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DIRECT AND REVERSE LOG-SOBOLEV INEQUALITIES IN μ-DEFORMED SEGAL–BARGMANN ANALYSIS

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Both direct and reverse log-Sobolev inequalities, relating the Shannon entropy with a μ-deformed energy, are shown to hold in a family of μ-deformed Segal–Bargmann spaces. This shows that the μ-deformed energy of a state is finite if and only if its Shannon entropy is finite. The direct inequality is a new result, while the reverse inequality has already been shown by the authors but using different methods. Next the μ-deformed energy of a state is shown to be finite if and only if its Dirichlet form energy is finite. This leads to both direct and reverse log-Sobolev inequalities that relate the Shannon entropy with the Dirichlet energy. We obtain that the Dirichlet energy of a state is finite if and only if its Shannon entropy is finite. The main method used here is based on a study of the reproducing kernel function of these spaces and the associated integral kernel transform.

Keywords: Segal–Bargmann analysis; log-Sobolev inequality; reverse log-Sobolev inequality; reproducing kernel Hilbert space.

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1. Definitions and Notation

We begin with some definitions and notation. We start with an introduction to μ-deformed Segal–Bargmann analysis (which is itself a realization of μ-deformed quantum mechanics, though we will not go into that here). For background on these subjects, see Refs. 20 and 25. For recent related work, see Refs. 2, 21–23, 31 and 32. The introductions of Refs. 21 and 22 provide more motivation for studying this topic.

First, we take $\mu > -1/2$ to be a fixed parameter throughout this paper.

Definition 1.1. Say $\lambda > 0$. We define measures in the complex plane $\mathbb{C}$ by

$$d\nu_{e,\mu,\lambda}(z) := \nu_{e,\mu,\lambda}(z)dxdy\,,$$

$$d\nu_{o,\mu,\lambda}(z) := \nu_{o,\mu,\lambda}(z)dxdy\,,$$

whose densities are defined by

$$\nu_{e,\mu,\lambda}(z) := \lambda\frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma(\mu+\frac{1}{2})}K_{\mu-\frac{1}{2}}(|\lambda^{\frac{1}{2}}z|^2)|\lambda^{\frac{1}{2}}z|^{2\mu+1}\,, \tag{1.1}$$

$$\nu_{o,\mu,\lambda}(z) := \lambda\frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma(\mu+\frac{1}{2})}K_{\mu+\frac{1}{2}}(|\lambda^{\frac{1}{2}}z|^2)|\lambda^{\frac{1}{2}}z|^{2\mu+1} \tag{1.2}$$

for $0 \neq z \in \mathbb{C}$, where Γ (the Euler gamma function) and K_α (the MacDonald function of order α) are defined in Ref. 19. Moreover, $dxdy$ is Lebesgue measure in $\mathbb{C}$.

The function K_α is also known as the modified Bessel function of the third kind or Basset's function (see p. 5 Ref. 8). But it is also simply known as a modified Bessel function (see p. 961 of Ref. 11, and p. 374 of Ref. 1). One way to identify the MacDonald function is to note the following useful property:

$$K_\alpha(x) = \int_0^\infty du e^{-x\cosh u}\cosh(\alpha u)\,, \tag{1.3}$$

for $x > 0$ and any $\alpha \in \mathbb{R}$ (see p. 119 of Ref. 19). An explanation of how the MacDonald functions come into this theory in a natural way is given in Ref. 36.

From the formulas (1.1) and (1.2), one can see why the case $\mu = -1/2$ has not been included. One should refer to the discussion of the Bose-like oscillator in Ref. 25 (especially, note Theorem 5.7) for motivation for the condition $\mu > -1/2$.

Let $\mathcal{H}(\mathbb{C})$ be the space of all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$. We note that $f_{\mathrm{e}} := (f + Jf)/2$ (respectively, $f_{\mathrm{o}} := (f - Jf)/2$) defines the even (respectively, odd) part of f, where $Jf(z) := f(-z)$ is the parity operator. So, $f = f_{\mathrm{e}} + f_{\mathrm{o}}$.

We use throughout the paper the standard notations for L^p spaces and their norms without further comment. All L^p spaces in this paper are complex. However, the ambiguous notation $\|\cdot\|_{p\to q}$ is used to denote the operator norm from some L^p space to some L^q space without specifying the measure spaces involved. The context will indicate which measures spaces are meant.

Definition 1.2. The *λ-dilated, μ-deformed Segal–Bargmann space* defined for $0 < p < \infty$ and $\lambda > 0$ is

$$\mathcal{B}^p_{\mu,\lambda} := \mathcal{H}(\mathbb{C}) \cap \{f : \mathbb{C} \to \mathbb{C} | f_{\mathrm{e}} \in L^p(\mathbb{C}, \nu_{\mathrm{e},\mu,\lambda}) \text{ and } f_{\mathrm{o}} \in L^p(\mathbb{C}, \nu_{\mathrm{o},\mu,\lambda})\}\,,$$

where $f = f_{\mathrm{e}} + f_{\mathrm{o}}$ is the decomposition of a function into its even and odd parts. Next we define

$$\|f\|_{\mathcal{B}^p_{\mu,\lambda}} := (\|f_{\mathrm{e}}\|^p_{L^p(\mathbb{C},\nu_{\mathrm{e},\mu,\lambda})} + \|f_{\mathrm{o}}\|^p_{L^p(\mathbb{C},\nu_{\mathrm{o},\mu,\lambda})})^{1/p}$$

for all $f \in \mathcal{B}^p_{\mu,\lambda}$. We also define the *even subspace* of $\mathcal{B}^p_{\mu,\lambda}$ by

$$\mathcal{B}^p_{\mathrm{e},\mu,\lambda} := \mathcal{B}^p_{\mu,\lambda} \cap \{f : f = f_{\mathrm{e}}\}$$

and the *odd subspace* of $\mathcal{B}^p_{\mu,\lambda}$ by

$$\mathcal{B}^p_{\mathrm{o},\mu,\lambda} := \mathcal{B}^p_{\mu,\lambda} \cap \{f : f = f_{\mathrm{o}}\}\,.$$

In these definitions we do not write the subscript λ when $\lambda = 1$.

As far as we know, the two Definitions 1.1 and 1.2 are due to us but first appeared in print in joint work of the second author with Pita in Ref. 21. Behind these definitions there is a lot of history which we will relate to the best of our knowledge. As is customary, we do offer our sincere apologies to those researchers whose work we have not mentioned merely due to our own ignorance. These definitions are due to the present authors in Ref. 2 in 2006 in the case when $0 < p < \infty$ and $\lambda = 1$ and $\mu > -1/2$ and to Marron[20] in 1994 in the case when $p = 2$ and $\lambda > 0$ and $\mu > -1/2$. However, Marron's work closely follows Rosenblum's in Ref. 26 (also in 1994) where the case $p = 2$ and $\lambda = 1$ and $\mu > -1/2$ is presented. The works of Rosenblum and Marron were most influential for our work on this topic. However, in Sharma *et al.*[30] formula (2.58) gives the inner product in Eq. (1.4) below up to a multiplicative constant. So these authors already had in 1981 the case $p = 2$ and $\lambda = 1$ and $\mu > -1/2$. This is the earliest reference that we are aware of. But slightly later in 1984 Cholewinski in Ref. 7 has the case $p = 2$, $\lambda = 1$ and $\mu \geq 0$, but only for the even subspace. Next Sifi and Soltani in Ref. 31 in 2002 have the case $p = 2$, $\lambda = 1$ and $\mu \geq 0$. Finally, we note that Ben Saïd and Ørsted in Ref. 4 in 2006 present in detail the case $p = 2$ and $\lambda = 1$ and $\mu \geq 0$ in Example 4.17, though they are aware of the case when μ is negative.

The next known result is elementary. We include it here since it seems not to have been proved in the literature before.

Proposition 1.1. *For $p \geq 1$ and $\lambda > 0$ we have that $\|\cdot\|_{\mathcal{B}^p_{\mu,\lambda}}$ is a norm and that $\mathcal{B}^p_{\mu,\lambda}$ is a Banach space which is the (internal) direct sum of the Banach subspaces $\mathcal{B}^p_{\mathrm{e},\mu,\lambda}$ and $\mathcal{B}^p_{\mathrm{o},\mu,\lambda}$.*

Proof. The proofs that $\|\cdot\|_{\mathcal{B}^p_{\mu,\lambda}}$ is a norm and that we have a direct sum are straightforward and left to the reader. It remains for us here to show that this space is complete. This argument is well known (for example, see Ref. 15), and we give a sketch of it.

We first note that by definition $f \in \mathcal{B}^p_{\mu,\lambda}$ is equivalent to these conditions:

(1) f_{e} and f_{o} are holomorphic in $\mathbb{C}$.
(2) $f_{\mathrm{e}} \in L^p(\mathbb{C}, \nu_{\mathrm{e},\mu,\lambda})$ and $f_{\mathrm{o}} \in L^p(\mathbb{C}, \nu_{\mathrm{o},\mu,\lambda})$.

Since f_{e} is holomorphic, we have by the theory of a complex variable that

$$f_{\mathrm{e}}(z) = \frac{1}{\pi r^2} \int_{B_r(z)} d\mu_{\mathrm{L}}(w) f_{\mathrm{e}}(w)$$

for any $z \in \mathbb{C}$ and any $r > 0$, where μ_{L} is Lebesgue measure and $B_r(z)$ is the ball of radius r and center z. So, using the fact that $\nu_{\mathrm{e},\mu,\lambda}$ has no zeros,

$$f_{\mathrm{e}}(z) = \frac{1}{\pi r^2} \int_{\mathbb{C}} d\mu_{\mathrm{L}}(w) \nu_{\mathrm{e},\mu,\lambda}(w) \left(\chi_{B_r(z)}(w) \frac{1}{\nu_{\mathrm{e},\mu,\lambda}(w)} \right) f_{\mathrm{e}}(w)\,,$$

where χ_S denotes the characteristic function of a set S. Applying Hölder's inequality, we get for all $z \in \mathbb{C}$ that

$$|f_{\mathrm{e}}(z)| \le C_{\mathrm{e}}(z) \|f_{\mathrm{e}}\|_{L^p(\nu_{\mathrm{e},\mu,\lambda})}\,,$$

where

$$C_{\mathrm{e}}(z) = \frac{1}{\pi r^2} \left\| \frac{\chi_{B_r(z)}}{\nu_{\mathrm{e},\mu,\lambda}} \right\|_{L^{p'}(\nu_{\mathrm{e},\mu,\lambda})}$$

is a finite real number that depends continuously on z. Here p' is the usual dual Lebesgue index. Similarly, we get

$$|f_{\mathrm{o}}(z)| \le C_{\mathrm{o}}(z) \|f_{\mathrm{o}}\|_{L^p(\nu_{\mathrm{o},\mu,\lambda})}\,,$$

where $C_{\mathrm{o}}(z)$ depends continuously on z. To show that $\mathcal{B}^p_{\mu,\lambda}$ is complete, we take a Cauchy sequence f_n in that space and show that it converges to an element of the space. But f_n Cauchy in $\mathcal{B}^p_{\mu,\lambda}$ implies that the sequence of even parts $(f_n)_{\mathrm{e}}$ is Cauchy in $L^p(\nu_{\mathrm{e},\mu,\lambda})$ and that the sequence of odd parts $(f_n)_{\mathrm{o}}$ is Cauchy in $L^p(\nu_{\mathrm{o},\mu,\lambda})$. Since $p \ge 1$, these two Lebesgue spaces are complete and so $(f_n)_{\mathrm{e}} \to g$ and $(f_n)_{\mathrm{o}} \to h$ as $n \to \infty$, where $g \in L^p(\nu_{\mathrm{e},\mu,\lambda})$ and $h \in L^p(\nu_{\mathrm{o},\mu,\lambda})$. Clearly, g is even and h is odd. Now by a standard argument, the above two inequalities imply that $(f_n)_{\mathrm{e}} \to g$ and $(f_n)_{\mathrm{o}} \to h$ uniformly on compact subsets of $\mathbb{C}$, and so g and h are holomorphic. This implies that $g + h \in \mathcal{B}^p_{\mu,\lambda}$ and that $f_n \to g + h$ in the norm of $\mathcal{B}^p_{\mu,\lambda}$. QED

Moreover, for $p = 2$ we have that $\mathcal{B}^2_{\mu,\lambda}$ is a Hilbert space (see Ref. 20) with inner product defined by

$$\langle f, g \rangle_{\mathcal{B}^2_{\mu,\lambda}} := \langle f_{\mathrm{e}}, g_{\mathrm{e}} \rangle_{L^2(\nu_{\mathrm{e},\mu,\lambda})} + \langle f_{\mathrm{o}}, g_{\mathrm{o}} \rangle_{L^2(\nu_{\mathrm{o},\mu,\lambda})}\,. \tag{1.4}$$

Of course, $f = f_{\mathrm{e}} + f_{\mathrm{o}}$ and $g = g_{\mathrm{e}} + g_{\mathrm{o}}$ are the representations of f and g as the sums of their even and odd parts. (We will often use such representations without explicit comment, letting the notation carry the burden of explanation.) In this case, $\mathcal{B}^2_{\mu,\lambda}$ is the Hilbert space (internal) direct sum of the subspaces $\mathcal{B}^2_{\mathrm{e},\mu,\lambda}$ and $\mathcal{B}^2_{\mathrm{o},\mu,\lambda}$. As we shall see in Sec. 3, each of the spaces $\mathcal{B}^2_{\mu,\lambda}$, $\mathcal{B}^2_{\mathrm{e},\mu,\lambda}$ and $\mathcal{B}^2_{\mathrm{o},\mu,\lambda}$ is a reproducing kernel Hilbert space. When $\mu = 0$ and $\lambda = 1$, this reduces to the usual Segal–Bargmann space, denoted here by $\mathcal{B}^2$. (See Refs. 3 and 28.) Further motivation for the nomenclature in Definition 1.2 is given in Ref. 36.

Note that $\nu_{\mathrm{e},\mu,\lambda}(z) = \lambda \nu_{\mathrm{e},\mu}(\lambda^{1/2} z)$ and $\nu_{\mathrm{o},\mu,\lambda}(z) = \lambda \nu_{\mathrm{o},\mu}(\lambda^{1/2} z)$, so that λ is a dilation parameter. Or, in other words, the dilation operator T_λ defined by

$$T_\lambda f(z) := f(\lambda^{1/2} z) \tag{1.5}$$

for $f \in \mathcal{B}^2_\mu$ and $z \in \mathbb{C}$ is a unitary transformation from $\mathcal{B}^2_{\mathrm{e},\mu}$ onto $\mathcal{B}^2_{\mathrm{e},\mu,\lambda}$ and from $\mathcal{B}^2_{\mathrm{o},\mu}$ onto $\mathcal{B}^2_{\mathrm{o},\mu,\lambda}$. Therefore, T_λ is also a unitary map from $\mathcal{B}^2_\mu$ onto $\mathcal{B}^2_{\mu,\lambda}$.

One can relate the parameter λ to Planck's constant $\hbar$ by considering the case $\mu = 0$. We first observe that for $z \in \mathbb{C}$, $z \neq 0$ and $\mu = 0$ we have that

$$\nu_{\mathrm{e},0,\lambda}(z) = \nu_{\mathrm{o},0,\lambda}(z) = \lambda \frac{2^{1/2}}{\pi \Gamma(1/2)} K_{1/2}(|\lambda^{1/2} z|^2) \cdot |\lambda^{1/2} z| = \frac{\lambda}{\pi} e^{-\lambda |z|^2}\,,$$

which is a normalized Gaussian, using $K_{1/2}(x) = K_{-1/2}(x) = (\pi/(2x))^{1/2} e^{-x}$. (see pp. 110 and 112 of Ref. 19). This should be compared with the Gaussian

$$\nu_{\mathrm{Gauss},\hbar}(z) := \frac{1}{\pi\hbar} e^{-|z|^2/\hbar}\,, \tag{1.6}$$

which is the density for the measure of the Segal–Bargmann space for any $\hbar > 0$. (See pp. 9 and 21 of Ref. 15. Note that the identification $t = \hbar$ was made in Ref. 15.) So it turns out that $\lambda = 1/\hbar$. (For those who are confused by the fact that $\hbar$ and $|z|^2$ have the same dimensions, let us note that there is a *normalized* harmonic oscillator Hamiltonian implicitly used here. So there is both a mass and a frequency which have been taken equal to the dimensionless constant 1.)

Definition 1.3. Let (Ω, ν) be a measure space with finite measure (meaning that $0 < \nu(\Omega) < \infty$). Define the *entropy* of any f in $L^2(\Omega, \nu)$ to be

$$S_{L^2(\Omega,\nu)}(f) := \int_\Omega d\nu(\omega) |f(\omega)|^2 \log |f(\omega)|^2 - \|f\|^2_{L^2(\Omega,\nu)} \log \|f\|^2_{L^2(\Omega,\nu)}\,, \tag{1.7}$$

where $\|\cdot\|_{L^2(\Omega,\nu)}$ means the norm in the Hilbert space $L^2(\Omega,\nu)$, log is the natural logarithm, and $0 \log 0 := 0$ (to make the function $0 \leq r \mapsto r \log r$ continuous from the right at $r = 0$).

This definition is due to Shannon[29] in his theory of communication. The requirement that the measure be finite is not necessary, but is imposed to avoid technical details which are not important for us, since all the measure spaces in this paper have finite measure. (See Ref. 17 for an example where $\nu(\Omega) = \infty$.) For a finite measure space we have that $S_{L^2(\Omega,\nu)}(f)$ is defined for *all* $f \in L^2(\Omega,\nu)$ and moreover that

$$(-\log W) \|f\|^2_{L^2(\Omega,\nu)} \leq S_{L^2(\Omega,\nu)}(f)\,,$$

where $W = \nu(\Omega)$, by applying Jensen's inequality to the probability space $(\Omega, \nu/W)$ and the convex function $r \mapsto r \log r$ for $r \geq 0$. It follows that $S_{L^2(\Omega,\nu)}(f) > -\infty$, though $S_{L^2(\Omega,\nu)}(f) = +\infty$ could occur.

Definition 1.4. If there is a distinguished quadratic form $Q(f)$ defined for all $f \in X$, a closed subspace of $L^2(\Omega,\nu)$ where (Ω,ν) is a measure space, we say that an inequality holding for all $f \in X$ of the form

$$S_{L^2(\Omega,\nu)}(f) \leq C_1 Q(f) + C_2 \|f\|^2_{L^2(\Omega,\nu)}\,,$$

for constants $C_1 > 0$ and $C_2 \geq 0$ is a (*direct*) log-*Sobolev inequality* in X. Similarly, an inequality holding for all $f \in X$ of the form

$$Q(f) \leq D_1 S_{L^2(\Omega,\nu)}(f) + D_2 \|f\|^2_{L^2(\Omega,\nu)} \,,$$

for constants $D_1 > 0$ and $D_2 \geq 0$ is a *reverse* log-*Sobolev inequality* in X.

Usually, $Q(f)$ in this definition is a Dirichlet form, but this is not so in the main results given in Sec. 4. We understand $Q : X \to [0, \infty]$ as a sort of energy. If $Q(f)$ is only densely defined, we put $Q(f) = +\infty$ for f not in the original domain of Q. Also, the entropy in this definition can be equal to $+\infty$. So, one way to think about a direct log-Sobolev inequality is that it tells us that finite energy implies finite entropy. It can also be thought of as a type of coercivity inequality. Similar comments apply to reverse log-Sobolev inequalities. There is an extensive literature on log-Sobolev inequalities, starting with the articles of Federbush[9] and of Gross.[12] For more recent references, see Ref. 14 and references therein. The first reverse log-Sobolev inequality appeared in Ref. 34. Further studies of such inequalities can be found in Refs. 2, 6, 10, 13 and 35.

We use the standard convention in analysis that C represents a positive, finite constant (i.e. a quantity not depending on the variable of interest in the context) which may change value with each usage.

The organization of the article is as follows. In Sec. 2, we review some basic properties of the measures introduced above. In Sec. 3 we analyze each reproducing kernel function of the various Hilbert spaces studied here as the kernel function of an integral transform. In Sec. 4, we present our main result, an energy-entropy inequality which in special cases is a direct log-Sobolev inequality and in other cases is a reverse log-Sobolev inequality. All of this is in terms of a quadratic form called the μ-deformed energy and introduced by the authors in Ref. 2. Then in Sec. 5 we present relations between the μ-deformed energy and the Dirichlet form energy. This allows us to prove all of our main inequalities in terms of the Dirichlet form energy as well as in terms of the μ-deformed energy.

2. Properties of the Measures

We note the following results (see p. 136 of Ref. 19) for the asymptotic behavior of the MacDonald function $K_\alpha(x)$ for $\alpha \in \mathbb{R}$ and $x > 0$:

$$K_\alpha(x) \cong \frac{2^{|\alpha|-1}\Gamma(|\alpha|)}{x^{|\alpha|}} \qquad \text{as } x \to 0^+ \quad \text{if } \alpha \neq 0 \,,$$

$$K_0(x) \cong \log \frac{2}{x} \qquad \text{as } x \to 0^+ \,,$$

$$K_\alpha(x) \cong \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \qquad \text{as } x \to +\infty \quad \text{for all } \alpha \in \mathbb{R} \,.$$

Here $f(x) \cong g(x)$ as $x \to a$ means $\lim_{x\to a} f(x)/g(x) = 1$, where a is a limit point of a common domain of definition of the positive functions f and g. While the usual definition of K_α (see Ref. 19, pp. 108–109) gives an analytic function defined on $\mathbb{C}\backslash(-\infty, 0]$, we are only interested in its values for real $x > 0$. Notice that the asymptotic behavior of $K_\alpha(x)$ as $x \to +\infty$ does not depend on α to first order. But the next order term does depend on α.

Written in polar coordinates $d\nu_{\mathrm{o},\mu,\lambda}$ has density (with respect to $drd\theta$)

$$\lambda^{\mu+\frac{3}{2}} \frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma(\mu+\frac{1}{2})} K_{\mu+\frac{1}{2}}(\lambda r^2) r^{2\mu+2}\,.$$

So, the behavior of the density of $d\nu_{\mathrm{o},\mu,\lambda}$ near zero ($r \to 0^+$) is asymptotic to

$$C \frac{1}{(r^2)^{(\mu+1/2)}} r^{2\mu+2} = Cr$$

for all $\mu > -1/2$. On the other hand, $d\nu_{\mathrm{e},\mu,\lambda}$ has density

$$\lambda^{\mu+\frac{3}{2}} \frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma(\mu+\frac{1}{2})} K_{\mu-\frac{1}{2}}(\lambda r^2) r^{2\mu+2}$$

in polar coordinates (again with respect to $drd\theta$), whose asymptotic behavior as $|z| = r \to 0^+$ is given by the following three cases:

(a) For $-1/2 < \mu < 1/2$, we have $\nu_{\mathrm{e},\mu,\lambda}(z) \cong Cr^{2\mu+2}/(r^2)^{(\frac{1}{2}-\mu)} = Cr^{4\mu+1}$.
(b) For $\mu = 1/2$, we have $\nu_{\mathrm{e},\mu,\lambda}(z) \cong C(|\log r^2|)r^3 = Cr^3|\log r|$. Note that this is *not* the limit when $\mu \uparrow 1/2$ of the previous case.
(c) For $\mu > 1/2$, we have $\nu_{\mathrm{e},\mu,\lambda}(z) \cong Cr^{2\mu+2}/(r^2)^{(\mu-\frac{1}{2})} = Cr^3$. So for this range of values of μ, the functional form of the asymptotic dependence on r (for r near zero) is independent of μ, namely r^3, though the constant does depend on μ. Also, this functional form is the limit when $\mu \uparrow 1/2$ of the first case.

Note that in all cases the singularity of the MacDonald function at zero in formulas (1.1) and (1.2) has been regularized into a locally integrable function of r near $r = 0$ by the factor $r^{2\mu+2}$, which comes from a factor of $r^{2\mu+1}$ given in the definition of the densities of the measures and another factor of r that comes from the change of variables $dxdy = rdrd\theta$.

Using (1.3) we see immediately that $|\alpha| < |\beta|$ implies that $K_\alpha(x) < K_\beta(x)$ for all $x > 0$. In particular, we have $K_{\mu-1/2}(x) < K_{\mu+1/2}(x)$ for all $x > 0$ provided that $|\mu - 1/2| < |\mu + 1/2|$. But this last condition is equivalent to $\mu > 0$. So, for all $\mu > 0$ and all $z \in \mathbb{C}$ with $z \neq 0$ we have that

$$\nu_{\mathrm{e},\mu,\lambda}(z) < \nu_{\mathrm{o},\mu,\lambda}(z)\,. \tag{2.1}$$

In the case $\mu = 0$, we have already seen that $\nu_{\mathrm{e},0,\lambda}(z) = \nu_{\mathrm{o},0,\lambda}(z)$. Finally, in the case $-1/2 < \mu < 0$ we have $K_{\mu+1/2}(x) < K_{\mu-1/2}(x)$ for all $x > 0$ since $|\mu + 1/2| < |\mu - 1/2|$, and so it follows for $0 \neq z \in \mathbb{C}$ that

$$\nu_{\mathrm{o},\mu,\lambda}(z) < \nu_{\mathrm{e},\mu,\lambda}(z)\,. \tag{2.2}$$

Since $\nu_{\mathrm{e},\mu,\lambda}(z)$ and $\nu_{\mathrm{o},\mu,\lambda}(z)$ are integrable near zero, continuous in $\mathbb{C}\backslash\{0\}$ and decay as $r = |z| \to +\infty$ as $Cr^{2\mu+1}e^{-\lambda r^2}$ (density with respect to $drd\theta$), it follows that the measures $d\nu_{\mathrm{e},\mu,\lambda}(z)$ and $d\nu_{\mathrm{o},\mu,\lambda}(z)$ are finite. It turns out that $d\nu_{\mathrm{e},\mu,\lambda}(z)$ is a probability measure. To show this we will use the identity

$$\frac{d}{dx}[x^\alpha K_\alpha(x)] = -x^\alpha K_{\alpha-1}(x)$$

for $x > 0$ (see p. 110 of Ref. 19). So we now evaluate that

$$\begin{aligned}
\nu_{\mathrm{e},\mu,\lambda}(\mathbb{C}) &= \int_{\mathbb{C}} d\nu_{\mathrm{e},\mu,\lambda}(z) = 2\pi \int_0^\infty dr \frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma(\mu+1/2)} r^{2\mu+2}\lambda^{\mu+\frac{3}{2}} K_{\mu-1/2}(\lambda r^2) \\
&= \frac{2^{\frac{1}{2}-\mu}}{\Gamma(\mu+1/2)} \int_0^\infty ds s^{\mu+\frac{1}{2}} K_{\mu-1/2}(s) \\
&= \frac{2^{\frac{1}{2}-\mu}}{\Gamma(\mu+1/2)} \int_0^\infty ds \frac{d}{ds}(-s^{\mu+\frac{1}{2}} K_{\mu+1/2}(s)) \\
&= \frac{2^{\frac{1}{2}-\mu}}{\Gamma(\mu+1/2)} (-s^{\mu+\frac{1}{2}} K_{\mu+1/2}(s)) \Big|_0^\infty \\
&= \frac{2^{\frac{1}{2}-\mu}}{\Gamma(\mu+1/2)} 2^{\mu-\frac{1}{2}}\Gamma(\mu+1/2) = 1\,,
\end{aligned}$$

where we used the definition of the measure $d\nu_{\mathrm{e},\mu,\lambda}(z)$, a change of variables, the above quoted identity, the fundamental theorem of calculus and the asymptotic behavior of $K_{\mu+1/2}$ at zero and at infinity. (Another way of thinking about this fact is given in Ref. 36.) It now follows from (2.1) or (2.2) that $d\nu_{\mathrm{o},\mu,\lambda}(z)$ is *not* a probability measure when $\mu \neq 0$.

The results of this paper hold for every value of the scaling parameter $\lambda > 0$. However, to keep the notation manageable, we will usually put $\lambda = 1$ hereafter. Of course, the case of general λ is implied by the case $\lambda = 1$ by applying a dilation.

3. The Reproducing Kernel and Its Associated Integral Transform

There is a reproducing kernel function K for $\mathcal{B}^2_\mu$ (see Refs. 20 and 4), which satisfies the usual reproducing property, namely, $\langle K(\cdot, w)^*, f\rangle_{\mathcal{B}^2_\mu} = f(w)$ for all $f \in \mathcal{B}^2_\mu$ and $w \in \mathbb{C}$. In fact, $K(z,w) = \exp_\mu(z^* w)$ for all z, $w \in \mathbb{C}$, where the μ-deformed exponential function (see Ref. 25) is defined by $\exp_\mu(z) := \sum_{k=0}^\infty z^k/\gamma_\mu(k)$ and the μ-deformed factorial is defined recursively for all integers $k \geq 0$ by

$$\gamma_\mu(0) := 1 \quad \text{and } \gamma_\mu(k) := (k + 2\mu\chi_{\mathrm{o}}(k))\gamma_\mu(k-1) \quad \text{if } k \geq 1\,. \tag{3.1}$$

Finally, $\chi_{\mathrm{o}}(k) = 0$ for k even and $\chi_{\mathrm{o}}(k) = 1$ for k odd, that is, χ_{o} is nothing but the characteristic function of odd integers. Other conventions in force here are that z^* is the complex conjugate of $z \in \mathbb{C}$ and that all inner products are anti-linear in the first argument and linear in the second.

Notice that for the case $\mu = 0$ we have $\gamma_0(k) = k!$ and so $\exp_0(z) = e^z$. In general, the idea is that for $\mu = 0$ we recover familiar objects and relations, while for $\mu \neq 0$ we obtain a deformation of the standard theory. But it can happen that the deformed theory $\mu \neq 0$ has properties identical to those in the case $\mu = 0$. For example, we have that $\gamma_\mu(0) = 1 = 0!$ and $\exp_\mu(0) = 1 = e^0$. See Ref. 36 for more details about this point of view.

The results of the following lemma are immediate consequences of these definitions. The proofs can be found in Ref. 20.

Lemma 3.1. *The function* $\exp_\mu(z)$ *satisfies the following properties*:

(a) *For all* $\mu > -1/2$ *the* μ*-deformed exponential* $\exp_\mu(z)$ *is a holomorphic function whose domain is the entire complex plane* $\mathbb{C}$, *i.e. it is an entire function.*
(b) *For any* $\mu > -1/2$ *and all* $z \in \mathbb{C}$ *we have* $|\exp_\mu(z)| \leq \exp_\mu(|z|)$.
(c) *If* $\mu \geq 0$, *then we have* $|\exp_\mu(z)| \leq e^{|z|}$ *for every* $z \in \mathbb{C}$.
(d) *If* $-1/2 < \mu < 0$, *there is a* $C_\mu > 0$ *so that* $|\exp_\mu(z)| \leq C_\mu(1 + |z|^{|\mu|})e^{|z|}$ *for every* $z \in \mathbb{C}$.

Definition 3.1. For a measurable function $f = f_\mathrm{e} + f_\mathrm{o} : \mathbb{C} \to \mathbb{C}$ we now define an integral kernel transform, denoted Kf, that is associated to the reproducing kernel function K for $\mathcal{B}^2_\mu$ as follows:

$$Kf(w) := \int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z) K_\mathrm{e}(z,w) f_\mathrm{e}(z) + \int_{\mathbb{C}} d\nu_{\mathrm{o},\mu}(z) K_\mathrm{o}(z,w) f_\mathrm{o}(z)\,, \tag{3.2}$$

provided both integrals converge absolutely, this being a restriction on f as well as on $w \in \mathbb{C}$.

Here, of course, $K_\mathrm{e}(z,w)$ and $K_\mathrm{o}(z,w)$ refer to the even and odd parts of $K(z,w)$ with respect to the first variable z, and each is a kernel function for an integral kernel transform that enters in the definition (3.2) as well as in the subsequent definition (3.5). Notice that the first integral in (3.2), if it exists, gives an even function in w, while the second integral in (3.2), if it exists, gives an odd function in w. This property depends on the explicit form of $K(z,w)$.

If $f \in \mathcal{B}^2_\mu$, the right-hand side of definition (3.2) reduces to $\langle K(\cdot,w)^*, f\rangle_{\mathcal{B}^2_\mu} = f(w)$, that is, $Kf = f$ in this case. Of course, this remark is the motivation for this definition.

The kernel function $K(z,w)$ appears in Ref. 20, while all three kernel functions $K(z,w)$, $K_\mathrm{e}(z,w)$ and $K_\mathrm{o}(z,w)$ appear in Ref. 4. (Note that explicit formulas for these reproducing kernels are given in Example 4.17 in Ref. 4, and they appear to disagree with our formulas given below. But they are indeed equal to ours, as they must be.)

Notice that $K_\mathrm{e}(z,w) = \exp_{\mu,\mathrm{e}}(z^*w)$ and that $K_\mathrm{o}(z,w) = \exp_{\mu,\mathrm{o}}(z^*w)$, where $\exp_{\mu,\mathrm{e}}$ and $\exp_{\mu,\mathrm{o}}$ are the even and odd parts, respectively, of $\exp_\mu$. Since $\exp_{\mu,\mathrm{e}}(z^*w)$ (resp., $\exp_{\mu,\mathrm{o}}(z^*w)$) as a function of z is in $L^q(\nu_{\mathrm{e},\mu})$ (resp., $L^q(\nu_{\mathrm{o},\mu})$) for $1 \leq q < \infty$ and any fixed $w \in \mathbb{C}$, (which is a consequence of Lemma 3.1 and the previously cited asymptotic behavior of the MacDonald function near infinity), it follows by Hölder's inequality that $Kf(w)$ is well defined for every $w \in \mathbb{C}$ provided that $f_{\mathrm{e}} \in L^{p_1}(\mathbb{C}, \nu_{\mathrm{e},\mu})$ and $f_{\mathrm{o}} \in L^{p_2}(\mathbb{C}, \nu_{\mathrm{o},\mu})$ for some $1 < p_1 \le \infty$ and $1 < p_2 \le \infty$. One can use Morera's Theorem to show that the resulting function $w \mapsto Kf(w)$ is holomorphic for all $w \in \mathbb{C}$.

When $p = 2$ the spaces $\mathcal{B}^p_{\mathrm{e},\mu}$ and $\mathcal{B}^p_{\mathrm{o},\mu}$ introduced in Definition 1.2 become Hilbert spaces of holomorphic functions with reproducing kernel functions given by $K_{\mathrm{e}}(z,w) = \exp_{\mu,\mathrm{e}}(z^*w)$ for $\mathcal{B}^2_{\mathrm{e},\mu}$ and $K_{\mathrm{o}}(z,w) = \exp_{\mu,\mathrm{o}}(z^*w)$ for $\mathcal{B}^2_{\mathrm{o},\mu}$, where z, $w \in \mathbb{C}$. These kernels then have associated integral transforms, given by

$$K_{\mathrm{e}}f(w) := \int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z) K_{\mathrm{e}}(z,w) f(z) \tag{3.3}$$

and

$$K_{\mathrm{o}}f(w) := \int_{\mathbb{C}} d\nu_{\mathrm{o},\mu}(z) K_{\mathrm{o}}(z,w) f(z) \tag{3.4}$$

for measurable $f : \mathbb{C} \to \mathbb{C}$, provided the integrals converge absolutely. These two integral kernel transforms will be basic for our analysis.

Notice that we follow here the very common convention of using the same symbol to denote both a kernel function as well as its associated integral kernel transform. We have already done this before in equation (3.2).

We also consider $K_{\mathrm{e}} \oplus K_{\mathrm{o}}$, which is defined for $w \in \mathbb{C}$ as

$$(K_{\mathrm{e}} \oplus K_{\mathrm{o}})(f \oplus g)(w) := K_{\mathrm{e}}f(w) + K_{\mathrm{o}}g(w) \tag{3.5}$$

where f, $g : \mathbb{C} \to \mathbb{C}$ are measurable functions, provided that both integrals in (3.3) and (3.4) converge absolutely. Again, suitable integrability conditions on f and g guarantee that the integrals exist for all $w \in \mathbb{C}$ and, in that case, the resulting functions $K_{\mathrm{e}}f$ and $K_{\mathrm{o}}f$ are holomorphic in the entire complex plane. Moreover, note that $K_{\mathrm{e}} \oplus K_{\mathrm{o}} : L^2(\nu_{\mathrm{e},\mu}) \oplus L^2(\nu_{\mathrm{o},\mu}) \to \mathcal{B}^2_{\mathrm{e},\mu} \oplus \mathcal{B}^2_{\mathrm{o},\mu} = \mathcal{B}^2_{\mu}$ is the orthogonal projection in the Hilbert space of the domain onto the codomain, where the latter, $\mathcal{B}^2_{\mu}$, is included in the former, $L^2(\nu_{\mathrm{e},\mu}) \oplus L^2(\nu_{\mathrm{o},\mu})$, by the map $f = f_{\mathrm{e}} + f_{\mathrm{o}} \mapsto f_{\mathrm{e}} \oplus f_{\mathrm{o}}$. Notice that $L^2(\nu_{\mathrm{e},\mu}) \oplus L^2(\nu_{\mathrm{o},\mu})$ is an external direct sum of Hilbert spaces, while $\mathcal{B}^2_{\mathrm{e},\mu} \oplus \mathcal{B}^2_{\mathrm{o},\mu}$ is an internal direct sum of Hilbert spaces.

For all $w \in \mathbb{C}$ we have the identity

$$Kf(w) = (K_{\mathrm{e}} \oplus K_{\mathrm{o}})(f_{\mathrm{e}} \oplus f_{\mathrm{o}})(w) = K_{\mathrm{e}}(f_{\mathrm{e}})(w) + K_{\mathrm{o}}(f_{\mathrm{o}})(w)\,.$$

So the study of $K_{\mathrm{e}} \oplus K_{\mathrm{o}}$ and of K reduces to the study of K_{e} and K_{o}.

Let us note in passing that, while the transforms defined in (3.2) and (3.5) can be viewed as the sum of two integral transforms (each with respect to its own measure space), one can easily rewrite these as *one* integral transform with respect to the measure space $(\mathbb{C} \times \mathbb{Z}_2, \nu_\mu)$, where $\mathbb{Z}_2 = \{-1, +1\}$ is a multiplicative group, $\nu_\mu|_{\mathbb{C}\times\{+1\}} := \nu_{\mathrm{e},\mu}$ and $\nu_\mu|_{\mathbb{C}\times\{-1\}} := \nu_{\mathrm{o},\mu}$. The group $\mathbb{Z}_2$ can be identified with the Coxeter group (see Refs. 4 and 27) of this formalism.

Now a natural problem is to identify all quadruples p_1, q_1, p_2, q_2 of Lebesgue indices such that

$$K_{\mathrm{e}} \oplus K_{\mathrm{o}} : L^{p_1}(\nu_{\mathrm{e},\mu}) \oplus L^{p_2}(\nu_{\mathrm{o},\mu}) \to \mathcal{B}^{q_1}_{\mathrm{e},\mu} \oplus \mathcal{B}^{q_2}_{\mathrm{o},\mu} \tag{3.6}$$

is bounded, i.e. the operator norm with respect to the indicated domain and codomain is finite. And given that this operator is bounded, another problem is to ascertain if it is compact. For example, if $p_1 = q_1 = 2$ and $p_2 = q_2 = 2$, then $K_{\rm e} \oplus K_{\rm o}$ is bounded (since it is an orthogonal projection), but is not compact (since it is an orthogonal projection with infinite dimensional range).

For the purposes of the present exposition it is better to start with the more general problem of identifying those quadruples p_1, q_1, p_2, q_2 for which $K_{\rm e} \oplus K_{\rm o}$ is a bounded (or compact) transformation of $L^{p_1}(\nu_{{\rm e},\mu}) \oplus L^{p_2}(\nu_{{\rm o},\mu})$ to $\mathcal{B}^{q_1}_{{\rm e},\mu,a_1} \oplus \mathcal{B}^{q_2}_{{\rm o},\mu,a_2}$ for some reals a_1 and a_2. So we wish to study when

$$K_{\rm e} \oplus K_{\rm o} : L^{p_1}(\nu_{{\rm e},\mu}) \oplus L^{p_2}(\nu_{{\rm o},\mu}) \to \mathcal{B}^{q_1}_{{\rm e},\mu,a_1} \oplus \mathcal{B}^{q_2}_{{\rm o},\mu,a_2} \tag{3.7}$$

is bounded or compact.

Here we are using a weighted modification of the previously defined spaces. Specifically,

$$\mathcal{B}^{q}_{{\rm e},\mu,a} := \mathcal{H}(\mathbb{C}) \cap \{f : \mathbb{C} \to \mathbb{C} | f = f_{\rm e} \in L^q(\mathbb{C}, \nu_{{\rm e},\mu,a})\}\,,$$

$$\mathcal{B}^{q}_{{\rm o},\mu,a} := \mathcal{H}(\mathbb{C}) \cap \{f : \mathbb{C} \to \mathbb{C} | f = f_{\rm o} \in L^q(\mathbb{C}, \nu_{{\rm o},\mu,a})\}\,,$$

where $a \in \mathbb{R}$ and

$$d\nu_{{\rm e},\mu,a}(z) := e^{-a|z|^2} d\nu_{{\rm e},\mu}(z) \quad \text{and } d\nu_{{\rm o},\mu,a}(z) := e^{-a|z|^2} d\nu_{{\rm o},\mu}(z)\,. \tag{3.8}$$

Notice that here we allow a to be negative. When $a = 0$, we recover the spaces of Definition 1.2 for the case $\lambda = 1$. Strictly speaking, the notation for the measures defined in (3.8) conflicts with the notation of Definition 1.1, but we use it to avoid even more complicated notation. The point is that in the notation of the measures $d\nu_{{\rm e},\mu,\lambda}(z)$ and $d\nu_{{\rm o},\mu,\lambda}(z)$ in Definition 1.1 the variable $\lambda > 0$ could be interpreted as the variable $a \in \mathbb{R}$ in the measures in (3.8). However, the measures in Definition 1.1 are dilations of the measures $d\nu_{{\rm e},\mu,1}(z) \equiv d\nu_{{\rm e},\mu}(z)$ and $d\nu_{{\rm o},\mu,1}(z) \equiv d\nu_{{\rm o},\mu}(z)$ (as we noted earlier), while the measures in (3.8) are given by a simple weight function (depending on the parameter a) times the measures $d\nu_{{\rm e},\mu}(z)$ and $d\nu_{{\rm o},\mu}(z)$, which do not depend on a. It follows that the measures in (3.8) are not those of Definition 1.1 when $\mu \neq 0$. However, for $\mu = 0$ the measures in (3.8) are related to those of Definition 1.1 by $e^{-a|z|^2} d\nu_{{\rm e},0}(z) = \lambda^{-1} d\nu_{{\rm e},0,\lambda}(z)$, where $\lambda = 1 + a$ provided that $a \neq -1$. (In our applications we always have $a > -1$. See for example Theorem 3.1 below.)

Of course, this problem naturally splits into two problems, since the first (resp., second) summand on the left of (3.7) maps to the first (resp., second) summand on the right of (3.7). An answer is given in the following theorem.

Theorem 3.1. *Let $1 < p \leq \infty$ and $1 \leq q < \infty$. Then for any $a > p'q/4 - 1$, the integral kernel transform $K_{\rm e}$ (respectively, $K_{\rm o}$) is a compact (and, hence, bounded) operator from $L^p(\nu_{{\rm e},\mu})$ to $\mathcal{B}^q_{{\rm e},\mu,a}$ (respectively, from $L^p(\nu_{{\rm o},\mu})$ to $\mathcal{B}^q_{{\rm o},\mu,a}$).*

Consequently, for $1 < p_j \leq \infty$ and $1 \leq q_j < \infty$ and $a_j > p'_j q_j/4 - 1$ for $j = 1, 2$ we have that $K_{\rm e} \oplus K_{\rm o}$ is a compact (and hence, bounded) operator from $L^{p_1}(\nu_{{\rm e},\mu}) \oplus L^{p_2}(\nu_{{\rm o},\mu})$ to $\mathcal{B}^{q_1}_{{\rm e},\mu,a_1} \oplus \mathcal{B}^{q_2}_{{\rm o},\mu,a_2}$.

Here, p' is the usual index conjugate to p, namely, $p' = p/(p-1)$ for $1 < p < \infty$ and $\infty' = 1$ and $1' = \infty$.

Proof. The proof is given for the case of K_{e}, since the other case of K_{o} has a quite similar proof. (However, occasional parenthetical comments are given about the latter case.) The tool to prove this result is the *Hille–Tamarkin norm* (see Refs. 16 and 18). For the Lebesgue indices $1 < p \le \infty$ and $1 \le q < \infty$ and the kernel function K_{e} this norm is given by

$$|||K_{\mathrm{e}}|||_{p,q} := \left(\int_{\mathbb{C}} d\nu_{\mathrm{e},\mu,a}(w) \left(\int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z) |K_{\mathrm{e}}(z,w)|^{p'} \right)^{q/p'} \right)^{1/q} . \tag{3.9}$$

(For K_{o}, one has to use the measures $d\nu_{\mathrm{o},\mu,a}$ and $d\nu_{\mathrm{o},\mu}$.)

In the following we continue to use the same symbol K_{e} to represent the kernel function as well as the operator defined by that kernel function. The main property of the Hille–Tamarkin norm that will be used here is given next (see Refs. 16 and 18).

> If $|||K_{\mathrm{e}}|||_{p,q}$ as given in (3.9) is finite, then the corresponding integral kernel transform K_{e} is a compact operator and, hence, bounded from $L^p(\nu_{\mathrm{e},\mu})$ to $\mathcal{B}^q_{\mathrm{e},\mu,a}$. Moreover, the operator norm from $L^p(\nu_{\mathrm{e},\mu})$ to $\mathcal{B}^q_{\mathrm{e},\mu,a}$ is bounded above by the Hille–Tamarkin norm, namely, $\|K_{\mathrm{e}}\|_{p\to q} \le |||K_{\mathrm{e}}|||_{p,q}$.

We remark that the notation $|||K_{\mathrm{e}}|||_{p,q}$ has the same ambiguity as does $\|K_{\mathrm{e}}\|_{p\to q}$, namely that the relevant measures are omitted from the notation. But again context will clarify this.

So, the first step is to estimate the inner integral $\int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z)|K_{\mathrm{e}}(z,w)|^{p'}$ in Eq. (3.9) in order to determine its dependence on w. To do this, note that we have the following estimate, which follows from the definition of the even part of a function and from Lemma 3.1:

$$\begin{aligned} |K_{\mathrm{e}}(z,w)| &= \frac{1}{2} | \exp_\mu(z^* w) + \exp_\mu(-z^* | w) \\ &\le \exp_\mu(|z||w|) \le C_\mu (1 + |z|^{|\mu|} |w|^{|\mu|}) e^{|z||w|} . \end{aligned} \tag{3.10}$$

We can take C_μ as in part (d) of Lemma 3.1 for $-1/2 < \mu < 0$ and $C_\mu = 1$ for $\mu \ge 0$. (The same estimate holds for K_{o}). In the following estimates, the reader should not confuse the kernel function K_{e} with the MacDonald function $K_{\mu-1/2}$. Also, in agreement with our convention mentioned earlier, the symbol C in the following is a positive finite constant (i.e. independent of w, but not necessarily of μ or p) which can change with each occurrence.

Using the estimate (3.10), the definition of the measure $d\nu_{\mathrm{e},\mu}$, the asymptotics of the MacDonald function near $+\infty$ and a completion of the square, we have

$$\begin{aligned}\int_{|z|\geq M} d\nu_{\mathrm{e},\mu}(z)|K_{\mathrm{e}}(w,z)|^{p'} &\leq C\int_{|z|\geq M} d\nu_{\mathrm{e},\mu}(z)(1+|z|^{|\mu|}|w|^{|\mu|})^{p'}e^{p'|z||w|}\\ &= C\int_M^\infty dr K_{\mu-1/2}(r^2)r^{2\mu+2}(1+r^{|\mu|}|w|^{|\mu|})^{p'}e^{p'r|w|}\\ &\leq C\int_M^\infty dr e^{-r^2}r^{2\mu+1}(1+r^{|\mu|}|w|^{|\mu|})^{p'}e^{p'r|w|}\\ &= Ce^{p'^2|w|^2/4}\int_M^\infty dr e^{-(r-p'|w|/2)^2}r^{2\mu+1}(1+r^{|\mu|}|w|^{|\mu|})^{p'}.\end{aligned} \tag{3.11}$$

For our present purposes the particular value of $0 < M < \infty$ is not relevant. To estimate the integral in (3.11), we first note that for any $\alpha \geq 0$ we have the estimate

$$e^{-(x-\alpha)^2} \leq e^{\frac{1}{4}}e^{\alpha}e^{-x}$$

for all $x \geq 0$, which can be shown by calculus. Also for any $r > 0$ we have the elementary inequality

$$(1+\alpha)^r \leq C(1+\alpha^r) \tag{3.12}$$

for all $\alpha \geq 0$, where C depends only on r, and not on α. Applying these two inequalities to the integral in (3.11), we have that

$$\begin{aligned}&\int_M^\infty dr e^{-(r-p'|w|/2)^2}r^{2\mu+1}(1+r^{|\mu|}|w|^{|\mu|})^{p'}\\ &\quad\leq Ce^{p'|w|/2}\int_M^\infty dr e^{-r}r^{2\mu+1}(1+r^{p'|\mu|}|w|^{p'|\mu|}) \leq Ce^{p'|w|/2}(1+|w|^{p'|\mu|}).\end{aligned}$$

Substituting this into (3.11) we have that

$$\int_{|z|\geq M} d\nu_{\mathrm{e},\mu}(z)|K_{\mathrm{e}}(w,z)|^{p'} \leq Ce^{p'^2|w|^2/4}e^{p'|w|/2}(1+|w|^{p'|\mu|}).$$

Now we consider the case $|z| \leq M$, for which we see that

$$\begin{aligned}\int_{|z|\leq M} d\nu_{\mathrm{e},\mu}(z)|K_{\mathrm{e}}(w,z)|^{p'} &\leq C\int_{|z|\leq M} d\nu_{\mathrm{e},\mu}(z)(1+|z|^{|\mu|}|w|^{|\mu|})^{p'}e^{p'|z||w|}\\ &\leq C\int_{|z|\leq M} d\nu_{\mathrm{e},\mu}(z)(1+M^{|\mu|}|w|^{|\mu|})^{p'}e^{p'M|w|}\\ &\leq C(1+M^{|\mu|}|w|^{|\mu|})^{p'}e^{p'M|w|}\\ &\leq C(1+|w|^{p'|\mu|})e^{p'M|w|},\end{aligned}$$

where we first used the estimate (3.10), applied $|z| \leq M$ to the integrand, estimated the integral by a constant, and finally used (3.12) and then made an elementary estimate.

Putting all this together we have that

$$\int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z)|K_{\mathrm{e}}(w,z)|^{p'} \leq C((1+|w|^{p'|\mu|})e^{p'M|w|} + e^{p'^2|w|^2/4}e^{p'|w|/2}(1+|w|^{p'|\mu|}))\,. \qquad (3.13)$$

But now each of the terms on the right (3.13) can obviously be bounded by $C\exp(\beta p'^2|w|^2)$ for any $\beta > 1/4$ and all $w \in \mathbb{C}$, where now the constant C can depend on β (as well as on p, μ and M), but not on w. So, the final estimate on the inner integral in (3.9) is

$$\int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z)|K_{\mathrm{e}}(w,z)|^{p'} \leq Ce^{\beta p'^2|w|^2}$$

for *any* $\beta > 1/4$ and all $w \in \mathbb{C}$. Continuing with the computation of the Hille–Tamarkin norm of K_{e} in Eq. (3.9), we have to take the last expression to the power q/p' and then integrate with respect to the measure $d\nu_{\mathrm{e},\mu,a}(w)$. (Using $d\nu_{\mathrm{o},\mu,a}(w)$ for K_{o} gives the same results.) But this gives us the estimate

$$\begin{aligned} |||K_{\mathrm{e}}|||_{p,q}^q &= \int_{\mathbb{C}} d\nu_{\mathrm{e},\mu,a}(w)\left(\int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z)|K_{\mathrm{e}}(w,z)|^{p'}\right)^{q/p'} \\ &\leq C\int_{\mathbb{C}} d\nu_{\mathrm{e},\mu,a}(w)e^{\beta p'q|w|^2} = C\int_0^\infty dr K_{\mu-1/2}(r^2)r^{2\mu+2}e^{-ar^2}e^{\beta p'qr^2}\,. \end{aligned}$$

Now this last integral converges if and only if it converges near infinity. But there it has the upper bound

$$C\int_{M'}^\infty dr e^{-r^2}r^{2\mu+1}e^{-ar^2}e^{\beta p'qr^2}\,,$$

for some $M' > 0$, which converges if and only if $-1-a+\beta p'q < 0$. This condition in turn is equivalent to $a > \beta p'q-1$. However, we have by hypothesis that $a > p'q/4-1$, which implies that we can pick *some* $\beta > 1/4$ such that

$$a > \beta p'q - 1 > p'q/4 - 1\,.$$

Using this value of β in the above argument shows that $|||K_{\mathrm{e}}|||_{p,q} < \infty$. The remaining assertions of the theorem now follow directly. QED

Remark. The argument in this proof can be refined in the case $\mu > 0$ with the aim of getting an improved estimate for the Hille–Tamarkin norm and, hence, for the operator norm. Clearly, one can use part (c) of Lemma 3.1 (instead of part (d)) in this case. But we can use an even better estimate, that follows directly from (2.3.5) in Ref. 25. This says that for all $z \in \mathbb{C}$ and $\mu > 0$ we have that $|\exp_\mu(z)| \leq \exp_\mu(\mathrm{Re}(z))$. However, we are not trying to find optimal constants, nor do we believe it to be likely that the Hille–Tamarkin norm will produce them.

Corollary 3.1. *Let $1 < p \le \infty$ and $1 \le q < \infty$ be given with $p'q < 4$. Then the integral kernel transform K_{e} (respectively, K_{o}) is a compact (and hence bounded) operator from $L^p(\nu_{\mathrm{e},\mu})$ to $\mathcal{B}^q_{\mathrm{e},\mu}$ (respectively, from $L^p(\nu_{\mathrm{o},\mu})$ to $\mathcal{B}^q_{\mathrm{o},\mu}$).*

Proof. This is the special case $a = 0$ of the theorem. One only has to note that $\mathcal{B}^q_{\mathrm{e},\mu,a} = \mathcal{B}^q_{\mathrm{e},\mu}$ and that $\mathcal{B}^q_{\mathrm{o},\mu,a} = \mathcal{B}^q_{\mathrm{o},\mu}$ when $a = 0$. QED

Theorem 3.1 and its corollary generalize results proved in Ref. 33 for the case $\mu = 0$. Notice that the relations $a > p'q/4 - 1$ of the theorem and $p'q < 4$ of the corollary do not depend on the parameter μ, and so are identical with the relations already found in Ref. 33. However, the Hille–Tamarkin and operator norms most likely do depend on μ, though only an analysis which calculates good lower bounds for these norms (or the norms themselves) can settle this question. Here we have presented only upper bounds. Also, notice that for the case $\mu = 0$ it is proved in Ref. 33 that the integral kernel transform K is unbounded if $p'q > 4$. It is reasonable to conjecture that this also holds for the case $\mu \ne 0$.

4. The Main Results

To obtain the main results of this paper we will use an interpolation theorem due to Stein (see Ref. 37 or Theorem 3.6 in Ref. 5). This theorem is a generalization of the well-known interpolation theorem of Riesz–Thorin (see Ref. 38). The reason interpolation theory is used here is to obtain operator norm estimates that vary smoothly as the pair of Lebesgue indices varies. This will allow us to take a derivative with respect to the interpolation parameter t as the reader will shortly see. This derivative is central to the argument that we use.

Since the Stein theorem is not so widely known, we now quote it. But first, let us recall that a *simple function* is a measurable function f having a finite range $R \subset \mathbb{C}$ such that $f^{-1}(z)$ is a set of finite measure for every $z \in R$, $z \ne 0$.

Theorem 4.1. (Stein[37]) *Let (Ω_j, ν_j) for $j = 1, 2$ be σ-finite measure spaces. Let T be a linear transformation which takes simple complex-valued functions on Ω_1 to measurable complex-valued functions on Ω_2. Let p_0, p_1, q_0, q_1 be in $[1, \infty]$. Then, for $0 \le t \le 1$, define p_t and q_t by*

$$p_t^{-1} = (1-t)p_0^{-1} + tp_1^{-1} \quad \text{and } q_t^{-1} = (1-t)q_0^{-1} + tq_1^{-1} .$$

Suppose that u_0, $u_1 : \Omega_1 \to [0, \infty)$ and k_0, $k_1 : \Omega_2 \to [0, \infty)$ are measurable functions such that for all simple $f : \Omega_1 \to \mathbb{C}$ we have

$$\|(Tf)k_0\|_{L^{q_0}(\Omega_2,\nu_2)} \le A_0 \|fu_0\|_{L^{p_0}(\Omega_1,\nu_1)} ,$$

and

$$\|(Tf)k_1\|_{L^{q_1}(\Omega_2,\nu_2)} \le A_1 \|fu_1\|_{L^{p_1}(\Omega_1,\nu_1)}$$

for some finite constants $A_0 \geq 0$ and $A_1 \geq 0$. (Note that for some simple f the right side of these inequalities can be equal to $+\infty$.) For $0 \leq t \leq 1$, define functions $u_t := u_0^{1-t} u_1^t : \Omega_1 \to [0, \infty)$ and $k_t := k_0^{1-t} k_1^t : \Omega_2 \to [0, \infty)$. Then the transformation T can be extended uniquely to a linear transformation defined on the space of all $f : \Omega_1 \to \mathbb{C}$ that satisfy $\| f u_t \|_{L^{p_t}(\Omega_1, \nu_1)} < \infty$ in such a way that for all such f we have

$$\| (Tf) k_t \|_{L^{q_t}(\Omega_2, \nu_2)} \leq A_0^{1-t} A_1^t \| f u_t \|_{L^{p_t}(\Omega_1, \nu_1)} .$$

Now we will apply Stein's Theorem in the context of Theorem 3.1. The next result, including its proof using Stein's Theorem, follows the presentation in Ref. 33 for the case $\mu = 0$. Moreover, the next result and its proof are valid for K_{o} provided that we change the subscript "e" to "o" throughout.

Theorem 4.2. *Let $1 < p \leq \infty$, $1 \leq q < \infty$ and $a > p'q/4 - 1$. Then we have $\| K_{\mathrm{e}} \|_{p \to q} \leq |\!|\!| K_{\mathrm{e}} |\!|\!|_{p,q} < \infty$ and also that for all $0 \leq t \leq 1$, K_{e} is a bounded linear map from $L^{p_t}(\mathbb{C}, \nu_{\mathrm{e},\mu})$ to $L^{q_t}(\mathbb{C}, \nu^t_{\mathrm{e},\mu,a})$, where*

$$d\nu^t_{\mathrm{e},\mu,a}(z) := \exp\left(-\frac{tq_t}{q} a |z|^2 \right) d\nu_{\mathrm{e},\mu}(z) \tag{4.1}$$

for $p_t^{-1} = (1-t)2^{-1} + tp^{-1}$ and $q_t^{-1} = (1-t)2^{-1} + tq^{-1}$. Moreover, the operator norm from $L^{p_t}(\mathbb{C}, \nu_{\mathrm{e},\mu})$ to $L^{q_t}(\mathbb{C}, \nu^t_{\mathrm{e},\mu,a})$ satisfies

$$\| K_{\mathrm{e}} \|_{p_t \to q_t} \leq (\| K_{\mathrm{e}} \|_{p \to q})^t < \infty ,$$

or equivalently,

$$\| (K_{\mathrm{e}} f) k_t \|_{L^{q_t}(\nu_{\mathrm{e},\mu})} \leq A_{\mathrm{e}}^t \| f \|_{L^{p_t}(\nu_{\mathrm{e},\mu})} , \tag{4.2}$$

for all $f \in L^{p_t}(\mathbb{C}, \nu_{\mathrm{e},\mu})$, where $A_{\mathrm{e}} = A_{\mathrm{e}}(p, q, a, \mu) := \| K_{\mathrm{e}} \|_{p \to q} < \infty$ is the operator norm from $L^p(\mathbb{C}, \nu_{\mathrm{e},\mu})$ to $L^q(\mathbb{C}, \nu_{\mathrm{e},\mu,a})$ and where

$$k_t(z) = \exp(-at|z|^2/q) \tag{4.3}$$

for all $z \in \mathbb{C}$ and $0 \leq t \leq 1$.

Proof. In the context of Stein's theorem, we take $(\Omega_1, \nu_1) = (\mathbb{C}, \nu_{\mathrm{e},\mu})$ and $(\Omega_2, \nu_2) = (\mathbb{C}, \nu_{\mathrm{e},\mu,a})$. Also take $p_0 = q_0 = 2$, $p_1 = p$, $q_1 = q$ and $k_0(z) = u_0(z) = u_1(z) = 1$ for all $z \in \mathbb{C}$. Finally, put $k_1(z) = \exp(-a|z|^2/q)$. Note that

$$\| (K_{\mathrm{e}} f) k_t \|_{L^{q_t}(\nu_{\mathrm{e},\mu})} = \| K_{\mathrm{e}} f \|_{L^{q_t}(\nu^t_{\mathrm{e},\mu,a})} .$$

Here $k_t(z) = k_0^{1-t}(z) k_1^t(z)$ comes from the statement of Stein's Theorem. Using the definitions for $k_0(z)$ and $k_1(z)$ just given, we get that $k_t(z) = k_1^t(z) = \exp(-at|z|^2/q)$, which is just Eq. (4.3). Note that k_t also depends on a and q, although this is suppressed from the notation. For $t = 0$, we have

$$\| (K_{\mathrm{e}} f) k_0 \|_{L^2(\nu_{\mathrm{e},\mu})} \leq \| f u_0 \|_{L^2(\nu_{\mathrm{e},\mu})}$$

for all $f \in L^2(\nu_{\mathrm{e},\mu})$, since K_e is an orthogonal projection when considered as an operator with domain $L^2(\nu_{\mathrm{e},\mu})$. For $t = 1$, we can apply Theorem 3.1 because of our hypotheses on p, q and a and so we have that

$$\|(K_\mathrm{e} f)k_1\|_{L^q(\nu_{\mathrm{e},\mu})} \le A_\mathrm{e} \|f u_1\|_{L^p(\nu_{\mathrm{e},\mu})} .$$

(Recall that $A_\mathrm{e} = \|K_\mathrm{e}\|_{p\to q}$.) So, Stein's Theorem allow us to conclude that

$$\|(K_\mathrm{e} f)k_t\|_{L^{q_t}(\nu_{\mathrm{e},\mu})} \le 1^{1-t} A_\mathrm{e}^t \|f u_t\|_{L^{p_t}(\nu_{\mathrm{e},\mu})} = A_\mathrm{e}^t \|f\|_{L^{p_t}(\nu_{\mathrm{e},\mu})} ,$$

or, equivalently,

$$\|K_\mathrm{e} f\|_{L^{q_t}(\nu^t_{\mathrm{e},\mu,a})} \le A_\mathrm{e}^t \|f\|_{L^{p_t}(\nu_{\mathrm{e},\mu})}$$

for all $f \in L^{p_t}(\nu_{\mathrm{e},\mu})$. Here we have used $u_t = u_0^{1-t} u_1^t \equiv 1$. QED

In the next theorem and its discussion we will see three expressions arising quite naturally. These have been basically identified by us in Ref. 2 and are given next. We give these definitions for the measures introduced in Definition 1.1.

Definition 4.1. Let $\lambda > 0$ be a given value throughout of the dilation parameter. For every $g \in \mathcal{B}^2_{\mathrm{e},\mu,\lambda}$ define its *μ-deformed energy* by

$$E_{\mathrm{e},\mu,\lambda}(g) := \int_{\mathbb{C}} d\nu_{\mathrm{e},\mu,\lambda}(z) \lambda |z|^2 |g(z)|^2 . \tag{4.4}$$

Similarly, for every $h \in \mathcal{B}^2_{\mathrm{o},\mu,\lambda}$ define its *μ-deformed energy* by

$$E_{\mathrm{o},\mu,\lambda}(h) := \int_{\mathbb{C}} d\nu_{\mathrm{o},\mu,\lambda}(z) \lambda |z|^2 |h(z)|^2 .$$

Finally, for every $f \in \mathcal{B}^2_{\mu,\lambda}$ define its *μ-deformed energy* by

$$E_{\mu,\lambda}(f) := E_{\mathrm{e},\mu,\lambda}(f_\mathrm{e}) + E_{\mathrm{o},\mu,\lambda}(f_\mathrm{o}) , \tag{4.5}$$

where $f = f_\mathrm{e} + f_\mathrm{o}$ is the representation of f as the sum of its even and odd parts.

See Ref. 2 for the case $\lambda = 1$ of this definition. With the normalization we have chosen, we have that $E_{\mu,1}(f) = E_{\mu,\lambda}(T_\lambda f)$ for all $f \in \mathcal{B}^2_\mu$, where T_λ is defined in Eq. (1.5). Having made this comment, we now revert to the situation where $\lambda = 1$ and λ is suppressed from the notation.

We note that all of these μ-deformed energies are non-negative quantities, although they can be equal to $+\infty$. We have given in Ref. 2 explicit formulas for these μ-deformed energies in terms of the coefficients of the Taylor series (centered in the origin) of the function. Unfortunately, those formulas are rather unenlightening and do not show an immediate relation with the Dirichlet form energy, which we introduce in the next section. Note that in the case $\mu = 0$ these μ-deformed energies are related to the Dirichlet energy in the Segal–Bargmann space $\mathcal{B}^2$ via an identity of Bargmann that is proved in Ref. 3 [Eq. (3.17)], namely, for all $f \in \mathcal{B}^2$ we have that

$$\int_{\mathbb{C}} d\nu_{\mathrm{Gauss}}(z) |z|^2 |f(z)|^2 = \|f\|^2_{\mathcal{B}^2} + \langle f, Nf \rangle_{\mathcal{B}^2} ,$$

where $d\nu_{\text{Gauss}}(= d\nu_{\text{e},0} = d\nu_{\text{o},0})$ is a Gaussian measure [cf. Eq. (1.6)] and N is the number operator which is associated with the Dirichlet form. See Ref. 3 for more details. In the next section we will discuss a μ-deformed number operator N_μ acting in $\mathcal{B}^2_\mu$ and its associated Dirichlet form as well as its relation with the μ-deformed energies of Definition 4.1.

We now continue with the main results of this paper.

Theorem 4.3. *Suppose that* $1 < p \le \infty$, $1 \le q < \infty$ *and* $a > p'q/4 - 1$. *Then the energy-entropy inequality*

$$(p^{-1} - q^{-1}) S_{L^2(\nu_{\text{e},\mu})}(f) \le (\log A_{\text{e}}) \|f\|^2_{L^2(\nu_{\text{e},\mu})} + \frac{a}{q} E_{\text{e},\mu}(f) \tag{4.6}$$

holds, where $A_{\text{e}} = A_{\text{e}}(p, q, a, \mu)$ *is the operator norm of* K_{e} *acting from* $L^p(\nu_{\text{e},\mu})$ *to* $\mathcal{B}^q_{\text{e},\mu,a}$, *provided that one of the following hypotheses is satisfied*:

Hypothesis 1 : $f \in \mathcal{B}^{2+\epsilon}_{\text{e},\mu}$ *for some* $\epsilon > 0$.
Hypothesis 2 : $f \in \mathcal{B}^2_{\text{e},\mu}$, $1 < p \le 2$, $1 \le q \le 2$ *and* $S_{L^2(\nu_{\text{e},\mu})}(f) < \infty$.

Moreover, for the coefficients of the principle terms in (4.6), *namely the energy term* $E_{\text{e},\mu}(f)$ *and the entropy term* $S_{L^2(\nu_{\text{e},\mu})}(f)$, *we have the following cases*:

Case 1: $p^{-1} > q^{-1}$. *This implies that* $p'q/4 - 1 > 0$ *and so* $a > 0$. *Thus the coefficients of both* $S_{L^2(\nu_{\text{e},\mu})}(f)$ *and* $E_{\text{e},\mu}(f)$ *are positive and consequently* (4.6) *is a* direct log-Sobolev inequality *in* $\mathcal{B}^2_{\text{e},\mu}$ *with respect to the* μ*-deformed energy* $E_{\text{e},\mu}$.
Case 2: $p^{-1} \le q^{-1}$ *and* $p'q/4 - 1 \ge 0$. *Again* $a > 0$ *follows so that the coefficient of* $E_{\text{e},\mu}(f)$ *is positive, but now the coefficient of the entropy is non-positive. Since* $d\nu_{\text{e},\mu}(z)$ *is a probability measure,* $S_{L^2(\nu_{\text{e},\mu})}(f) \ge 0$ *and so* (4.6) *is trivially true.*
Case 3: $p'q/4 - 1 < 0$. *This implies that* $p^{-1} < q^{-1}$, *namely, that the coefficient of the entropy is negative. Moreover, we choose* a *such that* $0 > a > p'q/4 - 1$, *which means that the energy term also has a negative coefficient.* (*Of course, we can also choose* $a \ge 0$ *in this case. But then* (4.6) *becomes trivial.*) *In this case by putting the energy term on the left and the entropy term on the right,* (4.6) *gives us a* reverse log-Sobolev inequality *in* $\mathcal{B}^2_{\text{e},\mu}$ *with respect to the* μ*-deformed energy* $E_{\text{e},\mu}$.
Since $K_{\text{e}}1 = 1$ (*where* 1 *is the constant function, which is holomorphic and even*), *we have that* $A_{\text{e}} \ge 1$ *and so the coefficient of the norm term in* (4.6) *is non-negative. Here, we use that* $a < 0$ *implies* $\|1\|_{\mathcal{B}^q_{\text{e},\mu,a}} \ge 1$.

Remark. The corresponding inequality holds for odd functions. One merely has to change the subscript "e" to "o" throughout. We simply note the result here. So, with the same hypotheses as in Theorem 4.3, we have that

$$(p^{-1} - q^{-1}) S_{L^2(\nu_{\text{o},\mu})}(f) \le (\log A_{\text{o}}) \|f\|^2_{L^2(\nu_{\text{o},\mu})} + \frac{a}{q} E_{\text{o},\mu}(f)\,, \tag{4.7}$$

where $A_{\mathrm{o}} = A_{\mathrm{o}}(p, q, a, \mu)$ is the operator norm of K_{o} acting from $L^p(\nu_{\mathrm{o},\mu})$ to $\mathcal{B}^q_{\mathrm{o},\mu,a}$. However, the comments about the three cases need some modification. In Case 2 we remark that for $\mu > 0$ we can have negative entropies and (4.7) could be nontrivial for some choices of f. Also, the second paragraph of Case 3 does not apply.

The proof (in either the even or odd case) is essentially identical to that given in Ref. 34, except for some notational changes some of which are due to the absence of a Bargmann identity for $E_{\mathrm{e},\mu}(f)$ and some to a difference in the normalization of the measures. Since the proof in Ref. 34 is rather long and technical, it will not be repeated in detail here. However, we now present a sketch of the main ideas of the proof.

We start with the formula (4.2), which we repeat here:

$$\|(K_{\mathrm{e}}f)k_t\|_{L^{q_t}(\nu_{\mathrm{e},\mu})} \le A^t_{\mathrm{e}}\|f\|_{L^{p_t}(\nu_{\mathrm{e},\mu})}\,. \tag{4.2}$$

This is valid with A_{e} *finite* because of the assumptions imposed on p, q and a. We have proved this formula for $f \in L^{p_t}(\nu_{\mathrm{e},\mu})$ and hence for $f \in L^{p_t}(\nu_{\mathrm{e},\mu}) \cap \mathcal{B}^2_{\mathrm{e},\mu}$. But we will now use it for $f \in \mathcal{B}^2_{\mathrm{e},\mu}$. In the rest of this sketch, such technical details about domain issues and their ensuing complications will be omitted. The idea is that (4.2) is an *equality* when $t = 0$, since $p_0 = q_0 = 2$, $k_0 \equiv 1$, $A^0_{\mathrm{e}} = 1$ and $K_{\mathrm{e}}f = f$ because $f \in \mathcal{B}^2_{\mathrm{e},\mu}$. So, using a technique that dates back at least to Hirschman in Ref. 17 but that is also important in Ref. 12, we take f, p and q fixed and regard each side of (4.2) as a real-valued function of the real variable $t \in [0, 1]$. The fact that equality obtains at $t = 0$ implies that we can take the derivative (from the right) at $t = 0$ on both sides of (4.2) and thereby get another valid inequality, namely

$$\frac{d}{dt}\bigg|_{t=0^+}(\|fk_t\|_{L^{q_t}(\nu_{\mathrm{e},\mu})}) \le \frac{d}{dt}\bigg|_{t=0^+}(A^t_{\mathrm{e}}\|f\|_{L^{p_t}(\nu_{\mathrm{e},\mu})})\,,$$

which simplifies to

$$\frac{d}{dt}\bigg|_{t=0^+}(\|fk_t\|_{L^{q_t}(\nu_{\mathrm{e},\mu})}) \le (\log A_{\mathrm{e}})\|f\|_{L^2(\nu_{\mathrm{e},\mu})} + \frac{d}{dt}\bigg|_{t=0^+}(\|f\|_{L^{p_t}(\nu_{\mathrm{e},\mu})})\,. \tag{4.8}$$

Note that derivation in general is *not* an order-preserving operator, but that in this particular instance, the operator $d/dt|_{t=0^+}$ is. Using elementary calculus, a differentiation under the integral sign (which we do not justify here) and the definition in Eq. (1.7) of entropy, we find that

$$\frac{d}{dt}\bigg|_{t=0^+}(\|f\|_{L^{p_t}(\nu_{\mathrm{e},\mu})}) = \frac{(2^{-1} - p^{-1})S_{L^2(\nu_{\mathrm{e},\mu})}(f)}{\|f\|_{L^2(\nu_{\mathrm{e},\mu})}}\,, \tag{4.9}$$

provided that $\|f\|_{L^2(\nu_{\mathrm{e},\mu})} \neq 0$. But (4.6) is trivially true if $f \equiv 0$, so hereafter we exclude that case. Similarly, we find that

$$\left.\frac{d}{dt}\right|_{t=0^+}(\|fk_t\|_{L^{q_t}(\nu_{\mathrm{e},\mu})}) = \frac{(2^{-1}-q^{-1})S_{L^2(\nu_{\mathrm{e},\mu})}(f)}{\|f\|_{L^2(\nu_{\mathrm{e},\mu})}} + \frac{1}{\|f\|_{L^2(\nu_{\mathrm{e},\mu})}}\int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z)\left(-\frac{a}{q}|z|^2\right)|f(z)|^2\,, \quad (4.10)$$

using the following immediate consequence of Eq. (4.3):

$$\left.\frac{dk_t}{dt}\right|_{t=0^+} = -\frac{a}{q}|z|^2\,.$$

Substituting (4.9) and (4.10) into (4.8) and using the definition in Eq. (4.4) of the μ-deformed energy $E_{\mathrm{e},\mu}(f)$, we obtain

$$\frac{(2^{-1}-q^{-1})S_{L^2(\nu_{\mathrm{e},\mu})}(f)}{\|f\|_{L^2(\nu_{\mathrm{e},\mu})}} - \frac{1}{\|f\|_{L^2(\nu_{\mathrm{e},\mu})}}\frac{a}{q}E_{\mathrm{e},\mu}(f) \le (\log A_{\mathrm{e}})\|f\|_{L^2(\nu_{\mathrm{e},\mu})} + \frac{(2^{-1}-p^{-1})S_{L^2(\nu_{\mathrm{e},\mu})}(f)}{\|f\|_{L^2(\nu_{\mathrm{e},\mu})}}\,.$$

Then, multiplying by $\|f\|_{L^2(\nu_{\mathrm{e},\mu})}$, putting the two entropy terms on the left and the energy term on the right, we obtain (4.6). This concludes the sketch of the proof. QED

As noted before, the missing details of the proof, which amount to some ten pages, can be found in Ref. 34. It is in those details that Hypotheses 1 and 2 play a role in justifying the differentiation under the integral sign, as mentioned earlier.

Now we state a corollary of a part of the proof that we have not presented here. Again, refer to Ref. 34 for details. Notice that this is not a consequence of the conclusion of the previous theorem.

Corollary 4.1. *The following relations between entropies and μ-deformed energies hold*:

(a) *For all $f_{\mathrm{e}} \in \mathcal{B}^2_{\mathrm{e},\mu}$ we have that the Shannon entropy $S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}})$ is finite if and only if the μ-deformed energy $E_{\mathrm{e},\mu}(f_{\mathrm{e}})$ is finite.*
(b) *For all $f_{\mathrm{o}} \in \mathcal{B}^2_{\mathrm{o},\mu}$ we have that the Shannon entropy $S_{L^2(\nu_{\mathrm{o},\mu})}(f_{\mathrm{o}})$ is finite if and only if the μ-deformed energy $E_{\mathrm{o},\mu}(f_{\mathrm{o}})$ is finite.*
(c) *For all $f \in \mathcal{B}^2_{\mu}$ we have that the μ-deformed entropy $S_\mu(f)$ (see Definition 4.2 below) is finite if and only if the μ-deformed energy $E_\mu(f)$ is finite.*

By adding the inequalities (4.6) and (4.7) for the even and odd cases, we get the next result.

Corollary 4.2. *Let $1 < p_{\mathrm{e}} \le \infty$, $1 \le q_{\mathrm{e}} < \infty$, $a_{\mathrm{e}} > p'_{\mathrm{e}}q_{\mathrm{e}}/4 - 1$, $1 < p_{\mathrm{o}} \le \infty$, $1 \le q_{\mathrm{o}} < \infty$ and $a_{\mathrm{o}} > p'_{\mathrm{o}}q_{\mathrm{o}}/4 - 1$. Then we have the energy-entropy inequality*

$$(p_{\mathrm{e}}^{-1}-q_{\mathrm{e}}^{-1})S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) + (p_{\mathrm{o}}^{-1}-q_{\mathrm{o}}^{-1})S_{L^2(\nu_{\mathrm{o},\mu})}(f_{\mathrm{o}}) \le (\log A_{\mathrm{e}})\|f_{\mathrm{e}}\|^2_{L^2(\nu_{\mathrm{e},\mu})} + (\log A_{\mathrm{o}})\|f_{\mathrm{o}}\|^2_{L^2(\nu_{\mathrm{o},\mu})} + \frac{a_{\mathrm{e}}}{q_{\mathrm{e}}}E_{\mathrm{e},\mu}(f_{\mathrm{e}}) + \frac{a_{\mathrm{o}}}{q_{\mathrm{o}}}E_{\mathrm{o},\mu}(f_{\mathrm{o}})\,,$$

where $f = f_{\rm e} + f_{\rm o}$ *is the representation of* f *as the sum of its even and odd parts, provided that* $f_{\rm e}$ (*resp.*, $f_{\rm o}$) *satisfies one of the two Hypotheses of Theorem* 4.3 (*resp.*, *one of the two Hypotheses of Theorem* 4.3 *with "o" instead of "e"*).

Remark. We now will make a comparison of the present results with our previous results in Ref. 2. Note that Theorem 4.3, Case 3, gives

$$E_{{\rm e},\mu}(f_{\rm e}) \le \frac{q}{a}(p^{-1} - q^{-1}) S_{L^2(\nu_{{\rm e},\mu})}(f_{\rm e}) + \sigma_{\rm e}(p,q,a,\mu)\|f_{\rm e}\|^2_{L^2(\nu_{{\rm e},\mu})}$$

for some constant $\sigma_{\rm e}(p,q,a,\mu)$. It is shown by the second author in Ref. 34 that the coefficient of the entropy term can achieve any number $c > 1$. So we have:

Theorem 4.4. (Reverse log-Sobolev inequalities in $\mathcal{B}^2_{{\rm e},\mu}$ and $\mathcal{B}^2_{{\rm o},\mu}$ for the μ-deformed energy) *For every* $f_{\rm e} \in \mathcal{B}^2_{{\rm e},\mu}$ *we have that*

$$E_{{\rm e},\mu}(f_{\rm e}) \le c S_{L^2(\nu_{{\rm e},\mu})}(f_{\rm e}) + \tau_{\rm e}(c,\mu)\|f_{\rm e}\|^2_{L^2(\nu_{{\rm e},\mu})} \tag{4.11}$$

for any $c > 1$, *where* $\tau_{\rm e}(c,\mu)$ *is some finite constant.*

For every $f_{\rm o} \in \mathcal{B}^2_{{\rm o},\mu}$ *we have that*

$$E_{{\rm o},\mu}(f_{\rm o}) \le c S_{L^2(\nu_{{\rm o},\mu})}(f_{\rm o}) + \tau_{\rm o}(c,\mu)\|f_{\rm o}\|^2_{L^2(\nu_{{\rm o},\mu})} \tag{4.12}$$

for any $c > 1$, *where* $\tau_{\rm o}(c,\mu)$ *is some finite constant.*

The inequality (4.11) (resp., (4.12)) holds for *all* elements in $\mathcal{B}^2_{{\rm e},\mu}$ (resp., $\mathcal{B}^2_{{\rm o},\mu}$) due to an argument based on Corollary 4.1. Again, see Ref. 34 for more details. Similar reasoning justifies the subsequent results which, at first glance, appear to hold only in a certain dense subspace of the relevant Hilbert space, but actually hold in all of that Hilbert space.

The inequality (4.11) should be compared with Theorem 5.1 in Ref. 2, which says in our notation that

$$E_{{\rm e},\mu}(f_{\rm e}) \le c S_{L^2(\nu_{{\rm e},\mu})}(f_{\rm e}) + \kappa_{\rm e}(c,\mu)\|f_{\rm e}\|^2_{L^2(\nu_{{\rm e},\mu})}\,.$$

We have shown in Ref. 2 that for each $c > 1$ we can take

$$\kappa_{\rm e}(c,\mu) = c\log\int_{\mathbb{C}} d\nu_{{\rm e},\mu}(z) e^{|z|^2/c} < \infty\,.$$

So we have proved the same type of reverse log-Sobolev inequality in $\mathcal{B}^2_{{\rm e},\mu}$ though with a possibly different coefficient for the norm term. Similarly, our result (4.12) in the odd case corresponds to Theorem 5.2 in Ref. 2 with the same caveats. The method of Ref. 2 is based on the Young inequality and is due to Gross (see Refs. 10 and 35). One advantage of the results in Ref. 2 is that formulas are produced for the coefficients of the norm terms. Our analysis here is incomplete in that regard. It remains an open problem to identify the optimal constants of the norm terms. They may even be equal to zero as far as we currently know.

Finally, Corollary 4.2 in the particular case that $p_{\mathrm{e}} = p_{\mathrm{o}}$, $q_{\mathrm{e}} = q_{\mathrm{o}}$ and $a_{\mathrm{e}} = a_{\mathrm{o}}$ with $p'_{\mathrm{e}}q_{\mathrm{e}}/4 - 1 < a_{\mathrm{e}} < 0$ reduces to

$$E_\mu(f) \le c(S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) + S_{L^2(\nu_{\mathrm{o},\mu})}(f_{\mathrm{o}})) + \tau_{\mathrm{e}}(c,\mu)\|f_{\mathrm{e}}\|^2_{L^2(\nu_{\mathrm{e},\mu})} + \tau_{\mathrm{o}}(c,\mu)\|f_{\mathrm{o}}\|^2_{L^2(\nu_{\mathrm{o},\mu})}\,,$$

for any $c > 1$, using the definition of $E_\mu(f)$ in Eq. (4.5). This is the first inequality in Theorem 5.3 in Ref. 2, modulo the coefficient of the norm term. By taking $\tau(c,\mu) := \max(\tau_{\mathrm{e}}(c,\mu), \tau_{\mathrm{o}}(c,\mu))$, we get the next result.

Theorem 4.5. (Reverse log-Sobolev inequality in $\mathcal{B}^2_\mu$ for μ-deformed energy) *For every $f = f_{\mathrm{e}} + f_{\mathrm{o}} \in \mathcal{B}^2_\mu$ we have that*

$$E_\mu(f) \le c\{S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) + S_{L^2(\nu_{\mathrm{o},\mu})}(f_{\mathrm{o}})\} + \tau(c,\mu)\|f\|^2_{\mathcal{B}^2_\mu} \tag{4.13}$$

for any $c > 1$, where $\tau(c,\mu)$ is a finite constant.

This is the second inequality in Theorem 5.3 of Ref. 2, again modulo the coefficient of the norm term. Note that this does not appear to be a reverse log-Sobolev inequality in the sense of Definition 1.4 given that the expression in brackets on the right of (4.13) may not be immediately seen to be a Shannon entropy. In fact, it is *not* a Shannon entropy of $f \in \mathcal{B}^2_\mu$, since $\mathcal{B}^2_\mu$ is not defined as a subspace of an L^2 space. And we stated just this in Ref. 2, but it turns out that there is another way of viewing this. Note that the isometry $f \to (f_{\mathrm{e}}, f_{\mathrm{o}})$ maps

$$\mathcal{B}^2_\mu \to L^2(\mathbb{C}, \nu_{\mathrm{e},\mu}) \oplus L^2(\mathbb{C}, \nu_{\mathrm{o},\mu}) \cong L^2(\mathbb{C}\times\mathbb{Z}_2, \nu_\mu)$$

as we remarked in Sec. 3 and so this canonically identifies $\mathcal{B}^2_\mu$ with a closed subspace of $L^2(\mathbb{C}\times\mathbb{Z}_2, \nu_\mu)$, which is an L^2 space. We use this fact in the next definition.

Definition 4.2. (see Ref. 2) For $f = f_{\mathrm{e}} + f_{\mathrm{o}} \in \mathcal{B}^2_\mu$ we define its *μ-deformed entropy* by

$$S_\mu(f) := S_{L^2(\mathbb{C}\times\mathbb{Z}_2,\nu_\mu)}(f_{\mathrm{e}}, f_{\mathrm{o}})\,.$$

Then we immediately calculate $S_\mu(f) = S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) + S_{L^2(\nu_{\mathrm{o},\mu})}(f_{\mathrm{o}})$, which agrees with the definition in Ref. 2. Now this allows us to write (4.13) as follows:

$$E_\mu(f) \le cS_\mu(f) + \tau(c,\mu)\|f\|^2_{\mathcal{B}^2_\mu}\,.$$

In summary, we have another method of proving the reverse log-Sobolev inequalities in Ref. 2. However, the coefficients of the norm terms that we obtain here are most likely different (they *are* different in the case $\mu = 0$. See Ref. 35).

An important point is that the reproducing kernel method also produces direct log-Sobolev inequalities in $\mathcal{B}^2_{\mathrm{e},\mu}$ and in $\mathcal{B}^2_{\mathrm{o},\mu}$, and these are new results. So, we have the next result, which is a restatement of Case 1 of Theorem 4.3.

Theorem 4.6. (Log-Sobolev inequalities in $\mathcal{B}^2_{\mathrm{e},\mu}$ and $\mathcal{B}^2_{\mathrm{o},\mu}$ for the μ-deformed energy) *For all $f_{\mathrm{e}} \in \mathcal{B}^2_{\mathrm{e},\mu}$ we have*

$$a_{\mathrm{e}} S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) \le b_{\mathrm{e}} E_{\mathrm{e},\mu}(f_{\mathrm{e}}) + c_{\mathrm{e}} \|f_{\mathrm{e}}\|^2_{L^2(\nu_{\mathrm{e},\mu})},$$

where $a_{\mathrm{e}} > 0$, $b_{\mathrm{e}} > 0$ *and* $c_{\mathrm{e}} \ge 0$ *are finite constants.*

For all $f_{\mathrm{o}} \in \mathcal{B}^2_{\mathrm{o},\mu}$ *we have*

$$a_{\mathrm{o}} S_{L^2(\nu_{\mathrm{o},\mu})}(f_{\mathrm{o}}) \le b_{\mathrm{o}} E_{\mathrm{o},\mu}(f_{\mathrm{o}}) + c_{\mathrm{o}} \|f_{\mathrm{o}}\|^2_{L^2(\nu_{\mathrm{o},\mu})},$$

where $a_{\mathrm{o}} > 0$, $b_{\mathrm{o}} > 0$ *and* $c_{\mathrm{o}} \ge 0$ *are finite constants.*

Obviously, one can divide both sides of the previous inequalities by the coefficient of the entropy term without changing the sense of the inequality. Then one would try to find the optimal constant for the norm term, given a fixed value for the coefficient of the energy term.

Next by summing these two direct log-Sobolev inequalities, we obtain an energy-entropy inequality in $\mathcal{B}^2_\mu$ with two entropy terms of the form:

$$\begin{aligned} a_{\mathrm{e}} S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) + a_{\mathrm{o}} S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{o}}) &\le b_{\mathrm{e}} E_{\mathrm{e},\mu}(f_{\mathrm{e}}) + b_{\mathrm{o}} E_{\mathrm{o},\mu}(f_{\mathrm{o}}) \\ &\quad + c_{\mathrm{e}} \|f_{\mathrm{e}}\|^2_{L^2(\nu_{\mathrm{e},\mu})} + c_{\mathrm{o}} \|f_{\mathrm{o}}\|^2_{L^2(\nu_{\mathrm{o},\mu})}, \end{aligned}$$

where $f = f_{\mathrm{e}} + f_{\mathrm{o}} \in \mathcal{B}^2_\mu$. By taking $a := \min(a_{\mathrm{e}}, a_{\mathrm{o}})$, $b := \max(b_{\mathrm{e}}, b_{\mathrm{o}})$ and $c := \max(c_{\mathrm{e}}, c_{\mathrm{o}})$, we get for all $f = f_{\mathrm{e}} + f_{\mathrm{o}} \in \mathcal{B}^2_\mu$ that

$$a\{S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) + S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{o}})\} \le b E_\mu(f) + c\|f\|^2_{\mathcal{B}^2_\mu}.$$

We can apply Definition 4.2 to the term in brackets on the left and obtain the next result.

Theorem 4.7. (Log-Sobolev inequality for $\mathcal{B}^2_\mu$ for the μ-deformed energy) *For all* $f \in \mathcal{B}^2_\mu$ *we have that*

$$a S_\mu(f) \le b E_\mu(f) + c\|f\|^2_{\mathcal{B}^2_\mu},$$

where $a > 0$, $b > 0$ *and* $c \ge 0$ *are finite constants.*

As a closing comment to this section, we note that some other rather strange looking inequalities can be obtained from these results. For example, we can add a direct log-Sobolev inequality for $\mathcal{B}^2_{\mathrm{e},\mu}$ with a reverse log-Sobolev inequality for $\mathcal{B}^2_{\mathrm{e},\mu}$. (Similarly, we can do this for $\mathcal{B}^2_{\mathrm{o},\mu}$.) We can also add a direct log-Sobolev inequality for $\mathcal{B}^2_{\mathrm{e},\mu}$ with a reverse log-Sobolev inequality for $\mathcal{B}^2_{\mathrm{o},\mu}$ and, vice versa, a direct log-Sobolev inequality for $\mathcal{B}^2_{\mathrm{o},\mu}$ with a reverse log-Sobolev inequality for $\mathcal{B}^2_{\mathrm{e},\mu}$. Of course, none of these inequalities is more fundamental than their antecedents, and they seem to be mere curiosities as far as we can tell.

5. Dirichlet and μ-Deformed Energies

The μ-deformed energies introduced by us in Ref. 2 can be related to a Dirichlet form energy in $\mathcal{B}^2_\mu$. So we proceed to a discussion that will lead us to a definition of this latter concept.

We first note that one can introduce creation and annihilation operators, A_μ^* and A_μ respectively, which act in $\mathcal{B}_\mu^2$. In terms of the standard orthonormal basis $\{\Psi_n^\mu\}_{n\geq 0}$ of $\mathcal{B}_\mu^2$, where $\Psi_n^\mu(z) = z^n/(\gamma_\mu(n))^{1/2}$ (see Ref. 20), the definitions are:

$$A_\mu \Psi_n^\mu := \left(\frac{\gamma_\mu(n)}{\gamma_\mu(n-1)}\right)^{1/2} \Psi_{n-1}^\mu\,, \tag{5.1}$$

$$A_\mu^* \Psi_n^\mu := \left(\frac{\gamma_\mu(n+1)}{\gamma_\mu(n)}\right)^{1/2} \Psi_{n+1}^\mu \tag{5.2}$$

for every integer $n \geq 0$, where $\Psi_{-1}^\mu \equiv 0$. Then, one can extend the definitions (5.1) and (5.2) linearly to the dense subspace $\mathcal{D}_\mu^2$ of $\mathcal{B}_\mu^2$, where $\mathcal{D}_\mu^2$ is defined to be the set of all finite linear combinations of the Ψ_n^μ. While we have given these definitions explicitly in Ref. 2, one can find them discussed in a quite general situation in Sec 5 of Rosenblum's article[25] and, in a form isomorphic to that given here, in formulas (3.7.1) and (3.7.2) of Ref. 25. Moreover, it can be easily checked that

$$A_\mu f(z) = D_\mu f(z) := f'(z) + \frac{\mu}{z}(f(z) - f(-z))\,, \tag{5.3}$$

$$A_\mu^* f(z) = (M_\mu f)(z) := z f(z)\,, \tag{5.4}$$

for all $f \in \mathcal{D}_\mu^2$ and all $z \in \mathbb{C}$. Here $f'(z)$ is the complex derivative of $f(z)$. (We thank C. Pita for bringing formula (5.3) to our attention.) Of course, the formulas (5.3) and (5.4) can be used to define D_μ and M_μ, and hence A_μ and A_μ^* as well, on much larger space than $\mathcal{D}_\mu^2$. For example, we will use these formulas for definitions on $\mathcal{B}_\mu^2$ with the warning that the range will not then be a subspace of $\mathcal{B}_\mu^2$. We also use these formulas for definitions on $\mathcal{H}(\mathbb{C})$, the space of all holomorphic functions on $\mathbb{C}$, which is a domain invariant under the actions of D_μ and M_μ. (Note that the singularity at $z = 0$ in the second term of (5.3) is removable since f is holomorphic.) The operators D_μ and M_μ already appear on p. 373 of Ref. 25. Moreover, D_μ is well known to be a special case of a Dunkl operator. (See Ref. 27 and references therein.) From Eqs. (5.3) and (5.4) one sees immediately that

$$[A_\mu, A_\mu^*] = I + 2\mu J \tag{5.5}$$

on $\mathcal{H}(\mathbb{C})$. Of course, $[A_\mu, A_\mu^*] = A_\mu A_\mu^* - A_\mu^* A_\mu$ is the usual commutator of the two operators A_μ and A_μ^*, I is the identity operator, and J is the parity operator as introduced earlier. The commutation relation (5.5), which differs from the canonical commutation relation in the second term on the right, was essentially introduced by Wigner in Ref. 39 in order to answer negatively the question whether the standard quantum mechanical equations of motion determine the canonical commutation relations. Actually, Wigner presented a commutation relation for μ-deformed position and momentum operators (Q_μ and P_μ) that is equivalent to (5.5). The paper[39] by Wigner is the starting point of all further research concerning operators like Q_μ, P_μ, A_μ and A_μ^* and the spaces on which they act.

Up to this point in the discussion, A_μ and A_μ^* are two operators, each with its own definition. More than anything else, the notation indicates a wish that A_μ and A_μ^* should be adjoints of each other. But to define adjoints, one needs an inner product, and such a structure is not available in $\mathcal{H}(\mathbb{C})$. However, we can realize A_μ and A_μ^* as densely defined, closed unbounded operators in the Hilbert space $\mathcal{B}_\mu^2$. Then we do have the adjointness relation

$$\langle A_\mu^* f, g\rangle_{\mathcal{B}_\mu^2} = \langle f, A_\mu g\rangle_{\mathcal{B}_\mu^2}$$

for all f in the domain of A_μ^* and for all g in the domain of A_μ. As discussed further in Ref. 36, this relation can be taken as the motivation for the definition of the inner product for $\mathcal{B}_\mu^2$.

The *μ-deformed number operator* (see Ref. 2) is defined by

$$N_\mu := A_\mu^* A_\mu = M_\mu D_\mu\,,$$

and its associated quadratic form is then

$$\langle f, N_\mu f\rangle_{\mathcal{B}_\mu^2} = \langle f, A_\mu^* A_\mu f\rangle_{\mathcal{B}_\mu^2} = \langle A_\mu f, A_\mu f\rangle_{\mathcal{B}_\mu^2} = \|D_\mu f\|^2_{\mathcal{B}_\mu^2}\,. \tag{5.6}$$

This last expression justifies our calling this a *Dirichlet form*.

While the left of (5.6) has a natural domain given by the domain of N_μ, the right has a natural domain given by the domain of D_μ, which is strictly larger. Specifically we have

$$\begin{aligned}\mathrm{Domain}(N_\mu) &= \{f \in \mathcal{B}_\mu^2 \colon N_\mu f \in \mathcal{B}_\mu^2\}\,,\\ \mathrm{Domain}(D_\mu) &= \{f \in \mathcal{B}_\mu^2 \colon D_\mu f \in \mathcal{B}_\mu^2\}\,.\end{aligned}$$

Definition 5.1. The *Dirichlet form energy* (or the *Dirichlet energy*) is defined as $\|D_\mu f\|^2_{\mathcal{B}_\mu^2}$ for all f in $\mathrm{Domain}(D_\mu)$ and as $+\infty$ otherwise.

We avoid the standard convention of writing $\langle f, N_\mu f\rangle_{\mathcal{B}_\mu^2}$ for the Dirichlet energy. In fact, the operator N_μ does not enter the discussion here in any essential way, and we will not make any further explicit reference to it.

Note that we can use the commutation relation (5.5) to obtain, at least formally,

$$\begin{aligned}\|D_\mu f\|^2_{\mathcal{B}_\mu^2} &= \langle A_\mu f, A_\mu f\rangle_{\mathcal{B}_\mu^2} = \langle f, A_\mu^* A_\mu f\rangle_{\mathcal{B}_\mu^2}\\ &= \langle f, (A_\mu A_\mu^* - I - 2\mu J)f\rangle_{\mathcal{B}_\mu^2}\\ &= \|A_\mu^* f\|^2_{\mathcal{B}_\mu^2} - \|f\|^2_{\mathcal{B}_\mu^2} - 2\mu\langle f, Jf\rangle_{\mathcal{B}_\mu^2}\,.\end{aligned}$$

To make this rigorous, we will use the next result, whose proof is elementary. (See Ref. 3 for a proof in the case $\mu = 0$.)

Proposition 5.1. *Suppose* $g(z) = \sum_{k=0}^\infty b_k z^k$ *for* $b_k \in \mathbb{C}$ *is an entire function, that is, it is holomorphic for all* $z \in \mathbb{C}$. *Then,*

$$\|g\|^2_{\mathcal{B}_\mu^2} = \sum_{k=0}^\infty |b_k|^2 \gamma_\mu(k)\,, \tag{5.7}$$

where both sides are defined to be elements in $[0, \infty]$. *In particular,* $g \in \mathcal{B}^2_\mu$ *if and only if the series on the right of* (5.7) *is convergent.*

We now prove the result which we derived formally above.

Proposition 5.2. *For all* $f \in \mathcal{B}^2_\mu$ *we have that*

$$\|D_\mu f\|^2_{\mathcal{B}^2_\mu} = A^*_\mu f\|^2_{\mathcal{B}^2_\mu}\| - \|f\|^2_{\mathcal{B}^2_\mu} - 2\mu\langle f, Jf\|\rangle_{\mathcal{B}^2_\mu} . \tag{5.8}$$

In particular, $\|D_\mu f\|_{\mathcal{B}^2_\mu} < \infty$ *if and only if* $\|A^*_\mu f\|_{\mathcal{B}^2_\mu} < \infty$.

Proof. First we write $f(z) = \sum_{k=0}^\infty a_k z^k$, and we then calculate that

$$D_\mu f(z) = \sum_{k=0}^\infty a_k(k + 2\mu\chi_{\mathrm{o}}(k))z^{k-1} ,$$

$$A^*_\mu f(z) = \sum_{k=0}^\infty a_k z^{k+1} ,$$

$$Jf(z) = \sum_{k=0}^\infty (-1)^k a_k z^k ,$$

where χ_{o} is the characteristic function of the odd integers. It then follows that

$$\|D_\mu f\|^2_{\mathcal{B}^2_\mu} = \sum_{k=0}^\infty |a_k|^2(k + 2\mu\chi_{\mathrm{o}}(k))^2\gamma_\mu(k-1) ,$$

$$\|A^*_\mu f\|^2_{\mathcal{B}^2_\mu} = \sum_{k=0}^\infty |a_k|^2\gamma_\mu(k+1) ,$$

$$\|f\|^2_{\mathcal{B}^2_\mu} = \sum_{k=0}^\infty |a_k|^2\gamma_\mu(k) ,$$

$$\langle f, Jf\rangle_{\mathcal{B}^2_\mu} = \sum_{k=0}^\infty (-1)^k|a_k|^2\gamma_\mu(k) .$$

Here we use the convention that $\gamma_\mu(-1) = 0$. So (5.8) is a direct consequence of

$$(k + 2\mu\chi_{\mathrm{o}}(k))^2\gamma_\mu(k-1) = \gamma_\mu(k+1) - \gamma_\mu(k) - (2\mu)(-1)^k\gamma_\mu(k)$$

for all integers $k \geq 0$, which in turn follows from the definition (3.1) of the μ-deformed factorial γ_μ. Note that we have proved (5.8) for *all* $f \in \mathcal{B}^2_\mu$ in the sense that one side is finite if and only if the other side is finite. Since the last two terms on the right of (5.8) are finite for all $f \in \mathcal{B}^2_\mu$, the last assertion of the theorem follows directly. QED

The previous two propositions also appear in Ref. 31.

Notice that

$$\|A_\mu^* f\|^2_{\mathcal{B}^2_\mu} = \|M_\mu f\|^2_{\mathcal{B}^2_\mu} = \int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z)|z|^2|f_{\mathrm{o}}(z)|^2 + \int_{\mathbb{C}} d\nu_{\mathrm{o},\mu}(z)|z|^2|f_{\mathrm{e}}(z)|^2 , \tag{5.9}$$

since $(zf(z))_{\mathrm{e}} = zf_{\mathrm{o}}(z)$ and $(zf(z))_{\mathrm{o}} = zf_{\mathrm{e}}(z)$. While the last two integrals in (5.9) are reminiscent of the μ-deformed energies, $E_{\mathrm{e},\mu}(f_{\mathrm{e}})$ and $E_{\mathrm{o},\mu}(f_{\mathrm{o}})$, they are in fact *new* quantities. One way to think of this is that the integrals in (5.9) are "mixed" in terms of parity in the sense that the expression involving f_{o} in the first integral is integrated with respect to $d\nu_{\mathrm{e},\mu}$ and, vice versa, the expression involving f_{e} in the second integral is integrated with respect to $d\nu_{\mathrm{o},\mu}$. However, in $E_{\mathrm{e},\mu}(f_{\mathrm{e}})$ an even function f_{e} is integrated with respect to $d\nu_{\mathrm{e},\mu}$, and in $E_{\mathrm{o},\mu}(f_{\mathrm{o}})$ an odd function f_{o} is integrated with respect to $d\nu_{\mathrm{o},\mu}$.

The question now is how to relate the μ-deformed energies to these new quantities on the right of (5.9), and hence to the Dirichlet energy. First consider the case $\mu > 0$. The inequality $\nu_{\mathrm{e},\mu}(z) < \nu_{\mathrm{o},\mu}(z)$ of densities for $0 \neq z \in \mathbb{C}$ given in (2.1) allows us to write for $0 \neq f_{\mathrm{e}} \in \mathcal{B}^2_{\mathrm{e},\mu}$ that

$$E_{\mathrm{e},\mu}(f_{\mathrm{e}}) = \int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z)|z|^2|f_{\mathrm{e}}(z)|^2 < \int_{\mathbb{C}} d\nu_{\mathrm{o},\mu}(z)|z|^2|f_{\mathrm{e}}(z)|^2 . \tag{5.10}$$

Similarly, for $0 \neq f_{\mathrm{o}} \in \mathcal{B}^2_{\mathrm{o},\mu}$ we have that

$$\int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z)|z|^2|f_{\mathrm{o}}(z)|^2 < \int_{\mathbb{C}} d\nu_{\mathrm{o},\mu}(z)|z|^2|f_{\mathrm{o}}(z)|^2 = E_{\mathrm{o},\mu}(f_{\mathrm{o}}) . \tag{5.11}$$

However, for $\mu > 0$, we do *not* have an inequality $\nu_{\mathrm{o},\mu}(z) \leq C\nu_{\mathrm{e},\mu}(z)$ as we can see from the asymptotic behavior near zero of each side. Nonetheless, we claim that reverse inequalities corresponding to (5.10) and (5.11) can be proved. The complete result for all the possible cases for μ is as follows.

Theorem 5.1. *For every $\mu > 0$ there exists positive constants $C_{\mathrm{e},\mu} > 1$ and $C_{\mathrm{o},\mu} < 1$ such that*

$$E_{\mathrm{e},\mu}(f_{\mathrm{e}}) < \int_{\mathbb{C}} d\nu_{\mathrm{o},\mu}(z)|z|^2|f_{\mathrm{e}}(z)|^2 \leq C_{\mathrm{e},\mu} E_{\mathrm{e},\mu}(f_{\mathrm{e}}) \tag{5.12}$$

for all $0 \neq f_{\mathrm{e}} \in \mathcal{B}^2_{\mathrm{e},\mu}$ and

$$C_{\mathrm{o},\mu} E_{\mathrm{o},\mu}(f_{\mathrm{o}}) \leq \int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z)|z|^2|f_{\mathrm{o}}(z)|^2 < E_{\mathrm{o},\mu}(f_{\mathrm{o}})$$

for all $0 \neq f_{\mathrm{o}} \in \mathcal{B}^2_{\mathrm{o},\mu}$.

For the case $\mu = 0$, we have that

$$E_{\mathrm{e},0}(f_{\mathrm{e}}) = \int_{\mathbb{C}} d\nu_{\mathrm{o},0}(z)|z|^2|f_{\mathrm{e}}(z)|^2$$

and

$$E_{\mathrm{o},0}(f_{\mathrm{o}}) = \int_{\mathbb{C}} d\nu_{\mathrm{e},0}(z)|z|^2|f_{\mathrm{o}}(z)|^2 .$$

Finally, for the case $-1/2 < \mu < 0$ *we have*

$$E_{\mathrm{e},\mu}(f_{\mathrm{e}}) > \int_{\mathbb{C}} d\nu_{\mathrm{o},\mu}(z)|z|^2|f_{\mathrm{e}}(z)|^2 \geq C_{\mathrm{e},\mu} E_{\mathrm{e},\mu}(f_{\mathrm{e}})$$

for all $0 \neq f_{\mathrm{e}} \in \mathcal{B}^2_{\mathrm{e},\mu}$ *and*

$$C_{\mathrm{o},\mu} E_{\mathrm{o},\mu}(f_{\mathrm{o}}) \geq \int_{\mathbb{C}} d\nu_{\mathrm{e},\mu}(z)|z|^2|f_{\mathrm{o}}(z)|^2 > E_{\mathrm{o},\mu}(f_{\mathrm{o}})$$

for all $0 \neq f_{\mathrm{o}} \in \mathcal{B}^2_{\mathrm{o},\mu}$, *where* $0 < C_{\mathrm{e},\mu} < 1$ *and* $C_{\mathrm{o},\mu} > 1$.

Proof. Assume that $f_{\mathrm{e}} \in \mathcal{B}^2_{\mathrm{e},\mu}$. We claim that $E_{\mathrm{e},\mu}(f_{\mathrm{e}}) < \infty$ if and only if $\int_{\mathbb{C}} d\nu_{\mathrm{o},\mu}(z)|z|^2|f_{\mathrm{e}}(z)|^2 < \infty$. Actually, $E_{\mathrm{e},\mu}(f_{\mathrm{e}}) < \infty$ if and only if

$$\int_{\mathbb{C}} dxdy|z|^{2\mu+3}\exp(-|z|^2)|f_{\mathrm{e}}(z)|^2 < \infty \tag{5.13}$$

by the asymptotic behavior of the MacDonald function $K_{\mu-1/2}$ near infinity. The point here is that f_{e} has no local singularities, being holomorphic, and so only its behavior near infinity matters for the convergence of the integral that defines $E_{\mathrm{e},\mu}(f_{\mathrm{e}})$. But $\int_{\mathbb{C}} d\nu_{\mathrm{o},\mu}(z)|z|^2|f_{\mathrm{e}}(z)|^2 < \infty$ if and only if (5.13) holds, since again only the asymptotic behavior near infinity matters, and the behavior of $K_{\mu+1/2}$ to first order near infinity is the same as that of $K_{\mu-1/2}$ near infinity. This establishes the claim. (Actually, in this part of the proof only the continuity of f_{e} plays a role.) The expressions

$$(\|f_{\mathrm{e}}\|^2 + E_{\mathrm{e},\mu}(f_{\mathrm{e}}))^{1/2}$$

and

$$\left(\|f_{\mathrm{e}}\|^2 + \int_{\mathbb{C}} d\nu_{\mathrm{o},\mu}(z)|z|^2|f_{\mathrm{e}}(z)|^2\right)^{1/2}$$

define Hilbert norms in $\mathcal{B}^2_{\mathrm{e},\mu}$, and the result of the previous paragraph says that they define the same finite norm subspace, say $\mathcal{F}$, of $\mathcal{B}^2_{\mathrm{e},\mu}$. Moreover, this subspace $\mathcal{F}$ is closed in the corresponding entire L^2 space with respect to either one of these norms, and so $\mathcal{F}$ is a Hilbert space with respect to either one of these norms. (It is at this point that the holomorphicity of the functions is used in a standard argument already seen in Proposition 1.1.) We now consider the case $\mu > 0$. But then the open mapping theorem (see p. 82 of Ref. 24) together with the first inequality in (5.12), which we proved just before stating this theorem, implies the second inequality in (5.12) for all $f \in \mathcal{F}$. But (5.12) is trivially true for all $f \in \mathcal{B}^2_{\mathrm{e},\mu} \backslash \mathcal{F}$, since all three expressions are then equal to $+\infty$.

The case when $-1/2 < \mu < 0$ follows by similar arguments. Finally, the case $\mu = 0$ follows from the fact that $d\nu_{\mathrm{e},0} = d\nu_{\mathrm{o},0}$, something that we have already noted. QED

Remark. It would be desirable to give a constructive proof of this theorem for the case $\mu \neq 0$ with explicit formulas for $C_{\mathrm{e},\mu}$ and $C_{\mathrm{o},\mu}$. It also remains an open problem to identify the optimal values for the constants $C_{\mathrm{e},\mu}$ and $C_{\mathrm{o},\mu}$ when $\mu \neq 0$.

Though we will not use the next result in the form stated, we feel that it is worthwhile to include it here since it is the idea behind the remaining results in this section. It is an immediate consequence of (5.8), (5.9) and Theorem 5.1.

Corollary 5.1. *We have the following equivalences of μ-deformed and Dirichlet energies*:

(1) *For all $g \in \mathcal{B}^2_{\mathrm{e},\mu}$ we have that the μ-deformed energy $E_{\mathrm{e},\mu}(g)$ is finite if and only if the Dirichlet energy $\|D_\mu g\|^2_{\mathcal{B}^2_\mu}$ is finite.*
(2) *For all $h \in \mathcal{B}^2_{\mathrm{o},\mu}$ we have that the μ-deformed energy $E_{\mathrm{o},\mu}(h)$ is finite if and only if the Dirichlet energy $\|D_\mu h\|^2_{\mathcal{B}^2_\mu}$ is finite.*
(3) *For all $f \in \mathcal{B}^2_\mu$ we have that the μ-deformed energy $E_\mu(f)$ is finite if and only if the Dirichlet energy $\|D_\mu f\|^2_{\mathcal{B}^2_\mu}$ is finite.*

We can now put together the results of Sec. 4 and Theorem 5.1 to get direct and reverse inequalities for the Dirichlet energy $\|D_\mu f\|^2_{\mathcal{B}^2_\mu}$ and Shannon entropy. We continue using the notation from Sec. 4 and Theorem 5.1. We only state the case $\mu \geq 0$. The case $-1/2 < \mu < 0$ is quite similar.

Theorem 5.2. (Reverse log-Sobolev inequalities in $\mathcal{B}^2_{\mathrm{e},\mu}$ and $\mathcal{B}^2_{\mathrm{o},\mu}$ for Dirichlet energy) *Suppose that $\mu \geq 0$ and that $c > 1$. For every $f_{\mathrm{e}} \in \mathcal{B}^2_{\mathrm{e},\mu}$ we have that*

$$\|D_\mu f_{\mathrm{e}}\|^2_{\mathcal{B}^2_\mu} \leq c C_{\mathrm{e},\mu} S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) + (C_{\mathrm{e},\mu}\tau_{\mathrm{e}}(c) - (1+2\mu))\|f_{\mathrm{e}}\|^2_{L^2(\nu_{\mathrm{e},\mu})} .$$

For every $f_{\mathrm{o}} \in \mathcal{B}^2_{\mathrm{o},\mu}$ we have that

$$\|D_\mu f_{\mathrm{o}}\|^2_{\mathcal{B}^2_\mu} \leq c S_{L^2(\nu_{\mathrm{o},\mu})}(f_{\mathrm{o}}) + (\tau_{\mathrm{o}}(c) - (1-2\mu))\|f_{\mathrm{o}}\|^2_{L^2(\nu_{\mathrm{o},\mu})} .$$

For every $f = f_{\mathrm{e}} + f_{\mathrm{o}} \in \mathcal{B}^2_\mu$ we have that

$$\begin{aligned}\|D_\mu f\|^2_{\mathcal{B}^2_\mu} &\leq c C_{\mathrm{e},\mu} S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) + c S_{L^2(\nu_{\mathrm{o},\mu})}(f_{\mathrm{o}}) \\ &\quad + (C_{\mathrm{e},\mu}\tau_{\mathrm{e}}(c) - (1+2\mu))\|f_{\mathrm{e}}\|^2_{L^2(\nu_{\mathrm{e},\mu})} + (\tau_{\mathrm{o}}(c) - (1-2\mu))\|f_{\mathrm{o}}\|^2_{L^2(\nu_{\mathrm{o},\mu})} .\end{aligned}$$

Proof. The first two inequalities follow immediately from Theorems 4.5 and 5.1 as well as the identities (5.8) and (5.9). The last inequality is the sum of the previous two inequalities. It can be simplified slightly by estimating the sum of the norm terms. QED

Theorem 5.3. (Log-Sobolev inequalities in $\mathcal{B}^2_{\mathrm{e},\mu}$ and $\mathcal{B}^2_{\mathrm{o},\mu}$ for Dirichlet energy) *Suppose that $\mu \geq 0$. Then there are real constants $a_{\mathrm{e}} > 0$, $b_{\mathrm{e}} > 0$ and $c_{\mathrm{e}} \geq 0$ such that for all $f_{\mathrm{e}} \in \mathcal{B}^2_{\mathrm{e},\mu}$ we have*

$$a_{\mathrm{e}} S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) \leq b_{\mathrm{e}} \|D_\mu f_{\mathrm{e}}\|^2_{\mathcal{B}^2_\mu} + (b_{\mathrm{e}}(1+2\mu) + c_{\mathrm{e}})\|f_{\mathrm{e}}\|^2_{L^2(\nu_{\mathrm{e},\mu})} .$$

Also there are real constants $a_{\mathrm{o}} > 0$, $b_{\mathrm{o}} > 0$ *and* $c_{\mathrm{o}} \geq 0$ *such that for all* $f_{\mathrm{o}} \in \mathcal{B}^2_{\mathrm{o},\mu}$ *we have*

$$a_{\mathrm{o}} S_{L^2(\nu_{\mathrm{o},\mu})}(f_{\mathrm{o}}) \leq b_{\mathrm{o}} C^{-1}_{\mathrm{o},\mu} \|D_\mu f_{\mathrm{o}}\|^2_{\mathcal{B}^2_\mu} + (b_{\mathrm{o}} C^{-1}_{\mathrm{o},\mu}(1-2\mu) + c_{\mathrm{o}}) \|f_{\mathrm{o}}\|^2_{L^2(\nu_{\mathrm{o},\mu})} .$$

Finally, for every $f = f_{\mathrm{e}} + f_{\mathrm{o}} \in \mathcal{B}^2_\mu$ *we have that*

$$\begin{aligned} a_{\mathrm{e}} S_{L^2(\nu_{\mathrm{e},\mu})}(f_{\mathrm{e}}) + a_{\mathrm{o}} S_{L^2(\nu_{\mathrm{o},\mu})}(f_{\mathrm{o}}) \leq{} & b_{\mathrm{e}} \|D_\mu f_{\mathrm{e}}\|^2_{\mathcal{B}^2_\mu} + b_{\mathrm{o}} C^{-1}_{\mathrm{o},\mu} \|D_\mu f_{\mathrm{o}}\|^2_{\mathcal{B}^2_\mu} \\ & + (b_{\mathrm{e}}(1+2\mu) + c_{\mathrm{e}}) \|f_{\mathrm{e}}\|^2_{L^2(\nu_{\mathrm{e},\mu})} \\ & + (b_{\mathrm{o}} C^{-1}_{\mathrm{o},\mu}(1-2\mu) + c_{\mathrm{o}}) \|f_{\mathrm{o}}\|^2_{L^2(\nu_{\mathrm{o},\mu})} . \end{aligned}$$

Proof. The first two inequalities follow immediately from Theorems 4.6 and 5.1 as well as the identities (5.8) and (5.9). The last inequality is the sum of the previous two inequalities. It can also be simplified in form by using appropriate trivial estimates. QED

It seems reasonable to conjecture that the inequalities in Theorem 5.3 hold without the norm term, since this is known to be true in the case $\mu = 0$. However, the situation is not as clear for Theorem 5.2. It remains an open problem to determine the optimal coefficient of the norm term for each of these inequalities in Theorems 5.2 and 5.3, given that the other coefficients are fixed.

Just as in the previous section, we obtain the next immediate but important consequence.

Corollary 5.2. *We have these equivalences of entropies and Dirichlet energies:*

(1) *For all* $g \in \mathcal{B}^2_{\mathrm{e},\mu}$ *we have that the Shannon entropy* $S_{L^2(\nu_{\mathrm{e},\mu})}(g)$ *is finite if and only if the Dirichlet energy* $\|D_\mu g\|^2_{\mathcal{B}^2_\mu}$ *is finite.*
(2) *For all* $h \in \mathcal{B}^2_{\mathrm{o},\mu}$ *we have that the Shannon entropy* $S_{L^2(\nu_{\mathrm{o},\mu})}(h)$ *is finite if and only if the Dirichlet energy* $\|D_\mu h\|^2_{\mathcal{B}^2_\mu}$ *is finite.*
(3) *For all* $f \in \mathcal{B}^2_\mu$ *we have that the* μ*-deformed entropy* $S_\mu(f)$ *is finite if and only if the Dirichlet energy* $\|D_\mu f\|^2_{\mathcal{B}^2_\mu}$ *is finite.*

In Ref. 21 another quadratic form, called the *dilation energy*, is introduced in the μ-deformed Segal–Bargmann space. It is shown there that this dilation energy is comparable to the μ-deformed energy. So it is straightforward to obtain results analogous to those in this section with the dilation energy replacing the μ-deformed energy. The details are left to the interested reader. Actually, the log-Sobolev inequality proved in Theorem 6.3 of Ref. 21 can be used to prove a log-Sobolev inequality in the Segal–Bargmann space, though those authors did not state this explicitly. Nor did we realize this until we concluded this paper. It turns out that the log-Sobolev inequality proved in Ref. 21 has a very different flavor to it, since in general it relates entropies in two different spaces to each other much in the manner of a Hirschman inequality.

6. Concluding Remarks

Besides the problem of determining the best constants for all the inequalities proved here, another open problem is to establish a hypercontractivity result for this scale of μ-deformed Segal–Bargmann spaces. Note that in Ref. 2 we have shown reverse hypercontractivity in this scale of spaces.

We can consider formulating this theory in terms of holomorphic functions defined on $\mathbb{C}^n$ instead of on $\mathbb{C}$. This can be done where one replaces the Coxeter group $\mathbb{Z}_2 = \{I, J\}$ used here with the Coxeter group $(\mathbb{Z}_2)^n$ generated by the reflections J_k in $\mathbb{C}^n$ given by $J_k(z_1, \ldots, z_k, \ldots, z_n) := (z_1, \ldots, -z_k, \ldots, z_n)$ for $k = 1, \ldots, n$. We thank C. Pita for telling us about this formulation, which is also discussed in Ref. 4. However, the resulting theory is in some sense trivial in that everything factorizes as an n-fold product of the structures discussed here. It may be the case that with other choices of Coxeter group the theory in dimension n could be nontrivial. Refer to Ref. 4 for more details. Of course, there is also the possibility of doing this sort of theory in infinite dimension.

Finally, there is a "configuration" space $L^2(\mathbb{R}, |x|^{2\mu}dx)$ associated with $\mathcal{B}^2_\mu$ via a μ-deformed Segal–Bargmann transform. (See Ref. 36 or Ref. 20 for more details.) In this space there is a naturally defined number operator and its associated quadratic form. It seems reasonable to conjecture that there is a log-Sobolev inequality in this space as well as a hypercontractivity result on the scale of Banach spaces $L^p(\mathbb{R}, |x|^{2\mu}dx)$ for $p > 1$. Moreover, we conjecture that neither a reverse log-Sobolev inequality nor a reverse hypercontractivity result holds in this context.

Dedication

This work owes much to Marvin Rosenblum at a purely scientific level. (See Ref. 25, a work chock full of interesting results.) But Marvin was also a wonderful teacher, from whom the second author learned a lot of analysis and operator theory, including his first ever introduction to the Segal–Bargmann space. References 33–36 indicate just how important that introduction was for the second author. And our work in Ref. 2 owes much to Ref. 25. The news of Marvin's death saddened us greatly. As a friend has remarked, "He was one of the good guys." He certainly was. That alone is more than reason enough to dedicate this paper to his memory.

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