the domain of the operators we are considering, i.e., we have more than a Lipschitz map since the embedding $\Sigma^{\beta}_{X_1 \ldots X_n} \to (X_1 \otimes \ldots \otimes X_n, \beta)$ let us take advantage of the weak topology induced by the normed space and, as we said before, this phenomenon depends on the norm β. Moreover, the ideal property *I3* requires linear properties.

Condition *I2* implies $|y^*(f_T(p) - f_T(q))| \leq \beta(p - q)\, \|y^*\|\, A(f_T)$. After taking supremum over all $\|y^*\| \leq 1$ and $\beta(p-q) \leq 1$ we obtain $Lip^{\beta}(f_T) \leq A(f_T)$ for all $f_T \in \mathcal{A}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$.

Actually, we began this section by showing that the collection of all bounded Σ-operators is an ideal of Σ-operators. Proposition 2.2 does the analogue for the collection of p-summing Σ-operators.

2.2 Σ-Tensor Norms

The most popular notion of tensor norms for n-fold tensor products is the introduced by Floret and Hunfeld in [56]. This approximation is a straightforward generalization of tensor norms for the case of two factors. Following this proposal, many examples of tensor norms and some theory has been developed, see for example [1, 17, 34, 42, 75, 83, 90]. On the side of Lipschitz theory, the authors of [24] construct the Lipschitz tensor product of a metric space and a Banach space. This product allows to define norms analogous to tensor norms. Examples of these type of norms and examples of duality can be found in [24, 25, 26, 36].

2.2.1 The Projective and Injective Case

In Chapter 1 we saw that every Σ-operator $f : \Sigma_{X_1 \dots X_n} \to Y^{\#}$ between linear spaces has an associated functional $\varphi_f : X_1 \otimes \dots \otimes X_n \otimes Y \to \mathbb{K}$. In the case of an election $(X_1, \dots, X_n, Y, \beta)$ in the class $\mathcal{BAN}$ we will try to norm the space $X_1 \otimes \dots \otimes X_n \otimes Y$ such that the boundedness of φ_f implies that of $f : \Sigma^{\beta}_{X_1 \dots X_n} \to Y^*$.

In this section we present the projective case of Σ-tensor norms in the version of spaces. We try to define the analogue of the linear projective tensor norm (see for instance [41, 93]) on the space $X_1 \otimes \dots \otimes X_n \otimes Y$.

Definition 2.4. *Given an election $(X_1, \dots, X_n, Y, \beta)$ in the class $\mathcal{BAN}$ we define the projective Σ-tensor norm on spaces on $X_1 \otimes \dots \otimes X_n \otimes Y$ by*

$$\pi^{\beta}(u; X_i Y) := \inf \sum_i \beta(p_i - q_i) \|y_i\|$$

where the infimum is taken over all representations of the form $u = \sum_i (p_i - q_i) \otimes y_i$ with p_i, $q_i \in \Sigma$ and $y_i \in Y$. If the context is clear we simply write $\pi^{\beta}(u)$.

Notice that for defining the projective Σ-tensor norm on spaces on $X_1 \otimes \dots \otimes X_n \otimes Y$ it is essential to specify the reasonable crossnorm β, that is, we have to fix an election in the class $\mathcal{BAN}$. The representations of u of the form $\sum_{i=1}^{m} (p_i - q_i) \otimes y_i$ are actually all the representations in $X_1 \otimes \dots \otimes X_n \otimes Y$. It is worth pointing out that although we are algebraically identifying the vector spaces $(X_1 \otimes \dots \otimes X_n, \beta) \otimes Y$ and $X_1 \otimes \dots \otimes X_n \otimes Y$ we are not taking the projective norm on the tensor product of normed spaces $(X_1 \otimes \dots \otimes X_n, \beta) \otimes Y$.

As a result of the Lipschitz equivalence of the metric spaces $\Sigma^{\beta}_{X_1 \dots X_n}$ and $\Sigma^{\theta}_{X_1 \dots X_n}$ we obtain that the normed spaces $\left(X_1 \otimes \dots \otimes X_n \otimes Y, \pi^{\beta}\right)$ and $\left(X_1 \otimes \dots \otimes X_n \otimes Y, \pi^{\theta}\right)$ are linearly isomorphic.

The next proposition contains the essence of a Σ-tensor norm on spaces.

Proposition 2.5. *For any election $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$ we have that:*

i) π^β *is a norm on* $X_1 \otimes \ldots \otimes X_n \otimes Y$.

ii) $\pi^\beta((p-q) \otimes y) \le \beta(p-q)\|y\|$ *for all* $p, q \in \Sigma$ *and* $y \in Y$.

iii) The functional

$$\begin{array}{rcl} \varphi \otimes y^* : \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \pi^\beta\right) & \to & \mathbb{K} \\ x^1 \otimes \ldots \otimes x^n \otimes y & \mapsto & \varphi(x^1 \otimes \ldots \otimes x^n) y^*(y) \end{array}$$

is bounded and $\|\varphi \otimes y^*\| \le \|\varphi\|_\beta \, \|y^*\|$ *for all* $\varphi \in \mathcal{L}^\beta\left(X_1, \ldots, X_n\right)$ *and* $y^* \in Y^*$.

iv) For any Σ-θ-*operator* $f_R : \Sigma^\theta_{Z_1 \ldots Z_n} \to \Sigma^\beta_{X_1 \ldots X_n}$ *and bounded linear operator* $S : W \to Y$, *the operator*

$$\begin{array}{rcl} f_R \otimes S : \left(Z_1 \otimes \ldots \otimes Z_n \otimes W, \pi^\theta\right) & \to & \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \pi^\beta\right) \\ z^1 \otimes \ldots \otimes z^n \otimes w & \mapsto & f_R(z^1 \otimes \ldots \otimes z^n) \otimes S(w) \end{array}$$

is bounded and $\|f_R \otimes S\| \le \|R\| \|S\|$.

Proof. The proof of (i) is routine and (ii) follows from definition. The triangle inequality implies

$$\begin{array}{rcl} |\left\langle \varphi \otimes y^* \ , \ u \right\rangle| & = & \left| \sum_i \varphi(p_i - q_i) \, y^*(y_i) \right| \\ & \le & Lip^\beta(\varphi) \, \|y^*\| \, \sum_i \beta(p_i - q_i) \, \|y_i\|. \end{array}$$

After taking infimum over all the representations of u and noticing that $Lip^\beta(\varphi) \le \|\varphi\|_\beta$ we obtain (iii).

Finally, (iv) follows from

$$\begin{array}{rcl} \pi^\beta(f_R \otimes S(u)) & = & \pi^\beta \left(\sum_i \left(f_R(p_i) - f_R(q_i)\right) \, Sy_i \right) \\ & \le & \sum_i \beta(f_R(p_i) - f_R(q_i)) \, \|S(y_i)\| \\ & \le & \|\widetilde{R}\| \, \|S\| \sum_i \theta(p_i - q_i) \, \|y_i\| \end{array}$$

for all representations $\sum_{i=1}^{m} (p_i - q_i) \otimes y_i$ of u. ∎

Notice that property (ii) lets us control the norm of not just simple tensors $x^1 \otimes \ldots \otimes x^n \otimes y$ but those of the form $(p-q) \otimes y$. Actually, in (ii) the equality is verified. To see this, it is enough to take, due to the Hahn-Banach theorem, two bounded linear functionals φ and y^* in the respective unit ball such that $\varphi(p-q) = \beta(p-q)$ and $y^*(y) = \|y\|$, then

$$\beta(p-q)\ \|y\| = \langle \varphi \otimes y^*\ ,\ (p-q) \otimes y \rangle \leq \pi^\beta((p-q) \otimes y).$$

On the other hand, the norm of a functional $\varphi \otimes y^*$ like in (iii) is exactly $Lip^\beta(\varphi)\ \|y\|$. To prove this affirmation, let $\eta > 0$, then chose p and q in $\Sigma^\beta_{X_1 \ldots X_n}$ and y in Y such that $(1-\eta)\, Lip^\beta(\varphi) \leq |\varphi(p-q)|$, $(1-\eta)\ \|y^*\| \leq |y^*(y)|$ and $\beta(p-q) \leq 1$, $\|y\| \leq 1$. Hence

$$(1-\eta)^2\, Lip^\beta(\varphi) \|y^*\| \leq |\, \langle \varphi \otimes y^*, (p-q) \otimes y \rangle\, | \leq \|\varphi \otimes y^*\|.$$

The first evidence of the duality between Σ-tensor norms and Σ-operators is presented in the following proposition. This is a generalization of the simplest representation of bounded linear (and multilinear) operators by functionals on a tensor product.

Proposition 2.6. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. The operator*

$$\begin{array}{rcl} \left(X_1 \otimes \ldots \otimes X_n \otimes Y,\ \pi^\beta\right)^* & \to & \mathcal{L}\left(\Sigma^\beta_{X_1 \ldots X_n}, Y^*\right) \\ \varphi & \mapsto & f_\varphi \end{array}$$

is a linear isometric isomorphism.

Proof. By definition of the Lipschitz norm of f_φ we have that

$$\begin{array}{rcl} |\langle f_\varphi(p) - f_\varphi(q)\ ,\ y \rangle| & = & |\varphi((p-q) \otimes y)| \\ & \leq & \|\varphi\|\ \pi^\beta((p-q) \otimes y) \\ & \leq & \|\varphi\|\ \beta(p-q)\ \|y\| \end{array}$$

implies $Lip^\beta(f_\varphi) \leq \|\varphi\|$. The converse inequality is deduced easily also since

$$\begin{array}{rcl} |\varphi_f(u)| & = & \left| \sum_i \langle f_\varphi(p_i) - f_\varphi(q_i)\ ,\ y_i \rangle \right| \\ & \leq & \sum_i \|f_\varphi(p_i) - f_\varphi(q_i)\|\ \|y_i\| \\ & \leq & Lip^\beta(f)\ \sum_i \beta(p_i - q_i)\ \|y_i\| \end{array}$$

holds for all representations $\sum_{i=1}^{m}(p_i - q_i) \otimes y_i$ of u. ■

Proposition 2.6 implies that $\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \pi^\beta\right)$ and $((X_1 \otimes \ldots \otimes X_n, \beta) \otimes Y, \pi)$ are not, in general, isomorphic. Otherwise, $\widetilde{T} : (X_1 \otimes \ldots \otimes X_n, \beta) \to Y$ must be bounded for every bounded multilinear operator $T : X_1 \times \cdots \times X_n \to Y$. The unique case of coincidence is $\beta = \pi$. Despite this, the identity $\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \pi^\beta\right) \to ((X_1 \otimes \ldots \otimes X_n, \beta) \otimes Y, \pi)$ is always bounded since $\pi \leq \pi^\beta$, see (i) of Proposition 2.5.

We know that the collection of all bounded Σ-operators is an ideal of Σ-operators. Thus, any component $\mathcal{L}\left(\Sigma^\beta_{X_1 \ldots X_n}, Y^*\right)$ contains $\mathcal{F}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y^*\right)$. Even more, we know that the space $\mathcal{F}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y^*\right)$ is linearly isomorphic to the tensor product $\mathcal{L}^\beta\left(X_1, \ldots, X_n\right) \otimes Y^*$. It is clear that a norm with the properties of π^β is not compatible with $\mathcal{L}^\beta\left(X_1, \ldots, X_n\right) \otimes Y^*$. If we want to represent the space $\mathcal{F}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y^*\right)$, endowed with the norm $\pi^\beta(\cdot; X_i, Y)$, in terms of $\mathcal{L}^\beta\left(X_1, \ldots, X_n\right) \otimes Y^*$ we should consider an accurate notion of tensor norm.

Definition 2.7. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. We define the injective Σ-tensor norm on duals on the space $\mathcal{L}^\beta\left(X_1, \ldots, X_n\right) \otimes Y$ as*

$$\varepsilon_\beta(v) := \sup\left\{ \, |\langle (p-q) \otimes y^*, v\rangle| \, \middle| \, \beta(p-q) \leq 1, \ \|y^*\| \leq 1 \right\}$$

for all $v \in \mathcal{L}^\beta\left(X_1, \ldots, X_n\right) \otimes Y$

For the definition of ε_β we have to fix an election in $\mathcal{BAN}$. That is, it is not enough to give the spaces X_i and Y. Notice that we are not taking all functionals on the space $\mathcal{L}^\beta\left(X_1, \ldots, X_n\right)$ but just all of the form $\delta_p - \delta_q$ with $p, q \in \Sigma_{X_1 \ldots X_n}$.

Proposition 2.8. *For any election $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$ we have*

i) ε_β *is a norm on* $\mathcal{L}^\beta\left(X_1, \ldots, X_n\right) \otimes Y$.

ii) $\varepsilon_\beta(\varphi \otimes y) \leq \|\varphi\|_\beta \|y\|$.

iii) $\sup_{\substack{\beta(p-q)\leq 1 \\ \|y^*\|\leq 1}} \left|\langle (p-q) \otimes y^* \, , \, v\rangle\right| \leq \varepsilon_\beta(v)$ *for every* $v \in \mathcal{L}^\beta\left(X_1, \ldots, X_n\right) \otimes Y$.

iv) If $A : \mathcal{L}^\beta\left(X_1, \ldots, X_n\right) \to \mathcal{L}^\theta\left(Z_1, \ldots, Z_n\right)$ *and* $B : Y \to W$ *are linear operators such that* A *preserves* Σ *then*

$$\begin{array}{rcl} A \otimes B : \left(\mathcal{L}^\beta\left(X_1, \ldots, X_n\right) \otimes Y, \varepsilon_\beta\right) & \to & \left(\mathcal{L}^\theta\left(Z_1, \ldots, Z_n\right) \otimes W, \varepsilon_\theta\right) \\ \varphi \otimes y & \mapsto & A(\varphi) \otimes B(y) \end{array}$$

is bounded and $\|A \otimes B\| \leq \|A\|\|B\|$.

Proof. The item (i) is routine. The proof of (ii) follows immediately from

$$\begin{aligned}\varepsilon_\beta(\varphi\otimes y) &= \sup_{\substack{\beta(p-q)\leq 1\\ \|y\|\leq 1}} |\langle (p-q)\otimes y^*, \varphi\otimes y\rangle| \\ &= \sup_{\substack{\beta(p-q)\leq 1\\ \|y\|\leq 1}} |\varphi(p-q)y^*(y)| \\ &= Lip^\beta(\varphi)\,\|y^*\| \\ &= \|\varphi\|_\beta\,\|y^*\|.\end{aligned}$$

The item (iii) follows from

$$\begin{aligned}\left|\langle (p-q)\otimes y^*, v\rangle\right| &= \left|\left\langle \frac{(p-q)\otimes y^*}{\beta(p-q)\,\|y^*\|}, v\right\rangle\right| \beta(p-q)\,\|y^*\| \\ &\leq \beta(p-q)\,\|y^*\|\,\varepsilon_\beta(v)\end{aligned}$$

with the obvious assumption that $p\neq q$ and $y^*\neq 0$.

For (iv) let $\sum_i \varphi_i\otimes y_i$ be a representation of v, then

$$\begin{aligned}\left|\langle (a-b)\otimes w^*, A\otimes B(v)\rangle\right| &= \left|\left\langle (a-b)\otimes w^*, \sum_i A(\varphi_i)\otimes By_i\right\rangle\right| \\ &= \left|\sum_i [A(\varphi_i)(a)-A(\varphi_i)(b)]\; w^*(By_i)\right| \\ &= \left|\sum_i [(A^*\delta_a - A^*\delta_b)\varphi_i]\; B^*(w^*)y_i\right| \\ &= \left|\langle [A^*(\delta_a)-A^*(\delta_b)]\otimes B^*(w^*), v\rangle\right| \\ &\leq \beta(A^*(\delta_a)-A^*(\delta_b))\,\|B^*(w^*)\|\,\varepsilon_\beta(v) \\ &\leq \|A\|\,\|B\|\,\theta(a-b)\,\|w^*\|\,\varepsilon_\beta(v).\end{aligned}$$

Hence $\varepsilon_\theta(A\otimes B(v))\leq \|A\|\,\|B\|\,\varepsilon_\beta(v)$. ∎

The next result exhibits the relation between the injective Σ-tensor norm on duals and the projective Σ-tensor norm on spaces.

Proposition 2.9. *Let $(X_1,\ldots,X_n,Y,\beta)$ be an election in $\mathcal{BAN}$. Then*

$$\left(\mathcal{L}^\beta\left(X_1,\ldots,X_n\right)\otimes Y, \varepsilon_\beta\right)\to\left(X_1\otimes\ldots\otimes X_n\otimes Y^*, \pi^\beta\right)^*$$

and

$$\left(\mathcal{L}^\beta\left(X_1,\ldots,X_n\right)\otimes Y^*, \varepsilon_\beta\right)\to\left(X_1\otimes\ldots\otimes X_n\otimes Y, \pi^\beta\right)^*$$

are linear into isometries.

Proof. First, let us prove that every v defines a bounded functional. For this end, let u be in $\left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \pi^\beta\right)$. We have that

$$\begin{aligned} |\langle v, u\rangle| &= \left|\left\langle v, \sum_i (p_i - q_i)\otimes y_i^*\right\rangle\right| \\ &\leq \sum_i |\langle v\,,\,(p_i - q_i)\otimes y_i^*\rangle| \\ &\leq \varepsilon_\beta(v)\sum_i \beta(p_i - q_i)\,\|y_i^*\| \end{aligned}$$

holds for all representations of u of the form $\sum_i (p_i - q_i)\otimes y_i^*$. Hence, v defines a bounded linear functional and $\|v : \left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \pi^\beta\right) \to \mathbb{K}\| \leq \varepsilon_\beta(v)$. On the other hand,

$$\begin{aligned} |\langle (p-q)\otimes y^*\,,\,v\rangle| &= |\langle v\,,\,(p-q)\otimes y^*\rangle| \\ &\leq \|v : (X_1 \otimes \ldots \otimes X_n \otimes Y^*, \pi^\beta) \to \mathbb{K}\|\,\pi^\beta((p-q)\otimes y) \end{aligned}$$

implies that $\varepsilon_\beta(v) \leq \|v : \left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \pi^\beta\right) \to \mathbb{K}\|$.

In addition, for each $v = \sum_j \varphi_j \otimes y_j^*$ in $\left(\mathcal{L}^\beta\left(X_1, \ldots, X_n\right)\otimes Y^*, \varepsilon_\beta\right)$ we have

$$\begin{aligned} \varepsilon_\beta(v) &= \sup\left\{\,\left|\left\langle (p-q)\otimes y^{**}, \sum_j \varphi_j \otimes y_j^*\right\rangle\right| \,\Big|\, \beta(p-q)\leq 1,\ \|y^{**}\|\leq 1\right\}. \\ &= \sup\left\{\,\left|\sum_j \varphi_j(p-q)\,y^{**}(y_j^*)\right| \,\Big|\, \beta(p-q)\leq 1,\ \|y^{**}\|\leq 1\right\}. \\ &= \sup\left\{\,\left|\left\langle y^{**}\,, \sum_j \varphi_j(p-q)\,y_j^*\right\rangle\right| \,\Big|\, \beta(p-q)\leq 1,\ \|y^{**}\|\leq 1\right\}. \\ &= \sup\left\{\,\left|\left\langle \sum_j \varphi_j(p-q)\,y_j^*\,,\,y\right\rangle\right| \,\Big|\, \beta(p-q)\leq 1,\ \|y\|\leq 1\right\}. \\ &= \sup\left\{\,|\langle (p-q)\otimes y\,,\,v\rangle| \,\Big|\, \beta(p-q)\leq 1,\ \|y\|\leq 1\right\}. \end{aligned}$$

Then,

$$\begin{aligned} |\langle v, u\rangle| &= \left|\left\langle v, \sum_i (p_i - q_i)\otimes y_i\right\rangle\right| \\ &\leq \sum_i |\langle v\,,\,(p_i - q_i)\otimes y_i\rangle| \\ &\leq \varepsilon_\beta(v)\sum_i \beta(p_i - q_i)\,\|y_i\| \end{aligned}$$

asserts that every v in $\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y^*,\varepsilon_\beta\right)$ defines a bounded linear functional and $\|v:\left(X_1\otimes\ldots\otimes X_n\otimes Y,\pi^\beta\right)\to\mathbb{K}\|\le\varepsilon_\beta(v)$. Finally,

$$\begin{aligned}|\left\langle (p-q)\otimes y\,,\,v\right\rangle| &= |\left\langle v\,,\,(p-q)\otimes y\right\rangle| \\ &\le \|v:\left(X_1\otimes\ldots\otimes X_n\otimes Y,\pi^\beta\right)\to\mathbb{K}\|\,\pi^\beta((p-q)\otimes y)\end{aligned}$$

implies that $\varepsilon_\beta(v)\le\|v:\left(X_1\otimes\ldots\otimes X_n\otimes Y,\pi^\beta\right)\to\mathbb{K}\|$. ∎

Within the proof of Proposition 2.9 we may find that

$$\varepsilon_\beta(v)=\sup\left\{\,|\left\langle\sum_j\varphi_j(p-q)\,y_j^*\,,\,y\right\rangle|\;\Big|\;\beta(p-q)\le 1,\ \|y\|\le 1\,\right\}$$

holds for all v in $\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y^*,\varepsilon_\beta\right)$. This identity proves that the finite rank Σ-operator defined by v verifies $Lip^\beta(v)=\varepsilon_\beta(v)$. In other words,

$$\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y^*,\varepsilon_\beta\right)=\mathcal{F}\left(\Sigma^\beta_{X_1\ldots X_n};Y^*\right)$$

holds linearly and isometrically when $\mathcal{F}\left(\Sigma^\beta_{X_1\ldots X_n};Y^*\right)$ is considered as a normed subspace of $\left(X_1\otimes\ldots\otimes X_n\otimes Y^*,\pi^\beta\right)^*$. The duality of the injective Σ-tensor norm on duals and the projective Σ-tensor norm on spaces provides a complete picture of the representation of bounded Σ-operators as is shown in the diagram

$$\begin{array}{ccc}\left(X_1\otimes\ldots\otimes X_n\otimes Y,\pi^\beta\right)^* & \longrightarrow & \mathcal{L}\left(\Sigma^\beta_{X_1\ldots X_n},Y^*\right)\\ \uparrow & & \uparrow\\ \left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y^*,\varepsilon_\beta\right) & \longrightarrow & \mathcal{F}\left(\Sigma^\beta_{X_1\ldots X_n};Y^*\right)\end{array}\tag{2.1}$$

where the horizontal arrows are linear isometric isomorphisms and the vertical arrows are linear isometries.

Actually, the second embedding of Proposition 2.9 still holds for Σ-tensor norms under certain conditions, see Theorem 3.9.

The combination of Propositions 2.6 and 2.9 gives us the isometry

$$\begin{aligned}\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y,\varepsilon_\beta\right) &\to \left(\mathcal{F}\left(\Sigma^\beta_{X_1\ldots X_n};Y\right),Lip^\beta(\cdot)\right)\\ \sum_i\varphi\otimes y_i &\mapsto \sum_i\varphi\cdot y_i\end{aligned}\tag{2.2}$$

We denote the completion of $\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y,\varepsilon_\beta\right)$ by $\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\widehat{\otimes}Y,\widehat{\varepsilon_\beta}\right)$. The operator (2.2) can be extended to the completion to obtain the isometry

$$\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\widehat{\otimes}Y,\widehat{\varepsilon_\beta}\right)\to\overline{\mathcal{F}\left(\Sigma^{\beta}_{X_1\ldots X_n};Y\right)},$$

where the closure on the operators is calculated in $\mathcal{L}\left(\Sigma^{\beta}_{X_1\ldots X_n},Y\right)$. All the Σ-operators in $\overline{\mathcal{F}\left(\Sigma^{\beta}_{X_1\ldots X_n};Y\right)}$ are called approximable. Therefore, the Σ-operator $f:\Sigma^{\beta}_{X_1\ldots X_n}\to Y$ is approximable if there exist a sequence of finite rank Σ-operators $f_n:\Sigma^{\beta}_{X_1\ldots X_n}\to Y$, whose linearization is bounded on $(X_1\otimes\ldots\otimes X_n,\beta)$, such that $Lip^{\beta}(f-f_n)\to 0$.

On the space $\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y$ it is possible to define a projective norm as follows.

Definition 2.10. *Let $(X_1,\ldots,X_n,Y,\beta)$ be an election in $\mathcal{BAN}$. The projective Σ-tensor norm on duals on $\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y$ is defined by*

$$\pi_\beta(v):=\inf\left\{\sum_i Lip_\beta(\varphi_i)\|y_i\|\;\middle|\;v=\sum_i\varphi_i\otimes y_i\right\}.$$

The projective Σ-tensor norm on duals enjoys the same properties as the injective Σ-tensor norm on duals as is shown in the next proposition.

Proposition 2.11. *For any election $(X_1,\ldots,X_n,Y,\beta)$ in $\mathcal{BAN}$ we have*

i) π_β *is a norm on* $X_1\otimes\ldots\otimes X_n\otimes Y$.

ii) $\pi_\beta(\varphi\otimes y)\le\|\varphi\|_\beta\|y\|$.

iii) $\sup\limits_{\substack{\beta(p-q)\le 1\\ \|y^*\|\le 1}}\left|\langle(p-q)\otimes y^*\,,\,v\rangle\right|\le\pi_\beta(v)$ *for every* $v\in\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y$.

iv) If $A:\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\to\mathcal{L}^{\theta}\left(Z_1,\ldots,Z_n\right)$ *and* $B:Y\to W$ *are linear operators such that* A *preserves* Σ *then*

$$\begin{array}{rcl}A\otimes B:\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y,\pi_\beta\right)&\to&\left(\mathcal{L}^{\theta}\left(Z_1,\ldots,Z_n\right)\otimes W,\pi_\theta\right)\\ \varphi\otimes y&\mapsto&A(\varphi)\otimes B(y)\end{array}$$

is bounded and $\|A\otimes B\|\le\|A\|\|B\|$.

Proof. As before, (i) is routine and (ii) is deduced directly from the definition. To prove (iii) let $v = \sum_i \varphi \otimes y_i$, then

$$\begin{aligned} |\langle (p-q)\otimes y^*\,,\, v\rangle| &= \Big|\sum_i \varphi(p-q) y^*(y_i)\Big| \\ &\leq \beta(p-q)\,\|y^*\|\,\sum_i Lip^\beta(\varphi_i)\,\|y_i\| \end{aligned}$$

holds for all $p, q \in \Sigma_{X_1 \ldots X_n}$ and $Y^* \in y^*$. The verification of (iv) is easy to see also. First, notice that $Lip^\beta(A\varphi) \leq \|A\|\, Lip^\beta(\varphi)$ since A preserves Σ. Then

$$\sum_i Lip^\beta(A\varphi_i)\,\|By_i\| \leq \|A\|\,\|B\| \sum_i Lip^\beta(\varphi_i)\,\|y_i\|$$

completes the proof. ∎

Going further, we may define an injective Σ-tensor norm on spaces as follows.

Definition 2.12. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. We define the injective Σ-tensor norm on spaces on $X_1 \otimes \ldots \otimes X_n \otimes Y$ by*

$$\varepsilon^\beta(u) := \sup\left\{ |\langle \varphi \otimes y^*\,,\; u\rangle| \;\middle|\; \|\varphi\|_\beta \leq 1,\; \|y^*\| \leq 1 \right\}$$

for all u in $X_1 \otimes \ldots \otimes X_n \otimes Y$.

It is clear that in this case the spaces

$$\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \varepsilon^\beta\right) = ((X_1 \otimes \ldots \otimes X_n, \beta) \otimes Y, \varepsilon)$$

are isometrically linearly isomorphic.

Proposition 2.13. *For any election $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$ we have that:*

i) ε^β is a norm on $X_1 \otimes \ldots \otimes X_n \otimes Y$.

ii) $\varepsilon^\beta((p-q)\otimes y) \leq \beta(p-q)\|y\|$ for all $p, q \in \Sigma$ and $y \in Y$.

iii) The functional

$$\begin{aligned} \varphi \otimes y^* : \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \varepsilon^\beta\right) &\to \mathbb{K} \\ x^1 \otimes \ldots \otimes x^n \otimes y &\mapsto \varphi(x^1 \otimes \ldots \otimes x^n) y^*(y) \end{aligned}$$

is bounded and $\|\varphi \otimes y^*\| \leq \|\varphi\|_\beta\, \|y^*\|$ for all $\varphi \in \mathcal{L}^\beta(X_1, \ldots, X_n)$ and $y^* \in Y^*$.

iv) For any Σ-θ-operator $f_R : \Sigma^{\theta}_{Z_1 \ldots Z_n} \to \Sigma^{\beta}_{X_1 \ldots X_n}$ and bounded linear operator $S : W \to Y$, the operator

$$\begin{array}{rcl} f_R \otimes S : \left(Z_1 \otimes \ldots \otimes Z_n \otimes W, \varepsilon^{\theta}\right) & \to & \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \varepsilon^{\beta}\right) \\ z^1 \otimes \ldots \otimes z^n \otimes w & \mapsto & f_R(z^1 \otimes \ldots \otimes z^n) \otimes S(w) \end{array}$$

is bounded and $\|R \otimes S\| \leq \|R\| \|S\|$.

Proof. In this situation (ii), (iii) and (iv) are particular cases of the injective tensor product of normed spaces $((X_1 \otimes \ldots \otimes X_n, \beta) \otimes Y, \varepsilon)$. ∎

In geometrical terms, the projective Σ-tensor norms π^{β} and π_{β} can be viewed as the Minkowski gauge functionals of certain closed convex hulls of subsets of $X_1 \otimes \ldots \otimes X_n \otimes Y$ and $\mathcal{L}^{\beta}(X_1, \ldots, X_n) \otimes Y$.

Proposition 2.14. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. Then:*

i) The unit ball of $\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \pi^{\beta}\right)$ coincides with the closed convex hull of the set

$$(\Sigma - \Sigma) \cap B_{(X_1 \otimes \ldots \otimes X_n, \beta)} \otimes B_Y = \{\, (p - q) \otimes y \mid \beta(p - q) \leq 1,\ \|y\| \leq 1 \,\}.$$

ii) The unit ball of the normed space $\left(\mathcal{L}^{\beta}(X_1, \ldots, X_n) \otimes Y, \pi_{\beta}\right)$ coincides with the convex hull of the set

$$\left\{ \varphi \otimes y \mid Lip^{\beta}(\varphi) \leq 1,\ \|y\| \leq 1 \right\}.$$

Proof. Let u be such that $\pi^{\beta}(u) < 1$. By definition of the projective Σ-tensor norm, there exists a representation of u of the form $\sum_{i=1}^{m} (p_i - q_i) \otimes y_i$ with $\sum_i \beta(p_i - q_i) \|y_i\| < 1$. Plainly, we may suppose that $p_i \neq q_i$ and $y_i \neq 0$. We have that

$$u = \sum_{i=1}^{m} (p_i - q_i) \otimes y_i = \sum_i \beta(p_i - q_i) \|y_i\| \frac{p_i - q_i}{\beta(p_i - q_i)} \otimes \frac{y_i}{\|y_i\|}.$$

This new representation of u asserts that $u \in \overline{con}\left((\Sigma - \Sigma) \cap B_{(X_1 \otimes \ldots \otimes X_n, \beta)} \otimes B_Y\right)$. As a consequence, the whole unit ball of $(X_1 \otimes \ldots \otimes X_n, \beta)$ is contained as well.

Conversely, suppose u lies in the closed convex hull of $(\Sigma - \Sigma) \cap B_{(X_1 \otimes \ldots \otimes X_n, \beta)} \otimes B_Y$. Thus, u admits a representation $\sum_i \lambda_i (p_i - q_i) \otimes y_i$ with $\sum_i |\lambda_i| \leq 1$. Hence

$$\pi^{\beta}(u) \leq \sum_i |\lambda_i| \, \beta(p_i - q_i) \|y_i\| \leq 1$$

finishes the proof of (i).

For (ii), let v with $\pi_\beta(v) < 1$. Choose a representation of v of the form $\sum\limits_i \varphi_i \otimes y_i$ such that $\sum\limits_i Lip^\beta(\varphi_i)\,\|y_i\| < 1$. We modify the representation as before to get

$$v = \sum_i Lip^\beta(\varphi_i)\,\|y_i\| = \sum_i Lip^\beta(\varphi_i)\,\|y_i\|\,\frac{\varphi}{Lip^\beta(\varphi_i)}\frac{y_i}{\|y_i\|},$$

and deduce that v lies in $\overline{con}\{\ \varphi \otimes y \mid Lip_\beta(\varphi) \leq 1,\ \|y\| \leq 1\ \}$. Hence, the whole unit ball is also contained. For the converse contention take $v = \sum\limits_i \lambda_i \varphi_i \otimes y_i$ with $\sum\limits_i |\lambda_i| \leq 1$, $Lip^\beta(\varphi_i) \leq 1$ and $\|y_i\| \leq 1$. Finally,

$$\pi_\beta(v) \leq \sum_i |\lambda_i|\,Lip^\beta(\varphi_i)\,\|y_i\| \leq 1$$

completes the proof. ■

A non trivial example of a norm on $X_1 \otimes \ldots \otimes X_n \otimes Y$ that enjoys the properties of the projective Σ-tensor norm on spaces was provided by Jorge Angúlo in his doctoral dissertation [6].

Definition 2.15. *Given an election $(X_1,\ldots,X_n,Y,\beta)$ in $\mathcal{BAN}$, and $p \in [1,\infty]$we define the norm d_p^β on $X_1 \otimes \ldots \otimes X_n \otimes Y$ by*

$$d_p^\beta(u) = \inf\left\{ \|\,(p_i - q_i)_i\,\|_{p^*}^{w\beta}\ \|\,(y_i)_i\,\|_p \;\middle|\; u = \sum_i (p_i - q_i) \otimes y_i \right\}.$$

Proposition 2.16. *For any election $(X_1,\ldots,X_n,Y,\beta)$ in $\mathcal{BAN}$ we have:*

i) d_p^β *is a norm on* $X_1 \otimes \ldots \otimes X_n \otimes Y$.

ii) $d_p^\beta((p-q)\otimes y) \leq \beta(p-q)\|y\|$ *for all* $p, q \in \Sigma$ *and* $y \in Y$.

iii) *The functional*

$$\begin{aligned} \varphi \otimes y^* : \left(X_1 \otimes \ldots \otimes X_n \otimes Y, d_p^\beta\right) &\rightarrow \mathbb{K} \\ x^1 \otimes \ldots \otimes x^n \otimes y &\mapsto \varphi(x^1 \otimes \ldots \otimes x^n) y^*(y) \end{aligned}$$

is bounded and $\|\varphi \otimes y^*\| \leq \|\varphi\|_\beta \|y^*\|$ *for all* $\varphi \in \mathcal{L}^\beta(X_1,\ldots,X_n)$ *and* $y^* \in Y^*$.

iv) *For any Σ-θ-operator* $f_R : \Sigma^\theta_{Z_1\ldots Z_n} \rightarrow \Sigma^\beta_{X_1\ldots X_n}$ *and bounded linear operator* $S : W \rightarrow Y$, *the operator*

$$\begin{aligned} f_R \otimes S : \left(Z_1 \otimes \ldots \otimes Z_n \otimes W, d_p^\theta\right) &\rightarrow \left(X_1 \otimes \ldots \otimes X_n \otimes Y, d_p^\beta\right) \\ z^1 \otimes \ldots \otimes z^n \otimes w &\mapsto f_R(z^1 \otimes \ldots \otimes z^n) \otimes S(w) \end{aligned}$$

is bounded and $\|f_R \otimes S\| \leq \|\widetilde{R}\|\|S\|$.

Proof. Part (i) was proved in the doctoral dissertation of Jorge Angulo [6].

We claim that

$$\varepsilon^{\beta}(u) \leq d_p^{\beta}(u) \leq \pi^{\beta}(u)$$

holds for all u in $X_1 \otimes \ldots \otimes X_n \otimes Y$. To see this affirmation, fix a representation of u of the form $\sum_{i=1}^{m} (p_i - q_i) \otimes y_i$, and let $\varphi \in \mathcal{L}^{\beta}(X_1, \ldots, X_n)$ and $y^* \in Y^*$. Then, the inequality of Hölder implies

$$\begin{aligned} |\langle \varphi \otimes y \,,\, u \rangle| &= \left| \sum_i \varphi(p_i - q_i)\, y^*(y_i) \right| \\ &\leq \left(\sum_i |\varphi(p_i - q_i)|^{p*} \right)^{1/p^*} \left(\sum_i |y^*(y_i)|^p \right)^{1/p} \\ &\leq \|\varphi\|_{\beta}\, \|y^*\|\, \|\,(p_i - q_i)_i\,\|_{p^*}^{w}\, \|\,(y_i)_i\,\|_p . \end{aligned}$$

Hence, $\varepsilon^{\beta}(u) \leq \|\,(p_i - q_i)_i\,\|_{p^*}^{w\beta}\, \|\,(y_i)_i\,\|_p$. After taking the infimum over all representations of u we obtain $\varepsilon^{\beta}(u) \leq d_p^{\beta}(u)$ for all u.

On the other hand,

$$d_p^{\beta}((p - q) \otimes y) \leq \beta(p - q)\, \|y\|$$

is obvious since we have sequences of just one element. In particular, the triangle inequality implies that $d_p^{\beta}(u) \leq \pi^{\beta}(u)$ is true for all u.

The condition $\varepsilon^{\beta}(u) \leq d_p^{\beta}(u)$ for all u implies that any bounded linear functional on $\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \varepsilon^{\beta}\right)$ is also bounded on $\left(X_1 \otimes \ldots \otimes X_n \otimes Y, d_p^{\beta}\right)$. In particular (iii) is proved.

Finally, to prove (iv) we use Proposition 1.6. For any representation $\sum_i (a_i - b_i) \otimes w_i$ of $u \in Z_1 \otimes \ldots \otimes Z_n \otimes W$ we have

$$\begin{aligned} d_p^{\beta}(f_R \otimes S(u)) &\leq \|(f_R(a_i) - f_R(b_i))\|_p^{w\beta}\, \|(Sw_i)\|_p \\ &\leq \|\widetilde{R}\|\, \|S\|\, \|(a_i - b_i)\|_p^{w\theta}\, \|(w_i)\|. \end{aligned}$$

After taking the infimum over all the representations of u we obtain

$$d_p^{\beta}(f_R \otimes S(u)) \leq \|\widetilde{R}\|\, \|S\|\, d_p^{\beta}(u)$$

for all u. ∎

In the doctoral dissertation of Jorge Angulo [6] the following theorem is proved for the case $\beta = \pi$. Slight modifications can be done for the general case.

Theorem. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in the class $\mathcal{BAN}$. Then*

$$\begin{array}{rcl} \left(X_1 \otimes \ldots \otimes X_n \otimes Y, d_p^\beta\right)^* & \to & \Pi_p\left(\Sigma^\beta_{X_1\ldots X_n}; Y^*\right) \\ \varphi & \mapsto & f_\varphi \end{array}$$

is a linear isometric isomorphism.

2.2.2 Σ-Tensor Norms on Spaces

The previous section provides the linear isometric isomorphisms

$$\left(X_1 \otimes \ldots \otimes X_n \otimes Y,\ \pi^\beta\right)^* = \mathcal{L}\left(\Sigma^\beta_{X_1\ldots X_n}, Y^*\right)$$

and

$$\left(X_1 \otimes \ldots \otimes X_n \otimes Y, d_p^\beta\right)^* = \Pi_p\left(\Sigma^\beta_{X_1\ldots X_n}; Y^*\right).$$

In this section we establish the theoretical framework for these type of norms. Here, the aim is to obtain a normed space such that the respective dual is linearly and isometrically isomorphic to a component of an ideal of Σ-operators.

Next, we present the definition of a Σ-tensor norm on spaces. Although it is defined in the class $\mathcal{BAN}$, the definition makes perfect sense if we restrict our attention to other classes such as $\mathcal{NORM}$, or the important class $\mathcal{FIN}$.

Definition 2.17. *A Σ-tensor norm on spaces α on the class of $\mathcal{BAN}$ assigns, to each election $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$, a norm α^β on the algebraic tensor product $X_1 \otimes \ldots \otimes X_n \otimes Y$ such that:*

S1 $\alpha^\beta((p-q)\otimes y) \leq \beta(p-q)\,\|y\|$ *for every* $p, q \in \Sigma_{X_1\ldots X_n}$ *and* $y \in Y$.

S2 For every $\varphi \in \mathcal{L}^\beta(X_1, \ldots, X_n)$ *and* $y^* \in Y$ *the linear functional*

$$\begin{array}{rcl} \varphi \otimes y^* : X_1 \otimes \ldots \otimes X_n \otimes Y & \to & \mathbb{K} \\ x^1 \otimes \ldots \otimes x^n \otimes y & \mapsto & \varphi(x^1 \otimes \ldots \otimes x^n) y^*(y) \end{array}$$

is bounded and $\|\varphi \otimes y^*\| \leq \|\varphi\|_\beta \|y^*\|$.

S3 If $f_R : \Sigma^\theta_{Z_1\ldots Z_n} \to \Sigma^\beta_{X_1\ldots X_n}$ *and* $S : W \to Y$ *denote a Σ-θ-operator and a bounded linear operator respectively, then the tensor product operator*

$$\begin{array}{rcl} f_R \otimes S : \left(Z_1 \otimes \ldots \otimes Z_n \otimes W, \alpha^\theta\right) & \to & \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^\beta\right) \\ z^1 \otimes \ldots \otimes z^n \otimes w & \mapsto & f_R(z^1 \otimes \ldots \otimes z^n) \otimes S(w) \end{array}$$

is bounded and $\|f_R \otimes S\| \leq \|\widetilde{R}\|\,\|S\|$.

First of all, notice that in order to define a normed space using the Σ-tensor norm α, an election $(X_1, \ldots, X_n, Y, \beta)$ is required. Every u in $X_1 \otimes \ldots \otimes X_n \otimes Y$ can be represented as a finite sum of tensors of the form $(p - q) \otimes y$ with p, q in $\Sigma_{X_1 \ldots X_n}$ and y in Y. Actually, every representation of u in $X_1 \otimes \ldots \otimes X_n \otimes Y$ can be transformed into a representation of this form. For this approach, these elements play the role of the simple tensors of the linear theory. Property *S1* says that the norm α^β takes into account the metric structure of $\Sigma^\beta_{X_1 \ldots X_n}$ by dominating these simple tensors. In recent metric theory has appeared the notion of a Lipschitz tensor product and a Lipschitz cross-norm on it. The principal reference for this notion is [24]. In this case, the condition of being a Lipschitz cross-norm [24, Def 3.1] is analogous to *S1* for Σ-tensor norms on spaces. In our proposal we say that a norm α^β on $X_1 \otimes \ldots \otimes X_n \otimes Y$ is crossed if it verifies *S1*.

On the other hand, *S2* is a generalization of being reasonable. In this setting, the resulting topological dual contains, at least, functionals of the form $\varphi \otimes y$ with φ being β-bounded. Since we are in the case of resonable crossnorms on $X_1 \otimes \ldots \otimes X_n$ then every tensor product of functionals as $x_1^* \otimes \ldots \otimes x_n^*$ is taken into account. In analogy with the metric theory of [24], *S2* plays the role of being dualizable for a Lipschitz cross-norm, see [24, Def 3.1]. In this dissertation we say that a norm α^β is reasonable if it satisfies *S2*.

Finally, *S3* generalizes the uniform property. It takes into account Σ-θ-operators which, as we said above, are compatible with the induced topologies on $\Sigma^\beta_{X_1 \ldots X_n}$ and $\Sigma^\theta_{Z_1 \ldots Z_n}$.

In other words, α is a Σ-tensor norm on spaces in the class $\mathcal{BAN}$ if for all elections $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$ it assigns a reasonable crossnorm α^β which verifies the uniform property.

Notice that the assignments $\pi : (X_1, \ldots, X_n, Y, \beta) \mapsto \pi^\beta$ and $\varepsilon : (X_1, \ldots, X_n, Y, \beta) \mapsto \varepsilon^\beta$ are Σ-tensor norms on spaces. Actually, just as in the linear case, these norms codify the fact of being reasonable and crossed for Σ-tensor norms on spaces as we prove next.

Proposition 2.18. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in the class $\mathcal{BAN}$. A norm α^β on $X_1 \otimes \ldots \otimes X_n \otimes Y$ is reasonable and crossed if and only if*

$$\varepsilon^\beta(u) \leq \alpha^\beta(u) \leq \pi^\beta(u)$$

for all u in $X_1 \otimes \ldots \otimes X_n \otimes Y$.

Proof. First, we prove the necessity. The combination of *S1* and the triangle inequality implies that

$$\alpha^\beta(u) \leq \sum_i \alpha^\beta((p_i - q_i) \otimes y_i) \leq \sum_i \beta(p - q) \, \|y_i\|$$

holds for all representations of u. Thus, $\alpha^\beta(u) \leq \pi^\beta(u)$. On the other hand, *S2* asserts that

$$|\, \langle \varphi \otimes y^* \, , \, u \rangle \,| \leq \|\varphi\|_\beta \, \|y^*\| \, \alpha^\beta(u).$$

Hence $\varepsilon^\beta(u) \leq \alpha^\beta(u)$.

The sufficiency is also direct. Condition *S1* is deduced form

$$\alpha^\beta((p-q)\otimes y) \leq \pi^\beta((p-q)\otimes y) = \beta(p-q)\,\|y\|$$

while, for *S2*, simply notice that $\varepsilon^\beta(u) \leq \alpha^\beta(u)$ implies

$$|\,\langle \varphi \otimes y\,,\, u\rangle\,| \leq \|\varphi\|_\beta\,\|y\|\,\varepsilon^\beta(u) \leq \|\varphi\|_\beta\,\|y\|\,\alpha^\beta(u).$$

■

In other words, Proposition 2.18 asserts that π is the greatest Σ-tensor norm on spaces and ε is the least Σ-tensor norm on spaces.

Corollary 2.19. *Let α be a Σ-tensor norm on spaces. Let $(X_1,\ldots,X_n,Y,\beta)$ be an election in $\mathcal{BAN}$. Then:*

i) $\alpha^\beta((p-q)\otimes y) = \beta(p-q)\,\|y\|$.

ii) $Lip^\beta(\varphi)\,\|y^*\| \leq \|\varphi\otimes y^* : \big(X_1\otimes\ldots\otimes X_n\otimes Y, \alpha^\beta\big) \to \mathbb{K}\| \leq \|\varphi\|_\beta\,\|y^*\|$.

Proof. Proposition 2.18 implies

$$\beta(p-q)\,\|y\| = \varepsilon^\beta((p-q)\otimes y) \leq \alpha^\beta((p-q)\otimes y) \leq \pi^\beta((p-q)\otimes y) = \beta(p-q)\,\|y\|$$

and

$$\begin{aligned} Lip^\beta(\varphi)\,\|y^*\| &= \|\varphi\otimes y^* : \Big(X_1\otimes\ldots\otimes X_n\otimes Y, \pi^\beta\Big) \to \mathbb{K}\| \\ &\leq \|\varphi\otimes y^* : \Big(X_1\otimes\ldots\otimes X_n\otimes Y, \alpha^\beta\Big) \to \mathbb{K}\| \\ &\leq \|\varphi\otimes y^* : \Big(X_1\otimes\ldots\otimes X_n\otimes Y, \varepsilon^\beta\Big) \to \mathbb{K}\| \\ &\leq \|\varphi\|_\beta\,\|y^*\|. \end{aligned}$$

■

In Corollary 2.19 affirmation (ii) is an equality for the case of $\beta = \pi(\cdot; X_1,\ldots,X_n)$.

2.2.3 Σ-Tensor Norms on Duals

In diagram (2.1) we saw that the space $\mathcal{F}\left(\Sigma^{\beta}_{X_1\ldots X_n};Y^*\right)$ endowed with the Lipschitz norm of the space $\mathcal{L}\left(\Sigma^{\beta}_{X_1\ldots X_n};Y^*\right)$ is linearly and isometrically isomorphic to the normed space $\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y^*,\varepsilon_\beta\right)$. In this section we present the theoretical framework of the norms like the injective Σ-tensor norm on duals ε.

Definition 2.20. *A Σ-tensor norm on duals ν on the class of Banach spaces assigns, to each election $(X_1,\ldots,X_n,Y,\beta)$ in $\mathcal{BAN}$, a norm ν_β on the tensor product $\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y$ with the following properties:*

D1 $\nu_\beta(\varphi\otimes y)\leq\|\varphi\|_\beta\|y\|$ *for every* $\varphi\in\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right),\ y\in Y$.

D2 $\sup\limits_{\substack{\beta(p-q)\leq 1\\ \|y^*\|\leq 1}}\left|\left\langle (p-q)\otimes y^*\ ,\ v\right\rangle\right|\leq\nu_\beta(v)$ *for every* $v\in\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y$.

D3 If $A:\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\rightarrow\mathcal{L}^{\theta}\left(Z_1,\ldots,Z_n\right)$ *and* $B:Y\rightarrow W$ *denote bounded linear operators where* A *preserves* Σ*, then the linear operator*

$$A\otimes B:\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y,\nu_\beta\right)\ \rightarrow\ \left(\mathcal{L}^{\theta}\left(Z_1,\ldots,Z_n\right)\otimes W,\nu_\theta\right)$$
$$\varphi\otimes y\ \mapsto\ A(\varphi)\otimes B(y)$$

is bounded and $\|A\otimes B\|\leq\|A\|\,\|B\|$.

Σ-tensor norms on duals, similar to the version on spaces, also makes sense for the classes $\mathcal{NORM}$ and $\mathcal{FIN}$.

As well as in the case of Σ-tensor norm on spaces (and ideals of Σ-operators), for defining a normed space using the Σ-tensor norm on duals ν we should fix an election $(X_1,\ldots,X_n,Y,\beta)$. In this case the simple tensors are those of $\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y$. Thus, *D1* means that ν_β is crossed in the usual sense. Notice that Σ-tensor norms on duals have no analogue neither in the case of tensor norms for 2-fold tensor products nor multilinear tensor norms in the sense of [56] since the case $n=1$ reduces to considering only spaces of the form $X^*\otimes Y$. Despite this fact, Σ-tensor norms on duals have more analogies with the metric theory of [24]. To be precise, with the Lipschitz cross-norms defined on the associated Lipschitz tensor product of a Lipschitz tensor product, see [24, Def 2.3, 3.5]. A norm ν_β on $\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y$ is said to be crossed if it satisfies *D1*.

Property *D2* tell us that the resulting topological dual contains the linear functionals defined by p,q in $\Sigma_{X_1\ldots X_n}$ and y^* in Y^*. Plainly, this is a weaker condition than being reasonable in the usual sense. We say that a norm ν_β on $\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y$ is reasonable if it satisfies *D2*.

Property *D3* is also a weaker condition of being uniform in the usual sense because we need Σ to be preserved.

In other words, a Σ-tensor norm on duals assigns to each election $(X_1, \ldots, X_n, Y, \beta)$ a reasonable crossnorm ν_β which verifies the uniform property.

Proposition 2.21. *A norm ν_β is a reasonable crossnorm if and only if*

$$\varepsilon_\beta(v) \leq \nu_\beta(v) \leq \pi(v;\, \mathcal{L}^\beta (X_1, \ldots, X_n)\, Y)$$

holds for all v.

Proof. Plainly, the inequality $\varepsilon_\beta(v) \leq \nu_\beta(v)$ is exactly condition *D2*. Now, if ν_β satisfies *D1* then the triangle inequality implies

$$\nu_\beta \left(\sum_i \varphi_i \otimes y_i \right) \leq \sum_i \|\varphi_i)\|_\beta \, \|y_i\|.$$

Hence $\nu_\beta(v) \leq \pi(v;\, \mathcal{L}^\beta (X_1, \ldots, X_n), Y)$. For the converse, $\nu_\beta(v) \leq \pi(v;\, \mathcal{L}^\beta (X_1, \ldots, X_n), Y)$ in the particular case $v = \varphi \otimes y$ asserts that $\nu_\beta(\varphi \otimes y) \leq \|\varphi\|_\beta \, \|y\|$. ∎

Corollary 2.22. *Let ν be a Σ-tensor norm on duals. For any election $(X_1, \ldots, X_n, Y, \beta)$ we have:*

i) $Lip^\beta(\varphi)\, \|y\| \leq \nu_\beta(\varphi \otimes y) \leq \|\varphi\|_\beta \, \|y\|$ *for all* $\varphi \in \mathcal{L}^\beta (X_1, \ldots, X_n)$ *and* $y \in Y$.

ii) $\|(p-q) \otimes y^* : \left(\mathcal{L}^\beta (X_1, \ldots, X_n) \otimes Y, \nu_\beta\right) \to \mathbb{K}\| = \beta(p-q)\, \|y^*\|$ *for all* $p, q \in \Sigma_{X_1 \ldots X_n}$ *and* $y^* \in Y^*$.

Proof. For (i) it is enough to notice that $\varepsilon_\beta(\varphi \otimes y) = Lip^\beta(\varphi)\, \|y\|$.

Condition *D2* implies $\|(p-q) \otimes y^* : \left(\mathcal{L}^\beta (X_1, \ldots, X_n) \otimes Y, \nu_\beta\right) \to \mathbb{K}\| \leq \beta(p-q)\, \|y^*\|$. For the converse, let $\eta > 0$. Choose y in B_Y such that $\|y^*\| < (1-\eta)\, |y^*(y)|$. On the other hand, apply the Hahn-Banach theorem to find a functional φ in the unit ball of $\mathcal{L}^\beta (X_1, \ldots, X_n)$ such that $\varphi(p-q) = \beta(p-q)$. Then,

$$\beta(p-q)\, \|y^*\| < (1-\eta)\, \varphi(p-q)\, |y^*(y)| = (1-\eta)\, |\, \langle (p-q) \otimes y \,,\, \varphi \otimes y \rangle \,|$$

ensures that $\beta(p-q) \leq \|(p-q) \otimes y^* : \left(\mathcal{L}^\beta (X_1, \ldots, X_n) \otimes Y, \nu_\beta\right) \to \mathbb{K}\|$. ∎

In this dissertation the most interesting usage of a Σ-tensor norm on duals occurs when we deal with spaces like $\mathcal{L}^\beta (X_1, \ldots, X_n) \otimes Y^*$. This space is naturally algebraically embedded into the algebraic dual of $X_1 \otimes \ldots \otimes X_n \otimes Y$, and is linearly isomorphic to $\mathcal{F}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y^*\right)$.

2.3 Association in the Class $\mathcal{FIN}$

The natural algebraic embedding

$$\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y^* \to \left(X_1\otimes\ldots\otimes X_n\otimes Y\right)^{\#}$$

given by evaluation becomes a linear isomorphism if the involved spaces are finite dimensional. That is,

$$\mathcal{L}^{\beta}\left(E_1,\ldots,E_n\right)\otimes F^* \cong \left(E_1\otimes\ldots\otimes E_n\otimes F\right)^{*}$$

is a linear isomorphism for all finite dimensional normed spaces E_i and F. Under these circumstances it is easy to see that every Σ-tensor norm on spaces defines a Σ-tensor norm on duals and vice versa in the class $\mathcal{FIN}$, compare with [41, Sec. 15.2], [93, 7.1] and [65].

Theorem 2.23. *Every Σ-tensor norm on spaces α on the class $\mathcal{FIN}$ defines a Σ-tensor norm on duals ν on the class $\mathcal{FIN}$ by*

$$\left(\mathcal{L}^{\beta}\left(E_1,\ldots,E_n\right)\otimes F,\nu_{\beta}\right) := \left(E_1\otimes\ldots\otimes E_n\otimes F^*,\alpha^{\beta}\right)^{*}.$$

Every Σ-tensor norm on duals ν on the class $\mathcal{FIN}$ defines a Σ-tensor norm on spaces α on the class $\mathcal{FIN}$ by

$$\left(E_1\otimes\ldots\otimes E_n\otimes F,\alpha^{\beta}\right) := \left(\mathcal{L}^{\beta}\left(E_1,\ldots,E_n\right)\otimes F^*,\nu_{\beta}\right)^{*}$$

Proof. We first prove that a Σ-tensor norm on spaces defines a Σ-tensor norm on duals. Let α be a Σ-tensor norm on spaces and let $(E_1,\ldots,E_n,F,\beta)$ be an election in $\mathcal{FIN}$.

ν_β **is crossed**: *D1* is equivalent to *S2* since

$$\nu_{\beta}(\varphi\otimes y) := \left\|\varphi\otimes y : \left(E_1\otimes\ldots\otimes E_n\otimes F^*,\alpha^{\beta}\right)\to\mathbb{K}\right\| \le \|\varphi\|_{\beta}\,\|y\|.$$

ν_β **is reasonable**: By the finite dimension hypothesis we have that the functionals defined by $(p-q)\otimes y^*$ are bounded. The definition of ν_β implies

$$\sup_{\nu_{\beta}(v)\le 1}\left|\left\langle (p-q)\otimes y^*\,,\,v\right\rangle\right| = \alpha^{\beta}((p-q)\otimes y^*) \le \beta(p-q)\ \|y^*\|.$$

In other words, ν_β verifies *D2*.

ν_β **verifies the uniform property**: Let $(M_1,\ldots,M_n,N,\theta)$ be an election in $\mathcal{FIN}$. Let $A:\mathcal{L}^{\beta}\left(E_1,\ldots,E_n\right)\to\mathcal{L}^{\theta}\left(M_1,\ldots,M_n\right)$ be a bounded linear operator that preserves Σ. The finite dimensional assumption lets us consider $A^*:(M_1\otimes\ldots\otimes M_n,\theta)\to(E_1\otimes\ldots\otimes E_n,\beta)$. The linearity of A^* lets us define the multilinear operator

$$\begin{aligned} T : M_1\times\cdots\times M_n &\to (E_1\otimes\ldots\otimes E_n,\beta)\\ (z^1,\ldots,z^n) &\mapsto A^*(z^1\otimes\ldots\otimes z^n). \end{aligned}$$

The universal property of the tensor product implies $\widetilde{T} = A^*$, so $\widetilde{T}$ is θ-bounded. Even more, the set $\{A^*(p)\,|\,p \in \Sigma_{M_1 \dots M_n}\}$ is contained in $\Sigma_{E_1 \dots E_n}$ since A preserves Σ. In other words, the associated Σ-operator of T, given by

$$\begin{array}{rcl} f_T : \Sigma^\theta_{M_1 \dots M_n} & \to & (E_1 \otimes \ldots \otimes E_n, \beta) \\ p & \mapsto & A^*(p) \end{array}$$

is a Σ-θ-operator.

On the other hand, let $B : N \to F$ be a linear operator. The uniform property of α implies that

$$\begin{array}{rcl} f_T \otimes B^* : \left(M_1 \otimes \ldots \otimes M_n \otimes N^*, \alpha^\theta\right) & \to & \left(E_1 \otimes \ldots \otimes E_n \otimes F^*, \alpha^\beta\right) \\ x^1 \otimes \ldots \otimes x^n \otimes y^* & \mapsto & f_T(x^1 \otimes \ldots \otimes x^n) \otimes B^* y^* \end{array}$$

is bounded and $\|f_T \otimes B^*\| \leq \|\widetilde{T}\|\,\|B^*\|$.

To prove the boundedness of

$$A \otimes B : \left(\mathcal{L}^\beta\left(E_1, \ldots, E_n\right) \otimes F, \nu_\beta\right) \to \left(\mathcal{L}^\theta\left(M_1, \ldots, M_n\right) \otimes N, \nu_\theta\right)$$

let us estimate

$$\begin{array}{rcl} \left|\,\langle A \otimes B(v)\,,\,u\rangle\,\right| & = & \left|\,\langle v\,,\,A^* \otimes B^*(u)\rangle\,\right| \\ & = & \left|\,\langle v\,,\,f_T \otimes B^*(u)\rangle\,\right| \\ & \leq & \nu_\beta(v)\,\alpha^\beta\left(f_T \otimes B^*(u); E_i F\right) \\ & \leq & \nu_\beta(v)\,\|A\|\,\|B\|\,\alpha^\theta(u; M_i N). \end{array}$$

After taking the supremum over all $\alpha^\theta(u; M_i N) \leq 1$ we obtain $\nu_\theta(A \otimes B(v)) \leq \nu_\beta(v)\,\|A\|\,\|B\|$.

Let ν be a Σ-tensor norm on duals and $(E_1, \ldots, E_n, F, \beta)$ be an election in $\mathcal{FIN}$.

α^β **is crossed**: Property *D2* asserts

$$\alpha^\beta((p-q) \otimes y) = \|(p-q) \otimes y : \left(\mathcal{L}^\beta\left(E_1, \ldots, E_n\right) \otimes F^*, \nu_\beta\right) \to \mathbb{K}\| \leq \beta(p-q)\,\|y\|.$$

α^β **is reasonable**: Once again, the finite dimensional hypothesis implies

$$\begin{array}{rcl} \|\varphi \otimes y^* : \left(E_1 \otimes \ldots \otimes E_n \otimes F, \alpha^\beta\right) \to \mathbb{K}\| & = & \sup\limits_{\alpha^\beta(u) \leq 1} \left|\,\langle \varphi \otimes y^*\,,\,u\rangle\,\right| \\ & = & \nu_\beta(\varphi \otimes y^*) \\ & \leq & \|\varphi\|_\beta\,\|y^*\|. \end{array}$$

α^β **verifies the uniform property**: Let $(M_1, \ldots, M_n, N, \theta)$ be another election in $\mathcal{FIN}$. Let $f_R : \Sigma^\theta_{M_1 \ldots M_n} \to (E_1 \otimes \ldots \otimes E_n, \beta)$ be a Σ-θ-operator. The bounded linear operator

$$\widetilde{R}^* : \mathcal{L}^\beta(E_1, \ldots, E_n) \to \mathcal{L}^\theta(M_1, \ldots, M_n)$$

preserves Σ since its adjoint operator $\widetilde{R}^{**}$ coincides with $\widetilde{R}$ and $\widetilde{R}(p) = f_R(p) \in \Sigma^\beta_{E_1 \ldots E_n}$ for all $p \in \Sigma^\theta_{M_1 \ldots M_n}$.

Let $S : N \to F$ be a bounded linear operator. The uniform property of ν implies that

$$\widetilde{R}^* \otimes S^* : \left(\mathcal{L}^\beta(E_1, \ldots, E_n) \otimes F^*, \nu_\beta\right) \to \left(\mathcal{L}^\theta(M_1, \ldots, M_n) \otimes N^*, \nu_\theta\right)$$

is bounded. Notice that the linear operators $f_R \otimes S$ and $\widetilde{R} \otimes S$ coincide. Therefore we obtain $(f_R \otimes S)^* = \left(\widetilde{R} \otimes S\right)^*$. Finally,

$$\begin{aligned}
|\langle f_R \otimes S(u), v\rangle| &\le |\langle u, (f_R \otimes S)^*(v)\rangle| \\
&\le \left|\left\langle u, \left(\widetilde{R} \otimes S\right)^*(v)\right\rangle\right| \\
&\le \left|\left\langle u, \widetilde{R}^* \otimes S^*(v)\right\rangle\right| \\
&\le \alpha^\theta(u)\, \nu_\theta(\widetilde{R}^* \otimes S^*(v)) \\
&\le \alpha^\theta(u)\, \|\widetilde{R}\|\, \|S\|\, \nu_\beta(v)
\end{aligned}$$

implies that $\alpha^\beta(f_R \otimes S(u)) \le \alpha^\theta(u)\, \|\widetilde{R}\|\, \|S\|$. So, α verifies the uniform property. ■

In other words, in the class $\mathcal{FIN}$, the definition of the Σ-tensor norm on duals ν defined by the Σ-tensor norm on spaces α as in Theorem 2.23, is the extension to the setting of Σ-tensor norms of the dual tensor norm (for the case of two factors). This is, for the case $n = 1$ we have $\nu = \alpha'$, see [41, Sec. 15.2], [93, Sec. 7.1].

We finish this chapter by relating Σ-tensor norms on duals and ideals of Σ-operators in the case of finite dimensional spaces, see [41, Sec 17].

Theorem 2.24. *Every Σ-tensor norm on duals ν on $\mathcal{FIN}$ defines an ideal $[\mathcal{A}, A]$ of Σ-operators on $\mathcal{FIN}$ by*

$$\mathcal{A}\left(\Sigma^\beta_{E_1 \ldots E_n}; F\right) := \left(\mathcal{L}^\beta(E_1, \ldots, E_n) \otimes F, \nu_\beta\right).$$

Conversely, every ideal of Σ-operators $[\mathcal{A}, A]$ on $\mathcal{FIN}$ defines a Σ-tensor norm on duals ν on $\mathcal{FIN}$, by

$$\left(\mathcal{L}^\beta(E_1, \ldots, E_n) \otimes F, \nu_\beta\right) := \mathcal{A}\left(\Sigma^\beta_{E_1 \ldots E_n}; F\right).$$

Proof. Let ν be a Σ-tensor norm on duals.

$[\mathcal{A}, A]$ **verifies *I1***: Let $\varphi \cdot y : \Sigma^{\beta}_{E_1 \ldots E_n} \to F$ be a rank-one Σ-operator with φ in $\mathcal{L}^{\beta}(E_1, \ldots, E_n)$ and y in F. Then $A(\varphi \otimes y) := \nu_\beta(\varphi \otimes y) \leq \|\varphi\|_\beta \|y\|$.

$[\mathcal{A}, A]$ **verifies *I2***: Let $p, q \in \Sigma_{E_1 \ldots E_n}$ and $y^* \in Y^*$. Then, if $f_T = \sum\limits_i \varphi_i \cdot y_i$ is associated with $v_f = \sum\limits_i \varphi_i \otimes y_i$ we have

$$\begin{aligned} |tr\left(\widetilde{T} L_{pqy^*}\right)| &= |\sum_i \varphi_i(p-q) y^*(y)| \\ &= |\left\langle (p-q) \otimes y^*, \sum_i \varphi_i \otimes y_i \right\rangle| \\ &\leq \beta(p-q)\, \|y^*\|\, \nu_\beta(v_T) \\ &= \beta(p-q)\, \|y^*\|\, A(T). \end{aligned}$$

$[\mathcal{A}, A]$ **verifies the ideal property**: Consider the composition

$$\Sigma^{\theta}_{M_1 \ldots M_n} \xrightarrow{f_R} \Sigma^{\beta}_{E_1 \ldots E_n} \xrightarrow{f} F \xrightarrow{S} N$$

where f_R is a Σ-θ-operator, $f = \sum\limits_i \varphi \cdot y_i$ is an element of $\mathcal{A}\left(\Sigma^{\beta}_{E_1 \ldots E_n}; F\right)$ and $S : F \to N$ is a bounded linear operator. Then,s Proposition 1.5 and uniformity of ν imply

$$\begin{aligned} A(S f f_R) &= \nu_\theta(v_{S f f_R}) \\ &= \nu_\theta\left(\widetilde{R}^* \otimes S(v_f)\right) \\ &\leq \|\widetilde{R}\|\, \|S\|\, \nu_\beta(v_f) \\ &= \|\widetilde{R}\|\, \|S\|\, A(f). \end{aligned}$$

Let $[\mathcal{A}, A]$ be an ideal of Σ-operators and $(E_1, \ldots, E_n, F, \beta)$ be an election in $\mathcal{FIN}$.

ν **is crossed**: Let $\varphi \otimes y \in \mathcal{L}^{\beta}(E_1, \ldots, E_n) \otimes F$ be a simple tensor. Then

$$\nu_\beta(\varphi \otimes y) := A(\varphi \otimes y) \leq \|\varphi\|_\beta \|y\|.$$

ν **is reasonable**: Let $p, q \in \Sigma_{E_1 \ldots E_n}$ and $y^* \in Y^*$. If $v \sum_i \varphi_i \otimes y_i$ and $T_v \in \mathcal{A}\left(\Sigma^{\beta}_{E_1 \ldots E_n}; F\right)$ is the associated operator, then

$$\begin{aligned} |\langle (p-q) \otimes y^* , v \rangle| &= |\sum_i \varphi_i (p-q)\, y^*(y_i)| \\ &= |tr\left(\widetilde{T_v} L_{pqy}\right)| \\ &\leq A^{\beta}(T_v)\, \beta(p-q)\, \|y^*\| \\ &= \nu_{\beta}(v)\, \beta(p-q)\, \|y^*\| \end{aligned}$$

ensures that $\|(p-q) \otimes y^* : \left(\mathcal{L}^{\beta}(E_1, \ldots, E_n) \otimes F, \nu_{\beta}\right) \to \mathbb{K}\| \leq \beta(p-q)\, \|y^*\|$.

ν **verifies the uniform property** : Let $(M_1, \ldots, M_n, N, \theta)$ be another election in $\mathcal{FIN}$ and let $A : \mathcal{L}^{\beta}(E_1, \ldots, E_n) \to \mathcal{L}^{\theta}(M_1, \ldots, M_n)$ be a bounded linear operators such that it preserves Σ. Let $B : F \to N$ be a bounded linear operator. If we look carefully, we are in the same situation of Theorem 2.23. That is,

$$\begin{aligned} f_A : \Sigma^{\theta}_{M_1 \ldots M_n} &\to (E_1 \otimes \ldots \otimes E_n, \beta) \\ p &\mapsto A^*(p) \end{aligned}$$

is a Σ-θ-operator. If $v = \sum_i \varphi_i \otimes y_i$, then $A \otimes B(v) = \left(\sum_i A\varphi_i \otimes By_i\right)$. Hence

$$\begin{aligned} \left\langle \sum_i A\varphi_i \cdot By_i \, , \, q \right\rangle &= \sum_i A\varphi_i(q)\, By_i \\ &= \sum_i \varphi_i\, A^*(q)\, By_i \\ &= B\left(\sum_i \varphi_i\, A^*(q)\, y_i\right) \\ &= B\left(f_v A^*(q)\right) \\ &= \langle B f_v f_A \, , \, q \rangle . \end{aligned}$$

Finally,

$$\begin{aligned} \nu_{\theta}\left(A \otimes B(v)\right) &= \nu_{\beta}\left(B f_v f_A\right) \\ &\leq \|A^*\|\, A(f_v)\, \|B\| \\ &= \|A\| \|B\|\, \nu_{\beta}(v) \end{aligned}$$

implies the boundedness of $A \otimes B : \left(\mathcal{L}^{\beta}(E_1, \ldots, E_n) \otimes F, \nu_{\beta}\right) \to \left(\mathcal{L}^{\theta}(M_1, \ldots, M_n) \otimes N, \nu_{\theta}\right)$. ■

Theorem 2.23 when combined with Theorem 2.24 tell us that, if we are provided with a Σ-tensor norm on duals then we may define a Σ-tensor norm on spaces and an ideal of Σ-operators $[\mathcal{A}, A]$ on the class $\mathcal{FIN}$ in order to get the linear isometries

$$\left(\mathcal{L}^{\beta}\left(E_1, \ldots, E_n\right) \otimes F^*, \nu_{\beta}\right) = \left(E_1 \otimes \ldots \otimes E_n \otimes F, \alpha^{\beta}\right)^* = \mathcal{A}^{\beta}\left(\Sigma^{\beta}_{E_1 \ldots E_n}; F^*\right). \tag{2.3}$$

Chapter 3

Main Theorems

This Chapter deals with the duality between ideals of Σ-operators and Σ-tensor norms in the general case of Banach spaces. Specifically, we state equation (2.3) in infinite dimensions under certain conditions on the ideal of Σ-operators and the Σ-tensor norms involved. We show how to extend Σ-tensor norms and ideals of Σ-operators from $\mathcal{FIN}$ to $\mathcal{BAN}$. These procedures allow us to prove the Duality Theorem for Σ-tensor norms and the Representation Theorem for Maximal Ideals of Σ-operators, see 3.9 and 3.13.

3.1 Extension From $\mathcal{FIN}$ to $\mathcal{BAN}$

Let α be a Σ-tensor norm on $\mathcal{BAN}$ and $(X_1, \ldots, X_n, Y, \beta)$ be an election. Consider E_i and F finite dimensional subspaces of X_i and Y respectively. Fix the projective tensor norm (in the sense of [41]) on $X_1 \otimes \ldots \otimes X_n$ and that of $E_1 \otimes \ldots \otimes E_n$. Since $\pi(\cdot; E_i)$ is the greatest reasonable crossnorm on $E_1 \otimes \ldots \otimes E_n$ we have that, in the diagram

$$\begin{array}{ccc} (E_1 \otimes \ldots \otimes E_n, \pi) & \longrightarrow & (X_1 \otimes \ldots \otimes X_n, \pi) \\ \uparrow & & \uparrow \\ \Sigma^{\pi}_{E_1 \ldots E_n} & \longrightarrow & \Sigma^{\pi}_{X_1 \ldots X_n} \end{array}$$

the vertical arrows are isometries and the horizontal are bounded and Lipschitz contractions respectively. On the other hand, for $\pi|$ we have that, in the next diagram

$$\begin{array}{ccc} (E_1 \otimes \ldots \otimes E_n, \pi|) & \longrightarrow & (X_1 \otimes \ldots \otimes X_n, \pi) \\ \uparrow & & \uparrow \\ \Sigma^{\pi|}_{E_1 \ldots E_n} & \longrightarrow & \Sigma^{\pi}_{X_1 \ldots X_n} \end{array}$$

all arrows are isometries. For each u in $E_1 \otimes \ldots \otimes E_n \otimes F$ two natural options arise, namely,

$$\alpha^{\pi(\cdot;E_i)}(u;E_iF) \qquad \text{and} \qquad \alpha^{\pi|}(u;E_iF) = \alpha^{\pi(\cdot;X_i)}(u;E_iF).$$

Since the projective tensor norm (in the sense of [41]) is not well behaved under subspaces, the topologies on $\Sigma_{E_1\ldots E_n}$ induced by $(E_1 \otimes \ldots \otimes E_n, \pi)$ and $(X_1 \otimes \ldots \otimes X_n, \pi)$ could be different. Our best option is to consider $\alpha^{\pi|}(u;E_iF)$ because the resulting topologies induced on $\Sigma_{E_1\ldots E_n}$ by $(X_1 \otimes \ldots \otimes X_n, \pi)$ and $(E_1 \otimes \ldots \otimes E_n, \pi|)$ coincide. In the general case we proceed in the same fashion.

For an arbitrary election $(X_1, \ldots, X_n, Y, \beta)$ we have that

$$\begin{array}{rcl} f_{E_i} : \Sigma^{\beta|}_{E_1\ldots E_n} & \to & \Sigma^{\beta}_{X_1\ldots X_n} \\ x^1 \otimes \ldots \otimes x^n & \mapsto & x^1 \otimes \ldots \otimes x^n \end{array}$$

is a Σ-$\beta|$-operator since its associated multilinear operator is the multilinear mapping given by $(x^1, \ldots, x^n) \mapsto x^1 \otimes \ldots \otimes x^n$ and $\Sigma_{E_1\ldots E_n}$ is a subset of $\Sigma_{X_1\ldots X_n}$. The uniform property of α applied to

$$f_{E_i} \otimes I_F : \left(E_1 \otimes \ldots \otimes E_n \otimes F, \alpha^{\beta|}\right) \to \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^{\beta}\right)$$

implies that

$$\alpha^{\beta}(u;X_iY) \leq \alpha^{\beta|}(u;E_iF)$$

holds for all u in $X_1 \otimes \ldots \otimes X_n \otimes Y$, $E_i \in \mathcal{F}(X_i)$ and $F \in \mathcal{F}(\mathcal{Y})$ such that $u \in E_1 \otimes \ldots \otimes E_n \otimes F$.

If we are in the presence of a Σ-tensor norm on spaces α on the class $\mathcal{FIN}$ then we define the finite hull of α as follows.

Definition 3.1. *If α is a Σ-tensor norm on spaces on the class $\mathcal{FIN}$, then its finite hull is defined as follows. For any election $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$ define*

$$\vec{\alpha}^{\beta}(u) := \inf \left\{ \alpha^{\beta|}(u;E_i,F) \;\middle|\; E_i \in \mathcal{F}(X_i), F \in \mathcal{F}(Y), u \in E_1 \otimes \ldots \otimes E_n \otimes F \right\}.$$

A Σ-tensor norm α on the class $\mathcal{BAN}$ is said to be finitely generated if it coincides with its finite hull, i.e. $\alpha^{\beta} = \vec{\alpha}^{\beta}$ for all elections $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$.

Plainly $\vec{\alpha}^{\beta}$ is homogeneous and verifies the triangle inequality. To see that $\vec{\alpha}^{\beta}(u) = 0$ implies $u = 0$ notice that $\varepsilon^{\beta}(u) = 0$ since $\varepsilon^{\beta|}(u;\, E_iF) \leq \alpha^{\beta|}(u;\, E_iF)$ holds for all $E_i \in \mathcal{F}(X_i)$ and $F \in \mathcal{F}(Y)$ such that $u \in E_1 \otimes \ldots \otimes E_n \otimes F$.

Proposition 3.2. *The injective and projective Σ-tensor norms on spaces are finitely generated.*

Proof. Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$ and $u \in X_1 \otimes \ldots \otimes X_n \otimes Y$. Choose finite dimensional subspaces E_i and F of X_i and Y respectively such that $u \in E_1 \otimes \ldots \otimes E_n \otimes F$. Let $\varphi \in \mathcal{L}^{\beta|}(E_1, \ldots, E_n)$ such that $\|\varphi\|_{\beta|} \leq 1$ and $y^* \in B_F$. Apply the Hahn-Banach theorem to find functionals ψ and z^* in $\mathcal{L}^{\beta}(X_1, \ldots, X_n)$ and Y^* such that $\psi = \varphi$ in $E_1 \otimes \ldots \otimes E_n$, $z^* = y^*$ in F, $\|\psi\| = \|\varphi\|$ and $\|z^*\| = \|y^*\|$. Then

$$\begin{aligned} |\langle \varphi \otimes y^*, u\rangle| &= |\langle \psi \otimes z^*, u\rangle| \\ &\leq \sup\{\, |\langle \psi \otimes z^*, u\rangle| \mid \|\psi\|_\beta \leq 1,\ \|z^*\| \leq 1\,\}. \\ &= \varepsilon^\beta(u;\ X_iY) \end{aligned}$$

ensures that, $\varepsilon^\beta(u;\ X_iY) = \varepsilon^{\beta|}(u;\ E_iF)$. In particular ε is a finitely generated Σ-tensor norm on spaces.

For the projective case, let $\eta > 0$ and choose a representation of u of the form $\sum_{i=1}^{m}(p_i - q_i) \otimes y_i$ such that $\sum_i \beta(p_i - q_i)\,\|y\| < (1+\eta)\,\pi^\beta(u;\ X_iY)$. Set $p_i = x_i^1 \otimes \ldots \otimes x_i^n$ and $q_i = z_i^1 \otimes \ldots \otimes z_i^n$ and define E_i as the finite dimensional subspace of X_i generated by $\{x_i, z_i\}$ and F as the finite dimensional subspace of Y generated by $\{y_i\}$. In particular $\sum_{i=1}^{m}(p_i - q_i) \otimes y_i$ is a representation of u in the space $E_1 \otimes \ldots \otimes E_n \otimes F$. Hence,

$$\begin{aligned} \pi^{\beta|}(u;\ E_iF) &\leq \sum_i \beta|(p_i - q_i)\,\|y\| \\ &= \sum_i \beta(p_i - q_i)\,\|y\| \\ &< (1+\eta)\,\pi^\beta(u;\ X_iY) \end{aligned}$$

holds for all $\eta > 0$ which implies that π is a finite generated Σ-tensor norm on spaces. ∎

The next proposition shows that taking the finite hull of a Σ-tensor norm on spaces is well behaved.

Proposition 3.3. *The finite hull of a Σ-tensor norm on spaces α defined on the class $\mathcal{FIN}$ is a Σ-tensor norm on spaces on the class $\mathcal{BAN}$. In general, if α is a Σ-tensor norm on spaces on the class $\mathcal{BAN}$ then $\alpha \leq \vec{\alpha}$.*

Proof. Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. To see that $\vec{\alpha}^\beta$ is a reasonable crossnorm, take u and finite dimensional subspaces E_i and F of X_i and Y respectively such that $u \in E_1 \otimes \ldots \otimes E_n \otimes F$. Then

$$\varepsilon^{\beta}(u;\ X_iY) \leq \varepsilon^{\beta|}(u;\ E_iF) \leq \alpha^{\beta|}(u;\ E_iF) \leq \pi^{\beta|}(u;\ E_iF)$$

asserts that $\varepsilon^{\beta}(u) \leq \overrightarrow{\alpha}^{\beta}(u) \leq \pi^{\beta}(u)$ holds in $X_1 \otimes \ldots \otimes X_n \otimes Y$ since π^{β} is finitely generated. Thus, $\vec{\alpha}^{\beta}$ is a reasonable crossed norm.

Let $(Z_1, \ldots, Z_n, W, \theta)$ be another election in the class $\mathcal{BAN}$. Let $f_R : \Sigma^{\theta}_{Z_1 \ldots Z_n} \to \Sigma^{\beta}_{X_1 \ldots X_n}$ and $S : W \to Y$ be a Σ-θ-operator and a bounded linear operator respectively. Given $u \in Z_1 \otimes \ldots \otimes Z_n \otimes W$ and finite dimensional subspaces M_i and N of Z_i and W respectively with $u \in M_1 \otimes \ldots \otimes M_n \otimes N$, there exist finite dimensional subspaces E_i and F of X_i and Y respectively such that $f_R(\Sigma_{M_1 \ldots M_n}) \subset \Sigma_{E_1 \ldots E_n}$ and $S(N) \subset F$. Consider the Σ-$\theta|$-operator associated to the multilinear operator $R| : M_1 \times \cdots \times M_n \to (E_1 \otimes \ldots \otimes E_n, \beta|)$

$$\begin{array}{rcl} f_{R|} : \Sigma^{\theta|}_{M_1 \ldots M_n} & \to & (E_1 \otimes \ldots \otimes E_n, \beta|) \\ p & \mapsto & f_R(p). \end{array}$$

The uniform property of α in the class $\mathcal{FIN}$ implies the boundedness of the operator

$$f_{R|} \otimes S : \left(M_1 \otimes \ldots \otimes M_n \otimes N, \alpha^{\theta|}\right) \to \left(E_1 \otimes \ldots \otimes E_n \otimes F, \alpha^{\beta|}\right).$$

Then,

$$\begin{array}{rcl} \vec{\alpha}^{\beta}(f_R \otimes S(u);\ X_iY) & = & \alpha^{\beta|}\left(f_{R|} \otimes S|(u);\ E_iF\right) \\ & \leq & \|\tilde{R}|\|\ \|S|\|\ \alpha^{\theta|}\ (u;\ M_iN) \\ & \leq & \|\widetilde{R}\|\ \|L\|\ \alpha^{\theta|}(u;\ M_iN) \end{array}$$

holds for all M_i and N. Thus,

$$\vec{\alpha}^{\beta}(f_R \otimes S(u);\ X_iY) \leq \|\widetilde{R}\|\ \|S\|\ \vec{\alpha}^{\theta}(u;\ Z_iY)$$

ensures the boundedness of

$$f_R \otimes S : \left(Z_1 \otimes \ldots \otimes Z_n \otimes W, \vec{\alpha}^{\theta}\right) \to \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \vec{\alpha}^{\beta}\right).$$

If α is actually a Σ-tensor norm on spaces on the class $\mathcal{BAN}$, then, to see $\alpha \leq \vec{\alpha}$ it is enough to remember that we have already proved that

$$\alpha^{\beta}(u; X_iY) \leq \alpha^{\beta|}(u; E_iF)$$

holds for all finite dimensional subspaces E_i and F. ■

In the case of Σ-tensor norms on duals defined on the class $\mathcal{FIN}$ we may also extend it to the class $\mathcal{BAN}$ as follows. Let ν be a Σ-tensor norm on duals and $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. Let $E_i \in \mathcal{F}(X_i)$ and $L \in \mathcal{CF}(Y)$. From the isometry

$$(E_1 \otimes \ldots \otimes E_n, \beta|) \to (X_1 \otimes \ldots \otimes X_n, \beta)$$

we have that

$$R_{E_i} : \mathcal{L}^{\beta}\left(X_1, \ldots, X_n\right) \to \mathcal{L}^{\beta|}\left(E_1, \ldots, E_n\right)$$

is a linear quotient operator, given by restriction, that preserves Σ. Let $Q_L : Y \to Y/L$ be the natural quotient map. The uniform property of ν applied to

$$R_{E_i} \otimes Q_L : \left(\mathcal{L}^{\beta}\left(X_1, \ldots, X_n\right) \otimes Y, \nu_{\beta}\right) \to \left(\mathcal{L}^{\beta|}\left(E_1, \ldots, E_n\right) \otimes Y/L, \nu_{\beta|}\right)$$

implies that

$$\nu_{\beta|}\left(R_{E_i} \otimes Q_L(v); \mathcal{L}^{\beta|}\left(E_1, \ldots, E_n\right) Y/L\right) \leq \nu_{\beta}\left(v; \mathcal{L}^{\beta}\left(X_1, \ldots, X_n\right), Y\right)$$

holds for all $v \in \mathcal{L}^{\beta}\left(X_1, \ldots, X_n\right) \otimes Y$, $E_i \in \mathcal{F}(X_i)$ and $L \in \mathcal{CF}(Y)$.

Definition 3.4. *Given a Σ-tensor norm on duals ν on the class $\mathcal{FIN}$, we define its cofinite hull as follows. For any election $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$ define*

$$\overleftarrow{\nu}_{\beta}(v) := \sup \nu_{\beta|}\left(R_{E_i} \otimes Q_L(v); \mathcal{L}^{\beta|}\left(E_1, \ldots, E_n\right) Y/L\right)$$

where the supremum is taken over all $E_i \in \mathcal{F}(X_i)$ and $L \in \mathcal{CF}(Y)$. A Σ-tensor norm on duals ν on $\mathcal{BAN}$ is named cofinitely generated if $\nu = \overleftarrow{\nu}$.

Proposition 3.5. *The injective Σ-tensor norm on duals is a cofinitely generated Σ-tensor norm on duals.*

Proof. Due to the comments above we have that

$$\varepsilon_{\beta|}\left(R_{E_i} \otimes Q_L(v); \mathcal{L}^{\beta|}\left(E_1, \ldots, E_n\right) Y/L\right) \leq \varepsilon^{\beta}\left(v; \mathcal{L}^{\beta}\left(X_1, \ldots, X_n\right) Y\right)$$

holds for all $E_i \in \mathcal{L}(X_i)$ and $L \in \mathcal{CF}(Y)$. This implies that

$$\overrightarrow{\varepsilon}_{\beta} \leq \varepsilon_{\beta}.$$

For the converse inequality let $\eta > 0$ and choose $p, q \in \Sigma^{\beta}_{X_1 \ldots X_n}$ and $y^* \in Y^*$ such that $\beta(p - q) \leq 1$, $\|y^*\| \leq 1$ and

$$(1 - \eta)\, \varepsilon_{\beta}(v; \mathcal{L}^{\beta}\left(X_1, \ldots, X_n\right) Y) \leq |\, \langle (p - q) \otimes y\, ,\, v \rangle\, |.$$

If $p = x^1 \otimes \ldots \otimes x^n$ and $q = z^1 \otimes \ldots \otimes z^n$ define $E_i := span\{x^i, z^i\}$ and $L := Ker(y^*)$. Thus, E_i is a finite dimensional subspace of X_i and L is a finite codimensional subspace of Y. The adjoint operator of the natural quotient map $Q_L : Y \to Y/L$ is an isometry that maps the functional

$$\begin{aligned} \psi : Y/L &\to \mathbb{K} \\ y + L &\mapsto y^*(y) \end{aligned}$$

into y^*, i. e. $Q_L^*(\psi) = y^*$. Then, if $v = \sum_i \varphi_i \otimes y_i$ we have

$$\begin{aligned} (1-\eta)\, \varepsilon_\beta(v; \mathcal{L}^\beta\,(X_1, \ldots, X_n)\, Y) &\leq |\,\langle (p-q) \otimes y\,,\, v\rangle\,| \\ &= |\sum_i \varphi_i(p-q)\, y^*(y_i)| \\ &= |\sum_i R_{E_i}(\varphi_i)(p-q)\, Q_L^*(\psi)(y_i)| \\ &= |\sum_i R_{E_i}(\varphi_i)(p-q)\, \psi(Q_L y_i)| \\ &= |\,\langle (p-q) \otimes \psi\,,\, R_{E_i} \otimes Q_L(v)\rangle\,| \\ &\leq \varepsilon_{\beta|}(R_{E_i} \otimes Q_L(v); \mathcal{L}^{\beta|}\,(E_1, \ldots, E_n)\, Y/L) \\ &\leq \overrightarrow{\varepsilon}_\beta(v; \mathcal{L}^\beta\,(X_1, \ldots, X_n)\, Y). \end{aligned}$$

Thus $\varepsilon_\beta(v; \mathcal{L}^\beta\,(X_1, \ldots, X_n)\, Y) \leq \overrightarrow{\varepsilon}_\beta(v; \mathcal{L}^\beta\,(X_1, \ldots, X_n)\, Y)$ holds for all v. ∎

The next proposition shows that taking the cofinite hull of a Σ-tensor norm on duals is well behaved, that is, the cofinite hull is again a Σ-tensor norm on duals.

Proposition 3.6. *The cofinite hull of a Σ-tensor norm on duals on the class $\mathcal{FIN}$ is a Σ-tensor norm on duals on the class $\mathcal{BAN}$. In general, if ν is a Σ-tensor norm on duals defined on the class $\mathcal{BAN}$, then $\overleftarrow{\nu} \leq \nu$.*

Proof. Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. Let E_i be a finite dimensional subspace of X_i and L be a finite codimensional subspace of Y. Then

$$\begin{aligned} \varepsilon_\beta(R_{E_i} \otimes Q_L(v); \mathcal{L}^{\beta|}\,(E_1, \ldots, E_n)\, Y/L) &\leq \nu_\beta(R_{E_i} \otimes Q_L(v); \mathcal{L}^{\beta|}\,(E_1, \ldots, E_n)\, Y/L) \\ &\leq \pi\left(R_{E_i} \otimes Q_L(v); \mathcal{L}^{\beta|}\,(E_1, \ldots, E_n)\, Y/L\right) \\ &\leq \pi(v; \mathcal{L}^\beta\,(X_1, \ldots, X_n)\, Y). \end{aligned}$$

After taking supremum over all E_i and L as above we obtain

$$\varepsilon_\beta(v) \leq \overrightarrow{\nu}_\beta(v) \leq \pi(v; \mathcal{L}^\beta\,(X_1, \ldots, X_n)\, Y).$$

This is, $\overrightarrow{\nu}_\beta$ is a reasonable crossnorm on $\mathcal{L}^\beta(X_1, \ldots, X_n) \otimes Y$.

To prove that $\overrightarrow{\nu}$ is uniform, let $(Z_1, \ldots, Z_n, W, \theta)$ be another election in $\mathcal{BAN}$ and let $A : \mathcal{L}^\beta(X_1, \ldots, X_n) \to \mathcal{L}^\theta(Z_1, \ldots, Z_n)$ and $B : Y \to W$ be bounded linear operators with A preserving Σ. Let M_i be a finite dimensional subspace of Z_i and G be a finite codimensional subspace of W.

We have that $A^*(M_1 \otimes \ldots \otimes M_n)$ is a finite dimensional subspace of $(X_1 \otimes \ldots \otimes X_n, \beta)^{**}$ but since A preserves Σ then it is actually a subspace of $(X_1 \otimes \ldots \otimes X_n, \beta)$. This way, there exist finite dimensional subspaces E_i such that $A^*(M_1 \otimes \ldots \otimes M_n) \subset E_1 \otimes \ldots \otimes E_n$. The linear inclusion $(E_1 \otimes \ldots \otimes E_n, \beta|) \to (X_1 \otimes \ldots \otimes X_n, \beta)$ implies that the restriction map $R_{E_i} : \mathcal{L}^\beta(X_1, \ldots, X_n) \to \mathcal{L}^{\beta|}(E_1, \ldots, E_n)$ is a quotient map. Then we may define

$$\begin{array}{rcl} \mathfrak{A} : \mathcal{L}^{\beta|}(E_1, \ldots, E_n) & \to & \mathcal{L}^{\theta|}(M_1, \ldots, M_n) \\ \varphi & \mapsto & R_{M_i} A(\psi) \end{array}$$

where ψ is any element in $\mathcal{L}^\beta(X_1, \ldots, X_n)$ such that $R_{E_i}\psi = \varphi$. A picture of this situation is the next diagram we have

$$\begin{array}{ccc} \mathcal{L}^\beta(X_1, \ldots, X_n) & \xrightarrow{R_{M_i}A} & \mathcal{L}^{\theta|}(M_1, \ldots, M_n) \\ {\scriptstyle R_{E_i}}\downarrow & \nearrow_{\mathfrak{A}} & \\ \mathcal{L}^{\beta|}(E_1, \ldots, E_n) & & \end{array}.$$

On the other hand, the quotient $Q_G : W \to W/G$ asserts that $Q_G^* : (W/G)^* \to W^*$ is an isometry. Hence $B^*Q_G^*((W/G)^*)$ is a finite dimensional subspace of Y^*. Then,

$$L := \{y \,|\, f(y) = 0 \,\text{for all} f \in B^*Q_G^*((W/G)^*)\} = Ker(Q_G B)$$

is a finite codimensional subspace of Y. Define

$$\begin{array}{rcl} \mathfrak{B} : Y/L & \to & W/G \\ y + L & \mapsto & Q_G B(z) \end{array}$$

where z is any element in Y such that $Q(z) = y + L$. Hence,

$$\begin{array}{ccc} Y & \xrightarrow{Q_G B} & W/G \\ {\scriptstyle Q_L}\downarrow & \nearrow_{\mathfrak{B}} & \\ Y/L & & \end{array}.$$

is commutative.

The uniformity of ν implies that

$$\mathfrak{A} \otimes \mathfrak{B} : \left(\mathcal{L}^{\beta|}\left(E_1, \ldots, E_n\right) \otimes Y/L, \nu_{\beta|}\right) \to \left(\mathcal{L}^{\theta|}\left(M_1, \ldots, M_n\right) \otimes W/G, \nu_{\theta|}\right)$$

is bounded and $\|\mathfrak{A} \otimes \mathfrak{B}\| \leq \|\mathfrak{A}\| \, \|\mathfrak{B}\| \leq \|A\| \, \|B\|$. Finally, for $v = \sum_i \varphi_i \otimes y_i$ we have

$$\begin{aligned}
\nu_{\theta|}\left(R_{M_i} \otimes Q_G\left(A \otimes B(v)\right); \mathcal{L}^{\theta|}\left(M_i\right) W/G\right) &= \nu_{\theta|}\left(\sum_i R_{M_i} A\varphi_i \otimes Q_G B y_i; \mathcal{L}^{\theta|}\left(M_i\right) W/G\right) \\
&= \nu_{\theta|}\left(\sum_i \mathfrak{A} R_{E_i}\varphi_i \otimes \mathfrak{B} Q_L y_i; \mathcal{L}^{\theta|}\left(M_i\right) W/G\right) \\
&\leq \|\mathfrak{A}\| \, \|\mathfrak{B}\| \, \nu_{\beta|}\left(R_{E_i} \otimes Q_L(v); \mathcal{L}^{\beta|}\left(E_i\right) Y/L\right) \\
&\leq \|A\| \, \|B\| \, \overrightarrow{\nu}_\beta(v; \mathcal{L}^\beta\left(X_1, \ldots, X_n\right) Y).
\end{aligned}$$

After taking suprema over all M_i and G as above we obtain

$$\overrightarrow{\nu}_\beta(A \otimes B(v); \mathcal{L}^\theta\left(Z_1, \ldots, Z_n\right) w) \leq \|A\| \, \|B\| \, \overrightarrow{\nu}_\beta(v; \mathcal{L}^\beta\left(X_1, \ldots, X_n\right) Y).$$

Thus, $\overrightarrow{\nu}$ is uniform. ∎

For ideals of Σ-operators let $[\mathcal{A}, A]$ be an ideal of Σ-operators and $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. Let $E_i \in \mathcal{F}(X_i)$ and $L \in \mathcal{CF}(Y)$. The inclusion map

$$f_{E_i} : \Sigma^{\beta|}_{E_1 \ldots E_n} \to (X_1 \otimes \ldots \otimes X_n, \beta)$$

is a Σ-$\beta|$-operator. For every $f_T \in \mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right)$ consider the composition

$$\begin{array}{ccc}
\Sigma^\beta_{X_1 \ldots X_n} & \xrightarrow{\quad f_T \quad} & Y \\
{\scriptstyle f_{E_i}} \uparrow & & \downarrow {\scriptstyle Q_L} \\
\Sigma^{\beta|}_{E_1 \ldots E_n} & \xrightarrow[\quad Q_L f_T f_{E_i} \quad]{} & Y/L
\end{array} \quad .$$

The ideal property of $[\mathcal{A}, A]$ implies that

$$A(Q_L f_T f_{E_i}) \leq A(f_T)$$

holds for all E_i and L as above.

If $[\mathcal{A}, A]$ is defined only in the class $\mathcal{FIN}$ then we may extend it to the class $\mathcal{BAN}$ as follows.

Definition 3.7. *Given an ideal of Σ-operators $[\mathcal{A}, A]$ on the class $\mathcal{FIN}$, we define for every Σ-operator $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$*

$$A^{max}(f) := \sup\left\{ A(Q_L f f_{E_i} : \Sigma^{\beta|}_{E_1 \ldots E_n} \to Y/L) \;\middle|\; E_i \in \mathcal{F}(X_i),\ L \in \mathcal{CF}(Y) \right\}.$$

The set

$$\mathcal{A}^{max}\left(\Sigma^{\beta}_{X_1 \ldots X_n}, Y\right) := \left\{ f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y \;\middle|\; A^{max}(f) < \infty \right\}$$

is defined as the maximal hull of $[\mathcal{A}, A]$. If the ideal $[\mathcal{A}, A]$ coincides with its maximal hull , then it is named maximal.

The next proposition show that the maximal hull of an ideal $[\mathcal{A}, A]$ is again an ideal of Σ-operators.

Proposition 3.8. *The maximal hull of an ideal of Σ-operators on the class $\mathcal{FIN}$ is an ideal of Σ-operators on the class $\mathcal{BAN}$.*

Proof. Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$.

I1: Let $\varphi \in \mathcal{L}^{\beta}(X_1, \ldots, X_n)$, $y \in Y$ and E_i and L as above. Then

$$\begin{aligned} A(Q_L \varphi \cdot y f_{E_i}) &= A(R_{E_i}(\varphi) \cdot Q_L(y)) \\ &\leq \|R_{E_i}(\varphi)\|_{\beta|} \, \|Q_L(y)\| \\ &\leq \|\varphi\|_{\beta} \, \|y\| \end{aligned}$$

asserts $A^{max}(\varphi \cdot y) < \infty$. In particular, $\mathcal{F}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ is contained in $\mathcal{A}^{max}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$

I2: Let $p, q \in \Sigma^{\beta}_{X_1 \ldots X_n}$, $y^* \in Y^*$ and $f_T \in \mathcal{A}\left(\Sigma^{\beta}_{X_1 \ldots X_n};\, Y\right)$. Then

$$\begin{aligned} |tr\left(L_{pqy^*}\widetilde{T}\right)| &= |y^*\left(f_T(p) - f_T(q)\right)| \\ &= |y_1^*\left(Q_L f_T f_{E_i}(p) - Q_L f_T f_{E_i}(q)\right)| \\ &\leq \|y_1^*\| \, \beta|(p-q) \, A\left(Q_L f_T f_{E_i} : \Sigma^{\beta|}_{E_1 \ldots E_n} \to Y/L\right) \\ &\leq \|y^*\| \, \beta(p-q) \, A^{max}(f_T) \end{aligned}$$

where $L := ker(y^*) \in \mathcal{CF}(Y)$, $E_i \in \mathcal{F}(X_i)$ and y_1^* are such that $p, q \in \Sigma^{\beta|}_{E_1 \ldots E_n}$ is such that $y^* = y_1^* \, Q_L$.

Ideal property: Let $(Z_1, \ldots, Z_n, W, \theta)$ be an election in $\mathcal{BAN}$. Consider the composition

$$\Sigma^{\theta}_{Z_1 \ldots Z_n} \xrightarrow{f_R} \Sigma^{\beta}_{X_1 \ldots X_n} \xrightarrow{f} Y \xrightarrow{S} W$$

where S is a bounded linear operator, f is a bounded Σ-operator and f_R is a Σ-θ-operator with associated multilinear operator R. Let M_i be a finite dimensional subspace of Z_i and G be a finite codimensional subspace of W. Since $f_R(\Sigma^{\theta}_{Z_1\ldots Z_n})$ is contained in $\Sigma^{\beta}_{X_1\ldots X_n}$ and $M_1 \otimes \ldots \otimes M_n$ is a finite dimensional space then there exist finite dimensional subspaces E_i of X_i such that $f_R f_{M_i} \subset \Sigma^{\beta|}_{E_1\ldots E_n}$. Set $L = Ker(Q_G S) \in \mathcal{CF}(Y)$. Consider the commutative diagram

$$
\begin{array}{ccccccc}
\Sigma^{\theta}_{Z_1\ldots Z_n} & \xrightarrow{f_R} & \Sigma^{\beta}_{X_1\ldots X_n} & \xrightarrow{f} & Y & \xrightarrow{S} & W \\
f_{M_i}\uparrow & \searrow^{f_R f_{M_i}} & \uparrow f_{E_i} & & Q_L \downarrow & & \downarrow Q_G \\
\Sigma^{\theta|}_{M_1\ldots M_n} & & \Sigma^{\beta|}_{E_1\ldots E_n} & & Y/L & \xrightarrow[\mathfrak{B}]{} & W/G
\end{array} .
$$

Then

$$
\begin{aligned}
A\left(Q_G(Sff_R)f_{M_i}\right) &\leq A\left(\mathfrak{B}(Q_L f f_{E_i})(f_R f_{M_i})\right) \\
&\leq \|S\|\, A\left(Q_L f f_{E_i}\right) \|\widetilde{R}\| \\
&\leq \|S\|\, A^{max}(f)\, \|\widetilde{R}\|.
\end{aligned}
$$

Finally, after taking suprema over all M_i and G as above we have

$$A^{max}(Sff_R) \leq \|S\|\, A^{max}(f)\, \|\widetilde{R}\|.$$

Complete norm: Let (f^n) be a sequence in $\mathcal{A}\left(\Sigma^{\beta}_{X_1\ldots X_n}; Y\right)$ such that $\sum A^{max}(f^n) < \infty$. Define $f = \sum f^n : \Sigma^{\beta}_{X_1\ldots X_n} \to Y$. Notice that

$$\sum \|f^n(p)\| \leq \sum Lip^{\beta}(f^n) \leq \sum A^{max}(f^n).$$

Hence, $\sum f^n(p)$ converges in Y for every $p \in \Sigma_{X_1\ldots X_n}$ since it is a Banach space. Moreover, $\sum Lip^{\beta}(f^n) \leq \sum A^{max}(f^n)$ also implies that$f$ is a Lipschitz function on $\Sigma^{\beta}_{X_1\ldots X_n}$. This is, f is a bounded Σ-operator. Finally,

$$
\begin{aligned}
A(Q_L f f_{E_i}) &= A(Q_L \left(\sum f^n\right) f_{E_i}) \\
&= A(\sum Q_L f^n f_{E_i}) \\
&\leq \sum A(Q_L f^n f_{E_i}) \\
&\leq \sum A^{max}(f^n)
\end{aligned}
$$

holds for all E_i and L. ■

The maximal ideal of Σ-operators $[\mathcal{A}, A]$ is said to be associated to the Σ-tensor norm on duals ν if

$$\left(\mathcal{L}^{\beta}\left(E_1, \ldots, E_n\right) \otimes F, \nu_{\beta}\right) = \mathcal{A}\left(\Sigma^{\beta}_{E_1\ldots E_n}; F\right)$$

holds isometrically for all elections $(E_1, \ldots, E_n, F, \beta)$ in $\mathcal{FIN}$, compare with [41, Sec. 17.3].

3.2 Duality Theorem for Σ-Tensor Norms

This section is dedicated to prove the Duality Theorem. It shows the behavior, in the general case of Banach spaces, of Σ-tensor norms which are related only in the class of finite dimensional normed spaces.

Assume that α and ν are two Σ-tensor norm on spaces and duals respectively related in the class of finite dimensional normed spaces as follows

$$\left(\mathcal{L}^{\beta}\left(E_1,\ldots,E_n\right)\otimes F,\nu_{\beta}\right)=\left(E_1\otimes\ldots\otimes E_n\otimes F^*,\alpha^{\beta}\right)^*.$$

For general Banach spaces we have the algebraic embedding

$$\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y\to\left(X_1\otimes\ldots\otimes X_n\otimes Y^*\right)^{\#}.$$

A natural question is the behavior of the norms α and ν under this circumstances. To answer this question.

Theorem 3.9. *Let ν be a cofinitely generated Σ-tensor norm on duals and α be the finitely generated Σ-tensor norm on spaces such that*

$$\left(\mathcal{L}^{\beta}\left(E_1,\ldots,E_n\right)\otimes F,\nu_{\beta}\right)=\left(E_1\otimes\ldots\otimes E_n\otimes F^*,\alpha^{\beta}\right)^*$$

holds isometrically for all elections $(E_1,\ldots,E_n,F,\beta)$ in $\mathcal{FIN}$. Then

$$\left(\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y,\nu_{\beta}\right)\to\left(X_1\otimes\ldots\otimes X_n\otimes Y^*,\alpha^{\beta}\right)^*$$

is an into isometry for all elections $(X_1,\ldots,X_n,Y,\beta)$ in $\mathcal{BAN}$.

Proof. Let $(X_1,\ldots,X_n,Y,\beta)$ be an election in $\mathcal{BAN}$.

Let $v=\sum_j\varphi_j\otimes y_j\in\mathcal{L}^{\beta}\left(X_1,\ldots,X_n\right)\otimes Y$. If $\alpha^{\beta}(u;\, X_iY)<1$ then there exist $E_i\in\mathcal{F}(X_i)$ and $F\in\mathcal{F}(Y^*)$ such that $\alpha^{\beta|}(u;E_iF)<1$. The space $L=\{y\,|\,y^*(y)=0 \text{ for all } y^*\in F\}$ is an element of $\mathcal{CF}(Y)$ such that F is isometric to $(Y/L)^*$. Then,

$$\begin{aligned}
|\langle v\,,\,u\rangle| &= |\sum_{ij}\varphi_j(p_i-q_i)y_i^*(y_j)| \\
&= |\sum_{ij}R_{E_i}(\varphi_j)(p_i-q_i)z_i^*(Q_L y_j)| \\
&= |\left\langle \sum_j R_{E_i}(\varphi_j)\otimes Q_L y_j\,,\,\sum_i (p_i-q_i)\otimes z_i^*\right\rangle| \\
&= |\left\langle R_{E_i}\otimes Q_L(v)\,,\,\sum_i (p_i-q_i)\otimes z_i^*\right\rangle| \\
&\leq \nu_{\beta|}\left(R_{E_i}\otimes Q_L(v);\,\mathcal{L}^{\beta|}\left(E_1,\ldots,E_n\right)Y/L\right)\,\alpha^{\beta|}\left(\sum_i (p_i-q_i)\otimes z_i^*;\,E_i(Y/L)^*\right) \\
&\leq \nu_\beta\left(v;\,\mathcal{L}^\beta\left(X_1,\ldots,X_n\right)Y\right).
\end{aligned}$$

After taking suprema over all $\alpha^\beta(u) < 1$ we obtain

$$\|v:\left(X_1\otimes\ldots\otimes X_n\otimes Y^*,\alpha^\beta\right)\to\mathbb{K}\| \leq \nu_\beta\left(v;\,\mathcal{L}^\beta\left(X_1,\ldots,X_n\right)Y\right).$$

For the converse inequality, let $E_i\in\mathcal{F}(X_i)$ and $L\in\mathcal{CF}(Y)$ and $\eta>0$. There exist $u\in E_1\otimes\ldots E_n\otimes(Y/L)^*$ such that $\alpha^{\beta|}(u;\,E_i(Y/L)^*)<1$ and $\nu_{\beta|}(R_{E_i}\otimes Q_L(v))\,(1-\eta)\leq|\langle R_{E_i}\otimes Q_L(v)\,,\,u\rangle|$. The space $(Y/L)^*$ is isometrically isomorphic to a finite dimensional subspace of F of Y^*. Then

$$\begin{aligned}
|\langle R_{E_i}\otimes Q_L(v)\,,\,u\rangle| &= |\sum_{ij}R_{E_i}(\varphi_j)(p_i-q_i)z_i^*(Q_L y_i)| \\
&= |\sum_{ij}\varphi_j(p_i-q_i)y_i^*(y_i)| \\
&= |\langle v\,,\,u\rangle| \\
&\leq \|v:\left(X_1\otimes\ldots\otimes X_n\otimes Y^*,\alpha^\beta\right)\to\mathbb{K}\|
\end{aligned}$$

after taking suprema over all E_i and L as above we obtain that

$$\nu_\beta(u;\;\mathcal{L}^\beta\left(X_1,\ldots,X_n\right)Y)(1-\eta)\leq\|v:\left(X_1\otimes\ldots\otimes X_n\otimes Y^*,\alpha^\beta\right)\to\mathbb{K}\|$$

holds for all $\eta>0$. ∎

3.3 Representation Theorem for Maximal Ideals of Σ-Operators

In this section we prove the most important result of the dissertation, namely the Representation Theorem for maximal ideals for Σ-operators (RT). For this end, we have to develop some theory of Σ-tensor norms and ideals of Σ-operators. We begin with the behavior of the Σ-tensor norms on spaces and the canonical extension of functionals. We then continue with the regular property of maximal ideals of Σ-operators. The relevance of the RT lies in the fact that it provides a tensorial representation of any component of a maximal ideal of Σ-operators.

The authors of [56] established a representation theorem for maximal multi-ideals by tensor norms (in their sense). As we said before, this popular approximation has plenty of examples of multi-ideals and tensor norms; however, just a few of them are related by duality. In references [1, 30, 32, 42, 75] we can find explicit examples of the duality of tensor norms and multi-ideals.

Proposition 3.10. *Let α be a finitely generated Σ-tensor norm on spaces on $\mathcal{BAN}$ and let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. Then $\big(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^\beta\big)$ is a normed subspace of $\big(X_1 \otimes \ldots \otimes X_n \otimes Y^{**}, \alpha^\beta\big)$.*

Proof. In this proof we have identified the spaces Y and its image under the canonical embedding K_Y. By uniformity of α it is clear that $\alpha^\beta(u; X_i Y^{**}) \leq \alpha^\beta(u; X_i Y)$. For the converse inequality let $u \in X_1 \otimes \ldots \otimes X_n \otimes Y$, $\varepsilon > 0$ and fix a representation of u of the form $\sum\limits_i x_i^1 \otimes \ldots \otimes x_i^n \otimes y_i$. There exist finite dimensional subspaces $E_i \subset X_i$ and $F \subset Y^{**}$ such that $u \in E_1 \otimes \ldots \otimes E_n \otimes F$ and

$$\alpha^{\beta|}(u; E_i F) \leq (1+\varepsilon)\, \alpha^\beta(u; X_i Y^{**}).$$

Without loss of generality we may assume that $y_i \in F$. The principle of local reflexivity ensures the existence of a finite dimensional subspace G of Y and an isomorphism $\psi : F \to G$ such that $\psi(y_i) = y_i$ and $\|\psi\| \leq (1+\varepsilon)$. Consider the Σ-$\beta|$-operator given by the identity $I : \Sigma^{\beta|}_{E_1 \ldots E_n} \to \Sigma^{\beta|}_{E_1 \ldots E_n}$. Again, uniformity of α implies that

$$\alpha^{\beta|}(I \otimes \psi(u); E_i G) \leq (1+\varepsilon)\, \alpha^{\beta|}(u; E_i F).$$

Finally,

$$\begin{aligned} \alpha^\beta(u; X_i Y) &\leq \alpha^{\beta|}(u; E_i G) \\ &= \alpha^{\beta|}(I \otimes \psi(u); E_i G) \\ &\leq (1+\varepsilon)\, \alpha^{\beta|}(u; E_i F) \\ &\leq (1+\varepsilon)^2\, \alpha^\beta(u; X_i Y^{**}) \end{aligned}$$

holds for all $\varepsilon > 0$. ■

In Chapter 1 we defined the canonical extension, for linear spaces, of any functional φ defined on $X_1 \otimes \ldots \otimes X_n \otimes Y$. If we are in the presence of a Σ-tensor norm on spaces we may fix the spaces X_i and the norm β and consider the normed space $\left(X_1 \otimes \ldots \otimes X_n \otimes Y^{**}, \alpha^\beta\right)$. It turns out that boundedness of φ is equivalent to that of the canonical extension $\overline{\varphi}$ if α is finitely generated.

Proposition 3.11. *Let α be a finitely generated Σ-tensor norm on spaces on the class $\mathcal{BAN}$ and $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. Then $\varphi \in \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^\beta\right)^*$ if and only if $\overline{\varphi} \in \left(X_1 \otimes \ldots \otimes X_n \otimes Y^{**}, \alpha^\beta\right)^*$.*

Proof. First, suppose $\overline{\varphi}$ is bounded. Let $u \in \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^\beta\right)$. Then $\varphi(u) = \overline{\varphi}(u)$ and $\alpha^\beta\,(u; X_iY) = \alpha^\beta\,(u; X_iY^{**})$. These facts together imply that φ is bounded and $\|\varphi\| \le \|\overline{\varphi}\|$.

Conversely, suppose φ is bounded. Fix u in $\left(X_1 \otimes \ldots \otimes X_n \otimes Y^{**}, \alpha^\beta\right)$ and let $\eta > 0$ a positive number. Since α is finitely generated there exist finite dimensional subspaces E_i and F of X_i and Y^{**} respectively such that $E_1 \otimes \ldots \otimes E_n \otimes F$ contains u and

$$\alpha^{\beta|}\,(u; E_iF) \le (1+\eta)\,\alpha^\beta\,(u; X_iY^{**})\,.$$

Let $\sum_i x_i^1 \otimes \ldots \otimes x_i^n \otimes y_i^{**}$ be a fixed representation of u in $E_1 \otimes \ldots \otimes E_n \otimes F$. Now, we may apply the Principle of Local Reflexivity to $F \subset Y^{**}$ and $span\{f_\varphi(x_i^1 \otimes \ldots \otimes x_i^n)\} \subset Y^*$ to find a finite dimensional subspace $G \subset Y$, an isomorphism $\psi : F \to G$ with norm less than $1+\eta$ and $\left\langle f_\varphi(x_i^1 \otimes \ldots \otimes x_i^n)\,,\, \psi(y_i^{**})\right\rangle = \left\langle y_i^{**}\,,\, f_\varphi(x_i^1 \otimes \ldots \otimes x_i^n)\right\rangle$. The last equality implies $\varphi(x_i^1 \otimes \ldots \otimes x_i^n \otimes \phi(y_i^{**})) = \overline{\varphi}(x_i^1 \otimes \ldots \otimes x_i^n \otimes y_i^{**})$, and so $\varphi \circ (I \otimes \phi)(u) = \overline{\varphi}(u)$. Finally,

$$\begin{aligned} |\overline{\varphi}(u)| &= |\varphi \circ (I \otimes \phi)(u)| \\ &\le \|\varphi\|\,\alpha^\beta\,(I \otimes \phi(u); X_iY) \\ &\le \|\varphi\|\,\alpha^{\beta|}\,(I \otimes \phi(u); E_iG) \\ &\le \|\varphi\|\,(1+\eta)\,\alpha^{\beta|}\,(u; E_iF) \\ &\le \|\varphi\|\,(1+\eta)^2\,\alpha^\beta\,(u; X_iY^{**}) \end{aligned}$$

completes the proof. ■

Let $f : \Sigma^\beta_{X_1 \ldots X_n} \to Y$ be a Σ-operator. Plainly, any ideal property of f depends on all the objects involved. If we fix all the spaces X_i and the norm β we may ask if the ideal property of f is preserved if $f(\Sigma_{X_1 \ldots X_n})$ is immersed in Y^{**}. This corresponds to consider the compositions $K_Y f : \Sigma^\beta_{X_1 \ldots X_n} \to Y^{**}$. If we are in the presence of a maximal ideal of Σ-operators then it turns out that this phenomenon is always true. Commonly, an ideal $[\mathcal{A}, A]$ provided this property is called regular.

Proposition 3.12. *Let $[\mathcal{A}, A]$ be a maximal ideal of Σ-operators. Then $f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1\ldots X_n}; Y\right)$ if and only if the composition $K_Y f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1\ldots X_n}; Y^{**}\right)$.*

Proof. The ideal property asserts that $K_Y f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1\ldots X_n}; Y^{**}\right)$ and $A(K_Y f) \leq A(f)$ whenever $f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1\ldots X_n}; Y\right)$.

Conversely, suppose $K_Y f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1\ldots X_n}; Y^{**}\right)$. We will prove that

$$A(Q_L f f_{E_i}) \leq A(K_Y T)$$

holds for all $E_i \in \mathcal{F}(X_i)$ and $L \in \mathcal{CF}(Y)$. Let E_i and L as above. Consider the subspace H of Y^{**} defined as the span of the set $K_Y f(\Sigma_{E_1\ldots E_n})$. On the other hand, there exist a finite dimensional subspace F of Y^* isometric to $(Y/L)^*$ via Q_L^*. Let $\varepsilon > 0$, then by the Principle of Local Reflexivity there exist a finite dimensional subspace G of Y and an isomorphism $\psi : H \to G$ with norm less than $1 + \varepsilon$ such that

$$\langle y^* \, , \, \psi(y^{**})\rangle = \langle y^{**} \, , \, y^*\rangle$$

for all $y^* \in F$ and $y^{**} \in H$. Then, for $\varphi \in (Y/L)^*$ and $p \in \Sigma_{E_1\ldots E_n}$ we obtain

$$\begin{aligned}
\langle \varphi \, , \, Q_L f f_{E_i}(p)\rangle &= \langle Q_L^*\varphi \, , \, f(p)\rangle \\
&= \langle K_Y f(p) \, , \, Q_L^*\varphi\rangle \\
&= \langle Q_L^*\varphi \, , \, \psi K_Y f(p)\rangle \\
&= \langle \varphi \, , \, Q_L \psi K_Y f(p)\rangle
\end{aligned}$$

which means that $Q_L f f_{E_i} = Q_L \psi K_Y f$. Finally, the ideal property implies

$$A(Q_L f f_{E_i}) = A(Q_L \psi K_Y f) \leq (1 + \varepsilon) \; A(K_Y f)$$

which ensures $A(Q_L f f_{E_i}) \leq A(K_Y T)$. The proof is complete after taking suprema over all E_i and L as above. ∎

Once we have developed all the needed language of Σ-operators and Σ-tensor norms we are ready to establish and prove the RT. It acquires the following form.

Theorem 3.13. *Let ν be a Σ-tensor norm on duals and $[\mathcal{A}, A]$ be the maximal ideal of Σ-operators associated to ν. Then for every election $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$ it is verified*

$$\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^{\beta}\right)^* = \mathcal{A}\left(\Sigma^{\beta}_{X_1\ldots X_n}; Y^*\right)$$

and

$$\left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \alpha^{\beta}\right)^* \bigcap \mathcal{L}\left(\Sigma^{\beta}_{X_1\ldots X_n}, Y\right) = \mathcal{A}\left(\Sigma^{\beta}_{X_1\ldots X_n}; Y\right)$$

where α is the finitely generated Σ-tensor norm on spaces defined by ν.

Proof. First, we will prove the second equality. Given finite dimensional subspaces E_i of X_i and a finite codimensional subspace L of Y we have, by hypothesis, that

$$\left(E_1 \otimes \ldots \otimes E_n \otimes (Y/L),^* \alpha^{\beta|}\right)^* = \mathcal{A}\left(\Sigma^{\beta|}_{E_1 \ldots E_n}; Y/L\right) \tag{3.1}$$

is a linear isometric isomorphism.

Let $f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ and $\eta > 0$ fixed. Let φ_f be the associated functional of f. For $u \in \left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \alpha^{\beta}\right)$ there exist $E_i \in \mathcal{F}(X_i)$ and $F \in \mathcal{F}(Y^*)$ such that $u \in \left(E_1 \otimes \ldots \otimes E_n \otimes F, \alpha^{\beta|}\right)$ and $\alpha^{\beta|}(u) \leq (1+\eta)\, \alpha^{\beta}(u)$. The space F defines a finite codimensional subspace L of Y such that $(Y/L)^* = F$ holds linearly and isometrically. Then, (3.1) ensures

$$\begin{aligned}
|\varphi_f(u)| &= |\varphi_f \circ (f_{E_i} \otimes Q_L^*)(u)| \\
&\leq \|\varphi_f \circ (f_{E_i} \otimes Q_L^*) : \left(E_1 \otimes \ldots \otimes E_n \otimes (Y/L)^*, \alpha^{\beta|}\right) \to \mathbb{K}\|\, \alpha^{\beta|}(u) \\
&= A(Q_L f f_{E_i})\, (1+\eta)\, \alpha^{\beta}(u) \\
&\leq A(f)\, (1+\eta)\, \alpha^{\beta}(u).
\end{aligned}$$

Hence, φ_f is bounded and $\|\varphi_f\| \leq A(f)$.

For the converse inequality let $\varphi \in \left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \alpha^{\beta}\right)^*$ such that its associated Σ-operator has range contained in Y. First, notice that

$$\sup\{\, A(Q_L f_\varphi f_{E_i}) \mid E_i \in \mathcal{F}(X_i)\ L \in \mathcal{CF}(Y)\,\} < \infty.$$

This is easy to see since

$$A(Q_L f_\varphi f_{E_i}) = \|\varphi \circ (f_{E_i} \otimes Q_L^*)\| \leq \|\varphi\| \tag{3.2}$$

holds for all $E_i \in \mathcal{F}(X_i)$ and $L \in \mathcal{CF}(Y)$. This means that $f_\varphi \in \mathcal{A}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$. Actually, if we take suprema in (3.2) over all E_i and L we obtain, by maximality, that $A(f_\varphi) \leq \|\varphi\|$.

For the first equality let $f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y^*\right)$. We will prove that

$$\begin{aligned}
\zeta_f : (X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^{\beta}) &\to \mathbb{K} \\
x^1 \otimes \ldots \otimes x^n \otimes y &\mapsto \left\langle f(x^1 \otimes \ldots \otimes x^n), y\right\rangle
\end{aligned}$$

is bounded. By Proposition 3.11 this occurs exactly when $\overline{\zeta_f} : (X_1 \otimes \ldots \otimes X_n \otimes Y^{**}, \alpha^{\beta}) \to \mathbb{K}$ is bonded. We just have proved that the functional $\varphi_f : (X_1 \otimes \ldots \otimes X_n \otimes Y^{**}, \alpha^{\beta}) \to \mathbb{K}$ is bounded and $\|\varphi_T\| = A(f)$. But

$$\varphi_f(x^1 \otimes \ldots \otimes x^n \otimes y^{**}) = \left\langle y^{**}\,,\, f(x^1 \otimes \ldots \otimes x^n)\right\rangle = \overline{\zeta_f}(x^1 \otimes \ldots \otimes x^n \otimes y^{**})$$

asserts that $\varphi_f = \overline{\zeta_f}$. Hence ζ_T is bounded and $\|\zeta_f\| = A(f)$.

Conversely, let $\varphi \in \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^\beta\right)^*$. Consider the associated Σ-operators of φ and its canonical extension $f_\varphi : \Sigma^\beta_{X_1 \ldots X_n} \to Y^*$ and $f_{\overline{\varphi}} : \Sigma^\beta_{X_1 \ldots X_n} \to Y^{***}$. By definition, we obtain

$$\begin{aligned} \left\langle f_{\overline{\varphi}}(x^1 \otimes \ldots \otimes x^n)\,,\, y^{**}\right\rangle &= \overline{\varphi}\left(x^1 \otimes \ldots \otimes x^n \otimes y^{**}\right) \\ &= \left\langle y^{**}\,,\, f_\varphi(x^1 \otimes \ldots \otimes x^n)\right\rangle \\ &= \left\langle K_{Y^*} f_\varphi(x^1 \otimes \ldots \otimes x^n)\,,\, y^{**}\right\rangle. \end{aligned}$$

This means that $f_{\overline{\varphi}}$ has range contained in Y^* and $f_{\overline{\varphi}} = K_{Y^*} f_\varphi$. Now, Proposition 3.11 implies

$$\overline{\varphi} \in \left(X_1 \otimes \ldots \otimes X_n \otimes Y^{**}, \alpha^\beta\right)^* \bigcap \mathcal{L}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y^*\right) = \mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n};\, Y^*\right).$$

Finally, $K_{Y^*} f_\varphi \in \mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n};\, Y^{***}\right)$ asserts that $f_\varphi \in \mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n};\, Y^*\right)$ and

$$A(f_\varphi) = A(K_{Y^*} f_\varphi) = \|\overline{\varphi}\| = \|\varphi\|.$$

■

We finish this chapter by proving a criterion that ensures the maximality of ideals of Σ-operators, see [41, Ex. 17.2] and Sections 4.4, 4.7 and 4.8.

Proposition 3.14. *Let $[\mathcal{A}, A]$ be an ideal of Σ-operators. Suppose there exist a finitely generated Σ-tensor norm on spaces α such that for any election of Banach spaces $(X_1, \ldots, X_n, Y, \beta)$ we have that*

$$\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^\beta\right)^* = \mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n};\, Y^*\right) \tag{3.3}$$

holds linearly and isometrically. Then, $[\mathcal{A}, A]$ is maximal.

Proof. Theorem 2.24 ensures that $[\mathcal{A}, A]$ is an ideal on the class $\mathcal{FIN}$. Proposition 3.8 tells us that its maximal hull is an ideal of Σ-operators on the class $\mathcal{BAN}$. The RT combined with (3.3) asserts that

$$\mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n};\, Y^*\right) = \mathcal{A}^{max}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y^*\right) \tag{3.4}$$

holds linearly and isometrically for all elections $(X_1, \ldots, X_n, Y, \beta)$ in $\mathcal{BAN}$. Another application of the RT combined with (3.4) leads us to

$$\mathcal{A}^{max}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right) = \mathcal{A}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y^{**}\right) \cap \mathcal{L}\left(\Sigma^\beta_{X_1 \ldots X_n}, Y\right).$$

Finally, for f in $\mathcal{A}^{max}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right)$ we have

$$A(f) = A(K_Y^{-1} K_Y f) \leq A(K_Y f) = A^{max}(f).$$

The proof is complete since the converse inequality $A^{max}(f) \leq A(f)$ is always true. ■

Chapter 4

Applications and Examples

In this chapter we present the notion of injective and surjective ideals of Σ-operators. Parallel, we precise the notion of injective Σ-tensor norm on duals and quotient Σ-tensor norm on spaces. In Proposition 4.8 we establish enough properties on the associated Σ-tensor norm in order to obtain an ideal either injective or surjective. The majority of the chapter is dedicated to generalize the most common ideals to the context of ideals of Σ-operators. In particular, we obtain factorizations of multilinear operators of diverse nature.

4.1 Injective and Surjective Ideals of Σ-Operators

As we will see next, properties in a maximal operator ideal $[\mathcal{A}, A]$ can be deduce from some properties on the associated Σ-tensor norm. To be precise, we define injective Σ-tensor norms on duals and quotient Σ-tensor norm on spaces.

In order to define a quotient Σ-tensor norm on spaces we need an accurate notion of quotient Σ-θ-operator. Aside, to precise the notion on injective Σ-tensor norm on dual we need appropriate assumptions on operators that preserves Σ. Let us recall that a Lipschitz $f : X \to Y$ between metric spaces is said to be 1-co-Lipschitz if for every $x \in X$ and $r > 0$ it is verified that

$$B_r\left(f(x)\right) \subset f\left(B_r(x)\right).$$

Recent research about Lipschitz quotient and co-Lipschitz functions can be find in [71, 72, 73]. Let us make some remarks we will use about 1-co-Lipschitz operators. First, notice that a 1-co-Lipschitz must be surjective. To see this affirmation, let a in Y and x in X. If $0 = d(f(x), a)$, then $f(x) = a$. Otherwise, $0 < d(f(x), a) =: r$. Then, y is an element of $B_{r+1}(f(x)) \subset f\left(B_{r+1}(x)\right)$. This asserts that f is surjective.

We claim that every 1-co-Lipschitz function verifies

$$\inf\{d(p,q) \mid f(q)=a,\ f(p)=b\} \leq d(a,b). \tag{4.1}$$

To prove this, let a and b in Y, and $\eta > 0$. Chose p in X with $f(p)=a$. Plainly, b is an element in $B_{(1+\eta)d(a,b)}(a) \subset f\left(B_{(1+\eta)d(a,b)}(p)\right)$. Hence, there exist q such that $f(q)=b$ and $d(p,q) \leq (1+\eta)\, d(a,b)$. As a consequence, (4.1) must be true.

Definition 4.1. *A Σ-tensor norm on duals ν is named injective if for every linear isometry $B : W \to Y$ and every isometric operator $A : \mathcal{L}^{\theta}(Z_1,\ldots,Z_n) \to \mathcal{L}^{\beta}(X_1,\ldots,X_n)$ which preserves Σ and $A^*|_{\Sigma^{\beta}_{X_1\ldots X_n}}$ is 1-co-Lipschitz we have that*

$$A \otimes B : \left(\mathcal{L}^{\theta}(Z_1,\ldots,Z_n) \otimes W, \nu_\theta\right) \to \left(\mathcal{L}^{\beta}(X_1,\ldots,X_n) \otimes Y, \nu_\beta\right)$$

is a linear isometry.

We are not interested in define a injective Σ-tensor norm on spaces since this is enough for our purposes.

Proposition 4.2. *The injective Σ-tensor norm on duals is injective.*

Proof. Let A and B as in Definition 4.1. Since ε_β verifies the uniform property then we just prove that $\varepsilon_\theta(v) \leq \varepsilon_\beta(A\otimes B(v))$ holds for all v in $\mathcal{L}^{\theta}(Z_1,\ldots,Z_n)\otimes$. Let a and b in $\Sigma^{\theta}_{Z_1\ldots Z_n}$ and w^* in w^* such that $\theta(a-b)<1$ and $\|w^*\|<1$. There exist p and q in $\Sigma^{\beta}_{X_1\ldots X_n}$ and y^* such that $A^*(p)=a$, $A^*(q)=b$, $B^*(y^*)=w^*$, $\beta(p-q)<1$ and $\|y^*\|<1$. Then

$$|\left\langle (a-b)\otimes w^*\,,\, v\right\rangle| = |\left\langle (p-q)\otimes y\,,\, A\otimes B(v)\right\rangle| \leq \varepsilon_\beta(A\otimes B(v))$$

completes the proof. ∎

Definition 4.3. *The Σ-θ-operator $f_Q : \Sigma^{\theta}_{Z_1\ldots Z_n} \to \Sigma^{\beta}_{X_1\ldots X_n}$ is said to be quotient if:*

1. *$\widehat{Q}\,(Z_1\otimes\ldots\otimes Z_n,\theta) \to (X_1\otimes\ldots\otimes X_n,\beta)$ is a linear quotient operator between normed spaces.*
2. *$f_Q : \Sigma^{\theta}_{Z_1\ldots Z_n} \to \Sigma^{\beta}_{X_1\ldots X_n}$ is 1-co-Lipschitz.*

This notion of quotient Σ-operator is a combination between Lipschitz and linear theory. The particular case $n=1$ reduces to the classical notion of quotient operators since the metric spaces $\Sigma^{\theta}_{Z_1\ldots Z_n}$ and $\Sigma^{\beta}_{X_1\ldots X_n}$ become Banach spaces; moreover, the Lipschitz and linear quotient notions coincide.

Notice that condition 1 implies that f_Q is a Lipschitz function with $Lip^{\theta}(f_Q)\leq 1$. This consequence in combination with (4.1) implies

$$\beta(p-q) = \inf\{\theta(a-b) \mid f_Q(a)=p,\ f_Q(b)=q\}. \tag{4.2}$$

Definition 4.4. *A Σ-tensor norm on spaces α is called quotient if for every quotient Σ-θ-operator $f_Q : \Sigma^{\theta}_{Z_1 \dots Z_n} \to \Sigma^{\beta}_{X_1 \dots X_n}$ and every quotient operator $P : W \to Y$ the linear bounded operator*

$$f_Q \otimes P : \left(Z_1 \otimes \ldots \otimes Z_n \otimes W, \alpha^{\theta}\right) \to \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^{\beta}\right)$$

is quotient.

We may pay attention just in one side of this property. The Σ-tensor norm on spaces is named domain quotient if $f_Q \otimes I_Y : \left(Z_1 \otimes \ldots \otimes Z_n \otimes Y, \alpha^{\theta}\right) \to \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^{\beta}\right)$ is quotient for all quotient Σ-θ-operator f_Q and the identity operator in Y I_Y.

Proposition 4.5. *The projective Σ-tensor norm π is domain quotient.*

Proof. Let $(X_1, \ldots, X_n, Y, \beta)$ and $(Z_1, \ldots, Z_n, W, \theta)$ be elections in $\mathcal{BAN}$. Let $\eta > 0$ and $u = \sum_{i=1}^{m} (p_i - q_i) \otimes y_i \in X_1 \otimes \ldots \otimes X_n \otimes Y$. Choose $a_i, b_i \in \Sigma^{\theta}_{Z_1 \dots Z_n}$ and $w_i \in W$ such that $f_Q(a_i) = p_i$, $f_Q(b) = q_i$, $P(w_i) = y_i$, and $\theta(a_i - b_i) \leq (1 + \eta\, \beta(p_i - q_i))$ and $\|w_i\| \leq (1+\eta)\, \|y_i\|$. Plainly, $f_Q \otimes P\left(\sum_i (a_i - b_i) \otimes w_i\right) = u$ and

$$\begin{aligned} \pi^{\theta}\left(\sum_i (a_i - b_i) \otimes w_i\right) &\leq \sum_i \theta(a_i - b_i)\, \|w_i\| \\ &\leq (1+\eta)^2 \sum_i \beta(p_i - q_i)\, \|y_i\|. \end{aligned}$$

Hence, $\pi^{\theta}\left(\sum_i (a_i - b_i) \otimes w_i\right) \leq \pi^{\beta}(u)$.

On the other hand, the uniform property of π implies that

$$f_Q \otimes P : \left(Z_1 \otimes \ldots \otimes Z_n \otimes W, \pi^{\theta}\right) \to \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \pi^{\beta}\right)$$

is bounded and $\|f_R \otimes P\| \leq 1$. Thus,

$$\pi^{\beta}(u) = \inf\{\pi^{\theta}(u') \,|\, f_Q \otimes P(u') = u\}$$

completes the proof. ■

Proposition 4.6. *If the tensor norm on duals ν is injective in $\mathcal{FIN}$ then the finite hull of the dual norm on spaces α associated to ν is domain quotient in $\mathcal{FIN}$.*

Proof. Let $f_Q : \Sigma^{\theta}_{Z_1 \dots Z_n} \to \Sigma^{\beta}_{X_1 \dots X_n}$ be a quotient Σ-θ-operator. The linear operator $\widetilde{Q}^* : \mathcal{L}^{\beta}(X_1, \dots, X_n) \to \mathcal{L}^{\theta}(Z_1, \dots, Z_n)$ is a linear isometry since $\widetilde{Q}$ is a linear quotient operator; moreover, it preserves Σ. We obtain the linear isometry

$$\widetilde{Q}^* \otimes I_{Y^*} : \left(\mathcal{L}^{\beta}(X_1, \dots, X_n) \otimes Y^*, \nu_{\beta}\right) \to \left(\mathcal{L}^{\theta}(Z_1, \dots, Z_n) \otimes Y^*, \nu_{\theta}\right)$$

whose adjoint operator

$$\left(\widetilde{Q}^* \otimes I_{Y^*}\right)^* : \left(\mathcal{L}^{\theta}(Z_1, \dots, Z_n) \otimes Y^*, \nu_{\theta}\right)^* \to \left(\mathcal{L}^{\beta}(X_1, \dots, X_n) \otimes Y^*, \nu_{\beta}\right)^*$$

is quotient. This is nothing that

$$\widetilde{Q} \otimes I_Y : \left(Z_1 \otimes \dots \otimes Z_n \otimes Y, \alpha^{\theta}\right) \to \left(X_1 \otimes \dots \otimes X_n \otimes Y, \alpha^{\beta}\right)$$

is a quotient operator. ∎

The following definitions clearly are generalizations of injective and quotient ideals in the sense of [41].

Definition 4.7. *The ideal of Σ-operators $[\mathcal{A}, A]$ is said to be:*

i) Injective if $A(if) = A(f)$ whenever $f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1 \dots X_n}; Y\right)$ and $i : Y \to W$ is a linear isometry.

ii) Surjective if $A(ff_R) = A(f)$ whenever $f_R : \Sigma^{\theta}_{Z_1 \dots Z_n} \to \Sigma^{\beta}_{X_1 \dots X_n}$ is a quotient Σ-θ-operator and $f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1 \dots X_n}; Y\right)$.

Proposition 4.8. *Let ν and $[\mathcal{A}, A]$ associated. Let α be the finitely generated Σ-tensor norm on spaces defined by ν. Then:*

i) If α is right quotient, then $[\mathcal{A}, A]$ is injective.

ii) If α is domain quotient, then $[\mathcal{A}, A]$ is surjective.

Proof. To prove (i), let $f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1 \dots X_n}; Y\right)$ and $i : Y \to W$ a linear isometry. Then $\varphi_{if} = \varphi_f \circ (I_{X_i} \otimes i^*)$. The linear operator

$$I_{X_i} \otimes i^* : \left(X_1 \otimes \dots \otimes X_n \otimes W^*, \alpha^{\beta}\right) \to \left(X_1 \otimes \dots \otimes X_n \otimes Y^*, \alpha^{\beta}\right)$$

is quotient since α is right quotient. In particular, $\|\varphi_f\| = \|\varphi_f \circ (I \otimes i^*)\|$. Finally, the RT implies

$$A(if) = \|\varphi_{if}\| = \|\varphi_f\| = A(f).$$

To prove (ii), let $f_Q : \Sigma^{\theta}_{Z_1 \ldots Z_n} \to \Sigma^{\beta}_{X_1 \ldots X_n}$ be a quotient Σ-θ-operator and $f \in \mathcal{A}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$. In this situation $\varphi_{f f_Q} = \varphi_f \circ (f_Q \otimes I_{Y^*})$. The linear operator

$$f_Q \otimes I_{y^*} : \left(Z_1 \otimes \ldots \otimes Z_n \otimes Y^*, \alpha^{\beta}\right) \to \left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \alpha^{\beta}\right)$$

is quotient since α is domain quotient. Hence, $\|\varphi_f\| = \|\varphi_f \circ (f_Q \otimes I_{Y^*})\|$. Finally, the RT implies

$$A(f f_Q) = \|\varphi_f \circ (f_Q \otimes I_{Y^*})\| = \|\varphi_f\| = A(f).$$

■

The ideal of bounded Σ-operators is a surjective while the ideal of p-summing Σ-operators is injective.

4.2 Compact and Weakly Compact Σ-Operators

In this section we deal with compact and weakly compact Σ-operators. First, recall that a linear operator $T : X \to Y$ is compact if every bounded subset of X if mapped into a relatively compact subset of Y. Due to linear properties, this notion is equivalent to say that the image of the closed unit ball is a relatively compact subset of Y. The same phenomenon occurs in the multilinear case. Recall that a multilinear bounded operator $T : X_1 \times \cdots \times X_n \to Y$ is compact if $T(B_{X_1} \times \cdots \times B_{X_n})$ is a relatively compact subset of Y (see [8, 10, 18, 19, 23, 28, 62, 91]). The case of Σ-operators preserves this property as we will see. Let us denote by $B^{\beta}(p, r)$ the ball of $\Sigma^{\beta}_{X_1 \ldots X_n}$ with center p and radius r.

Definition 4.9. *The Σ-operator $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ is called compact if the set $f\left(B^{\beta}(p, r)\right)$ is a relatively compact subset of Y for each $p \in \Sigma^{\beta}_{X_1 \ldots X_n}$ and $r > 0$.*

The collection of all compact Σ-operators from $\Sigma^{\beta}_{X_1 \ldots X_n}$ into Y is denoted by $\mathcal{K}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ and it is provided of the norm Lip^{β}.

Theorem 4.10. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$ and $T : X_1 \times \cdots \times X_n \to Y$ be a bounded multilinear operator with associated Σ-operator $f_T : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ then:*

i) $T : X_1 \times \cdots \times X_n \to Y$ is compact.

ii) $\widetilde{T} : (X_1 \otimes \ldots \otimes X_n, \pi) \to Y$ is compact.

iii) $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ is compact for all reasonable crossnorm β.

iv) $f\left(B^{\beta}(0, 1)\right) \subset Y$ is relatively compact for all reasonable crossnorm β.

v) $\widetilde{T}\left((\Sigma - \Sigma) \cap B_{(X_1 \otimes \ldots \otimes X_n, \beta)}\right) \subset Y$ is relatively compact.

are equivalent.

Proof. The implication (i)$\Rightarrow$ (ii) is a well known fact, see for instance [62].

(ii) implies (iii): Since $\pi(p-q) \leq 2^n\,\beta(p-q)$ holds for all $p, q \in \Sigma_{X_1 \ldots X_n}$, then $B^\beta(p,r) \subset B^\pi(p, 2^n r)$ for all $p \in \Sigma_{X_1 \ldots X_n}$ and $r > 0$. The contentions

$$B^\beta(p,r) \subset B^\pi(p, 2^n r) \subset p + 2^n B_{(X_1 \otimes \ldots \otimes X_n, \pi)}$$

asserts that $f(B^\beta(p,r))$ is mapped into a relatively compact subset of Y.

(iii) implies (iv) is obvious.

(iv) implies (i): It is clear since $T(B_{X_1} \times \cdots \times B_{X_n}) \subset f_T(B^\beta(0,1))$.

(ii) implies(v): The inequality $\pi(p-q) \leq 2^n\,\beta(p-q)$ asserts that

$$(\Sigma - \Sigma) \cap B_{(X_1 \otimes \ldots \otimes X_n, \beta)} \subset (\Sigma - \Sigma) \cap 2^n B_{(X_1 \otimes \ldots \otimes X_n, \pi)} \subset 2^n B_{(X_1 \otimes \ldots \otimes X_n, \pi)}.$$

Hence, the compactness of $\widetilde{T}$ implies that $\widetilde{T}\left((\Sigma - \Sigma) \cap B_{(X_1 \otimes \ldots \otimes X_n, \beta)}\right)$ is relatively compact.

(v) implies (i): Clearly, $T(B_{X_1} \times \cdots \times B_{X_n}) \subset \widetilde{T}\left((\Sigma - \Sigma) \cap B_{(X_1 \otimes \ldots \otimes X_n, \beta)}\right)$. ■

Theorem 4.10 implies that compactness of a Σ-operator $f_T : \Sigma^\beta_{X_1 \ldots X_n} \to Y$ is equivalent to the compactness of the associated multilinear operator $T : X_1 \times \cdots \times X_n \to Y$. Moreover, compactness of a Σ-operator is a property that not depends on the reasonable crossnorm β chosen. As we said above, it is enough to prove relatively compactness of the image of the set $B^\beta(0,1)$. If we carefully inspect (v), we deduce that T is compact if and only if the set

$$\left\{ \frac{f_T(p) - f_T(q)}{\beta(p-q)} \;\Big|\; p \neq q \right\}$$

is relatively compact. In other words, compactness of a multilinear operator T is equivalent to compactness of the associated Σ-operator $f_T : \Sigma^\beta_{X_1 \ldots X_n} \to Y$ considered as a Lipschitz function, see [61, 3].

Corollary 4.11. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. Then, $\mathcal{K}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right)$ and $\mathcal{K}(X_1, \ldots, X_n; Y)$ are linearly isomorphic.*

Proof. This fact is immediate from the equivalence of (i) and (ii) of Theorem 4.10 and the inequalities $2^n\, Lip^\beta(f_T) \leq Lip^\pi(f_T) \leq Lip^\beta(f_T)$. ■

Proposition 4.12. *The class of compact Σ-operators is an ideal of Σ-operators.*

Proof. Corollary 4.11 asserts that $\mathcal{K}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ is a Banach space. Even more, it proves that every finite rank Σ-operator is compact. On the other hand, the inclusion $f_R\left(B^{\theta}(p,r)\right) \subset B^{\beta}\left(f_R(p), \|\widetilde{R}\| r\right)$ asserts that the collection of compact Σ-operators verifies the ideal property. ∎

Consider the non compact Cesàro operator $C : \ell_2 \to \ell_2$ defined by

$$C(x) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)_n,$$

for all $x = (x_n)_n$ in ℓ_2 (the non compactness of C can be found in [4] and [92]). Define

$$\begin{aligned} T : \ell_2 \times \ell_2 &\to \ell_2 \\ (x,y) &\mapsto C\left(\sum_{n=1}^{\infty} x_n y_n e_n\right) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i y_i\right)_n . \end{aligned}$$

Notice that T is well defined since the Hölder inequality implies that

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\|x\|\,\|y\|\right)^2 .$$

Hence,

$$\begin{aligned} \|T(x,y)\|^2 &\leq \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i y_i\right)^2 \\ &\leq \sum_{i=1}^{\infty} \frac{1}{n^2}\|x\|_2\,\|y\|^2 \end{aligned}$$

implies that T is bounded and $\|T\| \leq \left(\sum\limits_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2}$. Now, consider the linearization of T to the projective tensor product, this is

$$\widetilde{T} : (\ell_2 \widehat{\otimes} \ell_2, \pi) \to \ell_2.$$

Define T_k as the k-th partial sums of T

$$\begin{aligned} T_k : \ell_2 \times \ell_2 &\to \ell_2 \\ (x,y) &\mapsto C\left(\sum_{n=1}^{k} x_n y_n e_n\right). \end{aligned}$$

Plainly, T_k has range contained in $span\{e_1, \ldots, e_k\}$. Hence, $\widetilde{T_k} : (\ell_2 \widehat{\otimes} \ell_2, \pi) \to \ell_2$ is a compact linear operator. Moreover, the inequality

$$\|(T - T_k)(x, y)\| \leq \left(\sum_{n=k+1}^{\infty} \frac{1}{n^2} \right)^{1/2} \|x\| \, \|y\|$$

implies that $\|(\widetilde{T} - \widetilde{T_k})(u)\| \leq \left(\sum\limits_{n=k+1}^{\infty} \frac{1}{n^2} \right)^{1/2} \pi(u)$ holds for all $u \in (\ell_2 \otimes \ell_2, \pi)$. Then,

$$\|(\widetilde{T} - \widetilde{T_k})(v)\| \leq \left(\sum_{n=k+1}^{\infty} \frac{1}{n^2} \right)^{1/2} \pi(v)$$

holds for all $v \in (\ell_2 \widehat{\otimes} \ell_2, \pi)$. We may conclude that $\widetilde{T}$ is compact since it is a uniform limit of compact operators.

However, $\widetilde{T} : (\ell_2 \widehat{\otimes} \ell_2, H) \to \ell_2$ is not compact. To see this, we will show $\overline{\widetilde{T}\left(B_{(\ell_2 \widehat{\otimes} \ell_2, H)}\right)} = \overline{C(B_{\ell_2})}$. To prove this affirmation let $u = \sum\limits_{i,j=1}^{\infty} \lambda_{ij} e_i \otimes e_j$ such that $H(u) \leq 1$. Then, $\widetilde{T}(u) = C(\lambda_{ii})$. But, $\|(\lambda_{ii})\| \leq \left(\sum\limits_{i,j=1}^{\infty} |\lambda_{ij}|^2 \right)^{1/2} = H(u)$. This way, $\widetilde{T}(B_{(\ell_2 \otimes \ell_2, H)}) \subset C(B_{\ell_2})$. As a consequence, $\overline{\widetilde{T}\left(B_{(\ell_2 \widehat{\otimes} \ell_2, H)}\right)} \subset \overline{C(B_{\ell_2})}$. On the other hand, for $x \in \ell_2$ with $\|x\| \leq 1$ define $u = \sum\limits_{i=1}^{\infty} x_i \, e_i \otimes e_i$. Hence, $\widetilde{T}(u) = \sum\limits_{i=1}^{\infty} x_i \, T(e_i, e_i) = C(x)$. Besides, $H(u) = \left(\sum\limits_{i=1}^{\infty} |x_i|^2 \right)^{1/2} = \|x\| \leq 1$. This asserts that $C(B_{\ell_2}) \subset \widetilde{T}\left(B_{(\ell_2 \widehat{\otimes} \ell_2, H)}\right)$. Finally, after taking closures we conclude that $\overline{\widetilde{T}\left(B_{(\ell_2 \widehat{\otimes} \ell_2, H)}\right)} = \overline{C(B_{\ell_2})}$. Since C is not compact we obtain that $\widetilde{T} : (\ell_2 \widehat{\otimes} \ell_2, H) \to \ell_2$ is not compact.

This example shows that compactness of the Σ-operator $f_T : \Sigma_{X_1 \ldots X_n}^{\beta} \to Y$ does not imply the compactness of $\widetilde{T} : (X_1 \otimes \ldots \otimes X_n, \beta) \to Y$.

For the weakly compact case we have a similar situation. Recall that a multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ is said to be weakly compact if every bounded subset of $X_1 \times \cdots \times X_n$ is mapped into a relatively weakly compact subset of Y. For T to be weakly compact it is enough that $T(B_{X_1} \times \cdots \times B_{X_n})$ to be relatively weakly compact.

Definition 4.13. *The Σ-operator $f : \Sigma_{X_1 \ldots X_n}^{\beta} \to Y$ is called weakly compact if $f(B^{\beta}(p, r))$ is a relatively weakly compact subset of Y for all $p \in \Sigma_{X_1 \ldots X_n}^{\beta}$ and $r > 0$. The weakly compact norm of f_T is defined as its Lipschitz norm Lip^{β}.*

The collection of all weakly compact Σ-operators from $\Sigma^{\beta}_{X_1 \dots X_n}$ into Y is denoted by $\mathcal{W}\left(\Sigma^{\beta}_{X_1 \dots X_n}; Y\right)$ and it is provided of the norm Lip^{β}.

Theorem 4.14. *Let $(X_1, \dots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$ and $T : X_1 \times \dots \times X_n \to Y$ be a bounded multilinear operator with associated Σ-operator $f_T : \Sigma^{\beta}_{X_1 \dots X_n} \to Y$ then:*

i) $T : X_1 \times \dots \times X_n \to Y$ *is weakly compact.*

ii) $\widetilde{T} : (X_1 \otimes \ldots \otimes X_n, \pi) \to Y$ *is weakly compact.*

iii) $f : \Sigma^{\beta}_{X_1 \dots X_n} \to Y$ *is weakly compact for all reasonable crossnorm* β*.*

iv) $f\left(B^{\beta}(0,1)\right) \subset Y$ *is relatively weakly compact for all reasonable crossnorm* β*.*

v) $\widetilde{T}\left((\Sigma - \Sigma) \cap B_{(X_1 \otimes \ldots \otimes X_n, \beta)}\right) \subset Y$ *is relatively weakly compact.*

are equivalent.

Proof. The equivalence of (i) and (ii) follows from the Krein-Šmulian theorem. The other equivalences are derived exactly as in Theorem 4.10. ■

Corollary 4.15. *Let $(X_1, \dots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. Then, $\mathcal{W}\left(\Sigma^{\beta}_{X_1 \dots X_n}; Y\right)$ and $\mathcal{W}(X_1, \dots, X_n; Y)$ are linearly isomorphic.*

We may argue exactly as in Proposition 4.12 to prove the following result.

Proposition 4.16. *The collection of all weakly compact Σ-operators is an ideal of Σ-operators.*

4.3 Nuclear Σ-Operators

The notion of a nuclear Σ-operator can be easily extended from the linear case. Recall that a linear operator $T : X \to Y$ between Banach spaces is called nuclear if it can be expressed as

$$T = \sum_{i=1}^{\infty} x_i^* \cdot y_i$$

where (x_i^*) and (y_i) are bounded sequences in X^* and Y such that $\sum_{i=1}^{\infty} \|x_i^*\| \, \|y_i\| < \infty$.

In the multilinear setting the notion of nuclear multilinear operator has a few versions, see for instance [5, 31, 33, 75, 77, 88, 91]. On the side of Lipschitz theory we can see [3, 27, 38].

Definition 4.17. *The Σ-operator $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ between Banach spaces is nuclear if it can be expressed in the form*

$$f = \sum_{i=1}^{\infty} \varphi_i \cdot y_i$$

where (φ_i) and (y_i) are bounded sequences in $\mathcal{L}^{\beta}(X_1, \ldots, X_n)$ and Y respectively such that $\sum\limits_{i=1}^{\infty} Lip^{\beta}(\varphi_i)\, \|y_i\| < \infty$. The nuclear norm of f is defined by

$$N(f) := \inf \left\{ \sum_{i=1}^{\infty} Lip_{\beta}(\varphi_i)\, \|y_i\| \right\}$$

where the infimum is taken over the all possible representations as above.

The collection of all nuclear Σ-operators from $\Sigma^{\beta}_{X_1 \ldots X_n}$ into Y is denoted by $\mathcal{N}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ and it is a Banach space with the nuclear norm N, see Proposition 4.20.

In the linear setting, the nuclear property is related with the projective tensor norm by the surjective linear operator

$$X^* \widehat{\otimes}_{\pi} Y \to N(X, Y). \tag{4.3}$$

In analogy with the linear case, the collection of nuclear Σ-operators is related with the projective Σ-tensor norm on duals. To present the extension of (4.3) to the setting of Σ-operators, first, we characterize the completion respective to the projective Σ-tensor norm on duals. We use a similar argument to the presented in [96, P. 94]

Proposition 4.18. *Let π be the projective Σ-tensor norm on duals and $(X_1, \ldots, X_n, Y, \beta)$ be an election on $\mathcal{BAN}$. Every v in the completion of $\left(\mathcal{L}^{\beta}(X_1, \ldots, X_n) \otimes Y, \pi_{\beta}\right)$ verifies*

$$\widehat{\pi_{\beta}}(v) = \inf \left\{ \sum_{i=1}^{\infty} Lip^{\beta}(\varphi_i)\, \|y_i\| \;\middle|\; v = \sum_{i=1}^{\infty} \varphi_i \otimes y_i,\ \sum_{i=1}^{\infty} Lip^{\beta}(\varphi_i)\, \|y_i\| < \infty \right\}.$$

Proof. First notice that every series $\sum\limits_{i=1}^{\infty} \varphi_i \otimes y_i$ such that $\sum\limits_{i=1}^{\infty} Lip^{\beta}(\varphi_i)\, \|y_i\| < \infty$ defines an element v in the completion of $\left(\mathcal{L}^{\beta}(X_1, \ldots, X_n) \otimes Y, \pi_{\beta}\right)$ since it is absolutely convergent. Moreover

$$\widehat{\pi_{\beta}}(v) \leq \sum_{i=1}^{\infty} Lip^{\beta}(\varphi_i)\, \|y_i\|.$$

On the other hand, let v be a non zero element in the completion and fix $\varepsilon > 0$. Choose a sequence (u_i) in $\left(\mathcal{L}^{\beta}(X_1, \ldots, X_n) \otimes Y, \pi_{\beta}\right)$ such that

$$\widehat{\pi_{\beta}}(v - u_i) < \frac{\varepsilon}{2^{i+1}}.$$

Since $\pi_\beta(u_1) < \varepsilon + \widehat{\pi_\beta}(v)$, there exist a representation $\sum_{j=1}^{j_1} \varphi_j \otimes y_j$ of u_1 with the property $\sum_{j=1}^{j_1} Lip^\beta(\varphi_j)\,\|y_j\| < \varepsilon + \widehat{\pi_\beta}(v)$. Define $v_i = u_{i+1} - u_i$ for all i in $\mathbb{N}$. Then

$$\pi_\beta(v_i) \le \pi_\beta(v - u_{i+1}) + \pi_\beta(v - u_i) \le \frac{\varepsilon}{2^{i+2}} + \frac{\varepsilon}{2^{i+1}} \le \frac{\varepsilon}{2^i}.$$

Therefore, there exists a representation $\sum_{j=j_i+1}^{j_{i+1}} \varphi_j \otimes y_j$ of v_i such that $\sum_{j=j_i+1}^{j_{i+1}} Lip^\beta(\varphi_j)\,\|y_j\| < \frac{\varepsilon}{2^i}$. Hence, $v = u_1 + \sum_{i=1}^{\infty} v_i = \sum_{j=1}^{\infty} \varphi_j \otimes y_j$ and

$$\sum_{j=1}^{\infty} Lip^\beta(\varphi_j)\,\|y_j\| \le \sum_{j=1}^{j_1} Lip^\beta(\varphi_i)\,\|y_i\| + \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \le 2\varepsilon + \widehat{\pi_\beta}(v).$$

The proof is complete since ε is arbitrary. ■

Slight modifications in the proof of Proposition 4.18 lead us to another useful formula for for the projective Σ-tensor norm on duals. We obtain

$$\widehat{\pi_\beta}(v) = \inf\left\{ \sum_{i=1}^{\infty} Lip^\beta(\varphi_i)\,\|y_i\| \;\middle|\; v = \sum_{i=1}^{\infty} \varphi_i \otimes y_i,\ \sum_{i=1}^{\infty} Lip^\beta(\varphi_i) < \infty,\ \|y_i\| \to 0 \right\}.$$

Proposition 4.19. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election on $\mathcal{BAN}$. Then the operator*

$$\begin{array}{rcl} \left(\mathcal{L}^\beta(X_1, \ldots, X_n) \otimes Y, \pi_\beta\right) & \to & \mathcal{N}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right) \\ \varphi \otimes y & \mapsto & \varphi \cdot y \end{array} \tag{4.4}$$

is bounded. Moreover, the unique extension to the completion

$$\begin{array}{rcl} J : \left(\mathcal{L}^\beta(X_1, \ldots, X_n) \widehat{\otimes} Y, \widehat{\pi_\beta}\right) & \to & \mathcal{N}\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right) \\ \sum_{i=1}^{\infty} \varphi_i \otimes y_i & \mapsto & \sum_{i=1}^{\infty} \varphi_i \cdot y_i \end{array}$$

is a surjective operator.

Proof. Plainly, the operator defined by (4.4) is well defined. Moreover, for any tensor v in $\mathcal{L}^\beta(X_1, \ldots, X_n) \otimes Y$, the triangle inequality implies

$$N\left(\sum_i \varphi_i \otimes y_i\right) \le \sum_i Lip^\beta(\varphi_i)\,\|y_i\|$$

for every representation. The conclusion follows from Proposition 4.18. ■

Proposition 4.20. *The collection of nuclear Σ-operators is an ideal of Σ-operators.*

Proof. Let $(X_1, \ldots, X_n, Y, \beta)$ be an election on $\mathcal{BAN}$. It is clear that $\mathcal{N}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ is a vector space and N is non-negative, homogeneous and verifies the triangle inequality. Besides,

$$\|f(p) - f(q)\| \leq \sum_{i=1}^{\infty} Lip^{\beta}(\varphi_i)\, \|y_i\|$$

holds for any representation of f. Therefore, $Lip^{\beta}(f) \leq N(F)$. As a consequence $N(f) = 0$ implies $f = 0$.

The rank-one Σ-operator $\varphi \cdot y$ is a nuclear Σ-operator; moreover, its nuclear norm satisfies $N(\varphi \cdot y) \leq Lip^{\beta}(\varphi)\, \|y\|$ by definition. As a consequence, every finite rank Σ-operator is nuclear.

Let p, q in $\Sigma^{\beta}_{X_1 \ldots X_n}$ and y^* in Y^*. We have

$$\begin{aligned} \left|tr\left(\widetilde{T} Lpqy^*\right)\right| &\leq \|y^*\|\, Lip^{\beta}(f_T)\, \beta(p-q) \\ &\leq \|y^*\|\, N(f_T)\, \beta(p-q) \end{aligned}$$

for all nuclear Σ-operator f_T.

To prove the ideal property consider the composition

$$\Sigma^{\theta}_{Z_1 \ldots Z_n} \xrightarrow{f_R} \Sigma^{\beta}_{X_1 \ldots X_n} \xrightarrow{f} Y \xrightarrow{S} W$$

where f_R is a Σ-θ-operator, S is a bounded linear operator and f is a nuclear Σ-operator. Let $\sum\limits_{i=1}^{\infty} \varphi_i \cdot y_i$ be a nuclear representation of f. A simple calculation shows

$$S f f_R = \sum_{i=1}^{\infty} (\varphi_i \circ \widetilde{R}) \cdot S(y_i).$$

Even more, the inequalities $Lip^{\theta}(\varphi_i \circ \widetilde{R}) \leq \|\widetilde{R}\|\, Lip^{\beta}(\varphi_i)$ and $\|S(y_i)\| \leq \|S\|\, \|y_i\|$ assert that $N(S f f_R) \leq \|\widetilde{R}\|\, \|S\|\, N(f)$.

The completeness of $\mathcal{N}\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ with the nuclear norm follows from Proposition 4.19. ■

In the linear context $J : \left(X^* \widehat{\otimes}_{\pi} Y\right) \to N(X, Y)$ is injective if either X^* or Y has the approximation property, see [93, Cor. 4.17]. In Proposition 4.21 we prove that J is injective under certain approximation property on $Y = \left(\widehat{\bigotimes}_{i=1}^{n} X_i, \widehat{\beta}\right)$, the completion of the normed space $(X_1 \otimes \ldots \otimes X_n, \beta)$.

Consider $(X_1, \ldots, X_n, Y, \beta)$ an election on $\mathcal{BAN}$. Notice that Lip^β defines a norm on $\mathcal{L}^\beta(X_1, \ldots, X_n)$. The identity

$$\mathcal{L}^\beta(X_1, \ldots, X_n) \to \left(\mathcal{L}^\beta(X_1, \ldots, X_n), Lip^\beta\right)$$

is a bounded linear operator with $\|I\| \leq 1$. In the particular case $\beta = \pi$, I is an isometry.

Let ψ be a bounded linear functional on $\left(\mathcal{L}^\beta(X_1, \ldots, X_n) \otimes Y, \pi_\beta\right)$. Define

$$\begin{array}{rcl} L_\psi : Y & \to & \left(\mathcal{L}^\beta(X_1, \ldots, X_n), Lip^\beta\right)^* \\ y & \mapsto & L_\psi(y) : \varphi \mapsto \psi(\varphi \otimes y). \end{array}$$

Since

$$|\langle L_\psi(y), \varphi\rangle| \leq \|\psi\| \, \pi_\beta(\varphi \otimes y) = \|\psi\| \, Lip^\beta(\varphi) \, \|y\|$$

holds for all φ and y, the operator L_ψ is bounded and $\|L_\psi\| \leq \|\psi\|$.

Proposition 4.21. *Let $X_1, \ldots, X_n$ be Banach spaces and let β be a reasonable crossnorm on $X_1 \otimes \ldots \otimes X_n$. Suppose that for every null sequence $(w_i)_i$ in $\left(\widehat{\bigotimes}_{i=1}^n X_i, \widehat{\beta}\right)$, for every bounded functional ψ on $\left(\mathcal{L}^\beta(X_1, \ldots, X_n) \otimes \left(\widehat{\bigotimes}_{i=1}^n X_i, \widehat{\beta}\right), \pi_\beta\right)$ and $\varepsilon > 0$ there exists a finite rank operator $S : \left(\widehat{\bigotimes}_{i=1}^n X_i, \widehat{\beta}\right) \to \left(\mathcal{L}^\beta(X_1, \ldots, X_n), Lip^\beta\right)^*$ such that S approximates L_ψ on the compact set $\{w_i\} \cup \{0\}$, that is*

$$\|S(w_i) - L_\psi(w_i)\| < \varepsilon$$

for all $i \in \mathbb{N}$. Then the linear map

$$J : \left(\mathcal{L}^\beta(X_1, \ldots, X_n) \widehat{\otimes} \left(\widehat{\bigotimes}_{i=1}^n X_i, \widehat{\beta}\right), \widehat{\pi_\beta}\right) \to \mathcal{N}\left(\Sigma^\beta_{X_1 \ldots X_n}; \left(\widehat{\bigotimes}_{i=1}^n X_i, \widehat{\beta}\right)\right)$$

is injective.

Proof. Let $v = \sum_{i=1}^\infty \varphi_i \otimes w_i$ where $\widehat{\pi_\beta}(w_i) \to 0$ and $\sum_{i=1}^\infty Lip^\beta(\varphi_i) < \infty$. Suppose v represents the zero Σ-operator, this is

$$0 = \sum_{i=1}^\infty \varphi_i(p) \, w_i$$

for all p in $\Sigma^{\beta}_{X_1 \dots X_n}$. Hence

$$0 = \sum_{i=1}^{\infty} \eta(\varphi_i)\, w_i$$

for all η $\left(\mathcal{L}^{\beta}\left(X_1, \dots, X_n\right), Lip^{\beta}\right)^*$.

Let ψ be a bounded functional on $\left(\mathcal{L}^{\beta}\left(X_1, \dots, X_n\right) \widehat{\otimes} \left(\overset{n}{\underset{i=1}{\widehat{\otimes}}} X_i, \widehat{\beta}\right), \pi_{\beta}\right)$ and $\varepsilon > 0$. There exist a finite rank operator $S : \left(\overset{n}{\underset{i=1}{\widehat{\otimes}}} X_i, \widehat{\beta}\right) \to \left(\mathcal{L}^{\beta}\left(X_1, \dots, X_n\right), Lip^{\beta}\right)^*$ such that

$$\left| \left\langle L_{\psi}(w_i) - S(w_i)\, ,\, \varphi \right\rangle \right| < \varepsilon\, Lip^{\beta}(\varphi) \qquad \forall\, \varphi \in \mathcal{L}^{\beta}\left(X_1, \dots, X_n\right),\, i \in \mathbb{N}.$$

Fix a representation $\sum_j f_j \otimes \eta_j$ of S with $f_j \in \left(\overset{n}{\underset{i=1}{\widehat{\otimes}}} X_i, \widehat{\beta}\right)^*$ and $\eta_j \in \left(\mathcal{L}^{\beta}\left(X_1, \dots, X_n\right), Lip^{\beta}\right)^*$. Notice that

$$\begin{aligned}
\sum_{i=1}^{\infty} \langle S(w_i), \varphi_i \rangle &= \sum_{i=1}^{\infty} \left\langle \sum_j f_j(w_i)\, \eta_j, \varphi_i \right\rangle \\
&= \sum_{i=1}^{\infty} \sum_j f_j(w_i)\, \eta_j(\varphi_i) \\
&= \sum_j \sum_{i=1}^{\infty} f_j(w_i)\, \eta_j(\varphi_i) \\
&= \sum_j f_j \left(\sum_{i=1}^{\infty} \eta_j(\varphi_i)\, w_i \right) \\
&= 0.
\end{aligned}$$

Then

$$\begin{aligned}
|\psi(v)| &= |\psi\left(\sum_{i=1}^{\infty}\varphi_i\otimes w_i\right)| \\
&= |\sum_{i=1}^{\infty}\psi\left(\varphi_i\otimes w_i\right)| \\
&= |\sum_{i=1}^{\infty}\langle L_\psi(w_i),\varphi_i\rangle| \\
&= |\sum_{i=1}^{\infty}\langle L_\psi(w_i)-S(w_i),\varphi_i\rangle+\sum_{i=1}^{\infty}\langle S(w_i),\varphi_i\rangle| \\
&\leq \sum_{i=1}^{\infty}|\langle L_\psi(w_i)-S(w_i),\varphi_i\rangle| \\
&\leq \varepsilon\sum_{i=1}^{\infty}Lip^\beta(\varphi_i)
\end{aligned}$$

asserts $|\psi(v)|=0$. Since ψ is arbitrary we conclude that $v=0$. ■

Definition 4.22. *We say that the tuple $(X_1,\ldots,X_n,\beta)$ has the approximation property if for every compact subset $K\subset\left(\widehat{\bigotimes}_{i=1}^{n}X_i,\widehat{\beta}\right)$ and $\varepsilon>0$ there exists a finite rank operator $S:\left(\widehat{\bigotimes}_{i=1}^{n}X_i,\widehat{\beta}\right)\to\left(\widehat{\bigotimes}_{i=1}^{n}X_i,\widehat{\beta}\right)$ such that*

$$|\varphi(w)-\varphi(S(w))|\leq\varepsilon\, Lip_\beta(\varphi)\qquad\forall\,\varphi\in\mathcal{L}^\beta\left(X_1,\ldots,X_n\right),\ w\in K.$$

Proposition 4.23. *Let $X_1,\ldots,X_n$ be Banach spaces and β be a reasonable crossnorm on $X_1\otimes\ldots\otimes X_n$. If $(X_1,\ldots,X_n,\beta)$ has the approximation property then*

$$J:\left(\mathcal{L}^\beta\left(X_1,\ldots,X_n\right)\widehat{\otimes}\left(\widehat{\bigotimes}_{i=1}^{n}X_i,\widehat{\beta}\right),\widehat{\pi_\beta}\right)\to\mathcal{N}\left(\Sigma^\beta_{X_1\ldots X_n};\left(\widehat{\bigotimes}_{i=1}^{n}X_i,\widehat{\beta}\right)\right)$$

is injective.

Proof. Let (w_i) be a null sequence in $\left(\widehat{\bigotimes}_{i=1}^{n}X_i,\widehat{\beta}\right)$. Let ψ be a non zero bounded functional on $\left(\mathcal{L}^\beta\left(X_1,\ldots,X_n\right)\otimes\left(\widehat{\bigotimes}_{i=1}^{n}X_i,\widehat{\beta}\right),\pi_\beta\right)$ and $\varepsilon>0$. Notice that $\{w_i\}\cup\{0\}$ is compact. The approximation property asserts the existence of a finite rank linear operators

$S : \left(\widehat{\bigotimes}_{i=1}^{n} X_i, \widehat{\beta} \right) \to \left(\widehat{\bigotimes}_{i=1}^{n} X_i, \widehat{\beta} \right)$ such that

$$|\phi(w_i) - \phi(S(w_i))| \leq \frac{\varepsilon}{\|\psi\|} \, Lip_\beta(\phi) \qquad \forall \, \phi \in \mathcal{L}^\beta \left(X_1, \ldots, X_n \right), \; i \in \mathbb{N}.$$

The condition

$$\|L_\psi(w_i) - L_\psi \circ S(w_i)\| \leq \|\psi\| \, \beta(w_i - S(w_i)) \qquad \forall \, i \in \mathbb{N}$$

implies

$$| \left\langle L_\psi(w_i) - L_\psi \circ S(w_i) \, , \, \varphi \right\rangle | \leq Lip^\beta(\varphi) \, \varepsilon \qquad \forall \, \varphi \in \mathcal{L}^\beta \left(X_1, \ldots, X_n \right) \; i \in \mathbb{N}$$

since $\beta(w_i - S(w_i)) \leq \sup_{Lip^\beta(\phi) \leq 1} |\phi(w_i) - \phi(S(w_i))|$. Proposition 4.21 completes the proof ■

If the multilinear operator $T : X_1 \times \cdots \times X_n \to Y$ is such that its associated Σ-operator $f_T : \Sigma^\beta_{X_1 \ldots X_n} \to Y$ is nuclear, then T is expressed in the form

$$T = \sum_{i=1}^{\infty} \varphi \cdot y_i$$

where (φ_i) and (y_i) are bounded sequences in $\mathcal{L}^\beta \left(X_1, \ldots, X_n \right)$ and Y with $\sum_{i=1}^{\infty} Lip^\beta(\varphi) \, \|y_i\|$ convergent.

4.4 p-Summing Σ-Operators

In this section we collect the most relevant facts about the collection of p-summing Σ-operator developed in [7] (and [6]), see Definition 2.1. It is worth to point out that these results were developed in the dissertation of Jorge Angúlo in the context of multilinear operators and $\beta = \pi$.

The attempts to establish summability for the multilinear case has given place to several generalizations; among others, we find [2, 12, 13, 14, 15, 43, 48, 76, 78, 79, 81, 82, 84, 89, 91, 98]. On the side of metric theory, in [53] is presented the notion of p-summability for Lipschitz functions. Following this reference, many research has been developed, see for instance [3, 24, 25, 26, 28, 35, 37, 38, 60, 94, 95].

The notion of p-summability for Σ-operators has the advantage that they admit analogues for Pietsch Domination and Factorization Theorems. Actually, this is the first factorization result for multilinear operators obtained by Σ-operators.

Theorem. *The* Σ*-operator* $f : \Sigma^\beta_{X_1\dots X_n} \to Y$ *is* p*-summing if and only if there exist* $C > 0$ *and a regular, Borel, probability measure* μ *on the compact set* $K := \big(B_{(X_1\otimes\dots\otimes X_n,\beta)^*}, w^*\big)$ *such that*

$$\|f(a) - f(b)\| \le C \left(\int_K |\varphi(a) - \varphi(b)|^p d\mu(\varphi) \right)^{\frac{1}{p}}$$

for all a, b *in* $\Sigma^\beta_{X_1\dots X_n}$*. Under these circumstances* $\pi_p(f) = \inf C$ *where the infimum is taken over all* C *as above.*

Using this result, it is easy to prove the Factorization Theorem for p-summing Σ-operators. In order to make the factorization theorem easier to read , we denote by $i : \Sigma^\beta_{X_1\dots X_n} \to C(B_{(X_1\otimes\dots\otimes X_n,\beta)^*}, w^*)$ the Σ-operator given by evaluation, this is, $\langle i(p)\,,\, h\rangle = h(p)$.

Theorem. *The* Σ*-operator* $f : \Sigma^\beta_{X_1\dots X_n} \to Y$ *is* p*-summing if and only if there exist a regular, Borel, probability measure on* $\big(B_{(X_1\otimes\dots\otimes X_n,\beta)^*}, w^*\big)$*, a subset* M *of* $L_p(\mu)$ *and a Lipschitz function* $u : M \to Y$ *such that* $M = j_p| \circ i(\Sigma^\beta_{X_1\dots X_n})$*,*

$$\begin{array}{ccc} \Sigma^\beta_{X_1\dots X_n} & \xrightarrow{\ f\ } & Y \\ \downarrow{\scriptstyle i} & & \uparrow{\scriptstyle u} \\ i(\Sigma^\beta_{X_1\dots X_n}) & \xrightarrow{\ j_p|\ } & M \\ \downarrow & & \downarrow \\ C(B_{(X_1\otimes\dots\otimes X_n,\beta)^*}) & \xrightarrow[\ j_p\ ]{} & L_p(\mu) \end{array}$$

commutes and $\pi_p(f) = Lip(u)$*.*

It is worth to notice that factorizations of p-summing Σ-operators are not linear anymore since they are obtained through subsets of spaces $L_p(\mu)$. This fact naturally leads us to the Lipschitz condition of the function $u : M \to Y$.

In his doctoral dissertation, Jorge Angúlo defines a generalization of the Chevet-Saphar tensor norm d_p. In this approach we present a general definition of d_p using reasonable crossnorms β.

Definition 4.24. *Let* $(X_1, \dots, X_n, Y, \beta)$ *be an election on* $\mathcal{BAN}$*. We define the norm* d^β_p *on* $X_1 \otimes \dots \otimes X_n \otimes Y$ *by*

$$d^\beta_p(u) = \inf \left\{ \|(p_i - q_i)\|^{w\beta}_{p^*} \, \|(y_i)\|_p \;\middle|\; u = \sum_{i=1}^m (p_i - q_i) \otimes y_i \right\}.$$

Proposition 4.25. *The norm* d_p *is a finitely generated* Σ*-tensor norm on spaces.*

Proof. In Jorge Angúlo's dissertation is proved that d_p is a reasonable crossnorm in the case $\beta = \pi$. The same proof works for the general case. Here we only prove the uniform and finitely generated properties.

Let $f_R : \Sigma^{\theta}_{Z_1 \ldots Z_n} \to \Sigma^{\beta}_{X_1 \ldots X_n}$ and $S : W \to Y$ be a Σ-θ-operator and linear operator respectively. Let $u = \sum_{i=1}^{m} (p_i - q_i) \otimes y_i$ in $Z_1 \otimes \ldots \otimes Z_n \otimes W$. Then

$$\begin{aligned} d_p^{\beta}\left(f_R \otimes S(v)\right) &= d_p^{\beta}\left(\sum_i (f_R(p_i) - f_R(q_i)) \otimes S(y_i)\right) \\ &\leq \|(f_R(p_i) - f_R(q_i))\|^{w\beta}_{p^*} \, \|(S(y_i))\|_p \\ &\leq \|\widetilde{R}\| \, \|S\| \, \|(p_i - q_i))\|^{w\theta}_{p^*} \, \|(y_i)\|_p. \end{aligned}$$

As a consequence, $d_p^{\beta}\left(f_R \otimes S(v)\right) \leq \|\widetilde{R}\| \, \|S\| \, d_p^{\theta}(u)$.

To see that d_p^{β} is finitely generated let u in $X_1 \otimes \ldots \otimes X_n \otimes Y$ and $\eta > 0$. There exists a representation of u, $\sum_{i=1}^{m} (p_i - q_i) \otimes y_i$ such that

$$\|(p_i - q_i)\|^{w\beta}_{p^*} \, \|(y_i)\|_p \leq d_p^{\beta}(u) + \eta.$$

It is clear that there exist finite dimensional subspaces E_i and F of X_i and Y respectively such that $p_i, q_i \in \Sigma^{\beta|}_{E_1 \ldots E_n}$ and $y_i \in F$. Hence

$$\|(p_i - q_i); E_1 \otimes \ldots \otimes E_n\|^{w\beta|}_{p^*} = \|(p_i - q_i)\|^{w\beta}_{p^*}$$

and

$$\|(y_i); F\|^{w}_{p} = \|(y_i)\|^{w}_{p}.$$

Finally

$$d_p^{\beta|}(u;\, E_i F) \leq d_p^{\beta}(u;\, X_i Y) + \eta$$

holds for all $\eta > 0$. ∎

The relation of p-summing Σ-operators with the Σ-tensor norm on spaces d_p is presented in the following result.

Theorem 4.26. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election on $\mathcal{BAN}$. Then*

$$\left(X_1 \otimes \ldots \otimes X_n \otimes Y, d^{\beta}_{p^*}\right)^* = \Pi_p\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y^*\right)$$

is a linear isometric isomorphism.

This result in combination with Theorem 3.13 implies that $[\Pi_p, \pi_p]$ fits in the context of ideals of Σ-operators.

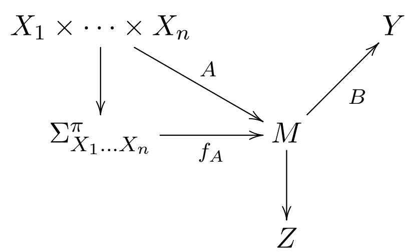
Proposition 4.27. *The collection* $[\Pi_p, \pi_p]$ *is a maximal ideal of* Σ*-operators.*

Proof. Apply Criterion 3.14. ■

4.5 Σ-Images

Factorizations of Σ-operators (hence, of multilinear operators) are obtained in terms of the image of a multilinear operator viewed as a subset of a Banach space. In order to have an easy language and a good manipulation of these factorizations, we present the notion of Σ-image.

Definition 4.28. *Consider the diagram*



where

i) $X_1, \ldots, X_n$, Y *and* Z *are Banach spaces.*

ii) M *is a subset of* Z*.*

iii) $A : X_1 \times \cdots \times X_n \to Z$ *is a bounded multilinear operator.*

iv) $f_A : \Sigma^{\pi}_{X_1 \ldots X_n} \to Z$ *is the associated bounded* Σ*-operator of* A*.*

v) $B : M \to Y$ *is a Lipschitz function.*

If $A(X_1 \times \cdots \times X_n) = f_A(\Sigma^{\pi}_{X_1 \ldots X_n}) = M$ *we say that* M *is the* Σ*-image associated to the tuple* $(X_1, \ldots, X_n, f_A)$*.*

The set

$$\mathcal{L}(M;Y) := \{ \, \psi : M \to Y \mid \psi A \text{ is multilinear and Lipschitz } \}$$

is a vector space endowed with the sum and product by scalars defined pointwise and it becomes a Banach space with the Lipschitz norm. For $\mathcal{L}(M;\mathbb{K})$ we simply write M^*.

For any $B \in \mathcal{L}(M;Y)$ we define its adjoint linear operator by

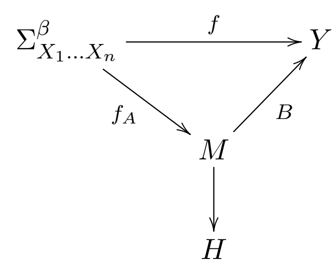
$$\begin{array}{rcl} B^* : Y^* & \to & M^* \\ y^* & \mapsto & y^* B. \end{array}$$

4.6 Factorization through Hilbert Spaces and the Σ-Tensor Norms on Spaces γ

This section is dedicated to present the collection of Σ-operators that admit a factorization through a Hilbert space. In order to have a good exposition we followed the steps of [87]. In this case we obtain three characterizations, the first is by definition of factorizations, see Definition 4.29, the second by domination, see Theorem 4.34 and the last by tensorial representation with the help of the Σ-tensor norm on spaces γ_2, see Definition 4.35.

The Factorization theorem of p-summing Σ-operators taught us that factorizations of Σ-operators (hence, of multilinear operators) cannote be expected linear. Even more, Lipschitz conditions are involved. Bearing this in mind, the proposal of Σ-operators that factor through a Hilbert spaces is the next (compare with [36]).

Definition 4.29. *We say that the Σ-operator $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ admits a factorization through a Hilbert space if there exist a Hilbert space H, a Σ-image M of H associated to $(X_1, \ldots, X_n, f_A)$ and a Lipschitz map $B : M \to Y^{**}$ such that the diagram*



commutes. Define $\Gamma(f) = \inf Lip^{\beta}(f_A)\, Lip(B)$ where the infimum is taken over all possible factorizations as above.

The collection of all Σ-operators form $\Sigma^{\beta}_{X_1 \ldots X_n}$ into Y that admit a factorization through a Hilbert space is denoted by $\Gamma\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$.

Proposition 4.30. *The pair $[\Gamma, \Gamma]$ is an ideal of Σ-operators.*

Proof. It is straightforward to prove that $\Gamma\left(\Sigma^{\beta}_{X_1 \ldots X_n}; Y\right)$ is vector space and Γ is a norm on it. Aside, let $f = \varphi \cdot y$ a rank-one Σ-operator. Then $f = B \circ \varphi$ where $B : \mathbb{K} \to Y$ is defined by $B(\lambda) = \lambda y$. Then, by definition, f factors through the Hilbert space $\mathbb{K}$ and $\Gamma(f) \leq Lip^{\beta}(\varphi)\, \|y\|$. Therefore, every finite rank Σ-operator factors through a Hilbert space. For p and q in $\Sigma^{\beta}_{X_1 \ldots X_n}$ and y^* in Y^* we have

$$\begin{aligned} |tr L_{pqy^*} \widetilde{T}| &\leq \|y^*\|\, \|f(p) - f(q)\| \\ &= \|y^*\|\, \|B f_A(p) - B f_A(q)\| \\ &\leq \|y^*\|\, Lip(B)\, Lip^{\beta}(f_A) \beta(p - q) \end{aligned}$$

for all f_T in $\Gamma\left(\Sigma^{\beta}_{X_1\ldots X_n};Y\right)$, and $f_T = Bf_A$.

To verify the ideal property consider the composition

$$\Sigma^{\theta}_{Z_1\ldots Z_n} \xrightarrow{f_R} \Sigma^{\beta}_{X_1\ldots X_n} \xrightarrow{f} Y \xrightarrow{S} W$$

where f_R is a Σ-θ-operator, $f = Bf_A$ factors through a Hilbert space and $S : Y \to W$ is a bounded linear operator. Then the composition Sff_R factors through the same Hilbert space than f. Even more

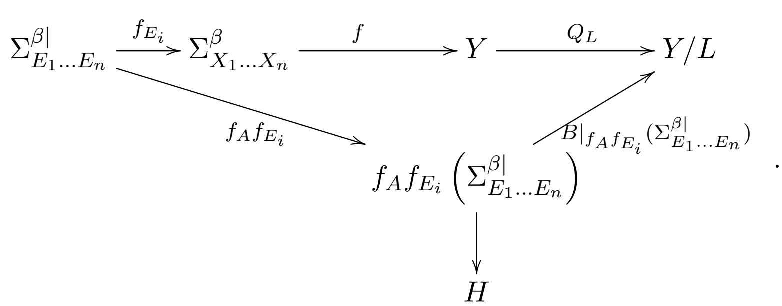
$$\Gamma(Sff_R) \leq Lip^{\theta}(f_A f_R)\, Lip(SB) \leq \|\widetilde{R}\|\, \|S\|\, Lip^{\theta}(f_A)\, Lip(B)$$

implies $\Gamma(Sff_R) \leq \|\widetilde{R}\|\, \|S\|\, \Gamma(f)$. ∎

Actually, $[\Gamma, \Gamma]$ is maximal. This result requires a bigger effort. The tool to prove it is the technique of ultrafilters.

Theorem 4.31. *The Σ-operator $f : \Sigma^{\beta}_{X_1\ldots X_n} \to Y$ admits a factorization through a Hilbert space and $\Gamma(f) \leq C$ if and only if for every finite dimensional subspace E_i of X_i and finite codimensional subspace L of Y the Σ-operator $Q_L f f_{E_i} : \Sigma^{\beta|}_{E_1\ldots E_n} \to Y/L$ admits a factorization through a Hilbert space and $\Gamma(Q_L f f_{E_i}) \leq C$. In other words, $[\Gamma, \Gamma]$ is maximal.*

Proof. Suppose that $f : \Sigma^{\beta}_{X_1\ldots X_n} \to Y$ factors through a Hilbert space. Let $f = Bf_A$ be a typical factorization for f. Let E_i and L as above. Consider the diagram



We have $Lip^{\beta|}(f_A f_{E_i}) \leq Lip^{\beta}(f_A)$ and $Lip(B|_{f_A f_{E_i}}(\Sigma^{\beta|}_{E_1\ldots E_n})) \leq Lip(B)$. Thus, $Q_L f f_{E_i}$ factors trough a Hilbert space and $\Gamma(Q_L f f_{E_i}) \leq \Gamma(f)$.

For the converse, define $\mathcal{P} = \mathcal{F}(X_1) \times \cdots \times \mathcal{F}(X_n) \times \mathcal{CF}(Y)$. The relation $\leq$ defined by $(E_1, \ldots, E_n, L) \leq (M_1, \ldots, M_n, N)$ if $E_i \subset M_i$ and $N \subset L$ defines a partial order on $\mathcal{P}$. Let $\mathfrak{A}$ be an ultrafilter on $\mathcal{P}$ containing the sets

$$(E_1, \ldots, E_n, L)^{\#} = \{(M_1, \ldots, M_n, N) \,|\, (E_1, \ldots, E_n, L) \leq (M_1, \ldots, M_n, N)\}.$$

For each $(E_1, \ldots, E_n, L) \in \mathcal{P}$ there exist a factorization as follows

$$\begin{array}{ccc} \Sigma^{\beta|}_{E_1 \ldots E_n} & \xrightarrow{Q_L f f_{E_i}} & Y/L \\ & {\scriptstyle f_{A^{E_i L}}} \searrow \quad \nearrow {\scriptstyle B^{E_i L}} & \\ & M^{E_i L} & \\ & \downarrow & \\ & H^{E_i L} & \end{array}$$

with $Lip^{\beta}(A^{E_i L}) \leq 1$ and $Lip(B^{E_i L}) \leq C$. By the finite dimensional hypothesis we may assume that $H^{E_i L} = \ell_2^{n(E_1, \ldots, E_n, L)}$ where $n(E_1, \ldots, E_n, L)$ is a natural number.

Define

$$\begin{array}{rcl} A_{E_i L} : \Sigma^{\beta}_{X_1 \ldots X_n} & \to & \ell_2^{n(E_1, \ldots, E_n, L)} \\ p & \mapsto & \left\{ \begin{array}{cc} A^{E_i L}(p) & p \in \Sigma^{\beta|}_{E_1 \ldots E_n} \\ 0 & otherwise \end{array} \right. . \end{array}$$

It is not difficult to see that

$$\begin{array}{rcl} A : X_1 \times \cdots \times X_n & \to & \left(\ell_2^{n(E_1, \ldots, E_n, L)} \right)_{\mathfrak{A}} \\ (x^1, \ldots, x^n) & \mapsto & (A_{E_i L}(x^1 \otimes \ldots \otimes x^n))_{\mathfrak{A}} \end{array}$$

is a multilinear mapping. The associated Σ-operator is given by

$$\begin{array}{rcl} f_A : \Sigma^{\beta}_{X_1 \ldots X_n} & \to & \left(\ell_2^{n(E_1, \ldots, E_n, L)} \right)_{\mathfrak{A}} \\ p & \mapsto & (A_{E_i L}(p))_{\mathfrak{A}}. \end{array}$$

Moreover,

$$\begin{array}{rcl} \|f_A(p) - f_A(q)\|_{\mathfrak{A}} & = & \|(A_{E_i L}(p) - A_{E_i L}(q))\|_{\mathfrak{A}} \\ & = & \lim_{\mathfrak{A}} \|A_{E_i L}(p) - A_{E_i L}(q)\|_{\ell_2^{n(E_1, \ldots, E_n, L)}} \\ & \leq & \beta(p - q) \end{array}$$

asserts that f_A is a bounded Σ-operator with $Lip^{\beta}(f_A) \leq 1$.

Aside, consider the functions

$$\begin{array}{rcl} \overline{(B^{E_i L})^*} : Y^* & \to & (M^{E_i L})^* \\ y^* & \mapsto & \left\{ \begin{array}{ll} (B^{E_i L})^*(\zeta) & y^* = Q_L^*(\zeta) \in Q_L^*((Y/L)^*) \\ 0 & otherwise \end{array} \right. . \end{array}$$

Define

$$\begin{array}{rcl} B : f_A(\Sigma^{\beta}_{X_1 \ldots X_n}) & \rightarrow & Y^{**} \\ f_A(p) & \mapsto & Bf_A(p) \end{array}$$

where

$$\begin{array}{rcl} Bf_A(p) : Y^* & \rightarrow & \mathbb{K} \\ y^* & \mapsto & \lim\limits_{\mathfrak{A}} \left\langle \overline{(B^{E_iL})^*}(y^*) \, , \, A_{E_iL}(p) \right\rangle . \end{array}$$

The inequality

$$| \left\langle \overline{(B^{E_iL})^*}(y^*) \, , \, A_{E_iL}(p) \right\rangle - \left\langle \overline{(B^{E_iL})^*}(y^*) \, , \, A_{E_iL}(q) \right\rangle | \quad \leq \quad C \, \|y^*\| \, \|A_{E_iL}(p) - A_{E_iL}(q)\|$$

in the case $q = 0$ implies that $Bf_A(p)$ is well defined. Even more, $(A^{E_iL}(p))_{\mathfrak{A}} = (A^{E_iL}(q))_{\mathfrak{A}}$ asserts that B does not depend on representants since $\lim\limits_{\mathfrak{A}} \|A^{E_iL}(p) - A^{E_iL}(q)\| = 0$. Moreover, the general case ensures that B is a Lipschitz function and $Lip(B) \leq C$.

Finally, notice that for every $p \in \Sigma^{\beta}_{X_1 \ldots X_n}$ and $y^* \in Y^*$ there exists $(E_1, \ldots, E_n, L) \in \mathcal{P}$ such that $p \in \Sigma^{\beta|}_{E_1 \ldots E_n}$ and $y^* \in Q^*_L(Y/L)^*$. Then $(E_1, \ldots, E_n, L)^{\#} \in \mathfrak{A}$ ensures that

$$\lim_{\mathfrak{A}} \left\langle \overline{(B^{E_iL})^*}(y^*) \, , \, A_{E_iL}(p) \right\rangle = y^*(f(p)).$$

As a consequence $Bf_A(p) = K_Y f(p)$ for all p in $\Sigma^{\beta}_{X_1 \ldots X_n}$. This means that

$$\begin{array}{ccccc} \Sigma^{\beta}_{X_1 \ldots X_n} & & \xrightarrow{\quad f \quad} & & Y \\ & \searrow^{f_A} & & \nearrow_{K_Y^{-1}B} & \\ & & f_A(\Sigma^{\beta}_{X_1 \ldots X_n}) & & \\ & & \downarrow & & \\ & & \left(\ell_2^{n(E_1, \ldots, E_n, L)}\right)_{\mathfrak{A}} & & \end{array}$$

is commutative. Now, if we considering all the spaces $\ell_2^{n(E_1, \ldots, E_n, L)}$ as abstract L_2-spaces then there exist a measure space (Ω, μ) such that $\left(\ell_2^{n(E_1, \ldots, E_n, L)}\right)_{\mathfrak{A}}$ is (order) isometric to $L_2(\mu)$, see [67, Th. 1.b.2]. This means that f factors through a Hilbert space and $\Gamma(f) \leq C$. ■

The Σ-operators that factor through a Hilbert space can be characterized under their behavior on finite sequences.

Definition 4.32. *Given finite sequences (p_i), (q_i), (a_j), (b_j) contained in $\Sigma_{X_1\ldots X_n}$ we will write $(p_i,q_i)\leq_\beta (a_j,b_j)$ if*

$$\sum_i |\varphi(p_i)-\varphi(q_i)|^2 \leq \sum_j |\varphi(a_j)-\varphi(b_j)|^2$$

holds for all $\varphi\in\mathcal{L}^\beta(X_1,\ldots,X_n)$.

In last definition it is clear that we may assume that all the sequences have the same length. This domination is characterized in terms of scalar matrices as follows.

Proposition 4.33. *$(p_i,q_i)_{1\leq i\leq m}\leq_\beta (a_j,b_j)_{1\leq j\leq m}$ if and only if there exists an $m\times m$ scalar matrix (a_{ij}) such that*

$$\sum_i \Big|\sum_j a_{ij}\alpha_j\Big|^2 \leq \sum_j |\alpha_j|^2$$

for all $(\alpha_j)\in\mathbb{K}^m$ and

$$\varphi(p_i)-\varphi(q_i)=\sum_j a_{ij}\left(\varphi(a_j)-\varphi(b_j)\right)$$

for all φ in $\mathcal{L}^\beta(X_1,\ldots,X_n)$. As a consequence,

$$p_i-q_i=\sum_j a_{ij}(a_j-b_j).$$

Proof. For the if part define

$$S:=\left\{\left(\varphi(a_j)-\varphi(b_j)\right)_j \mid \varphi\in\mathcal{L}^\beta(X_1,\ldots,X_n)\right\}.$$

We may consider S as a subspace of ℓ_2^m. The operator $A:S\to\ell_2^m$ defined by

$$A\left(\left(\varphi(a_j)-\varphi(b_j)\right)_j\right):=\left(\varphi(p_i)-\varphi(q_i)\right)_i$$

is bounded and $\|A\|\leq 1$ since

$$\left\|A\left(\left(\varphi(a_j)-\varphi(b_j)\right)_j\right)\right\|_{\ell_2^m}^2=\sum_i|\varphi(p_i)-\varphi(q_i)|^2\leq\sum_j|\varphi(a_j)-\varphi(b_j)|^2.$$

It is possible to extend A to an operator $\tilde{A}:\ell_2^m\to\ell_2^m$ with $\|\tilde{A}\|\leq 1$. Let (a_{ij}) the $m\times m$ matrix associated to $\tilde{A}$, then

$$\begin{aligned}
\left(\varphi(p_i)-\varphi(q_i)\right)_i &= \tilde{A}\left(\left(\varphi(a_j)-\varphi(b_j)\right)_i\right)\\
&= (a_{ij})\left(\left(\varphi(a_j)-\varphi(b_j)\right)_i\right)\\
&= \left(\sum_j a_{ij}\left(\varphi(a_j)-\varphi(b_j)\right)\right)_i.
\end{aligned}$$

Conversely, since the norm of (a_{ij}) as operator is less than 1 we conclude

$$\sum_i |\varphi(p_i) - \varphi(q_i)|^2 = \sum_i \Big| \sum_j a_{ij} \left(\varphi(a_j) - \varphi(b_j)\right) \Big|^2 \le \sum_j |\varphi(a_j) - \varphi(b_j)|^2.$$

In other words, $(p_i, q_i) \le_\beta (a_j, b_j)$. ■

Theorem 4.34. *The Σ-operator* $f : \Sigma^\beta_{X_1 \dots X_n} \to Y$ *admits a factorization through a Hilbert space if and only if there exists a constant* $C > 0$ *such that*

$$\sum_i \|f(p_i) - f(q_i)\|^2 \le C^2 \sum_i \beta(a_i - b_i)^2, \tag{4.5}$$

for all finite sequences such that $(p_i, q_i) \le_\beta (a_i, b_i)$*. In this case* $\Gamma(f)$ *coincides with the smallest constant* C *as above.*

Proof. First, let us prove that (4.5) implies that f admits a factorization through a Hilbert space. For this end, we will use Theorem 4.31. Let E_i be a finite dimensional subspace of X_i. Set

$$K := \left\{ \zeta \in (E_1 \otimes \ldots \otimes E_n, \beta|)^* \mid \|\zeta\|_{\beta|} = 1 \right\}.$$

Since the spaces E_i are finite dimensional K is compact. Define S the subset of $C(K)$ given by the functions of the form

$$\phi(\zeta) = \sum_i |\zeta(p_i) - \zeta(q_i)|^2 - \sum_i |\zeta(a_i) - \zeta(b_i)|^2$$

where (a_i), (b_i), (p_i) and (q_i) are finite sequences in $\Sigma^{\beta|}_{E_1 \dots E_n}$ such that

$$C^2 \sum_i \beta|(a_i - b_i)^2 \le \sum_i \|f(p_i) - f(q_i)\|^2.$$

Every element ϕ in S satisfy $\|\phi\| > 0$ since there exist ζ in K such that $\phi(\zeta) > 0$. Moreover, S is a convex cone disjoint of the negative open cone $C_- := \{ \phi \mid \sup \phi < 0 \}$. An application of the Hahn-Banach theorem ensures the existence of a measure μ on K which separates C_- and S. It is possible to adjust μ to be a positive measure such that

$$0 \le \int_K \phi(\zeta)\, d\mu(\zeta)$$

for all ϕ in S. Since E_i is a finite dimensional space

$$D = \sup \left\{ \left(\int_K |\zeta(a) - \zeta(b)|^2 d\mu(\zeta) \right)^{\frac{1}{2}} \,\Big|\, \beta|(a - b) \le 1,\ a, b \in \Sigma^{\beta|}_{E_1 \dots E_n} \right\} > 0.$$

Thus, we may adjust μ such that $D = C$.

For every a, b, p and q in $\Sigma^{\beta|}_{E_1\ldots E_n}$ such that $C\ \beta|(a-b) \leq \|f(p) - f(q)\|$ we have

$$\int_K |\zeta(a) - \zeta(b)|^2 d\mu(\zeta) \leq \int_K |\zeta(p) - \zeta(q)|^2 d\mu(\zeta).$$

In particular, given p and q in $\Sigma^{\beta|}_{E_1\ldots E_n}$ such that $C < \|f(p) - f(q)\|$ and a, b in $\Sigma^{\beta|}_{E_1\ldots E_n}$ with $\beta|(a-b) < 1$, then

$$\int_K |\zeta(a) - \zeta(b)|^2 d\mu \leq \int_K |\zeta(p) - \zeta(q)|^2 d\mu.$$

As a consequence

$$C \leq \left(\int_K |\zeta(p) - \zeta(q)|^2 d\mu \right)^{1/2}$$

for all p, q in $\Sigma^{\beta|}_{E_1\ldots E_n}$ with $C \leq \|f(p) - f(q)\|$. Take $c = \|f(p) - f(q)\|$ and $\varepsilon > 0$. The homogeneous property of f asserts that

$$C < (C+\varepsilon)\frac{c}{c} = \left\| f\left(\frac{C+\varepsilon}{c} p\right) - f\left(\frac{C+\varepsilon}{c} q\right) \right\|.$$

Hence,

$$\frac{C}{C+\varepsilon}\|f(p) - f(q)\| \leq \left(\int_K |\zeta(p) - \zeta(q)|^2 d\mu \right)^{\frac{1}{2}}$$

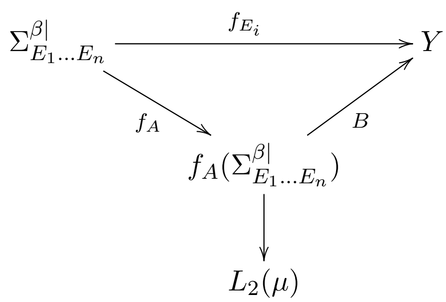
holds for all $\varepsilon > 0$. This way,

$$\|f(p) - f(q)\| \leq \left(\int_K |\zeta(p) - \zeta(q)|^2 d\mu \right)^{\frac{1}{2}} \qquad \forall\, p, q \in \Sigma^{\beta|}_{E_1\ldots E_n}. \tag{4.6}$$

On the other hand, it is clear that

$$\left(\int_K |\zeta(a) - \zeta(b)|^2 d\mu \right)^{\frac{1}{2}} \leq C\ \beta|(a-b) \qquad \forall\ a, b \in \Sigma^{\beta|}_{E_1\ldots E_n}. \tag{4.7}$$

Finally, we obtain a factorization as follows



where

$$\begin{array}{rcl} A : E_1 \times \cdots \times E_n & \to & L_2(\mu) \\ (x^1, \dots, x^n) & \mapsto & A(x^1, \dots, x^n) : \zeta \mapsto \zeta(x^1, \dots, x^n) \end{array}$$

and $B(f_A(p)) := f(p)$. Boundedness of f_A is deduced by (4.7); moreover, $Lip^{\beta}(f_A) \leq C$. Aside, (4.6) asserts that B is a well defined Lipschitz function and $Lip(B) \leq 1$. This way, the Σ-operators $Q_L f_{E_i}$ factors through a Hilbert space and $\Gamma(Q_L f_{E_i}) \leq C$. Theorem 4.31 implies that f factors through a Hilbert space and $\Gamma(f) \leq \inf C$.

Conversely, suppose that $f : \Sigma^{\beta}_{X_1\dots X_n} \to Y$ admits a factorizations through a Hilbert space. Let $f = Bf_A$ be a typical factorization of f through the Hilbert space H. If $(p_i - q_i) \leq_{\beta} (a_i - b_i)$, then it is clear that $(f_A(p_i) - f_A(q_i)) \leq (f_A(a_i) - f_A(a_i))$ in H. Given an orthonormal basis $(e_\alpha)_{\alpha\in I}$ of H we have $\|h\|^2 = \sum_{\alpha} |\langle h, e_\alpha\rangle|^2$. Then

$$\begin{array}{rcl} \displaystyle\sum_i \sum_{\alpha\in F} |\, \langle f_A(p_i) - f_A(q_i)\, ,\, e_\alpha \rangle \,|^2 & = & \displaystyle\sum_{\alpha\in F} \sum_i |\, \langle f_A(p_i) - f_A(q_i)\, ,\, e_\alpha \rangle \,|^2 \\ & \leq & \displaystyle\sum_{\alpha\in F} \sum_i |\, \langle f_A(a_i) - f_A(b_i)\, ,\, e_\alpha \rangle \,|^2 \end{array}$$

for all finite subset F of I. Thus

$$\sum_i \|f_A(p_i) - f_A(q_i)\|^2 \leq \sum_i \|f_A(a_i) - f_A(b_i)\|^2.$$

Finally

$$\begin{array}{rcl} \displaystyle\sum_i \|f(p_i) - f(q_i)\|^2 & = & \displaystyle\sum_i \|Bf_A(p_i) - Bf_A(q_i)\|^2 \\ & \leq & \displaystyle Lip(B)^2 \sum_i \|f_A(p_i) - f_A(q_i)\|^2 \\ & \leq & \displaystyle Lip(B)^2 \sum_i \|f_A(a_i) - f_A(b_i)\|^2 \\ & \leq & \displaystyle Lip(B)^2 Lip^{\beta}(f_A)^2 \sum_i \beta|(a_i - b_i)^2 \end{array}$$

implies that (4.5) holds and $\inf C \leq Lip(B) Lip^{\beta}(f_A)$. As a consequence, $\inf C \leq \Gamma(f)$. ∎

The tensorial representation of the maximal ideal $[\Gamma, \Gamma]$ is presented next. First, we define the involved Σ-tensor norm on spaces (compare with [36, Def 4.2] and [87, Ch. 2 Sec. b]).

Definition 4.35. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. For u in $X_1 \otimes \ldots \otimes X_n \otimes Y$ define*

$$\gamma_2^{\beta}(u) = \inf \left\{ \|(a_j - b_j)\|_2^{\beta} \, \|(y_i)\|_2 \;\middle|\; u = \sum_{i=1}^{m} (p_i - q_i) \otimes y_i, \ (p_i, q_i) \leq_{\beta} (a_j, b_j) \right\}.$$

Proposition 4.36. *γ_2 is a finitely generated Σ-tensor norm on spaces.*

Proof. First, let us prove that $\varepsilon^{\beta} \leq \gamma_2^{\beta}$. Let $u = \sum_{i=1}^{m} (p_i - q_i) \otimes y_i$. For finite sequences $(a_j), (b_j)$ such that $(p_i, q_i) \leq_{\beta} (a_j, b_j)$, the Holder inequality implies

$$\begin{aligned}
|\langle \varphi \otimes y^*, u \rangle| &= |\sum_i \varphi(p_i - q_i) \, y^*(y_i)| \\
&\leq \left(\sum_i |\varphi(p_i) - \varphi(q_i)|^2 \right)^{\frac{1}{2}} \left(\sum_i |y^*(y_i)|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_j |\varphi(a_j) - \varphi(b_j)|^2 \right)^{\frac{1}{2}} \left(\sum_i |y^*(y_i)|^2 \right)^{\frac{1}{2}} \\
&\leq \|\varphi\|_{\beta} \, \|y_i\| \left(\sum_j |\beta(a_j - b_j)|^2 \right)^{\frac{1}{2}} \left(\sum_i \|(y_i)\|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

Hence, $\varepsilon^{\beta} \leq \gamma_2^{\beta}$. In particular $\gamma_2^{\beta}(u) = 0$ implies $u = 0$. The homogeneous property of γ_2^{β} is clear.

For the triangle inequality take u, v in $X_1 \otimes \ldots \otimes X_n \otimes Y$ and $\eta > 0$. There exist representations $\sum_{i=1}^{m} (p_i - q_i) \otimes y_i$ and $\sum_{l} (s_j - t_j) \otimes z_j$ of u and v respectively, and sequences (a_i), (b_i), (x_j) and (w_j) such that $(p_i, q_i) \leq_{\beta} (a_i, b_i)$, $(s_j, t_j) \leq_{\beta} (x_j, w_j)$ and

$$\begin{aligned}
\|(a_i - b_i)\|_2^{\beta} &\leq (\gamma_2^{\beta}(u) + \eta)^{1/2} \\
\|(y_i)\|_2 &\leq (\gamma_2^{\beta}(u) + \eta)^{1/2} \\
\|(x_j - w_j)\|_2^{\beta} &\leq (\gamma_2^{\beta}(v) + \eta)^{1/2} \\
\|(z_j)\|_2 &\leq (\gamma_2^{\beta}(v) + \eta)^{1/2}.
\end{aligned}$$

Then $\sum_{i=1}^{m}(p_i - q_i)\otimes y_i + \sum_{l}(s_j - t_j)\otimes z_j$ is a representation of $u+v$. Moreover,

$$\sum_i |\varphi(p_i)-\varphi(q_i)|^2 + \sum_j |\varphi(s_j)-\varphi(t_j)|^2 \leq \sum_i |\varphi(a_i)-\varphi(b_i)|^2 + \sum_j |\varphi(x_j)-\varphi(w_j)|^2.$$

Then, the inequalities

$$\sum_i \beta(a_i-b_i)^2 + \sum_j \beta(x_j-w_j)^2 \leq \gamma_2^\beta(u)+\gamma_2^\beta(v)+2\eta$$

and

$$\sum_i \|(y_i)\|^2 + \sum_j \|(z_j)\|^2 \leq \gamma_2^\beta(u)+\gamma_2^\beta(v)+2\eta$$

asserts that

$$\gamma_2^\beta(u+v) \leq \gamma_2^\beta(u)+\gamma_2^\beta(v)+2\eta$$

holds for all $\eta > 0$. This way, γ_2^β verifies the triangle inequality.

Plainly, $\gamma_2^\beta((p-q)\otimes y) \leq \beta(p-q)\,\|y\|$. As a result, $\gamma_2^\beta \leq \pi^\beta$. Hence, γ_2^β is a reasonable crossnorm on $X_1\otimes\ldots\otimes X_n\otimes Y$.

To prove the uniform property let $(Z_1,\ldots,Z_n,W,\theta)$ be another election in $\mathcal{BAN}$. Let $f_R : \Sigma^\theta_{Z_1\ldots Z_n} \to \Sigma^\beta_{X_1\ldots X_n}$ be a Σ-θ-operator and $S: W \to Y$ be a bounded linear operator. For $u=\sum_{i=1}^{m}(p_i-q_i)\otimes y_i$ and $(a_j),(b_j)$ such that $(p_i,q_i)\leq_\theta (a_j,b_j)$ we have that

$$\begin{aligned}\sum_i |\varphi(f_R(p_i))-\varphi(f_R(q_i))|^2 &= \sum_i |\varphi\widetilde{R}(p_i)-\varphi\widetilde{R}(q_i)|^2\\ &\leq \sum_j |\varphi\widetilde{R}(a_j)-\varphi\widetilde{R}(b_j)|^2\\ &\leq \sum_j |\varphi(f_R(a_j))-\varphi(f_R(b_j))|^2\end{aligned}$$

is verified for all $\varphi\in\mathcal{L}^\beta(X_1,\ldots,X_n)$. This is, $(f_R(p_i),f_R(q_i))\leq_\beta (f_R(a_j),f_R(b_j))$. Then

$$\begin{aligned}\gamma_2^\beta(f_R\otimes S(u)) &\leq \|(f_R(p_i)-f_R(q_i))\|_2^\beta\,\|(S(y_i))\|_2\\ &\leq \|\widetilde{R}\|\,\|S\|\,\|(p_i-q_i)\|_2^\theta\,\|(y_i)\|_2.\end{aligned}$$

Hence, $\gamma_2^\beta(f_R\otimes S(u)) \leq \|\widetilde{R}\|\,\|S\|\,\gamma_2^\theta(u)$. In other words, the linear operator

$$f_R\otimes S : (Z_1\otimes\ldots\otimes Z_n\otimes W,\gamma_2^\theta)\to(X_1\otimes\ldots\otimes X_n\otimes Y,\gamma_2^\beta)$$

is bounded and $\|f_R \otimes S\| \leq \|\widetilde{R}\| \, \|S\|$.

To see that γ_2^β is finitely generated, first notice that $\gamma_2^\beta(u;\, X_iY) \leq \gamma_2^{\beta|}(u;\, E_iF)$ holds for all u and finite dimensional subspaces E_i and F of X_i and Y such that $u \in E_1 \otimes \ldots \otimes E_n \otimes F$.

Aside, let $\eta > 0$. Then, there exists a representation $\sum_{i=1}^{m} (p_i - q_i) \otimes y_i$ of u and finite sequences (a_j) and (b_j) such that $(p_i, q_i) \leq_\beta (a_j, b_j)$ and

$$\|(a_j - b_j)\|_2^\beta \, \|(y_i)\|_2 \leq (1+\eta)\, \gamma_2^\beta(u).$$

Let us denote $p_i = x_i^1 \otimes \ldots \otimes x_i^n$, $p_i = z_i^1 \otimes \ldots \otimes z_i^n$, $a_j = s_j^1 \otimes \ldots \otimes s_j^n$ and $b_j = t_j^1 \otimes \ldots \otimes t_j^n$. We set $E_i := span\{x_j^i, z_j^i, s_j^i, t_j^i\} \subset X_i$ and $F = span\{y_i\} \subset Y$. Hence, $u \in E_1 \otimes \ldots \otimes E_n \otimes F$, and $p_i, q_i, a_j, b_j \in \Sigma_{E_1 \ldots E_n}^{\beta|}$. Moreover, the Hahn-Banach theorem implies $(p_i, q_i) \leq_{\beta|} (a_j, b_j)$. Hence,

$$\begin{aligned} \gamma_2^{\beta|}(u;\, E_iF) &\leq \|(a_j - b_j)\|_2^{\beta|} \, \|y_i\|_2 \\ &= \|(a_j - b_j)\|_2^{\beta} \, \|y_i\|_2 \\ &\leq (1+\eta)\, \gamma_2^\beta(u) \end{aligned}$$

asserts that γ_2^β is finitely generated. ∎

Theorem 4.37. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. The operator*

$$\begin{aligned} \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \gamma_2^\beta\right)^* &\to \Gamma\left(\Sigma_{X_1 \ldots X_n}^\beta; Y^*\right) \\ \varphi &\mapsto f_\varphi \end{aligned}$$

is a linear isometric isomorphism.

Proof. We will use the linear isometry $(Y \oplus_{\ell_2} \cdots \oplus_{\ell_2} Y)^* = Y^* \oplus_{\ell_2} \cdots \oplus_{\ell_2} Y^*$.

Suppose that f factors through a Hilbert space. The combination of the above isometry and Theorem 4.34, implies that for all $(y_i)_i$ and $(p_i, q_i) \leq_\beta (a_j, b_j)$,

$$\left| \sum_i \langle f(p_i) - f(q_i)\,,\, y_i \rangle \right| \leq \Gamma(f)\, \|(a_j - b_j)\|_2^\beta \, \|(y_i)\|_2.$$

So, if $u = \sum_{i=1}^{m} (p_i - q_i) \otimes y_i$ is an element in $X_1 \otimes \ldots \otimes X_n \otimes Y$ and $(p_i, q_i) \leq_\beta (a_j, b_j)$ then

$$|\varphi_f(u)| \leq \Gamma(f)\, \|(a_j - b_j)\|_2^\beta \, \|(y_i)\|_2$$

which means that φ_f is bounded and $\|\varphi_f\| \leq \Gamma(f)$.

Conversely, suppose $\varphi \in \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \gamma_2^\beta\right)^*$. Let $(p_i, q_i) \leq_\beta (a_j, b_j)$ and $(y_i)_i$. Define $u = \sum_{i=1}^{m} (p_i - q_i) \otimes y_i$, then

$$\left| \sum_i \langle f_\varphi(p_i) - f_\varphi(q_i), y_i \rangle \right| = |\varphi(u)| \leq \|\varphi\| \, \|(a_j - b_j)\|_2^\beta \, \|(y_i)\|_2.$$

After taking suprema over $\sum_i \|y_i\|^2 \leq 1$ we obtain

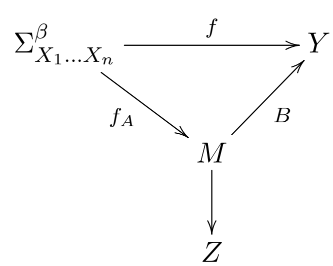
$$\left(\sum_i \|f_\varphi(p_i) - f_\varphi(q_i)\|^2 \right)^{\frac{1}{2}} \leq \|\varphi\| \, \|(a_j - b_j)\|_2^\beta.$$

This means that $f_\varphi : \Sigma_{X_1 \ldots X_n}^\beta \to Y^*$ factors through a Hilbert space and $\Gamma(f_\varphi) \leq \|\varphi\|$. ■

4.7 2-Dominated Σ-Operators and the Σ-Tensor Norm on Spaces ω_2

This section is dedicated to the case of 2-dominated Σ-operators. We present three characterizations for a Σ-operator f to be 2-dominated. The first, by definition, is generalizing the factorization of a typical 2-dominated linear operators. The second is by domination of the values $y^*(f(p))$ uniformly by 2-summing linear operators and Σ-operators. The third is the tensorial representation where we extend the Hilbert tensor norm ω_2 to the setting of Σ-tensor norms on spaces.

Definition 4.38. *A Σ-operator $f : \Sigma_{X_1 \ldots X_n}^\beta \to Y$ is named 2-dominated if there exist a Banach space Z, a Σ-image M of Z associated to $(X_1, \ldots, X_n, f_A)$ and a Lipschitz function $B : M \to Y$ such that $f_A : \Sigma_{X_1 \ldots X_n}^\beta \to Z$ is a 2-summing Σ-operator, B^* is a 2-summing linear operator and the diagram*



commutes. The 2-dominated norm of f is defined to be

$$D_2(f) = \inf \pi_2(f_A)\, \pi_2(B^*)$$

where the infimum is taken over all possible factorizations as above.

Let $\mathcal{D}_2\left(\Sigma^\beta_{X_1 \ldots X_n}; Y\right)$ denote the Banach space of all 2-dominated Σ-operators from $\Sigma^\beta_{X_1 \ldots X_n}$ into Y endowed with D_2.

In the definition of a 2-dominated Σ-operator we may chose Z as a Hilbert space. Thus, every 2-dominated Σ-operator factors through a Hilbert space. This time, the Σ-operator $f_A : \Sigma^\beta_{X_1 \ldots X_n} \to Z$ is not just bounded but 2-summing in the sense of Section 4.4. The same is true for the Lipschitz function $B : M \to Y$, $B^* : Y^* \to M^*$ is 2-summing. It is worth to notice that B is defined just in the set M. Aside, this definition can be interpreted as a composition of an ideal of Σ-operators and an ideal of Lipschitz functions (those whose adjoint linear operator is 2-summing).

The Hilbertian tensor norm ω_2 (see for instance [93, Sec. 7.4]) is easily extended to the case of Σ-tensor norm as following.

Definition 4.39. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in $\mathcal{BAN}$. Define*

$$\omega_2^\beta(u) := \inf \left\{ \|\, (p_i - q_i)\, \|_2^{w\beta} \; \|(y_i)\|_2^w \;\middle|\; u = \sum_{i=1}^m (p_i - q_i) \otimes y_i \right\}$$

for all u in $X_1 \otimes \ldots \otimes X_n \otimes Y$.

Proposition 4.40. *ω_2 is a finitely generated Σ-tensor norm on spaces.*

Proof. First, observe that given $\varphi \in \mathcal{L}^\beta(X_1, \ldots, X_n)$ and $y^* \in Y^*$ we have by definition of ω_2^β and Holder inequality

$$\begin{aligned} |\langle \varphi \otimes y\,,\, u\rangle| &\leq \sum_i |\varphi(p_i - q_i)\; y^*(y_i)| \\ &\leq \left(\sum_i |\varphi(p_i - q_i)|^2\right)^{1/2} \left(\sum_i |y^*(y_i)|^2\right)^{1/2} \\ &\leq \|\varphi\|_\beta\, \|y^*\|\; \|(p_i - q_i)\|_2^{w\beta}\; \|(y_i)\|_2^w. \end{aligned}$$

After taking infimum over all the representations of u we obtain

$$|\langle \varphi \otimes y, u\rangle| \leq \|\varphi\|_\beta\, \|y^*\|\, \omega_2^\beta(u).$$

Hence, $\varepsilon^\beta(u) \leq \omega_2^\beta(u)$. This way, $\omega_2^\beta(u) = 0$ implies $u = 0$ since ε^β is a norm. Condition $\omega_2^\beta(\lambda u) = |\lambda|\, \omega_2^\beta(u)$ is clear by definition.

For the triangle inequality take u, v in $X_1 \otimes \ldots \otimes X_n \otimes Y$ and $\eta > 0$. There exist representations $\sum_{i=1}^{m} (p_i - q_i) \otimes y_i$ and $\sum_j (a_j - b_j) \otimes z_j$ of u and v respectively such that

$$\begin{aligned} \|(p_i - q_i)\|_2^{w\beta} &\leq (\omega_2^\beta(u) + \eta)^{1/2} \\ \|(y_i)\|_2^{w} &\leq (\omega_2^\beta(u) + \eta)^{1/2} \\ \|(a_j - b_j)\|_2^{w\beta} &\leq (\omega_2^\beta(v) + \eta)^{1/2} \\ \|(z_j)\|_2^{w} &\leq (\omega_2^\beta(v) + \eta)^{1/2}. \end{aligned}$$

Then $\sum_{i=1}^{m} (p_i - q_i) \otimes y_i + \sum_j (a_j - b_j) \otimes z_j$ is a representation of $u + v$ and satisfies

$$\left(\sum_i |\varphi(p_i - q_i)|^2 + \sum_j |\varphi(a_j - b_j)|^2 \right)^{1/2} \leq (\omega_2^\beta(u) + \omega_2^\beta(v) + 2\eta)^{1/2}$$

and

$$\left(\sum_i |y^*(y_i)|^2 + \sum_j |y^*(z_j)|^2 \right)^{1/2} \leq (\omega_2^\beta(u) + \omega_2^\beta(v) + 2\eta)^{1/2}$$

for all $\varphi \in \mathcal{L}^\beta(X_1, \ldots, X_n)$ and $y^* \in Y^*$ both with norm less than one. Then

$$\omega_2^\beta(u + v) \leq (\omega_2^\beta(u) + \omega_2^\beta(v) + 2\eta)$$

holds for all $\eta > 0$. This way, ω_2^β verifies the triangle inequality.

It is clear that $\omega_2^\beta((p - q) \otimes y) \leq \beta(p - q)\,\|y\|$ since we are in the case of two sequences of just one term. We may conclude that ω_2^β is reasonable and crossed.

To prove the uniform property let $(Z_1, \ldots, Z_n, W, \theta)$ be another election in $\mathcal{BAN}$. Let $f_R : \Sigma^\theta_{Z_1 \ldots Z_n} \to \Sigma^\beta_{X_1 \ldots X_n}$ be a Σ-θ-operator and $S : W \to Y$ be a bounded linear operator. Then, Proposition 1.6 ensures

$$\begin{aligned} \omega_2^\beta(f_R \otimes S(u)) &\leq \|(f_R(p_i) - f_R(q_i)))\|_2^{w\beta}\, \|(S(y_i))\|_2^{w} \\ &\leq \|\widetilde{R}\|\, \|S\|\, \|(p_i - q_i)\|_2^{w\beta}\, \|(y_i)\|_2^{w}. \end{aligned}$$

Hence, $\omega_2^\beta(f_R \otimes S(u)) \leq \|\widetilde{R}\|\, \|S\|\, \omega_2^\beta(u)$. In other words, the linear operator

$$f_R \otimes S : (Z_1 \otimes \ldots \otimes Z_n \otimes W, \omega_2^\theta) \to (X_1 \otimes \ldots \otimes X_n \otimes Y, \omega_2^\beta)$$

is bounded and $\|f_R \otimes S\| \leq \|\widetilde{R}\|\, \|S\|$.

The uniform property asserts that $\omega_2^{\beta}(u;\, X_iY) \leq \omega_2^{\beta|}(u;\, E_iF)$ holds for all finite dimensional subspaces E_i and F of X_i and Y respectively whose tensor product contains u. Reciprocally, given $\eta > 0$ there exists a representation $\sum_{i=1}^{m}(p_i - q_i) \otimes y_i$ of u such that

$$\|(p_i - q_i)\|_2^{w\beta}\ \|(y_i)\|_2^{w} \leq (1+\eta)\, \omega_2^{\beta}(u;\, X_iY).$$

If we denote $p_i = x_i^1 \otimes \ldots \otimes x_i^n$ and $q_i = z_i^1 \otimes \ldots \otimes z_i^n$ we may consider the finite dimensional subspaces $E_i := span\{x_j^i\} \subset X_i$ and $F := span\{y_i\} \subset Y$. These spaces clearly satisfy $u \in E_1 \otimes \ldots \otimes E_n \otimes F$. Hence,

$$\begin{aligned} \|(p_i - q_i); E_i\|_2^{w\beta|} &\leq \|(p_i - q_i)\|_2^{wb} \\ \|(y_i); F\|_2^{w} &\leq \|(y_i)\|_2^{w} \end{aligned}$$

ensures

$$\begin{aligned} \omega_2^{\beta|}(u;\, E_iF) &\leq \|(p_i - q_i); E_i\|_2^{w\beta|}\|(y_i); F\|_2^{w} \\ &\leq \|(p_i - q_i)\|_2^{wb}\|(y_i)\|_2^{w} \\ &\leq \omega_2^{\beta}(u;\, X_iY)\ (1+\eta). \end{aligned}$$

Finally, ω_2^{β} is finitely generated. ∎

Since Σ-operators can be represented by functional on a tensor product, we characterize those functional which are bounded with the norm ω_2^{β}. This result is a generalization of the Kwapień Domination Theorem (see [93, Th. 7.32] and [41, Sec. 19.2]).

Theorem 4.41. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election in the class of Banach spaces. Suppose $\varphi : X_1 \otimes \ldots \otimes X_n \otimes Y \to \mathbb{K}$ is a linear functional. Then the following conditions are equivalent:*

i) φ is bounded on $\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \omega_2^{\beta}\right)$.

ii) There exists a regular Borel probability measure μ on $K := B_{(X_1 \otimes \ldots \otimes X_n, \beta)^*} \times B_{Y^*}$ and $C > 0$ such that

$$|\varphi((p-q)\otimes y)| \leq C \left(\int_K |\phi(p) - \phi(q)|^2 d\mu(\phi, y^*)\right)^{\frac{1}{2}} \left(\int_K |y^*(y)|^2 d\mu(\phi, y^*)\right)^{\frac{1}{2}}$$

for all $(p-q)\otimes y \in X_1 \otimes \ldots \otimes X_n \otimes Y$.

iii) There exists a Banach space Z, a Σ-image M of Z associated to the tuple $(X_1, \ldots, X_n, f_S)$ and a linear operator $R : Y \to M^*$ such that f_S and R are 2-summing and

$$\varphi(x^1 \otimes \ldots \otimes x^n \otimes y) = \left\langle\ R(y), f_S(x^1 \otimes \ldots \otimes x^n)\ \right\rangle.$$

Under this circumstances $\|\varphi\| = \inf C = \inf \pi_2(f_S)\ \pi_2(R)$ *where the first infimum is taken over all constants* C *of (ii) and the other over the all possible factorizations as in (iii).*

Proof. (i)⇒(ii). Boundedness of φ implies that for every $u = \sum_{i=1}^{m} (p_i - q_i) \otimes y_i$

$$\begin{aligned}
|\varphi(u)| &\leq \sum_i |\varphi((p_i - q_i) \otimes y_i)| \\
&\leq \|\varphi\|\ \|(p_i - q_i)\|_2^w\ \|(y_i)\|_2^w \\
&\leq \|\varphi\| \sup_{\substack{\phi \in B_{(X_1 \otimes \ldots \otimes X_n, \beta)^*} \\ y^* \in B_{Y^*}}} \left(\sum_i |\phi(p_i) - \phi(q_i)|^2 \right)^{\frac{1}{2}} \left(\sum_i |y^*(y_i)|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{\|\varphi\|}{2} \sup_{\substack{\phi \in B_{(X_1 \otimes \ldots \otimes X_n, \beta)^*} \\ y^* \in B_{Y^*}}} \sum_i |\phi(p_i) - \phi(q_i)|^2 + \sum_i |y^*(y_i)|^2.
\end{aligned}$$

Define the subset L of $C\left(B_{(X_1 \otimes \ldots \otimes X_n, \beta)^*} \times B_{Y^*}\right)$ as the functions f^A of the form

$$f^A(\phi, y^*) = \left| \sum_i \varphi\left((p_i - q_i) \otimes y_i\right) \right| - \frac{\|\varphi\|}{2} \sum_i |\phi(p_i) - \phi(q_i)|^2 + |y^*(y_i)|^2$$

where A is a finite subset of elements of the form $(p - q) \otimes y$. The set L is convex and has at least a non positive value. By the Hahn-Banach theorem there exist a Borel measure μ on $B_{(X_1 \otimes \ldots \otimes X_n, \beta)^*} \times B_{Y^*}$ such that there exist a constant α with

$$\mu(f) \leq \alpha \leq \mu(g)$$

for all f in L and g in the positive cone P. Since the vector zero is an element of L we obtain $0 \leq \alpha$ and by the cone property of P we have $\alpha \leq 0$. This way we may assume that μ is a nonnegative probability measure such that $\mu(f) \leq 0$ for all f in L.

For the set $A := \{(p - q) \otimes y\}$ we have

$$|\varphi((p - q) \otimes y)| \leq \frac{\|\varphi\|}{2} \left(\int_K |\phi(p) - \phi(q)|^2 d\mu + \int_K |y^*(y)|^2 d\mu \right).$$

Recall that for all real numbers a and b

$$ab = \inf \left\{ \frac{(ta)^2 + (t^{-1}b)^2}{2} \,\Big|\, t > 0 \right\}.$$

Notice that for all $t > 0$ we have $(p - q) \otimes y = tp \otimes t^{-1}y - tq \otimes t^{-1}y$. Hence

$$|\varphi((p - q) \otimes y)| \leq \frac{\|\varphi\|}{2} \left(\int_K |\phi(tp) - \phi(tq)|^2 d\mu + \int_K |y^*(t^{-1}y)|^2 d\mu \right).$$

After taking infimum over $t > 0$ we conclude (2) and $\inf C \leq \|\varphi\|$.

(ii) $\Rightarrow$ (iii). Consider the space $L_2(\mu)$ and the multilinear operator

$$\begin{array}{rcl} S : X_1 \times \cdots \times X_n & \to & \mathcal{L}_2(\mu) \\ (x^1, \dots, x^n) & \mapsto & S(x^1, \dots, x^n) : (\phi, y^*) \mapsto \phi(x^1 \otimes \ldots \otimes x^n). \end{array}$$

We claim that $f_S : \Sigma^{\beta}_{X_1 \ldots X_n} \to \mathcal{L}_2(\mu)$ is 2-summing. To see this, let $\underline{1} : B_{Y^*} \to \mathbb{K}$ the constant function with value 1. The Σ-operator f_S factors as follows

$$\begin{array}{ccccccc} \Sigma^{\beta}_{X_1 \ldots X_n} & \xrightarrow{I} & C(B_{(X_1 \otimes \ldots \otimes X_n, \beta)^*}) & \xrightarrow{\cdot \times \underline{1}} & C(K) & \xrightarrow{j_2} & \mathcal{L}_2(\mu) \\ \\ p & \longmapsto & I(p) & \longmapsto & I(p) \times \underline{1} & \longmapsto & j_2(I(p) \times \underline{1}) \end{array}$$

where I is the evaluation map and j_2 is the 2-summing linear identity. Then, for finite sequences (p_i) and (q_i) in $\Sigma^{\beta}_{X_1 \ldots X_n}$ we have

$$\begin{aligned} \sum_i \|f_S(p_i) - f_S(q_i)\|^2 &= \sum_i \|j_2(I(p_i) \times \underline{1}) - j_2(I(q_i) \times \underline{1})\|^2 \\ &\leq \sup_{(\phi, y^*) \in K} \sum_i |I(p_i) \times \underline{1}(\phi, y^*) - I(q_i) \times \underline{1}(\phi, y^*)|^2 \\ &\leq \sup_{(\phi, y^*) \in K} \sum_i |\phi(p_i) - \phi(q_i)|^2 \\ &= \sup_{\phi \in B_{(X_1 \otimes \ldots \otimes X_n, \beta)^*}} \sum_i |\phi(p_i) - \phi(q_i)|^2. \end{aligned}$$

As a consequence, f_S is 2-summing and $\pi_2(f_S) \leq 1$.

Aside, consider the evaluation map $i : Y \to C(B_{y^*})$. Set $M := f_S(\Sigma^{\beta}_{X_1 \ldots X_n}) \subset \mathcal{L}_2(\mu)$. Condition (ii) implies that for each $y \in Y$ we have a well defined Lipschitz function

$$\begin{array}{rcl} R_y : M & \to & \mathbb{K} \\ f_S(p) & \mapsto & \varphi(p \otimes y) \end{array}$$

with $Lip(R_y) \leq C \, \|j_2 i(y)\|_{\mathcal{L}2}$. Define R by the composition

$$\begin{array}{ccc} Y & \xrightarrow{R} & M^* \\ \downarrow & & \uparrow b \\ \underline{1} \times i(Y) & \xrightarrow[j_2|]{} & j_2(\underline{1} \times i(Y)) \\ \downarrow & & \downarrow \\ C(K) & \xrightarrow[j_2]{} & \mathcal{L}_2(\mu) \end{array}$$

where $b(j_2(\underline{1}\times i(y))) = R_y$. Condition (ii) implies that b is a well defined linear operator. The 2-summing property of j_2 implies that of R and $\pi_2(R) \le C$.

Finally, it is clear that $\varphi(p\otimes y) = R_y f_S(p) = \langle R(y), f_S(p)\rangle$. Moreover, $\pi_2^\beta(f_S)\,\pi_2(R) \le C$. This means that $\inf \pi_2^\beta(f_S)\,\pi_2(R) \le C$.

(iii)$\Rightarrow$ (i). Given $u = \sum_{i=1}^m (p_i - q_i)\otimes y_i$ in $X_1\otimes\ldots\otimes X_n\otimes Y$ we have

$$\begin{aligned}
|\varphi(u)| &\le \sum_i |\varphi((p_i-q_i)\otimes y_i)| \\
&= \sum_i |\langle\, R(y_i)\,,\, f_S(p_i)\,\rangle - \langle\, R(y_i)\,,\, f_S(q_i)\,\rangle| \\
&\le \sum_i Lip(R(y_i))\,\|f_S(p_i)-f_S(q_i)\| \\
&\le \left(\sum_i Lip(R(y_i))^2\right)^{\frac{1}{2}} \left(\sum_i \|f_S(p_i)-f_S(q_i)\|^2\right)^{\frac{1}{2}} \\
&\le \pi_2(R)\,\pi_2(S)\,\|(y_i)\|_2^w\,\|(p_i-q_i)\|_2^{w\beta}.
\end{aligned}$$

After taking infimum over all the representations of u of the form $\sum_{i=1}^m (p_i-q_i)\otimes y_i$ we obtain that $\varphi : \left(X_1\otimes\ldots\otimes X_n\otimes Y, \omega_2^\beta\right) \to \mathbb{K}$ is bounded and $\|\varphi\| \le \inf \pi_2(R)\pi_2(f_S)$. ∎

Theorem 4.42. *For every election* $(X_1,\ldots,X_n,Y,\beta)$ *in* $\mathcal{BAN}$

$$\left(X_1\otimes\ldots\otimes X_n\otimes Y, \omega_2^\beta\right)^* = \mathcal{D}_2\left(\Sigma^\beta_{X_1\ldots X_n}; Y^*\right)$$

holds linearly isomorphic and isometric.

Proof. Suppose $f : \Sigma^\beta_{X_1\ldots X_n} \to Y^*$ is such that φ_f is bounded on $\left(X_1\otimes\ldots\otimes X_n\otimes Y, \omega_2^\beta\right)$. We know that the canonical extension $\overline{\varphi_f}$ of φ_f is bounded on $\left(X_1\otimes\ldots\otimes X_n\otimes Y^{**}, \omega_2^\beta\right)$ with the same norm. By the last proposition there exist a regular Borel probability measure μ on $K := B_{(X_1\otimes\ldots\otimes X_n,\beta)^*}\times B_{Y^{***}}$ such that

$$|\,\langle\overline{\varphi_f}\,,\,(p-q)\otimes y^{**}\rangle\,| \le \|\overline{\varphi_f}\|\;\|p-q\|_{\mathcal{L}_2(\mu)}\;\|y^{**}\|_{\mathcal{L}_2(\mu)},$$

or equivalently

$$|\,\langle\, y^{**}, f(p)-f(q)\,\rangle\,| \le \|\overline{\varphi_f}\|\;\|p-q\|_{\mathcal{L}_2(\mu)}\;\|y^{**}\|_{\mathcal{L}_2(\mu)}. \tag{4.8}$$

Define

$$\begin{array}{rcl} A : X_1 \times \cdots \times X_n & \to & \mathcal{L}_2(\mu) \\ (x^1, \ldots, x^n) & \mapsto & A(x^1, \ldots, x^n) : (\varphi, y^*) \mapsto \varphi(x^1 \otimes \ldots \otimes x^n) \end{array}$$

and consider its associated Σ-operator $f_A : \Sigma^{\beta}_{X_1 \ldots X_n} \to \mathcal{L}_2(\mu)$. We claim that f_A is 2-summing. To see this, let us denote $\mathcal{L}^{\beta}(X_1, \ldots, X_n)$ by $\mathcal{L}^{\beta}$. Let $\underline{1} : B_{Y^{***}} \to \mathbb{K}$ the constant function with value 1. The Σ-operator f_A factors as follows

$$\Sigma^{\beta}_{X_1 \ldots X_n} \xrightarrow{I} C(B_{\mathcal{L}^{\beta}}) \xrightarrow{\cdot \times \underline{1}} C(K) \xrightarrow{j_2} \mathcal{L}_2(\mu)$$

$$p \longmapsto I(p) \longmapsto I(p) \times \underline{1} \longmapsto j_2(I(p) \times \underline{1})$$

where I is the evaluation map and j_2 is the 2-summing linear identity. Then, for finite sequences (p_i) and (q_i) in $\Sigma^{\beta}_{X_1 \ldots X_n}$ we have

$$\begin{array}{rcl} \displaystyle\sum_i \|f_A(p_i) - f_A(q_i)\|^2 & = & \displaystyle\sum_i \|j_2(I(p_i) \times \underline{1}) - j_2(I(q_i) \times \underline{1})\|^2 \\ & \leq & \displaystyle\sup_{(\varphi, y^{***}) \in K} \sum_i |I(p_i) \times \underline{1}(\varphi, y^{***}) - I(q_i) \times \underline{1}(\varphi, y^{***})|^2 \\ & \leq & \displaystyle\sup_{(\varphi, y^{***}) \in K} \sum_i |\varphi(p_i) - \varphi(q_i)|^2 \\ & = & \displaystyle\sup_{\varphi \in B_{\mathcal{L}^{\beta}}} \sum_i |\varphi(p_i) - \varphi(q_i)|^2. \end{array}$$

As a consequence, f_A is 2-summing and $\pi_2(f_A) \leq 1$.

Aside, define

$$\begin{array}{rcl} B : f_A(\Sigma^{\beta}_{X_1 \ldots X_n}) & \to & Y^* \\ f_A(p) & \mapsto & f(p). \end{array}$$

By (4.8), B is well defined. Moreover, the adjoint operator $B^* = Y^{**} \to M^*$ verifies

$$\begin{array}{rcl} |B^*(y^{**})(f_A(p)) - B^*(y^{**})(f_A(q))| & = & |\langle y^{**}, f(p) - f(q) \rangle| \\ & \leq & \|\overline{\varphi_f}\| \, \|p - q\|_{\mathcal{L}_2(\mu)} \, \|y^{**}\|_{\mathcal{L}_2(\mu)}. \end{array}$$

Last inequality asserts that $B^*(y^{**})$ is a Lipschitz function on M for each y^{**}, and its Lipschitz norm satisfies $L(B^*(y^{**})) \leq \|\overline{\varphi_f}\| \, \|y^{**}\|_{\mathcal{L}_2(\mu)}$. The linear operator B^* factors as follows

$$
\begin{array}{ccc}
Y^{**} & \xrightarrow{\;B^*\;} & M^* \\
\downarrow & & \uparrow b \\
\underline{1} \times i(Y^{**}) & \xrightarrow[j_2|]{} & j_2(\underline{1} \times i(Y^{**})) \\
\downarrow & & \downarrow \\
C(K) & \xrightarrow[j_2]{} & \mathcal{L}_2(\mu)
\end{array}
$$

where $i : Y^{**} \to C(B_{Y^{***}})$ is the evaluation map and

$$
\begin{array}{rcl}
b : j_2(\underline{1} \times i(Y^{**})) & \to & M^* \\
j_2(\underline{1} \times i(y^{**})) & \mapsto & B^*(y^{**}).
\end{array}
$$

Once again, (4.8) implies that b is well defined. In this case, the 2-summability of j_2 implies that of B^* and $\pi_2(B^*) \leq \|\overline{\varphi_f}\|$.

Finally, $f = Bf_A : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y^*$ is a 2-dominated Σ-operator with

$$D^{\beta}_2(f) \leq \pi^{\beta}_2(f_A)\, \pi(B^*) \leq \|\overline{\varphi_f}\| = \|\varphi_f\|.$$

Conversely, if φ is such that f_φ is 2-summing, take a factorization $f_\varphi = BA$ with f_A and B^* 2-summing. Then,

$$
\begin{array}{rcl}
\overline{\varphi}(p \otimes y^{**}) & = & \langle y^{**} \,,\, f_\varphi(p) \rangle \\
& = & \langle y^{**} \,,\, Bf_A(p) \rangle \\
& = & \langle B^* y^{**} \,,\, f_A(p) \,\rangle
\end{array}
$$

means that the canonical extension of φ is bounded on $\left(X_1 \otimes \ldots \otimes X_n \otimes Y^{**}, \omega^{\beta}_2\right)$. This way, φ is bounded on $\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \omega^{\beta}_2\right)$ and

$$\|\varphi\| = \|\overline{\varphi}\| \leq \pi_2(f_A)\, \pi_2(B^*)$$

holds for all factorizations of f_φ. This ensures that $\|\varphi\| \leq D^{\beta}_2(f_\varphi)$. ■

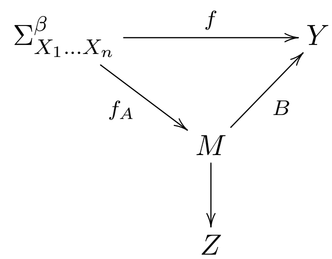
Proposition 4.43. *The collection of* 2*-dominated* Σ*-operators is a maximal ideal.*

Proof. Apply Criterion 3.14 ■

4.8 (p, q)-Dominated Σ-Operators and Laprest\'e Σ-Tensor Norms on Spaces

In the literature, dominated multilinear operators are defined as a particular case of absolutely $(s; r_1 \dots r_n)$-summing operators, see [31, 32, 59, 77, 80, 90]. In this section we define the collection of (p, q)-dominated operators by duality with the Lapresté Σ-tensor norm on spaces, see Definition 4.44 below.

If we consider the factorization diagrams of p-summing Σ-operators and those of factors through Hilbert space and 2-dominated Σ-operators, a reasonable proposal for factorizations of (p, q)-dominated Σ-operator is



(4.9)

where Z is Banach space, M is a Σ-image of Z associated to $(X_1, \dots, X_n, f_A)$ and $B : M \to Y$ is a Lipschitz function such that f_A is p-summing and B^* is q-summing. Even more, a proposal for the (p, q)-dominated norm of f is $D_{p,q}(f) = \inf \pi_p^{\beta}(A)\ \pi_q(B^*)$ where the infimum is taken over all the possible factorizations.

In this section we define the analogous of the Lapresté tensor norms for the case of Σ-tensor norm on spaces $\alpha_{p,q}$. Then, we define the ideal of (p, q)-dominated Σ-operators as the maximal ideal associated to the Σ-tensor norm on duals defined by α_{q^*,p^*}. In Theorem 4.49 we will see that, indeed, a (p, q)-dominated Σ-operator admits a factorization as in (4.9). Other approximations for tensor norms of Lapresté type can be find in [68, 69]

Definition 4.44. *Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} \geq 1$. Take the unique $r \in [1, \infty]$ with the property $1 = \frac{1}{r} + \frac{1}{q^*} + \frac{1}{p^*}$. For any election $(X_1, \dots, X_n, Y, \beta)$ in $\mathcal{BAN}$ we define the Lapresté norm on $X_1 \otimes \dots \otimes X_n \otimes Y$ by*

$$\alpha^{\beta}_{p,q}(u) := \inf \left\{ \|(\lambda_i)\|_r \ \|(p_i - q_i)\|^{w\beta}_{q^*} \ \|(y_i)\|^{w}_{p^*} \ \middle| \ u = \sum_i \lambda_i (p_i - q_i) \otimes y_i \right\}.$$

Proposition 4.45. *The Lapresté norm $\alpha_{p,q}$ is a finitely generated Σ-tensor norm on spaces.*

Proof. Let $(X_1, \dots, X_n, Y, \beta)$ be an election on $\mathcal{BAN}$. It is clear that $\alpha^{\beta}_{p,q}$ is a no-negative function and $\alpha^{\beta}_{p,q}(\lambda u) = |\lambda|\ \alpha^{\beta}_{p,q}$. Let $\varphi \in \mathcal{L}^{\beta}(X_1, \dots, X_n)$ and $y^* \in Y^*$. Then, the Hölder inequality implies that

$$\begin{aligned}
|\langle \varphi \otimes y^* \, , \, u\rangle | &\leq \sum_i |\lambda_i| \, |\varphi(p_i - q_i)| \, |y^*(y_i)| \\
&\leq \left(\sum_i |\lambda_i|^r\right)^{\frac{1}{r}} \left(\sum_i |\varphi(p_i - q_i)|^{q^*}\right)^{\frac{1}{q^*}} \left(\sum_i |y^*(y_i)|^{p^*}\right)^{\frac{1}{p^*}} \\
&\leq \|\varphi\|_\beta \, \|y^*\| \, \|(\lambda_i)\|_r \, \|(p_i - q_i)\|_{q^*}^{w\beta} \, \|(y_i)\|_{p^*}^{w}.
\end{aligned}$$

Therefore, $\varepsilon_\beta(u) \leq \alpha_{p,q}^\beta(u)$ for all u in $X_1 \otimes \ldots \otimes X_n \otimes Y$. In particular, $\alpha_{p,q}^\beta(u) = 0$ implies $u = 0$.

For the triangle inequality let u and v in $X_1 \otimes \ldots \otimes X_n \otimes Y$ and fix $\eta > 0$. By definition of $\alpha_{p,q}^\beta$ there exist representations $\sum_i \lambda_i(p_i - q_i) \otimes y_i$ and $\sum_j \mu_j(a_j - b_j) \otimes z_i$ of u and v respectively such that

$$\begin{aligned}
\|(\lambda_i)\|_r &\leq (\alpha_{p,q}^\beta(u) + \eta)^{\frac{1}{r}} \\
\|(p_i - q_i)\|_{q^*}^{w\beta} &\leq (\alpha_{p,q}^\beta(u) + \eta)^{\frac{1}{q^*}} \\
\|(y_i)\|_{p^*}^{w} &\leq (\alpha_{p,q}^\beta(u) + \eta)^{\frac{1}{p^*}}
\end{aligned}$$

and

$$\begin{aligned}
\|(\mu_j)\|_r &\leq (\alpha_{p,q}^\beta(v) + \eta)^{\frac{1}{r}} \\
\|(a_j - b_j)\|_{q^*}^{w\beta} &\leq (\alpha_{p,q}^\beta(v) + \eta)^{\frac{1}{q^*}} \\
\|(z_j)\|_{p^*}^{w} &\leq (\alpha_{p,q}^\beta(v) + \eta)^{\frac{1}{p^*}}.
\end{aligned}$$

Notice that $\sum_i \lambda_i(p_i - q_i) \otimes y_i + \sum_j \mu_i(a_j - b_j) \otimes z_i$ is a representation of $u + v$. Moreover

$$\begin{aligned}
\left(\sum_i |\lambda_i|^r + \sum_j |\mu_j|^r\right)^{\frac{1}{r}} &= (\|(\lambda_i)\|_r^r + \|(\mu_j)\|_r^r)^{\frac{1}{r}} \\
&\leq \Big((\alpha_{p,q}^\beta(u) + \eta) + (\alpha_{p,q}^\beta(v) + \eta)\Big)^{\frac{1}{r}} \\
&= \Big(\alpha_{p,q}^\beta(u) + \alpha_{p,q}^\beta(v) + 2\eta\Big)^{\frac{1}{r}}.
\end{aligned}$$

The same reasoning produces

$$\|(p_1 - q_1, \ldots, p_m - q_m, a_1 - b_1, \ldots, a_m - b_m)\|_{q^*}^{w\beta} \leq \Big(\alpha_{p,q}^\beta(u) + \alpha_{p,q}^\beta(v) + 2\eta\Big)^{\frac{1}{q^*}}$$

and

$$\|(y_1 - y_1, \ldots, y_m - y_m, z_1 - z_1, \ldots, z_m - z_m)\|_{p^*}^{w\beta} \leq \Big(\alpha_{p,q}^{\beta}(u) + \alpha_{p,q}^{\beta}(v) + 2\eta\Big)^{\frac{1}{p^*}}.$$

This way,

$$\alpha_{p,q}^{\beta}(u+v) \leq \alpha_{p,q}^{\beta}(u) + \alpha_{p,q}^{\beta}(v) + 2\eta$$

holds for every $\eta > 0$. This means that $\alpha_{p,q}^{\beta}$ verifies the triangle inequality.

We have already proved that $\alpha_{p,q}^{\beta}$ is reasonable. For $u = (p-q) \otimes y$ we have that

$$\alpha_{p,q}^{\beta}(u) \leq 1 \cdot \beta(p-q)\ \|y\|.$$

In other words, $\alpha_{p,q}^{\beta}$ is a reasonable crossnorm.

To see that $\alpha_{p,q}$ verifies the uniform property let $f_R : \Sigma_{Z_1 \ldots Z_n}^{\theta} \to \Sigma_{X_1 \ldots X_n}^{\beta}$ be a Σ-θ-operator operator and $S : W \to Y$ be a bounded linear operator. For any $v = \sum_i \lambda_i (p_i - q_i) \otimes y_i$ in $Z_1 \otimes \ldots \otimes Z_n \otimes W$ we have

$$\begin{aligned}
\alpha_{p,q}^{\beta}\left(f_R \otimes S(v)\right) &= \alpha_{p,q}^{\beta}\left(\sum_i \lambda_i (f_R(p_i) - f_R(q_i)) \otimes S(y_i)\right) \\
&\leq \|(\lambda_i)\|_r\ \|(f_R(p_i) - f_R(q_i))\|_{q^*}^{w\beta}\ \|(S(y_i))\|_{p^*}^{w} \\
&\leq \|\widetilde{R}\|\ \|S\|\ \|(\lambda_i)\|_r\ \|(p_i - q_i))\|_{q^*}^{w\theta}\ \|(y_i)\|_{p^*}^{w}.
\end{aligned}$$

This asserts that $R \otimes S : \left(Z_1 \otimes \ldots \otimes Z_n \otimes W,\ \alpha_{p,q}^{\theta}\right) \to \left(X_1 \otimes \ldots \otimes X_n \otimes Y,\ \alpha_{p,q}^{\beta}\right)$ is bounded and $\|f_R \otimes S\| \leq \|\widetilde{R}\|\ \|S\|$.

It only remains to prove that $\alpha_{p,q}$ is finitely generated. This is easy since, for any u and $\eta > 0$ there exists a representation $\sum_i \lambda_i (p_i - q_i) \otimes y_i$ of u with the property

$$\|(\lambda_i)\|_r\ \|(p_i - q_i)\|_{q^*}^{w\beta}\ \|(y_i)\|_{p^*}^{w} \leq \alpha_{p,q}^{\beta}(u) + \eta.$$

It is clear that there exist finite dimensional subspaces E_i and F of X_i and Y respectively such that $p_i, q_i \in \Sigma_{E_1 \ldots E_n}^{\beta|}$ and $y_i \in F$. Hence

$$\|(p_i - q_i); E_1 \otimes \ldots \otimes E_n\|_{q^*}^{w\beta|} = \|(p_i - q_i)\|_{q^*}^{w\beta}$$

and

$$\|(y_i); F\|_{p^*}^{w} = \|(y_i)\|_{p^*}^{w}.$$

Finally

$$\alpha_{p,q}^{\beta|}(u;\ E_iF) \leq \alpha_{p,q}^{\beta}(u;\ X_iY) + \eta$$

holds for all $\eta > 0$. ∎

Definition 4.46. *Let $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$. We define the maximal ideal of (p,q)-dominated Σ-operators, denoted by $[\mathcal{D}_{p,q}, D_{p,q}]$, as the maximal ideal associated with the Σ-tensor norm on duals defined by $\alpha_{q^*p^*}$.*

Notice that $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$ implies $\frac{1}{p^*} + \frac{1}{q^*} \geq 1$. Hence $\alpha_{q^*p^*}$ makes sense. Aside, the components of $[\mathcal{D}_{p,q}, D_{p,q}]$ are defined by

$$\begin{aligned}
\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha_{q^*,p^*}^{\beta}\right)^* &= \mathcal{D}_{pq}\left(\Sigma_{X_1 \ldots X_n}^{\beta}; Y^*\right) \\
\left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \alpha_{q^*,p^*}^{\beta}\right)^* \cap \mathcal{L}\left(\Sigma_{X_1 \ldots X_n}^{\beta}; Y\right) &= \mathcal{D}_{pq}\left(\Sigma_{X_1 \ldots X_n}^{\beta}; Y\right).
\end{aligned}$$

The (p,q)-dominated norm of the Σ-operator $f : \Sigma_{X_1 \ldots X_n}^{\beta} \to Y$ is given by

$$D_{p,q}(f) = \|\varphi_f : \left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \alpha_{q^*,p^*}^{\beta}\right) \to \mathbb{K}\|.$$

Naturally, if we want to formulate an equivalent form for (p,q)-dominated Σ-operators we have to characterize those functionals bounded on $\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha_{p,q}^{\beta}\right)$.

Proposition 4.47. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election of Banach spaces. The following are equivalent:*

i) $\varphi \in \left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha_{p,q}^{\beta}\right)^*$.

ii) There exists $C > 0$ such that

$$\|(\varphi((p_i - q_i) \otimes y_i))\|_{r^*} \leq C \,\|(p_i - q_i)\|_{q^*}^{w\beta}\, \|(y_i)\|_{p^*}^{w}.$$

In this case $\|\varphi\| = \inf C$ where the infimum is taken over all the constants C as above.

Proof. If (i) holds, then $u = \sum\limits_i \lambda_i (p_i - q_i) \otimes y_i$ implies

$$\begin{aligned}
\left|\sum_i \lambda_i \varphi((p_i - q_i) \otimes y_i)\right| &= |\varphi(u)| \\
&\leq \|\varphi\| \,\|(\lambda_i)\|_r \,\|(p_i - q_i)\|_{q^*}^{w\beta}\, \|(y_i)\|_{p^*}^{w}.
\end{aligned}$$

After taking suprema over $\|(\lambda_i)\|_r \leq 1$ we obtain

$$\|(\varphi((p_i - q_i) \otimes y_i))\|_{r^*} \leq \|\varphi\| \,\|(p_i - q_i)\|_{q^*}^{w\beta}\, \|(y_i)\|_{p^*}^{w}$$

and $\inf C \leq \|\varphi\|$.

In the opposite direction, (ii) implies that

$$\begin{aligned} |\varphi(u)| &= \left|\sum_i \lambda_i \varphi((p_i - q_i) \otimes y_i)\right| \\ &\leq C\,\|(\lambda_i)\|_r\,\|(p_i - q_i)\|_{q^*}^{w\beta}\,\|(y_i)\|_{p^*}^{w} \end{aligned}$$

holds for any representation of u. Then $|\varphi(u)| \leq C\,\alpha_{p,q}^{\beta}(u)$ and $\|\varphi\| \inf C$. ∎

The next result is a generalization of the Kwapień Domination Theorem, see [41, Sec 19.2] and [63].

Theorem 4.48. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election of Banach spaces and $\alpha_{p,q}$ be the Lapresté Σ-tensor norm on spaces. The following are equivalent:*

i) φ is a bounded linear functional on $\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha_{p,q}^{\beta}\right)$.

ii) For any w^*-compact subsets $K \subset B_{(X_1 \otimes \ldots \otimes X_n, \beta)^*}$ and $L \subset B_{Y^*}$ there exist a nonnegative constant C and probability regular Borel measures μ and ν on K and L respectively such that

$$|\langle \varphi, (p-q) \otimes y \rangle| \leq C \left(\int_K |\psi(p) - \psi(q)|^{q^*} d\mu\right)^{1/q^*} \left(\int_L |y^*(y)|^{p^*} d\nu\right)^{1/p^*}.$$

Under these circumstances $\|\varphi\| = \inf C$.

Proof. Condition (ii) when combined with Hölder inequality for $\frac{q^*}{r^*}$ and $\frac{p^*}{r^*}$ implies

$$\begin{aligned} \|(\varphi((p_i - q_i) \otimes y_i))\|_{r^*} &= \left(\sum_i |\varphi((p_i - q_i) \otimes y_i)|^{r^*}\right)^{\frac{1}{r^*}} \\ &\leq C \left(\sum_i \left(\int_K |\psi(p_i) - \psi(q_i)|^{q^*} d\mu(\psi)\right)^{\frac{r^*}{q^*}} \left(\int_L |y^*(y_i)|^{p^*} d\nu(y^*)\right)^{\frac{r^*}{p^*}}\right)^{\frac{1}{r^*}} \\ &\leq C \left(\sum_i \int_K |\psi(p_i) - \psi(q_i)|^{q^*} d\mu(\psi)\right)^{\frac{1}{q^*}} \left(\sum_i \int_L |y^*(y_i)|^{p^*} d\nu(y^*)\right)^{\frac{1}{p^*}} \\ &= C \left(\int_K \sum_i |\psi(p_i) - \psi(q_i)|^{q^*} d\mu(\psi)\right)^{\frac{1}{q^*}} \left(\int_L \sum_i |y^*(y_i)|^{p^*} d\nu(y^*)\right)^{\frac{1}{p^*}} \\ &\leq C\,\|(p_i - q_i)\|_{q^*}^{w\beta}\,\|(y_i)\|_{p^*}^{w}. \end{aligned}$$

Proposition 4.47 ensures that φ is bounded and $\|\varphi\| \leq \inf C$.

For the converse, assume $\|\varphi\| = 1$. Let $M_1^+(K) \subset C(K)^*$ be the set of probability measures on K. Analogously, $M_1^+(L) \subset C(L)^*$. Define $C := M_1^+(K) \times M_1^+(L) \subset C(K)^* \times C(L)^*$.

Consider the continuous functions

$$\begin{array}{rcl} I_{pq} : (K, w^*) & \to & \mathbb{R} \\ \psi & \mapsto & |\psi(p) - \psi(q)|^{q^*}, \end{array}$$

$$\begin{array}{rcl} I_y : (L, w^*) & \to & \mathbb{R} \\ y^* & \mapsto & |y^*(y)|^{p^*} \end{array}$$

and the canonical embeddings $K_{C(K)} : C(K) \to C(K)^{**}$ and $K_{C(L)} : C(L) \to C(L)^{**}$. Then

$$\begin{array}{rcl} K_{C(K)}(I_{pq}) : C(K)^* & \to & \mathbb{R} \\ \mu & \mapsto & \mu(I_{pq}) = \int\limits_K |\psi(p) - \psi(q)|^{q^*} d\mu(\psi). \end{array}$$

and

$$\begin{array}{rcl} K_{C(L)}(I_y) : C(L)^* & \to & \mathbb{R} \\ \nu & \mapsto & \nu(I_y) = \int\limits_L |y^*(y)|^{p^*} d\nu(y^*). \end{array}$$

Hence, the operator

$$\begin{array}{rcl} H_{pqy} : C & \to & \mathbb{R} \times \mathbb{R} \\ (\mu, \nu) & \mapsto & (\, K_{C(K)}(I_{pq})(\mu) \, , \, K_{C(L)}(I_y)(\nu) \,) \end{array}$$

is bounded. Define

$$f_{pq} = \pi_1 \circ H_{pqy}$$

and

$$g_y = \pi_2 \circ H_{pqy}$$

where $\pi_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the j-th projection. Also, consider the constant function

$$\begin{array}{rcl} c_{pqy} : C & \to & \mathbb{R} \times \mathbb{R} \\ (\mu, \nu) & \mapsto & |\, \langle \varphi \, , \, (p - q) \otimes y \rangle \,|^{r^*}. \end{array}$$

The functions f_{pq}, g_y and c_{pqy} are continuous by construction. Moreover, it is a simple matter to prove that they are affine.

Let $\mathfrak{F}$ be the set of all functions $f : C \to \mathbb{R}$ for which there exist finite sequences (p_i), (q_i) in $\Sigma^{\beta}_{X_1\ldots X_n}$ and (y_i) in Y such that

$$f = \sum_i \frac{r^*}{q^*} f_{p_i q_i} + \frac{r^*}{p^*} g_{y_i} + c_{p_i q_i y_i}.$$

In particular, every f in $\mathfrak{F}$ is upper semicontinuous and concave.

We claim that $\mathcal{F}$ is a convex set. To see this let

$$f_1 = \sum_i \frac{r^*}{q^*} f_{p_i q_i} + \frac{r^*}{p^*} g_{y_i} + c_{p_i q_i y_i}$$

and

$$f_2 = \sum_i \frac{r^*}{q^*} f_{a_i b_i} + \frac{r^*}{p^*} g_{w_i} + c_{a_i b_i w_i}.$$

A simple calculation shows that

$$\lambda_1 f^1 + \lambda_2 f^2 = \sum_i \frac{r^*}{q^*} \left(f_{\lambda^{\frac{1}{q^*}} p_i \lambda^{\frac{1}{q^*}} q_i} + f_{\lambda^{\frac{1}{q^*}} a_i \lambda^{\frac{1}{q^*}} b_i} \right) + \frac{r^*}{p^*} \left(g_{\lambda^{\frac{1}{p}} y_i} + g_{\lambda^{\frac{1}{p^*}} w_i} \right)$$
$$+ c_{\lambda^{\frac{1}{q^*}} p_i \lambda^{\frac{1}{q^*}} q_i \lambda^{\frac{1}{p^*}} y_i} + c_{\lambda^{\frac{1}{q^*}} a_i \lambda^{\frac{1}{q^*}} b_i \lambda^{\frac{1}{p^*}} w_i}.$$

The set C consider with the product topology of $(C(K)^*, w^*) \times (C(L)^*, w^*)$ is compact. Even more, it is convex.

We claim that any f in $\mathcal{F}$ is nonnegative in at least one point. To prove this, notice that every sequence (y_i) defines a w^*-continuous function

$$\begin{aligned} Y^* &\to \mathbb{R} \\ y^* &\mapsto \left(\sum_i |y^*(y_i)|^{p^*} \right)^{\frac{1}{p^*}}. \end{aligned}$$

Compactness of L ensures the existence of y^*_o such that

$$\|(y_i)\|^w_{p^*} = \left(\sum_i |y^*_0(y_i)|^{p^*} \right)^{\frac{1}{p^*}}.$$

Analogously, there exists $\psi_0 \in K$ such that

$$\|(p_i - q_i)\|^{w\beta}_{q^*} = \left(\sum_i |\psi_0(p_i) - \psi_0(q_i)|^{q^*} \right)^{\frac{1}{q^*}}.$$

For the Dirac measures δ_{ψ_0} on K and $\delta_{y_0^*}$ in L we have

$$\begin{aligned} f(\delta_{\psi_0}, \delta_{y_0^*}) &= \frac{r^*}{q^*}\left(\|(p_i - q_i)\|_{q^*}^{w\beta}\right)^{q^*} + \frac{r^*}{p^*}\left(\|(y_i)\|_{p^*}^{w}\right)^{q^*} - \sum_i |\langle \varphi, (p_i - q_i) \otimes y_i \rangle|^{r^*} \\ &\geq \left(\|(p_i - q_i)\|_{q^*}^{w\beta}\right)^{\frac{q^* r^*}{q^*}} \left(\|(y_i)\|_{p^*}^{w}\right)^{\frac{p^* r^*}{p^*}} - \sum_i |\langle \varphi, (p_i - q_i) \otimes y_i \rangle|^{r^*} \\ &= \left(\|(p_i - q_i)\|_{q^*}^{w\beta}\right)^{r^*} \left(\|(y_i)\|_{p^*}^{w}\right)^{r^*} - \sum_i |\langle \varphi, (p_i - q_i) \otimes y_i \rangle|^{r^*} \\ &\geq 0. \end{aligned}$$

Where the first inequality follows from the fact $\frac{a}{s} + \frac{b}{s^*} \geq a^{\frac{1}{s}} b^{\frac{1}{s}}$ for all $a \geq 0$ and $b \geq 0$ and $1 < s < \infty$ and the second from $\|\varphi\| = 1$.

We may apply the lemma of Ky Fan (see [41, A3]) to obtain $(\mu, \nu) \in M_1^+(K) \times M_1^+(L)$ such that

$$0 \leq f(\mu, \nu) \qquad \forall f \in \mathcal{F}.$$

As a consequence

$$|\langle \varphi, (p - q) \otimes y \rangle|^{r^*} \leq \frac{r^*}{q^*} \int_K |\psi(p) - \psi(q)|^{q^*} d\mu(\psi) + \frac{r^*}{p^*} \int_L |y^*(y)|^{p^*} d\nu(y^*).$$

Notice that for any $a, b > 0$ we have

$$\begin{aligned} |\langle \varphi, (p - q) \otimes y \rangle| &= ab\,|\langle \varphi, a^{-1}(p - q) \otimes b^{-1} y \rangle| \\ &\leq ab \left(\frac{r^*}{a^{q^*} q^*} \int_K |\psi(p) - \psi(q)|^{q^*} d\mu(\psi) + \frac{r^*}{b^{p^*} p^*} \int_L |y^*(y)|^{p^*} d\nu(y^*) \right)^{\frac{1}{r^*}}. \end{aligned}$$

Taking $a = \left(\int_K |\psi(p) - \psi(q)|^{q^*} d\mu(\psi) \right)^{\frac{1}{q^*}}$ and $b = \left(\int_L |y^*(y)|^{p^*} d\nu(y^*) \right)^{\frac{1}{p^*}}$ we obtain

$$\begin{aligned} |\langle \varphi, (p - q) \otimes y \rangle| &\leq ab \\ &= \left(\int_K |\psi(p) - \psi(q)|^{q^*} d\mu(\psi) \right)^{\frac{1}{q^*}} \left(\int_L |y^*(y)|^{p^*} d\nu(y^*) \right)^{\frac{1}{p^*}} \end{aligned}$$

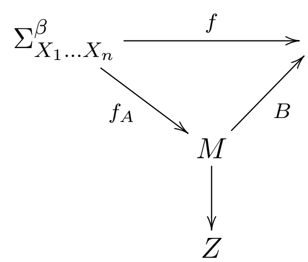
For the general case, a normalization of φ leads us to

$$|\langle \varphi, (p - q) \otimes y \rangle| \leq \|\varphi\| \left(\int_K |\psi(p) - \psi(q)|^{q^*} d\mu(\psi) \right)^{\frac{1}{q^*}} \left(\int_L |y^*(y)|^{p^*} d\nu(y^*) \right)^{\frac{1}{p^*}}$$

and $\inf C \leq \|\varphi\|$. ■

The next theorem is a generalization of the Kwapień's factorization theorem [41, Sec. 19.3] (see also [63]).

Theorem 4.49. *Let $(X_1, \ldots, X_n, Y, \beta)$ be an election on the class $\mathcal{BAN}$. The Σ-operator $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ is (p, q)-dominated if and only if f factors as follows*



(4.10)

where Z is a Banach space, M is a Σ-image of Z associated with $(X_1, \ldots, X_n, f_A)$ and $B : M \to Y$ is a Lipschitz function such that f_A is p-summing and $B^* : Y^* \to M^*$ is q-summing.

Proof. First, suppose $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ is a (p, q)-dominated Σ-operator. Then, by definition, φ_f is a bounded functional on $\left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, \alpha^{\beta}_{q^*, p^*}\right)$.

By Theorem 4.48 there exist measures μ, ν on $K := B_{(X_1 \otimes \ldots \otimes X_n, \beta)^*}$ and $L := B_{Y^{**}}$ respectively such that

$$\begin{aligned} |\langle y^*, f(p) - f(q) \rangle| &= |\langle \varphi, (p - q) \otimes y^* \rangle| \\ &\leq D_{p,q}(f) \left(\int_K |\psi(p) - \psi(q)|^p d\mu(\psi) \right)^{\frac{1}{p}} \left(\int_L |y^{**}(y^*)|^q d\nu(y^{**}) \right)^{\frac{1}{q}}. \end{aligned}$$

Define

$$\begin{aligned} A : X_1 \times \cdots \times X_n &\to L_p(\mu) \\ (x^1, \ldots, x^n) &\mapsto j_p \iota(x^1, \ldots, x^n) \end{aligned}$$

where $\iota : X_1 \times \cdots \times X_n \to C(K)$ acts by evaluation on K and $j_p : C(K) \to L_p(\mu)$ is the canonical map. The bounded Σ-operator f_A is p-summing since j_p so is, and K is the unit ball of $(X_1 \otimes \ldots \otimes X_n, \beta)$. Even more, $\pi^{\beta}_p(A) \leq \|\iota\| \, \pi_p(j_p) = 1$ since μ is a probability measure.

Aside, define

$$\begin{aligned} B : f_A(\Sigma^{\beta}_{X_1 \ldots X_n}) &\to Y \\ f_A(p) &\mapsto f(p). \end{aligned}$$

The inequality

$$|\langle y^*, Bf_A(p) - Bf_A(q)\rangle| \leq D_{p,q}(f)\,\|f_A(p) - f_A(q)\| \left(\int\limits_L |y^{**}(y^*)|^q d\nu(y^{**})\right)^{\frac{1}{q}}$$

ensures that B is well defined. Furthermore, it implies

$$|\langle B^*y^*, f_A(p) - f_A(q)\rangle| \leq D_{p,q}(f)\,\|f_A(p) - f_A(q)\| \left(\int\limits_L |y^{**}(y^*)|^q d\nu(y^{**})\right)^{\frac{1}{q}}.$$

As a consequence, $B^*(y^{**})$ is a Lipschitz function with

$$Lip(B^*(y^*)) \leq D_{p,q}(f) \left(\int\limits_L |y^{**}(y^*)|^q d\nu(y^{**})\right)^{\frac{1}{q}}.$$

Pietsch Domination Theorem asserts that $B^* : Y^* \to f_A(\Sigma^{\beta}_{X_1 \ldots X_n})^*$ is a linear q-summing operator with $\pi_q(B^*) \leq D_{p,q}(f)$. This way, f factors as in (4.10) and $\pi_p(f_A)\,\pi_q(B^*) \leq \mathcal{D}^{\beta}_{p,q}(f)$.

Conversely, let f_A and B as in (4.10). The Hölder inequality implies

$$\begin{aligned}
\|(\varphi_f((p_i - q_i)\otimes y_i^*))\|_r &= \|\,(\langle y_i^*\,,\, f(p_i) - f(q_i)\rangle)\,\|_r \\
&= \|\,(B^* y_i^* f_A(p_i) - B^* y_i^* f_A(q_i))\,\|_r \\
&= \left(\sum_i |\,\langle B^* y_i^*(f_A(p_i)) - B^* y_i^*(f_A(q_i))\rangle\,|^r\right)^{\frac{1}{r}} \\
&\leq \left(\sum_i Lip(B^* y_i^*)^r \|f_A(p_i) - f_A(q_i)\|^r\right)^{\frac{1}{r}} \\
&\leq \left(\sum_i Lip(B^* y_i^*)^q\right)^{\frac{1}{q}} \left(\sum_i \|f_A(p_i) - f_A(q_i)\|^p\right)^{\frac{1}{p}} \\
&\leq \pi_q(B^*)\,\pi_p^{\beta}(f_A)\,\|(p_i - q_i)\|_p^{w\beta}\,\|(y_i^*)\|_q^w
\end{aligned}$$

Proposition 4.47 asserts that φ_f is bounded on $\left(X_1 \otimes \ldots \otimes X_n \otimes Y, \alpha^{\beta}_{q^*,p^*}\right)$ and its norm is dominated by $\pi_p(B^*)\,\pi_q^{\beta}(f_A)$. Finally, since the factorization was arbitrary, we obtain $\mathcal{D}^{\beta}_{q^*p^*}(f) \leq \inf \pi_p(B^*)\,\pi_q^{\beta}(f_A)$ ∎

4.9 (p,q)-Factorable Σ-Operators and Lapresté Σ-Tensor Norms on Duals

In this section we present the generalization of the Lapresté tensor norms to the setting of Σ-tensor norms on duals. In this case, we define the maximal ideal of (p,q)-factorable Σ-operators as the associated with the Lapresté Σ-tensor norm on duals. In Theorem 4.57 we present the resulting factorization of a typical (p,q)-factorable Σ-operator.

The definition of the Lapresté Σ-tensor norms on duals is presented next. We will use the same symbol $\alpha_{p,q}$ than we used in the case of spaces.

Definition 4.50. *Let $p,q \in [1,\infty]$ such that $\frac{1}{p}+\frac{1}{q} \geq 1$. Take the unique $r \in [1,\infty]$ with the property $1 = \frac{1}{r}+\frac{1}{q^*}+\frac{1}{p^*}$. For any election $(X_1,\ldots,X_n,Y,\beta)$ in $\mathcal{BAN}$ we define the Lapresté norm on duals on $\mathcal{L}^\beta(X_1,\ldots,X_n)\otimes Y$ by*

$$\alpha_{p,q,\beta}(v) := \inf\left\{ \|(\lambda_i)\|_r \, \|(\varphi_i)\|_{q^*}^{wd} \, \|(y_i)\|_{p^*}^{w} \;\middle|\; v = \sum_i \lambda_i \, \varphi_i \otimes y_i \right\}.$$

Proposition 4.51. *The Lapresté norm $\alpha_{p,q}$ is a Σ-tensor norm on duals.*

Proof. Given p,q in $\Sigma_{X_1\ldots X_n}$, y^* in Y^* and $v \in \mathcal{L}^\beta(X_1,\ldots,X_n)\otimes Y$, the Hölder inequality implies

$$\begin{aligned}
|\langle p-q\otimes y\,,\,v\rangle| &= \sum_i |\lambda_i\,(\varphi_i(p)-\varphi_i(q))\,y^*(y_i)| \\
&\leq \left(\sum_i |\lambda_i|^r\right)^{1/r} \left(\sum_i |\varphi_i(p)-\varphi_i(q)|^{q^*}\right)^{1/q^*} \left(\sum_i |y^*(y_i)|^{p^*}\right)^{1/p^*} \\
&\leq \beta(p-q)\,\|y^*\|\,\|(\lambda_i)\|_r\,\|(\varphi_i)\|_{q^*}^{wd}\,\|(y_i)\|_{p^*}^{w}.
\end{aligned}$$

This means that

$$\varepsilon_\beta(v) \leq \alpha_{p,q,\beta}(v)$$

holds for all $v \in \mathcal{L}^\beta(X_1,\ldots,X_n)\otimes Y^*$. In particular,$\alpha_{p,q,\beta}(v)=0$ implies $v=0$. It is clear that $\alpha_{p,q,\beta}(\lambda v) = |\lambda|\,\alpha_{p,q,\beta}(v)$ for any scalar λ.

For the triangle inequality take representations $\sum_i \lambda_i\varphi_i\otimes y_i$ and $\sum_i \mu_i\phi_i\otimes z_i$ of v and u respectively such that

$$\begin{aligned}
\|(\lambda_i)\|_r &\leq (\alpha_{pqb}(v)+\eta)^{\frac{1}{r}} \\
\|(\varphi_i)\|_{q^*}^{wd} &\leq (\alpha_{pqb}(v)+\eta)^{\frac{1}{q^*}} \\
\|(y_i)\|_{p^*}^{w} &\leq (\alpha_{pqb}(v)+\eta)^{\frac{1}{p^*}}
\end{aligned}$$

and

$$\begin{aligned} \|(\mu_i)\|_r &\leq (\alpha_{pqb}(u)+\eta)^{\frac{1}{r}} \\ \|(\phi_i)\|^{wd}_{q^*} &\leq (\alpha_{pqb}(u)+\eta)^{\frac{1}{q^*}} \\ \|(z_i)\|^{w}_{p^*} &\leq (\alpha_{pqb}(u)+\eta)^{\frac{1}{p^*}} \end{aligned}$$

The sum $\sum \lambda_i \varphi_i \otimes y_i^* + \mu_j \phi_j \otimes z_j^*$ is a representation of $v+u$ that verifies

$$\| (\lambda_i, \mu_i) \|_r \, \| (\varphi_i, \phi_i) \|^{wd}_{q^*} \, \| (y_i^*, z_i^*) \| \leq \alpha_{p,q,\beta}(v) + \alpha_{p,q,\beta}(u) + 2\eta$$

where (λ_i, μ_i) stands for the finite sequence obtained by the terms of (λ_i) and (μ_i). Analogously for the remaining two. Last inequality ensures that $\alpha_{p,q,\beta}$ verifies the triangle inequality.

For $\varphi \otimes y$ it is immediate that

$$\alpha_{p,q,\beta}(\varphi \otimes y) \leq Lip^{\beta}(\varphi) \, \|y^*\| \leq \|\varphi\|_{\beta} \, \|y^*\|.$$

In other words, $\alpha_{p,q,\beta}$ is reasonable and crossed.

To prove that $\alpha_{p,q}$ is uniform let $A : \mathcal{L}^{\theta}(Z_1, \ldots, Z_n) \to \mathcal{L}^{\beta}(X_1, \ldots, X_n)$ be a bounded operator that preserves Σ and $B : W \to Y$ be a bounded linear operator. For $v = \sum_i \lambda_i \, \varphi_i \otimes y_i$ we have $A \otimes B(v) = \sum_i \lambda_i \, A(\varphi_i) \otimes B(y_i)$. Hence

$$\begin{aligned} \alpha_{p,q,\beta}(A \otimes B(v)) &\leq \|(\lambda_i)\|_r \, \|A(\varphi_i)\|^{wd}_{q^*} \, \|(B(y_i))\|^{w}_{p^*} \\ &\leq \|A\| \, \|B\| \, \|(\lambda_i)\|_r \, \|(\varphi_i)\|^{wd}_{q^*} \, \|(y_i)\|^{w}_{p^*}. \end{aligned}$$

After taking infimum over the representations of v we obtain

$$\alpha_{p,q,\beta}(A \otimes B(v)) \leq \|A\| \, \|B\| \, \alpha_{p,q,\theta}(v).$$

■

Definition 4.52. *The maximal ideal of (p,q)-factorable Σ-operators, denoted by $[\mathcal{L}_{pq}, L_{pq}]$, is defined as the associated with the Lapresté Σ-tensor norm on duals $\alpha_{p,q}$.*

Under this definition, any component of $[\mathcal{L}_{pq}, L_{pq}]$ in the class $\mathcal{FIN}$ has the form

$$\begin{aligned} \mathcal{L}_{pq}\left(\Sigma^{\beta}_{E_1 \ldots E_n}; F\right) &= \left(\mathcal{L}^{\beta}(E_1, \ldots, E_n) \otimes F, \alpha_{p,q,\beta}\right) \\ &= \left(E_1 \otimes \ldots \otimes E_n \otimes F^*, (\alpha'_{p,q})^{\beta}\right)^*. \end{aligned}$$

where $\alpha'_{p,q}$ is the finitely generated Σ-tensor norm on spaces defined by the Σ-tensor norm on duals $\alpha_{p,q}$, see Theorem 2.23 and Definition 3.1. Be definition, $L_{p,q}(f) = \alpha_{p,q,\beta}(v_f)$, where v_f

is the correspondent tensor of f in $\mathcal{L}^{\beta}(E_1, \ldots, E_n) \otimes F$. Furthermore, every representation of v_f induces a representation of f of the form $\sum_i \lambda_i \varphi_i \cdot y_i$. It is easy to verify that f can be factored as

$$\begin{array}{ccc} \Sigma^{\beta}_{E_1 \ldots E_n} & \xrightarrow{\ f\ } & F \\ {\scriptstyle f_R}\downarrow & & \uparrow{\scriptstyle S} \\ \ell^N_{q^*} & \xrightarrow[D_\lambda]{} & \ell^N_p \end{array} \tag{4.11}$$

where

$$\begin{array}{rcl} f_R : \Sigma^{\beta}_{E_1 \ldots E_n} & \to & \ell^N_{q^*} \\ p & \mapsto & \displaystyle\sum_i \varphi_i(p)\, e_i \\ D_\lambda : \ell^N_{q^*} & \to & \ell^N_p \\ (a_i) & \mapsto & (\lambda_i\, a_i) \\ S : \ell^N_p & \to & F \\ (b_i) & \mapsto & b_i\, y_i \end{array}$$

are a bounded Σ-operator and bounded linear operators respectively. Moreover, it is verified that

$$\begin{array}{rcl} Lip^{\beta}(f_R) & = & \|(\varphi_i)\|^{wd}_{q^*} \\ \|D_\lambda\| & = & \|(\lambda_i)\|_r \\ \|S\| & = & \|(y_i)\|^{w}_{p^*}. \end{array}$$

Conversely, every factorization as (4.11) induces a representation of v_f. As a consequence,

$$L_{pq}(f) = \inf Lip^{\beta}(f_R)\, \|D_\lambda\|\, \|S\|$$

where the infimum is taken over all possible factorizations as (4.11). In other words, the collection of (p, q)-factorable Σ-operators in the class $\mathcal{FIN}$ are those which admits a factorization as in (4.11).

Now, let φ be a bounded linear operator on $\left(E_1 \otimes \ldots \otimes E_n \otimes F^*, (\alpha'_{p,q})^{\beta}\right)$. Then, f_φ can be factored as $f_\varphi = S D_\lambda f_R$. This decomposition induces a factorization of φ as we see next. Let $x^1 \otimes \ldots \otimes x^n \otimes y^*$ in $E_1 \otimes \ldots \otimes E_n \otimes F^*$. We have

$$\begin{array}{rcl} \varphi(x^1 \otimes \ldots \otimes x^n \otimes y^*) & = & \left\langle y^*\,,\, f(x^1 \otimes \ldots \otimes x^n)\right\rangle \\ & = & \left\langle S^* y^*\,,\, D_\lambda f_R(x^1 \otimes \ldots \otimes x^n)\right\rangle \\ & = & \delta_\lambda(\widetilde{R} \otimes S^*(x^1 \otimes \ldots \otimes x^n \otimes y^*)). \end{array}$$

where δ_λ is the bounded functional on $\left(\ell_{q^*}^N \otimes \ell_{p^*}^N, \pi\right)$ associated to the linear operator D_λ. In diagram, we have obtained

$$E_1 \otimes \ldots \otimes E_n \otimes F^* \xrightarrow{\widetilde{R}\otimes S^*} \ell_{q^*}^N \otimes \ell_{p^*}^N \xrightarrow{\delta_\lambda} \mathbb{K} \tag{4.12}$$

In this formulation, $\|\varphi\| = \inf Lip^\beta(f_R)\,\|\delta_\lambda\|\,\|S\|$ where the infimum is taken over all the possible factorizations. Notice that without loss of generality we may suppose that all λ_i is non-negative.

In Theorem 4.54 we extend the diagram (4.12) to the general case of Banach spaces. The tool we use to achieve this goal is the ultraproduct technique. For general results about ultraproducts of Banach spaces the reader may see [58] and [99, Sec. 2].

Proposition 4.53. *Let $\mathcal{I}$ be a directed set and $\mathcal{U}$ be an ultrafilter on it. Suppose that for each $\iota \in \mathcal{I}$ we have a bounded Σ-operator*

$$\varphi_\iota : \Sigma^{\beta_\iota}_{E^1_\iota \ldots E^n_\iota} \to G_\iota.$$

If there exists a constant $C > 0$ such that $Lip(\varphi_\iota) \leq C$ for all $\iota \in I$, then the well defined Σ-operator

$$\begin{array}{rcl} (\varphi_\iota)_\mathcal{U} : \Sigma_{(E^1_\iota)_\mathcal{U} \cdots (E^n_\iota)_\mathcal{U}} & \to & (G_\iota)_\mathcal{U} \\ (x^1_\iota)_\mathcal{U} \otimes \ldots \otimes (x^n_\iota)_\mathcal{U} & \mapsto & (\varphi_\iota(x^1_\iota \otimes \ldots \otimes x^n_\iota))_\mathcal{U}. \end{array}$$

is bounded and Lipschitz with the metric induced by the ultraproduct $\left((E^1_\iota \otimes \ldots \otimes E^n_\iota, \beta_\iota)\right)_\mathcal{U}$ with $Lip\left((\varphi_\iota)_\mathcal{U}\right) \leq C$.

Proof. First, notice that the application

$$\begin{array}{rcl} \ell_\infty(E^1_\iota) \times \cdots \times \ell_\infty(E^n_\iota) & \to & (G_\iota)_\mathcal{U} \\ \left((x^1_\iota)_\iota, \ldots (x^n_\iota)_\iota\right) & \mapsto & (\varphi_\iota(x^1_\iota \otimes \ldots \otimes x^n_\iota))_\mathcal{U} \end{array}$$

does not depend of the representation. To see this, fix $(x^j_\iota)_\iota$ with $2 \leq j \leq n$ and let $(x^1_\iota)_\iota$ and $(y^1_\iota)_\iota$ such that $(x^1_\iota)_\iota - (y^1_\iota)_\iota$ belongs to $c_0(E^1_\iota)$. Then

$$\begin{aligned} \|(\varphi_\iota(x^1_\iota \otimes \ldots \otimes x^n_\iota))_\mathcal{U} - (\varphi_\iota(y^1_\iota \otimes \ldots \otimes x^n_\iota))_\mathcal{U}\|_\mathcal{U} &= \lim_\mathcal{U} \|\varphi_\iota(x^1_\iota \otimes \ldots \otimes x^n_\iota) - \varphi_\iota(y^1_\iota \otimes \ldots \otimes x^n_\iota)\| \\ &\leq C \lim_\mathcal{U} \beta_\iota((x^1_\iota - y^1_\iota) \otimes \ldots \otimes x^n_\iota)) \\ &= C \lim_\mathcal{U} \|x^1_\iota - y^1_\iota\|\,\|x^2_\iota\| \ldots \|x^n_\iota\| \\ &\leq C \lim_\mathcal{U} \|x^1_\iota - y^1_\iota\| \sup_{\iota\in\mathcal{I}} \|x^2_\iota\| \ldots \sup_{\iota\in\mathcal{I}} \|x^n_\iota\|. \end{aligned}$$

implies $(\varphi_\iota(x^1_\iota \otimes \ldots \otimes x^n_\iota))_{\mathcal{U}} = (\varphi_\iota(y^1_\iota \otimes \ldots \otimes x^n_\iota))_{\mathcal{U}}$. The same phenomenon occurs in the other entries.

Define the multilinear map

$$\begin{array}{rcl} (E^1_\iota)_{\mathcal{U}} \times \cdots \times (E^n_\iota)_{\mathcal{U}} & \to & (G_\iota)_{\mathcal{U}} \\ \big((x^1_\iota)_{\mathcal{U}}, \ldots, (x^n_\iota)_{\mathcal{U}}\big) & \mapsto & (\varphi_\iota(x^1_\iota \otimes \ldots \otimes x^n_\iota))_{\mathcal{U}}. \end{array}$$

The algebraic embedding

$$\begin{array}{rcl} (E^1_\iota)_{\mathcal{U}} \otimes \ldots \otimes (E^n_\iota)_{\mathcal{U}} & \to & \big((E^1_\iota \otimes \ldots \otimes E^n_\iota, \beta_\iota)\big)_{\mathcal{U}} \\ (x^1_\iota)_{\mathcal{U}} \otimes \ldots \otimes (x^n_\iota)_{\mathcal{U}} & \mapsto & (x^1_\iota \otimes \ldots \otimes x^n_\iota)_{\mathcal{U}} \end{array}$$

Allows us to define a norm on $(E^1_\iota)_{\mathcal{U}} \otimes \ldots \otimes (E^n_\iota)_{\mathcal{U}}$. Let us denote this norm by $(\beta_\iota)_{\mathcal{U}}$ the norm of $\big((E^1_\iota \otimes \ldots \otimes E^n_\iota, \beta_\iota)\big)_{\mathcal{U}}$. Then

$$\begin{aligned} &\big\|(\varphi_\iota)_{\mathcal{U}}((x^1_\iota)_{\mathcal{U}} \otimes \ldots \otimes (x^n_\iota)_{\mathcal{U}}) - (\varphi_\iota)_{\mathcal{U}}((y^1_\iota)_{\mathcal{U}} \otimes \ldots \otimes (y^n_\iota)_{\mathcal{U}})\big\|_{\mathcal{U}} \\ &\qquad = \lim_{\mathcal{U}} \|\varphi_\iota(x^1_\iota \otimes \ldots \otimes x^n_\iota) - \varphi_\iota(y^1_\iota \otimes \ldots \otimes y^n_\iota)\| \\ &\qquad \le C \lim_{\mathcal{U}} \beta_\iota \left(x^1_\iota \otimes \ldots \otimes x^n_\iota - y^1_\iota \otimes \ldots \otimes y^n_\iota\right) \\ &\qquad = C(\beta_\iota)_{\mathcal{U}}((x^1_\iota)_{\mathcal{U}} \otimes \ldots \otimes (x^n_\iota)_{\mathcal{U}} - (y^1_\iota)_{\mathcal{U}} \otimes \ldots \otimes (y^n_\iota)_{\mathcal{U}}) \end{aligned}$$

completes the proof. ∎

Theorem 4.54. *Any functional φ bounded on $\Big(X_1 \otimes \ldots \otimes X_n \otimes Y, \big(\alpha'_{p,q}\big)^\beta\Big)$ can be factored as*

$$X_1 \otimes \ldots \otimes X_n \otimes Y \xrightarrow{f_R \otimes S} L_{q^*}(\mu) \otimes_\pi L_{p^*}(\nu) \xrightarrow{\delta} \mathbb{K}$$

where μ and ν are strictly localizable measures, $f_R : \Sigma^\beta_{X_1 \ldots X_n} \to L_{q^*}(\mu)$ is a bounded Σ-operator, $S : Y \to L_p(\nu)$ is bounded and δ is a positive functional such that $Lip^\beta(R)\,\|S\|\,\|\delta\| \le \|\psi\|$.

Proof. Let $\mathcal{I} := \mathcal{F}(X_1) \times \cdots \times \mathcal{F}(X_n) \times \mathcal{F}(Y) \times]0,1]$ and $\mathcal{U}$ be an ultrafilter on $\mathcal{I}$ containing the set $\{\iota \,|\, \iota_0 \le \iota\}$ for all ι_0 in $\mathcal{I}$.

For each $\iota = (M_1, \ldots, M_n, N, \varepsilon)$ let φ_ι be the restriction of φ to $M_1 \otimes \ldots \otimes M_n \otimes N$, and take a factorization as follows

$$M_1 \otimes \ldots \otimes M_n \otimes N \xrightarrow{f_{R_\iota} \otimes S_\iota} \ell^{n_\iota}_{q^*} \otimes_\pi \ell^{n_\iota}_{p^*} \xrightarrow{\delta_\iota} \mathbb{K}$$

such that $\|\delta\iota\| \le 1$, $\|S_\iota \le 1\|$ and $Lip^{\beta_\iota}(R_\iota) \le \|\varphi_\iota\|\,(1+\varepsilon) \le \|\varphi\|\,(1+\varepsilon)$, where $\beta_\iota := \beta|$. Set $M_{i\iota} := M_i$. Define, for any $x^i \in X_i$

$$x_{i\iota} := \begin{cases} x^i & \text{if } x^i \in M_i \\ 0 & \text{otherwise} \end{cases}$$

Then,

$$\begin{array}{rcl} J_i : X_i & \to & (M_{i\iota})_{\mathcal{U}} \\ x_i & \mapsto & (x_{i\iota})_{\mathcal{U}} \end{array}$$

is a linear isometry. Analogously, define

$$\begin{array}{rcl} J_Y : Y & \to & (N_\iota)_{\mathcal{U}} \\ y & \mapsto & (N_\iota)_{\mathcal{U}}. \end{array}$$

Proposition 4.53 ensures that the bounded Σ-operator

$$(f_{R_\iota})_{\mathcal{U}} : \Sigma_{(M_{1\iota})_{\mathcal{U}},\ldots,\times(M_{n\iota})_{\mathcal{U}}} \to (\ell_{q^*}^{n_\iota})_{\mathcal{U}}.$$

is Lipschitz with the norm induced by $((M_{1\iota} \otimes \ldots \otimes M_{n\iota}, \beta_\iota))_{\mathcal{U}}$.

Consider the compositions

$$\begin{array}{rcl} R := (R_\iota)_{\mathcal{U}} \circ J_1 \times \cdots \times J_n : & X_1 \times \cdots \times X_n & \to (\ell_{q^*}^{n_\iota})_{\mathcal{U}} \\ S := (S_\iota)_{\mathcal{U}} \circ J_Y : & Y & \to (\ell_{p^*}^{n_\iota})_{\mathcal{U}}. \end{array}$$

The Σ-operator f_R is bounded. To see this let $p = x^1 \otimes \ldots \otimes x^n$ and $q = z^1 \otimes \ldots \otimes z^n$ in $\Sigma^\beta_{X_1 \ldots X_n}$. Then

$$\begin{array}{rcl} \|f_R(p) - f_R(q)\|_{\mathcal{U}} & = & \lim\limits_{\mathcal{U}} \|(f_{R_\iota})_{\mathcal{U}}((x_{1\iota})_{\mathcal{U}} \otimes \ldots \otimes (x_{n\iota})_{\mathcal{U}}) - (f_{R_\iota})_{\mathcal{U}}((z_{1\iota})_{\mathcal{U}} \otimes \ldots \otimes (z_{n\iota}))_{\mathcal{U}}\| \\ & \leq & \|\varphi\| \, (1+\varepsilon) \lim\limits_{\mathcal{U}} \beta_\iota(x_{1\iota} \otimes \ldots \otimes x_{n\iota} - z_{1\iota} \otimes \ldots \otimes z_{n\iota}) \\ & \leq & \|\varphi\| \, (1+\varepsilon) \, \beta(x^1 \otimes \ldots \otimes x^n - z^1 \otimes \ldots \otimes z^n). \end{array}$$

Aside, the ultraproduct $\delta := (\delta_\iota)_{\mathcal{U}} : (\ell_{q^*}^{n_\iota})_{\mathcal{U}} \otimes (\ell_{p^*}^{n_\iota})_{\mathcal{U}} \to \mathbb{K}$ is bounded and positive. Finally, consider the composition

$$X_1 \otimes \ldots \otimes X_n \otimes Y \xrightarrow{\widetilde{R} \otimes S} (\ell_{q^*}^{n_\iota})_{\mathcal{U}} \otimes (\ell_{p^*}^{n_\iota})_{\mathcal{U}} \xrightarrow{\delta} \mathbb{K}$$

factors φ since

$$\begin{array}{rcl} \left\langle \delta \, , \, \widetilde{R} \otimes S(x^1 \otimes \ldots \otimes x^n \otimes y) \right\rangle & = & \lim\limits_{\mathcal{U}} \langle \delta_\iota \, , \, R_\iota(x_{1\iota} \otimes \ldots \otimes x_{n\iota}) \otimes S_\iota(y_\iota) \rangle \\ & = & \lim\limits_{\mathcal{U}} \varphi_\iota(x_{1\iota} \otimes \ldots \otimes x_{n\iota} \otimes y_\iota) \\ & = & \varphi(x^1 \otimes \ldots \otimes x^n \otimes y). \end{array}$$

Hence linearity asserts $\varphi = \delta \circ \widetilde{R} \otimes S$. Moreover, $Lip^\beta(f_R) \, \|S\| \, \|\delta\| \leq \|\varphi\| \, (1+\varepsilon)$ ensures that

$$Lip_\beta(f_R) \, \|S\| \, \|\delta\| \leq \|\varphi\|.$$

The proof is complete since the ultraproducts $(\ell_{q^*}^{n_\iota})_{\mathcal{U}}$, $(\ell_{p^*}^{n_\iota})_{\mathcal{U}}$ are linearly (order) isometric to some $L_{q^*}(\mu)$ and $L_{p^*}(\nu)$ respectively where μ and ν are strictly localizable measures, see[67, Th. 1.b.2]. ∎

An immediate consequence of this Theorem is that every (p,q)-factorable Σ-operator admits a factorization through some $L_{q^*}(\mu)$ and $L_p(\nu)$.

Corollary 4.55. *For any (p,q)-factorable Σ-operator $f : \Sigma^\beta_{X_1 \dots X_n} \to Y$ there exist two strictly localizable measures μ and ν, a Σ-operator $f_R : \Sigma^\beta_{X_1 \dots X_n} \to L_{q^*}(\mu)$, a bounded linear operator $S : L_p(\nu) \to Y^{**}$ and a positive bounded linear operator $D : L_{q^*}(\mu) \to L_p(\nu)$ such that $K_Y f$ factors as*

$$\begin{array}{ccccc} \Sigma^\beta_{X_1 \dots X_n} & \xrightarrow{f} & Y & \xrightarrow{K_Y} & Y^{**} \\ \Big\downarrow{\scriptstyle R} & & & & \Big\uparrow{\scriptstyle S} \\ L_{q^*}(\mu) & & \xrightarrow[D]{\qquad\qquad} & & L_p(\nu) \end{array}$$

with $Lip^\beta(f_R)\,\|S\|\,\|D\| \leq L_{pq}(f)$.

Proof. By definition, the functional φ_f is bounded on $\left(X_1 \otimes \ldots \otimes X_n \otimes Y^*, (\alpha'_{p,q})^\beta\right)^*$ is bounded. Theorem 4.54 implies the existence of $f_R : \Sigma^\beta_{X_1 \dots X_n} \to L_{q^*}(\mu)$, $S_0 : Y^* \to L_{p^*}(\nu)$ and $\delta : L_{q^*}(\mu) \otimes_\pi L_{p^*}(\mu) \to \mathbb{K}$ such that $\varphi_f = \delta \circ \widetilde{R} \otimes S_0$. Hence

$$\begin{aligned} \langle K_Y f(p)\,,\, y^* \rangle &= \langle y^*\,,\, f(p) \rangle \\ &= \varphi_f(p \otimes y^*) \\ &= \delta \circ \widetilde{R} \otimes S_0(p \otimes y^*) \\ &= \delta(f_R(p) \otimes S_0 y^*) \\ &= \langle D f_R(p)\,,\, S_0 y^* \rangle \\ &= \langle S^* D f_R(p)\,,\, y^* \rangle \end{aligned}$$

where D is the linear operator associated to the functional δ and $S = S_0^*$. ∎

In section [41, Sec. 18.2] the authors prove that the inclusion operator $I : L_{q^*}(\mu) \to L_p(\mu)$ is (p,q)-factorable with (p,q)-factorable norm equal to the bounded norm. Moreover, in [41, Sec. 18.10] it is proved that a positive operator $D : L_{q^*}(\mu) \to L_p(\nu)$ factors through an inclusion $I : L_{q^*}(\mu_0) \to L_p(\mu_0)$ for some finite measure μ_0 with $L_{pq}(D) = \|D\|$.

Before presenting the complete characterization of a (p,q)-factorable operators we prove the next proposition.

Proposition 4.56. *Let $\alpha_{p,q}$ be the Lapresté Σ-tensor norm on duals. Let $(X_1,\ldots,X_n,Y,\beta)$ be an election of Banach spaces. Let $R : X_1\times\cdots\times X_n \to X$ be a bounded multilinear operator. Then the operator*

$$\widetilde{R}\otimes I_Y : \left(X_1\otimes\ldots\otimes X_n\otimes Y, (\alpha'_{p,q})^{\beta}\right) \to \left(X\otimes Y, (\alpha'_{p,q})^{\beta}\right)$$

is bounded.

Proof. Let $u \in X_1\otimes\ldots\otimes X_n\otimes Y$. Let E_i and F be finite dimensional subspaces such that $u\in E_1\otimes\ldots\otimes E_n\otimes F$. Choose $E\subset X$ such that $\widetilde{R}\otimes I(u)\in E\otimes F$. Let $v=\sum_i \lambda_i\, x_i^*\otimes y_i^*\in E^*\otimes F^*$. Then

$$|\left\langle \widetilde{R}\otimes I(u)\,,\, v\right\rangle| = |\left\langle \sum_i \lambda_i\,(x_i^* f_R f_{E_i})\otimes y_i^*\,,\, u\right\rangle|$$

where $\sum_i \lambda_i\,(x_i^* f_R f_{E_i})\otimes y_i^* \in \mathcal{L}^{\beta|}(E_1,\ldots,E_n)\otimes F^*$. The isometry

$$\left(\mathcal{L}^{\beta|}(E_1,\ldots,E_n)\otimes F^*, (\alpha_{p,q})_{\beta|}\right) = \left(E_1\otimes\ldots\otimes E_n\otimes F, (\alpha'_{p,q})^{\beta|}\right)^*$$

implies

$$\begin{aligned} |\left\langle \widetilde{R}\otimes I(u)\,,\, v\right\rangle| &\leq (\alpha_{p,q})_{\beta|}\left(\sum_i \lambda_i (x_i^* f_R f_{E_i})\otimes y_i^*\right)(\alpha'_{p,q})^{\beta|}(u; E_iF) \\ &\leq \|(\lambda_i)\|_r\,\|(x_i^* f_R f_{E_i})\|_{q^*}\,\|(y_i^*)\|\,(\alpha'_{p,q})^{\beta|}(u; E_iF) \\ &\leq Lip^{\beta}(f_R)\,\|(\lambda_i)\|_r\,\|(x_i^*)\|_{q^*}\,\|(y_i^*)\|\,(\alpha'_{p,q})^{\beta|}(u; E_iF). \end{aligned}$$

After taking infimum over all the representation of v we obtain

$$|\left\langle \widetilde{R}\otimes I(u)\,,\, v\right\rangle| \leq Lip^{\beta}(f_R)\,\alpha_{p,q}(v, E^*F^*)(\alpha'_{p,q})^{\beta|}(u;\, E_iF).$$

Hence

$$\alpha'_{p,q}\left(\widetilde{R}\otimes I(u);\, XY\right) \leq \alpha'_{p,q}(\widetilde{R}\otimes I(u);\, EF) \leq Lip^{\beta}(f_R)\,(\alpha'_{p,q})^{\beta}(u;\, E_iF).$$

Finally, since $(\alpha'_{p,q})^{\beta}$ is finitely generated

$$\alpha'_{p,q}(\widetilde{R}\otimes I(u);\, XY) \leq Lip^{\beta}(f_R)\,(\alpha'_{p,q})^{\beta}(u;\, X_iY).$$

But this is nothing than $\widetilde{R}\otimes I$ is bounded and $\|\widetilde{R}\otimes I\| \leq Lip^{\beta}(f_R)$. ∎

Proposition 4.56 asserts that the composition $\Sigma^{\beta}_{X_1\ldots X_n} \xrightarrow{f} X \xrightarrow{T} Y$ is (p,q)-factorable whether the linear operator T is. Even more, $L_{pq}(Tf) \leq Lip^{\beta}(f)\,\|T\|$. With this result in hand we establish the characterization of (p,q)-factorable Σ-operators.

Theorem 4.57. *For every Σ-operator $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ the following are equivalent:*

i) f is (p,q)-factorable.

ii) There exist a finite measure μ, a Σ-operator $f_R : \Sigma^{\beta}_{X_1 \ldots X_n} \to L_{q^*}(\mu)$, a bounded linear operator $S : L_p(\mu) \to Y^{**}$ such that

$$\begin{array}{ccccc} \Sigma^{\beta}_{X_1 \ldots X_n} & \xrightarrow{f} & Y & \xrightarrow{K_Y} & Y^{**} \\ {\scriptstyle f_R}\downarrow & & & & \uparrow{\scriptstyle S} \\ L_{q^*}(\mu) & & \xrightarrow[I]{\qquad\qquad} & & L_p(\mu) \end{array}$$

commutes.

Under these circumstances $L_{pq}(f) = \inf Lip^{\beta}(f_R)\,\|I\|\,\|S\|$ where the infimum is taken over all the factorizations as above.

Proof. If f is (p,q)-factorable, then Corollary 4.55 implies that $K_Y f = SDf_R$ where D is a positive linear operator and $L_{pq}(f) \geq Lip^{\beta}(f_R)\,\|S\|\,\|D\|$. By the comments above, D admits a factorization as BIA where $I : L_{q^*}(\mu) \to L_p(\mu)$ is an inclusion for some finite measure μ with $L_{pq}(D) = \|D\| = \|A\|\,\|I\|\,\|B\|$. Hence $K_Y f = SDf_R = (SB)I(Af_R)$ and

$$L_{pq}(f) \geq Lip^{\beta}(f_R)\,\|S\|\,\|D\| \geq Lip^{\beta}(f_R)\,\|S\|\,\|A\|\,\|I\|\,\|B\| \geq Lip^{\beta}(Af_R)\,\|SB\|\,\|I\|.$$

Conversely, if f admits a factorization as in (ii) then Proposition 4.56 implies that f is (p,q)-factorable since $I : L_{q^*}(\mu) \to L_p(\mu)$ is a (p,q)-factorable linear operator. Moreover

$$L_{pq}(f) = Lpq(K_Y f) \leq Lip^{\beta}(f_R)\, L_{pq}(I)\,\|S\| = Lip^{\beta}(f_R)\,\|I\|\,\|S\|.$$

■

Following this approximation, we may consider the maximal ideal of p-integrable Σ-operators, defined as $(p,1)$-factorable Σ-operators. Thus, we extend the notion of p-integrable operators (see [41, 18.7]) to the setting of Σ-operators and so, to the multilinear context. A typical factorization induced by a p-integral Σ-operator $f : \Sigma^{\beta}_{X_1 \ldots X_n} \to Y$ is

$$\begin{array}{ccccc} \Sigma^{\beta}_{X_1 \ldots X_n} & \xrightarrow{f} & Y & \xrightarrow{K_Y} & Y^{**} \\ {\scriptstyle f_R}\downarrow & & & & \uparrow{\scriptstyle S} \\ L_{\infty}(\mu) & & \xrightarrow[I]{\qquad\qquad} & & L_p(\mu) \end{array} .$$

Other approximations of integral (or 1-integral) multilinear operators can be found in [5, 14, 27, 31, 32, 77, 100]. On the side of the metric theory the reader could be interested in [38].

List of Symbols

General Symbols

$\mathbb{N}$	Natural numbers
$\mathbb{R}$	Field of real numbers
$\mathbb{C}$	Field of complex numbers
$\mathbb{K}$	Field of real or complex numbers
$\Sigma_{X_1 \ldots X_n}$	Subset of simple tensors of $X_1 \otimes \ldots \otimes X_n$, p. 1
$\Sigma^{\beta}_{X_1 \ldots X_n}$	The metric space (Σ, β), p. 8
$\langle f, a \rangle$	Image of the function f at a
$\mathcal{FIN}$	Class of finite dimensional normed spaces
$\mathcal{NORM}$	Class of normed spaces
$\mathcal{BAN}$	Class of Banach spaces
$\mathcal{F}(X)$	Collection of finite dimensional subspaces of X
$\mathcal{CF}(X)$	Collection of finite codimensional subspaces of X

Let X be a normed space

$X^{\#}$	Algebraic dual of the vector space Y
X^{*}	Topological dual of the normed space Y
B_X	Closed unit ball
$K_X : X \to X^{**}$	Canonical embedding (even in the algebraic case)

Associated functions

$\widetilde{T}$	Linearization of $T : X_1 \times \cdots \times X_n \to Y$
f_T	Σ-operator associated to T, p. 2
φ_f	Associated functional of the Σ-operator f, p. 2
$\overline{\varphi}$	Canonical extension of φ, p. 3
f_{φ}	Σ-operator associated to the functional φ, p. 3

Spaces of Multilinear Functions

Σ-tensor norms

Ideals of Σ-operators

Bibliography

[1] D. Achour, E. Dahia, P. Rueda, and E.A. Sánchez-Pérez, *Factorization of strongly (p,σ)-continuous multilinear operators*, Linear and Multilinear Algebra **62** (2014), no. 12, 1649–1670.

[2] D. Achour and L. Mezrag, *On the Cohen strongly p-summing multilinear operators*, Journal of Mathematical Analysis and Applications **327** (2007), no. 1, 550–563.

[3] D. Achour, P. Rueda, E.A. Sánchez Pérez, and R. Yahi, *Lipschitz operator ideals and the approximation property*, Journal of Mathematical Analysis and Applications **436** (2016), no. 1, 217 – 236.

[4] A. A. Albanese, J. Bonet, and W. J. Ricker, *Spectrum and compactness of the Cèsaro operator on weighted ℓ_p spaces*, Journal of the Australian Mathematical Society **99** (2015), no. 3, 287–314.

[5] R. Alencar, *Multilinear mapping of nuclear type and integral type*, Proceedings of the American Mathematical Society **94** (1985), no. 1, 33–38.

[6] J. Angulo, *Operadores multilineales p-sumantes*, Ph.D. thesis, Centro de Investigación en Matemáticas, 2010.

[7] J. Angulo and M. Fernández-Unzueta, *A geometric approach to study p-summability in multilinear mappings*, arXiv:1805.02115v1 (2018).

[8] R. Arens, *The adjoint of a bilinear operation*, Proceedings of the American Mathematical Society **2** (1951), no. 6, 839–848.

[9] R. M. Aron and P. D. Berner, *A Hahn-Banach extension theorem for analytic mappings*, Bulletin de la Société Mathématique de France **106** (1978), 3–24.

[10] R. M. Aron and P. Galindo, *Weakly compact multilinear mappings*, Proceedings of the Edinburgh Mathematical Society **40** (1997), no. 1, 181–192.

[11] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis, vol. 1*, Colloquium Publications, vol. 48, American Mathematical Society, Providence R. I., 2000.

[12] A. T. Bernardino, D. Pellegrino, J. B. Seoane-Sepúlveda, and M. L. V. Souza, *Nonlinear absolutely summing operators revisited*, Bulletin of the Brazilian Mathematical Society, New Series **46** (2015), no. 2, 205–249.

[13] F. Bombal, D. Pérez-García, and I. Villanueva, *Multilinear extensions of Grothendieck's theorem*, The Quarterly Journal of Mathematics **55** (2004), no. 4, 441–450.

[14] F. Bombal and I. Villanueva, *Integral operators on the product of C(K) spaces*, Journal of Mathematical Analysis and Applications **264** (2001), no. 1, 107 –121.

[15] G. Botelho, H. A. Braunss, Junek H., and D. Pellegrino, *Holomorphy types and ideals of multilinear mappings*, Studia Mathematica **177** (2006), 43–65.

[16] G. Botelho and J. R. Campos, *Type and cotype of multilinear operators*, Revista Matemática Complutense **29** (2016), no. 3, 659–676.

[17] G. Botelho, E. Çalişkan, and D. Pellegrino, *On the representation of multi-ideals by tensor norms*, Journal of the Australian Mathematical Society **90** (2011), no. 2, 253–269.

[18] G. Botelho, P. Galindo, and L. Pellegrini, *Uniform approximation on ideals of multilinear mappings*, Mathematica Scandinavica **106** (2010), no. 2, 301–319.

[19] G. Botelho and K. P. Ling, *On compact extensions of multilinear operators*, Bulletin of the Brazilian Mathematical Society, New Series **45** (2014), no. 2, 343–353.

[20] G. Botelho and D. M. Pellegrino, *On symmetric ideals of multilinear mappings between Banach spaces*, Journal of the Australian Mathematical Society **81** (2006), no. 1, 141–148.

[21] G. Botelho and D.M. Pellegrino, *Two new properties of ideals of polynomials and applications*, Indagationes Mathematicae **16** (2005), no. 2, 157 – 169.

[22] G. Botelho and E. R. Torres, *Hyper-ideals of multilinear operators*, Linear Algebra and its Applications **482** (2015), no. Supplement C, 1 – 20.

[23] A. Brown, Halmos P. R., and Shields A. L., *Cesàro operators*, Acta Scientiarum Mathematicarum **26** (1965), 125–137.

[24] M. G. Cabrera-Padilla, J. A. Chávez-Domínguez, A. Jiménez-Vargas, and M. Villegas-Vallecillos, *Lipschitz tensor product*, Khayyam Journal of Mathematics **1** (2015), no. 2, 185–218.

[25] ______, *Maximal Banach ideals of Lipschitz maps*, Annals of Functional Analysis **7** (2016), no. 4, 593–608.

[26] ______, *Duality for ideals of Lipschitz maps*, Banach Journal of Mathematical Analysis **11** (2017), no. 1, 108–129.

[27] M. G. Cabrera-Padilla and A. Jiménez-Vargas, *Lipschitz Grothendieck-integral operators*, Banach Journal of Mathematical Analysis **9** (2015), no. 4, 34–57.

[28] M. G. Cabrera-Padilla and A. Jiménez-Vargas, *A new approach on Lipschitz compact operators*, Topology and its Applications **203** (2016), 22–31.

[29] J. R. Campos, *Cohen and multiple Cohen strongly summing multilinear operators*, Linear and Multilinear Algebra **62** (2014), no. 3, 322–346.

[30] D. Carando, V. Dimant, and S. Muro, *Coherent sequences of polynomial ideals on Banach spaces*, Mathematische Nachrichten **282** (2009), no. 8, 1111–1133.

[31] D. Carando, V. Dimant, and P. Sevilla-Peris, *Limit orders and multilinear forms on ℓ_p spaces*, Publications of the Research Institute for Mathematical Sciences **42** (2006), no. 2, 507–522.

[32] ______, *Ideals of multilinear forms – a limit order approach*, Positivity **11** (2007), no. 4, 589–607.

[33] D. Carando, V. Dimant, P. Sevilla-Peris, and R. Villafañe, *Diagonal extendible multilinear operators between ℓ_p-spaces*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas **108** (2014), no. 2, 541–555.

[34] D. Carando and D. Galicer, *Unconditionality in tensor products and ideals of polynomials, multilinear forms and operators*, The Quarterly Journal of Mathematics **62** (2011), no. 4, 845–869.

[35] J. A. Chávez-Domínguez, *Duality for Lipschitz p-Summing Operators*, Journal of Functional Analysis **261** (2011), no. 2, 387–407.

[36] ______, *Lipschitz factorization through subsets of Hilbert space*, Journal of Mathematical Analysis and Applications **418** (2014), no. 1, 344 – 356.

[37] D. Chen and B. Zheng, *Remarks on Lipschitz p-summing operators*, Proceedings of the American Mathematical Society **139** (2011), no. 8, 2891–2898.

[38] D Chen and B. Zheng, *Lipschitz p-integral operators and Lipschitz p-nuclear operators*, Nonlinear Analysis: Theory, Methods & Applications **75** (2012), no. 13, 5270–5282.

[39] D. L. Cohn, *Measure theory*, 2 ed., Birkhäuser Advanced Texts Basler Lehrbücher, Birkhäuser Basel, 2013.

[40] W. J. Davis, T. Figiel, W. B. Johnson, and A. Pelczynski, *Factoring weakly compact operators*, Journal of Functional Analysis **17** (1974), no. 3, 311–327.

[41] A. Defant and K. Floret, *Tensor norms and operator ideals*, vol. 176, North-Holland Mathematics Studies, 1993.

[42] A. Defant and D. Pérez-García, *A tensor norm preserving unconditionality for $\mathcal{L}_p$-spaces*, Transactions of the American Mathematical Society **360** (2008), no. 6, 3287–3306.

[43] A. Defant, D. Popa, and U. Schwarting, *Coordinatewise multiple summing operators in Banach spaces*, Journal of Functional Analysis **259** (2010), no. 1, 220–242.

[44] J. Diestel, *Sequences and series in Banch spaces*, 1 ed., Graduate Text in Mathematics, vol. 92, Springer-Verlag New York, 1984.

[45] J. Diestel, J.H. Fourie, and J. Swart, *The metric theory of tensor products. Grothendieck's résumé revisited*, American Mathematical Society, Providence R.I., 2008.

[46] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge University Press, 1995.

[47] J. Diestel and Uhl J. J. Jr., *Vector measures*, Mathematical Surveys, vol. 15, American Mathematical Society, Providence R.I., 1977.

[48] V. Dimant, *Strongly p-summing multilinear operators*, Journal of Mathematical Analysis and Applications **278** (2003), no. 1, 182–193.

[49] S. Dineen, *Complex analysis on infinite dimensional spaces*, 1 ed., Springer Monographs in Mathematics, Springer-Verlag London, 1999.

[50] N. Dunford and J. T. Schwartz, *Linear Operators part I General Theory*, Pure and Applied Mahtematics, vol. 7, Interscience Publishers New York, 1971.

[51] ______, *Linear Operators part II Spectral Theory*, Pure and Applied Mahtematics, vol. 7, Interscience Publishers New York, 1971.

[52] ______, *Linear Operators part III Spectral Operators*, Pure and Applied Mahtematics, vol. 7, Interscience Publishers New York, 1971.

[53] J. D. Farmer and W. B. Johnson, *Lipschitz p-summing operators*, Proceedings of the American Mathematical Society **137** (2009), no. 9, 2989–2995.

[54] M. Fernández-Unzueta, *The Segre cone of Banach spaces and multilinear operators*, arXiv:1804.10641v1 (2018).

[55] K. Floret and D. García, *On ideals of polynomials and multilinear mappings between Banach spaces*, Archiv der Mathematik **81** (2003), no. 3, 300–308.

[56] K. Floret and S. Hunfeld, *Ultrastability of ideals of homogeneous polynomials and multilinear mappings on Banach spaces*, Proceedings of the American Mathematical Society **130** (2001), no. 5, 1425–1435.

[57] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Boletim Sociedade de Matemática de São Paulo **8** (1953), 1–79.

[58] S. Heinrich, *Ultraproducts in Banach spade theory*, Journal fÃ¼r die reine und angewandte Mathematik (Crelles Journal) **313** (1980), 72–104.

[59] H. Jarchow, C. Palazuelos, D. Pérez-García, and I. Villanueva, *Hahn-Banach extension of multilinear forms and summability*, Journal of Mathematical Analysis and Applications **336** (2007), no. 2, 1161–1177.

[60] A. Jiménez-Vargas and J. A. Chávez-Domínguez, *Ideals of Extendible Lipschitz Maps*, Researchgate Publication (2017).

[61] A. Jiménez-Vargas, J.M. Sepulcre, and M. Villegas-Vallecillos, *Lipschitz compact operators*, Journal of Mathematical Analysis and Applications **415** (2014), no. 2, 889–901.

[62] N. Krikorian, *Compact multilinear transformations*, Proceedings of the American Mathematical Society **33** (1972), no. 2, 373–376.

[63] S. Kwapień, *On operators factorizable through L_p space*, Mémoires de la Société Mathématique de France **31** (1972), 215–225.

[64] H. E. Lacey, *The isometric theory of banach spaces*, 1 ed., Grundlehren der mathematischen Wissenschaften, vol. 208, Springer-Verlag Berlin Heidelberg, 1974.

[65] D. R. Lewis, *Duals of tensor products*, Banach spaces of analytic functions, Lecture Notes in Mathematics, vol. 604, Springer, 1977, pp. 57–66.

[66] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Mathematica **29** (1968), 275–326.

[67] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I and II*, Springer Berlin Heidelberg, 1996.

[68] J. A. López-Molina, *Multilinear operator ideals associated to Saphar type $(n+1)$-tensor norms*, Positivity **11** (2007), no. 1, 95–117.

[69] ______, *$(n+1)$-tensor norms of Laprest́é type*, Glasgow Mathematical Journal **54** (2012), no. 3, 665–692.

[70] Landsberg J. M., *Tensors: Geometry and applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence R. I., 2012.

[71] O. Maleva, *Lipschitz quotient mappings with good ratio of constants*, Mathematika **49** (2002), no. 1-2, 159–165.

[72] ______, *Components of level sets of uniform co-lipschitz functions on the plane*, Proceedings of the American Mathematical Society **133** (2005), 841–850.

[73] ______, *On Lipschitz ball noncollapsing functions and uniform co-Lipschitz mappings of the plane*, Abstract and Applied Analysis **2005** (2005), no. 5, 543–562.

[74] M. Marcus and B. N. Moyls, *Transformations on tensor product spaces*, Pacific Journal of Mathematics **9** (1959), no. 4, 1215–1221.

[75] M. C. Matos, *On multilinear mappings of nuclear type*, Revista Matematica de la Universidad Complutense de Madrid **6** (1993), no. 1, 61–81.

[76] ______, *Fully absolutely summing and Hilbert-Schmidt multilinear mappings*, Collectanea Mathematica **54** (2003), no. 2, 111–136.

[77] D. Pellegrino and J. Ribeiro, *On multi-ideals and polynomial ideals of Banach spaces: a new approach to coherence and compatibility*, Monatshefte für Mathematik **173** (2014), no. 3, 379–415.

[78] D. Pellegrino and J. Santos, *Absolutely summing multilinear operators: A panorama*, Quaestiones Mathematicae **34** (2011), no. 4, 447–478.

[79] D. Pérez-García, *The inclusion theorem for multiple summing operators*, Studia Mathematica **165** (2004), 275–290.

[80] ______, *Comparing different clasees of absolutely suming multilinear operators*, Archiv der Mathematik **85** (2005), 258–267.

[81] D. Pérez-García and I. Villanueva, *Multiple summing operators on Banach spaces*, Journal of Mathematical Analysis and Applications **285** (2003), no. 1, 86–96.

[82] ______, *Multiple summing operators on $C(K)$ spaces*, Arkiv för Matematik **42** (2004), no. 1, 153–171.

[83] ______, *Unconditional bases in tensor products of Hilbert spaces*, Mathematica Scandinavica **96** (2005), 280–288.

[84] A. Pietsch, *Absolut p-summierende abbildungen in normierten räumen*, Studia Mathematica **28** (1967), 333–353.

[85] ______, *Operator ideals*, Mathematische Monographien, vol. 16, Deutscher Verlag der Wissenschaften, 1978.

[86] ______, *Ideals of multilinear functionals*, Proceedings of the Second International Conferece on operator Algebras, Ideals and Their Applications in Theoretical Phisycs, Taubern-Texte. Math., vol. 62, 1983, pp. 185–199.

[87] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, no. 60, 1986.

[88] D. Popa, *Nuclear multilinear operators with respect to a partition*, Reconditi del Circolo Matematico di Palermo **61** (2012), 307–319.

[89] ______, *Almost summing and multiple almost summing multilinear operators on lp spaces*, Archiv der Mathematik **103** (2014), no. 3, 291–300.

[90] ______, *Composition results for strongly summing and dominated multilinear operators*, Open Mathematics **12** (2014), no. 10, 1433–1446.

[91] M. S. Ramanujan and E. Schock, *Operator ideals and spaces of bilinear operators*, Linear and Multilinear Algebra **18** (1985), no. 4, 307–318.

[92] J. B. Reade, *On the spectrum of the Cesaro operator*, Bulletin of the London Mathematical Society **17** (1985), no. 3, 263–267.

[93] R.A. Ryan, *Introduction to tensor product of Banach spaces*, Springer-Verlag London, 2002.

[94] K. Saadi, *Some properties of Lipschitz strongly p-summing operators*, Journal of Mathematical Analysis and Applications **423** (2015), no. 2, 1410 –1426.

[95] ______, *On the composition ideals of Lipschitz mappings*, Banach Journal of Mathematical Analysis **11** (2017), no. 4, 825–840.

[96] H. H. Schaefer, *Topological vector spaces*, 2 ed., Graduate Texts in Mathematics, vol. 3, Springer-Verlag New York, 1999.

[97] R. Schatten, *A theory of cross-spaces*, Annals of Mathematis Studies, vol. 26, Princeton University Press, 1950.

[98] D. M. Serrano-Rodríguez, *Absolutely γ-summing multilinear operators*, Linear Algebra and its Applications **439** (2013), no. 12, 4110–4118.

[99] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals*, Chapman and Hall /Crc Monographs and Surveys in Pure and Applied Mmahtematics, vol. 38, Longman Sc & Tech, 1989.

[100] I. Villanueva, *Integral mappings between Banach spaces*, Journal of Mathematical Analysis and Applications **279** (2003), no. 1, 56 –70.

[101] N. Weaver, *Lipschitz algebras*, 1 ed., World Scientific Pub. Co. Inc., 1999.

[102] R. Westwick, *Transformations on tensor spaces.*, Pacific Journal of Mathematics **23** (1967), no. 3, 613–620.

[103] Wojtaszczyk, *Banach spaces for analysts*, Cambridge Studies in Advanced Mathematics, no. 25, Cambridge University Press, 1991.