**Integral Equations
and Operator Theory**

Commutative C^*-Algebras of Toeplitz Operators on the Unit Ball, I. Bargmann-Type Transforms and Spectral Representations of Toeplitz Operators

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Abstract. Extending known results for the unit disk, we prove that for the unit ball $\mathbb{B}^n$ there exist $n+2$ different cases of commutative C^*-algebras generated by Toeplitz operators, acting on weighted Bergman spaces. In all cases the bounded measurable symbols of Toeplitz operators are invariant under the action of certain commutative subgroups of biholomorphisms of the unit ball.

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1. Introduction

The commutative C^*-algebras of Toeplitz operators acting on the (weighted) Bergman spaces over the unit disk as well as various properties of the operators from these algebras have been intensively studied recently (see, for example, [5, 6, 7, 8, 11, 15, 16]). It turned out that the smoothness properties of symbols do not play any essential role in order that the corresponding Toeplitz operators generate a commutative C^*-algebra. Surprisingly the deep reason lies in the geometry of the underlying manifold (the hyperbolic plane $\equiv$ unit disk endowed with the standard hyperbolic metric, for the discussed case). The commutativity properties are governed only by the geometric configuration of the level lines of symbols, while the symbols themselves can by merely measurable. As it turns out these level lines have to be the cycles of a pencil of hyperbolic geodesics.

In this connection recall that there are three different types of pencils of hyperbolic geodesics on the unit disk: an elliptic pencil, which is formed by geodesics intersecting in a single point, a parabolic pencil, which is formed by parallel geodesics, and a hyperbolic pencil, which is formed by disjoint geodesics, i.e., by all geodesics orthogonal to a given one. The orthogonal trajectories to geodesics forming a pencil are called cycles. The cycles are always equidistant in the hyperbolic metric.

The main result of [8] states that assuming some natural conditions on the "richness" of the symbol set, *the C*-algebra generated by Toeplitz operators is commutative on each (commonly considered) weighted Bergman space if and only if there is a pencil of hyperbolic geodesics such that the symbols of the Toeplitz operators are constant on the cycles of this pencil.*

We mention that there is a natural one-to-one correspondence between the pencils of hyperbolic geodesics and the maximal commutative subgroups of the movements (conformal isometries) of the hyperbolic plane. Each such subgroup is just the one-parametric group generated by a (non identical) Möbius transformation. Given any such subgroup, the cycles of the corresponding pencil are precisely the sets which remain invariant under the action of this subgroup.

That is, the main result of [8] admits the following equivalent reformulation: assuming some natural conditions on the "richness" of the symbol set, *the C*-algebra generated by Toeplitz operators is commutative on each (commonly considered) weighted Bergman space if and only if there is a maximal commutative subgroup of the Möbius transformation such that the symbols of the Toeplitz operators are invariant under the action of this subgroup.*

The present paper is the first part of a work aimed to extend the results from the unit disk in $\mathbb{C}$ to the unit ball in $\mathbb{C}^n$. Our approach is based on the classification of the maximal commutative subgroups of the biholomorphic automorphisms of the unit ball. In Section 3 of this Part I we list five different types of commutative subgroups of the biholomorphisms of the unit ball $\mathbb{B}^n$ or its unbounded realization, the Siegel domain D_n. In the final Section 10 we show that, given any such subgroup, the C^*-algebra, generated by Toeplitz operators with (bounded measurable) symbols which are invariant under the action of this subgroup, is commutative on each (commonly considered) weighted Bergman space. Moreover we show that in each case the corresponding Toeplitz operators T_a admit the spectral type representations, i.e., all of them are unitary equivalent to certain multiplication operators $\gamma_a I$. The ex plicit form of γ_a is given for each of the five cases under consideration.

It is worth mentioning that such a spectral representation gives an easy access to the important properties of a Toeplitz operators: boundedness, compactness, spectral properties, invariant subspaces, etc.

We note that one of the above types, namely the *quasi-nilpotent*, depends on a parameter $k = 1, 2, \ldots, n-2$. Thus for the unit ball $\mathbb{B}^n$ of (complex) dimension n we have in total $n + 2$ different types of commutative C^*-algebras of Toeplitz

operators. For $n = 1$ these algebras coincide exactly with the three known types of the commutative algebras on the unit disk.

To achieve these results we construct in Sections 5 - 9 the analogues of the classical Bargmann transform and its inverse, which we use then as the unitary multiples in the representation $R T_a R^* = \gamma_a I$. A general scheme to construct such analogues is presented in Section 4, while its concrete realizations for each of the above five different cases constitute the content of Sections 5 - 9.

In the forthcoming Part II of the work we will show that the above five commutative subgroups are maximal commutative ones, and that each maximal commutative subgroup of biholomorphisms is conjugate to one from our list, while neither two from the list are conjugate. That is, we will classify the maximal commutative subgroups of biholomorphisms of the unit ball $\mathbb{B}^n$. Thus we will arrive to the following extension of the sufficiency condition of the existence of commutative algebras of Toeplitz operators: *given any maximal commutative subgroups of biholomorphisms of the unit ball $\mathbb{B}^n$, the C^*-algebra, generated by Toeplitz operators with measurable bounded symbols which are invariant under the action of this group, is commutative.* Another aim of Part II will be to describe the distinguished geometry of the level sets of such symbols (the orbits of maximal commutative subgroups of biholomorphisms of the unit ball), presenting thus the multidimensional generalization of a pencil of hyperbolic geodesics on the unit disk.

2. Weighted Bergman spaces and Bergman projections

Denote by $\mathbb{B}^n$ the unit ball in $\mathbb{C}^n$, that is,
$$\mathbb{B}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \ |z|^2 = |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$
Later on we will use the following notation for the points of $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$:
$$z = (z', z_n), \quad \text{where} \ \ z' = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}, \ \ z_n \in \mathbb{C}.$$
Denote by D_n the following Siegel domain in $\mathbb{C}^n$,
$$D_n = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \operatorname{Im} z_n - |z'|^2 > 0\}.$$
It is well known (and easy to check directly) that the Cayley transform $\zeta = \omega(z)$, where
$$\zeta_k = i\frac{z_k}{1 + z_n}, \quad k = 1, \ldots, n-1,$$
$$\zeta_n = i\frac{1 - z_n}{1 + z_n},$$
maps biholomorphically the unit ball $\mathbb{B}^n$ onto the Siegel domain D_n.

The inverse transform $z = \omega^{-1}(\zeta)$ is given by
$$z_k = -\frac{2i\zeta_k}{1 - i\zeta_n}, \quad k = 1, \ldots, n-1,$$
$$z_n = \frac{1 + i\zeta_n}{1 - i\zeta_n}.$$

Let $\mathcal{D} = \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$. The mapping

$$\kappa \, : \, (z', u, v) \in \mathcal{D} \longmapsto (z', u + iv + i|z'|^2) \in D_n, \tag{2.1}$$

is obviously a diffeomorphism between $\mathcal{D}$ and D_n.

Denote by $dv(z) = dx_1 dy_1 \cdots dx_n dy_n$, where $z_k = x_k + iy_k$, $k = 1, \ldots, n$, the standard Lebesgue measure in $\mathbb{C}^n$. We introduce the following one-parameter family of weights (see, for example, [17]),

$$\mu_\lambda(z) = c_\lambda \, (1 - |z|^2)^\lambda, \quad \lambda > -1,$$

where the normalizing constant

$$c_\lambda = \frac{\Gamma(n + \lambda + 1)}{\pi^n \Gamma(\lambda + 1)} \tag{2.2}$$

is chosen so that $\mu_\lambda(z) dv(z)$ is a probability measure in $\mathbb{B}^n$.

It is easy to see that under the inverse Cayley transform $z = \omega^{-1}(\zeta)$ we have

$$dv(z) \quad = \quad \frac{2^{2n}}{|1 - i\zeta_n|^{2n+2}} \, dv(\zeta), \tag{2.3}$$

$$1 - |z|^2 \quad = \quad \frac{2^2}{|1 - i\zeta_n|^2} \, (\operatorname{Im} \zeta_n - |z'|^2), \tag{2.4}$$

$$1 + z_n \quad = \quad \frac{2}{1 - i\zeta_n}. \tag{2.5}$$

Given a function $f \in L_2(\mathbb{B}^n, \mu_\lambda)$, changing the variables $z = \omega^{-1}(\zeta)$ and using (2.3), (2.4), we have

$$\begin{aligned}
\|f\|^2 \quad &= \quad \int_{\mathbb{B}^n} |f(z)|^2 c_\lambda \, (1 - |z|^2)^\lambda \, dv(z) \tag{2.6} \\
&= \quad \int_{D_n} |f(\omega^{-1}(\zeta))|^2 c_\lambda \frac{2^{2\lambda}}{|1 - i\zeta|^{2\lambda}} (\operatorname{Im} \zeta_n - |\zeta'|^2)^\lambda \frac{2^{2n}}{|1 - i\zeta|^{2n+2}} \, dv(\zeta) \\
&= \quad \int_{D_n} |f(\omega^{-1}(\zeta))|^2 \frac{2^{2n+2\lambda}}{|1 - i\zeta|^{2n+2\lambda+2}} c_\lambda \, (\operatorname{Im} \zeta_n - |\zeta'|^2)^\lambda \, dv(\zeta).
\end{aligned}$$

Introduce now the space $L_2(D_n, \widetilde{\mu}_\lambda)$, where

$$\widetilde{\mu}_\lambda(\zeta) = \frac{c_\lambda}{4} \, (\operatorname{Im} \zeta_n - |\zeta'|^2)^\lambda,$$

and the operator

$$U_\lambda : L_2(\mathbb{B}^n, \mu_\lambda) \longrightarrow L_2(D_n, \widetilde{\mu}_\lambda),$$

which acts as

$$(U_\lambda f)(\zeta) = \left(\frac{2}{1 - i\zeta_n} \right)^{n+\lambda+1} f(\omega^{-1}(\zeta)).$$

Then (2.6) can be rewritten as

$$\|f\|^2_{L_2(\mathbb{B}^n, \mu_\lambda)} = \|U_\lambda f\|^2_{L_2(D_n, \widetilde{\mu}_\lambda)}.$$

It is easy to check that the operator U_λ is unitary, and its inverse (and adjoint) operator

$$U_\lambda^{-1} : L_2(D_n, \widetilde{\mu}_\lambda) \longrightarrow L_2(\mathbb{B}^n, \mu_\lambda)$$

has the form

$$(U_\lambda^{-1} f)(z) = \frac{1}{(1 + z_n)^{n+\lambda+1}} f(\omega(z)).$$

Denote by $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ and by $\mathcal{A}_\lambda^2(D_n)$ the (weighted) Bergman subspaces of $L_2(\mathbb{B}^n, \mu_\lambda)$ and of $L_2(D_n, \widetilde{\mu}_\lambda)$, respectively. Recall that, as always, the Bergman space is the subspace of the corresponding L_2-space which consists of all analytic functions.

It is well known (see, for example, [17]) that the weighted Bergman kernel for the unit ball is given by

$$K_{\mathbb{B}^n, \lambda}(z, \zeta) = \frac{1}{(1 - z \cdot \overline{\zeta})^{n+\lambda+1}} = \frac{1}{(1 - \sum_{k=1}^n z_k \overline{\zeta}_k)^{n+\lambda+1}},$$

and that the (weighted) Bergman projection $B_{\mathbb{B}^n, \lambda}$ of $L_2(\mathbb{B}^n, \mu_\lambda)$ onto $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ has the form

$$(B_{\mathbb{B}^n, \lambda} f)(z) = \int_{\mathbb{B}^n} f(\zeta) \frac{(1 - |\zeta|^2)^\lambda}{(1 - z \cdot \overline{\zeta})^{n+\lambda+1}} c_\lambda \, dv(\zeta).$$

Let $z = \omega^{-1}(w)$ and $\zeta = \omega^{-1}(\eta)$, then

$$1 - z \cdot \overline{\zeta} = \frac{2^2}{(1 - iw_n)(1 + i\overline{\eta}_n)} \left(\frac{w_n - \overline{\eta}_n}{2i} - w' \cdot \overline{\eta'} \right). \tag{2.7}$$

We note that the unitary operator U_λ, being the isomorphism between $L_2(\mathbb{B}^n, \mu_\lambda)$ and $L_2(D_n, \widetilde{\mu}_\lambda)$, maps isomorphically $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ onto $\mathcal{A}_\lambda^2(D_n)$ as well. Changing the variables $z = \omega^{-1}(w)$, $\zeta = \omega^{-1}(\eta)$ and using (2.3)–(2.5), (2.7), we have

$$
\begin{aligned}
(Bf)(w) &= (U_\lambda B_{\mathbb{B}^n, \lambda} U_\lambda^{-1} f)(w) \\
&= \left(\frac{2}{1 - iw_n} \right)^{n+\lambda+1} \int_{\mathbb{B}^n} \frac{f(\omega(\zeta))}{(1 + \zeta_n)^{n+\lambda+1}} \frac{(1 - |\zeta|^2)^\lambda c_\lambda \, dv(\zeta)}{(1 - \omega^{-1}(w) \cdot \overline{\zeta})^{n+\lambda+1}} \\
&= \left(\frac{2}{1 - iw_n} \right)^{n+\lambda+1} \int_{D_n} \frac{(1 - i\eta_n)^{n+\lambda+1}}{2^{n+\lambda+1}} f(\eta) \\
&\quad \cdot c_\lambda \frac{2^{2\lambda} (\operatorname{Im} \eta_n - |\eta'|^2)^\lambda}{|1 - i\eta_n|^{2\lambda}} \frac{(1 - iw_n)^{n+\lambda+1}(1 + i\overline{\eta}_n)^{n+\lambda+1}}{2^{2n+2\lambda+2}} \\
&\quad \cdot \frac{1}{\left(\frac{w_n - \overline{\eta}_n}{2i} - w' \cdot \overline{\eta'} \right)^{n+\lambda+1}} \frac{2^{2n}}{|1 - i\eta_n|^{2n+2}} \, dv(\eta) \\
&= \int_{D_n} f(\eta) \frac{(\operatorname{Im} \eta_n - |\eta'|^2)^\lambda}{\left(\frac{w_n - \overline{\eta}_n}{2i} - w' \cdot \overline{\eta'} \right)^{n+\lambda+1}} \frac{c_\lambda}{4} \, dv(\eta).
\end{aligned}
$$

That is, the (weighted) Bergman kernel for D_n has the form (see, for example, [2, 3, 9])

$$K_{D_n,\lambda}(z,\zeta) = \frac{1}{\left(\frac{z_n-\overline{\zeta}_n}{2i} - z' \cdot \overline{\zeta'}\right)^{n+\lambda+1}},$$

and the weighted Bergman projection $B_{D_n,\lambda}$ of $L_2(D_n,\widetilde{\mu}_\lambda)$ onto the Bergman space $\mathcal{A}_\lambda^2(D_n)$ is given by

$$(B_{D_n,\lambda}f)(z) = \int_{D_n} f(\zeta) \frac{(\mathrm{Im}\,\zeta_n - |\zeta'|^2)^\lambda}{\left(\frac{z_n-\overline{\zeta}_n}{2i} - z' \cdot \overline{\zeta'}\right)^{n+\lambda+1}} \frac{c_\lambda}{4}\,dv(\zeta).$$

Return now to the domain $\mathcal{D} = \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$ whose points we will denote by $w = (z',u,v)$. Introduce the space $L_2(\mathcal{D},\eta_\lambda)$, where the weight η_λ is given by the formula

$$\eta_\lambda(w) = \eta_\lambda(v) = \frac{c_\lambda}{4}\,v^\lambda, \quad \lambda > -1,$$

and the constant c_λ is given by (2.2).

Introduce the operator $U_0 : L_2(D_n,\widetilde{\mu}_\lambda) \longrightarrow L_2(\mathcal{D},\eta_\lambda)$ as follows,

$$(U_0 f)(w) = f(\kappa(w)),$$

where the mapping κ is given by (2.1). The operator U_0 is obviously unitary, and the inverse operator has the form

$$(U_0^{-1}f)(z) = f(\kappa^{-1}(z)).$$

The (weighted) Bergman space $\mathcal{A}_\lambda^2(D_n)$ on the Siegel domain D_n can be characterized alternatively as the (closed) subspace of $L_2(D_n,\widetilde{\mu}_\lambda)$ which consists of all functions φ satisfying the equations

$$\frac{\partial}{\partial \overline{z}_k}\varphi = 0, \quad k = 1,\ldots,n.$$

Then the image $\mathcal{A}_0(\mathcal{D}) = U_0(\mathcal{A}_\lambda^2(D_n))$ is the subspace of $L_2(\mathcal{D},\eta_\lambda)$ which consists of all functions φ satisfying the equations

$$U_0 \frac{\partial}{\partial \overline{z}_k} U_0^{-1}\varphi = 0, \quad k = 1,\ldots,n.$$

It is obvious that

$$U_0 \frac{\partial}{\partial \overline{z}_n} U_0^{-1} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \tag{2.8}$$

While for $k = 1,\ldots,n-1$, we have

$$\begin{aligned}
U_0 \frac{\partial}{\partial \overline{z}_k} U_0^{-1}\varphi(z',u,v) &= U_0 \frac{\partial}{\partial \overline{z}_k}\varphi(z',\mathrm{Re}\,z_n,\mathrm{Im}\,z_n - |z'|^2) \\
&= U_0\left(\frac{\partial \varphi}{\partial \overline{z}_k} - \frac{\partial \varphi}{\partial v}\,z_k\right) \\
&= \left(\frac{\partial}{\partial \overline{z}_k} - \frac{\partial}{\partial v}\,z_k\right)\varphi.
\end{aligned}$$

That is,

$$U_0 \frac{\partial}{\partial \overline{z}_k} U_0^{-1} = \frac{\partial}{\partial \overline{z}_k} - \frac{\partial}{\partial v} z_k,$$

or, expressing $\frac{\partial}{\partial v}$ in terms of $\frac{\partial}{\partial u}$ using (2.8),

$$U_0 \frac{\partial}{\partial \overline{z}_k} U_0^{-1} = \frac{\partial}{\partial \overline{z}_k} - i \frac{\partial}{\partial u} z_k.$$

Thus the space $\mathcal{A}_0(\mathcal{D})$ coincides with the set of all $L_2(\mathcal{D}, \eta_\lambda)$-functions which satisfy the equations

$$\frac{1}{2} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \varphi = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial \overline{z}_k} - \frac{\partial}{\partial v} z_k \right) \varphi = 0, \quad k = 1, \ldots, n-1, \quad (2.9)$$

or the equations

$$\frac{1}{2} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \varphi = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial \overline{z}_k} - i \frac{\partial}{\partial u} z_k \right) \varphi = 0, \quad k = 1, \ldots, n-1. \quad (2.10)$$

3. Commutative subgroups of biholomorphisms

We list here five essentially different types of commutative subgroups of biholomorphisms of the unit ball $\mathbb{B}^n$, or its unbounded realization, the Siegel domain D_n. In Part II of the paper we will show that, first, these subgroups are maximal commutative subgroups of biholomorphisms, and second, each maximal commutative subgroup of biholomorphisms is conjugate to one from this list, while neither two from the list are conjugate. That is, in a sense, this list classifies the maximal commutative subgroups of biholomorphisms of the unit ball $\mathbb{B}^n$.

Quasi-elliptic group of biholomorphisms of the unit ball $\mathbb{B}^n$ is isomorphic to $\mathbb{T}^n$ with the following group action:

$$t: \ z = (z_1, \ldots, z_n) \in \mathbb{B}^n \longmapsto tz = (t_1 z_1, \ldots, t_n z_n) \in \mathbb{B}^n,$$

for each $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$.

Quasi-parabolic group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$ with the following group action:

$$(t, h): \ (z', z_n) \in D_n \longmapsto (tz', z_n + h) \in D_n,$$

for each $(t, h) \in \mathbb{T}^{n-1} \times \mathbb{R}$.

Quasi-hyperbolic group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}_+$ with the following group action:

$$(t, r): \ (z', z_n) \in D_n \longmapsto (r^{1/2} t z', r z_n) \in D_n,$$

for each $(t, r) \in \mathbb{T}^{n-1} \times \mathbb{R}_+$,

Nilpotent group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}$ with the following group action:

$$(b, h): \ (z', z_n) \in D_n \mapsto (z' + b, z_n + h + 2i z' \cdot b + i|b|^2) \in D_n,$$

for each $(b, h) \in \mathbb{R}^{n-1} \times \mathbb{R}$;

Quasi-nilpotent group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$, $0 < k < n-1$, with the following group action:

$$(t, b, h) : (z', z'', z_n) \in D_n \longmapsto (tz', z'' + b, z_n + h + 2iz'' \cdot b + i|b|^2)) \in D_n,$$

for each $(t, b, h) \in \mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$.

Note that setting in the quasi-nilpotent case $k = n-1$ we obtain the quasi-parabolic group, while for $k = 0$ we obtain the nilpotent group. At the same time we prefer to distinguish these three cases in order to make our calculations more transparent.

4. Bargmann type transform

We describe here a scheme which has been already successfully used, for example, in [12, 13, 14, 15] and which will be used in the subsequent sections.

Although the scheme is very simple, a considerable amount of work is required in each particular case in order to define and calculate all necessary data. The purpose of it is to give a description of the space of analytic functions under study in "real analysis terms", i.e., as an appropriate L_2 space, and to construct the operator R which being restricted onto the analytic space maps it isometrically onto the corresponding L_2 space, and which together with its adjoint R^* provide the factorization of the identity operator on the L_2 space and the orthogonal projection of the initial Hilbert space onto the space of analytic functions.

We note that the operator R restricted onto the space of analytic functions together with its inverse R^* can be considered as the analogs of the classical Bargmann transform (and its inverse) [1], moreover they do give the classical Bargmann transform for the corresponding case, see [14].

Let H be a separable Hilbert space and $\mathcal{A}$ be its closed subspace. Denote by P the orthogonal projection of H onto $\mathcal{A}$. We assume as well that there exist

(i) measurable spaces X and Y with measures μ and η respectively,

(ii) a unitary operator

$$U : H \longrightarrow L_2(X, \mu) \otimes L_2(Y, \eta),$$

(iii) a measurable subspace X_1 of X and a function $g_0 = g_0(x, y)$ on $X_1 \times Y$, such that
   – for each $x \in X_1$ the function $g_0(x, \cdot) \in L_2(Y, \eta)$,
     and $\|g_0(x, \cdot)\|_{L_2(Y, \eta)} = 1$,
   – the operator U maps $\mathcal{A}$ onto $g_0 L_2(X_1, \mu) \subset L_2(X, \mu) \otimes L_2(Y, \eta)$:

$$U : \mathcal{A} \longrightarrow g_0 L_2(X_1, \mu).$$

Then, for each $\varphi = g_0 f \in U(\mathcal{A}) = g_0 L_2(X_1, \mu)$, where $f \in L_2(X_1, \mu)$, one has obviously

$$\|\varphi\|_{U(\mathcal{A})} = \|f\|_{L_2(X_1, \mu)}.$$

We introduce now the isometric imbedding

$$R_0 : L_2(X_1, \mu) \longrightarrow L_2(X, \mu) \otimes L_2(Y, \eta)$$

by the formula

$$R_0 : f \in L_2(X_1, \mu) \longmapsto \widetilde{f} g_0 \in L_2(X, \mu) \otimes L_2(Y, \eta),$$

where

$$\widetilde{f} = \left\{ \begin{array}{ll} f, & x \in X_1 \\ 0, & x \in X \setminus X_1 \end{array} \right. .$$

Then the adjoint operator

$$R_0^* : L_2(X, \mu) \otimes L_2(Y, \eta) \longrightarrow L_2(X_1, \mu)$$

is given by

$$(R_0^* \varphi)(x) = \int_Y \varphi(x, y) \, \overline{g_0(x, y)} \, d\eta, \quad x \in X_1.$$

It is easy to check that

$$R_0^* R = I \quad : \quad L_2(X_1, \mu) \longrightarrow L_2(X_1, \mu),$$
$$R R_0^* = Q \quad : \quad L_2(X, \mu) \otimes L_2(Y, \eta) \longrightarrow U(\mathcal{A}),$$

where Q is the orthogonal projection of $L_2(X, \mu) \otimes L_2(Y, \eta)$ onto the image of the space $\mathcal{A}$ under the unitary operator U, i.e., $U(\mathcal{A}) = g_0 L_2(X_1, \mu)$.

Combining all the above we come to the following result.

Theorem 4.1. *The operator $R = R_0^* U$ maps the Hilbert space H onto $L_2(X_1, \mu)$, and the restriction*

$$R|_{\mathcal{A}} : \mathcal{A} \longrightarrow L_2(X_1, \mu)$$

is an isometric isomorphism.

The adjoint operator

$$R^* = U^* R_0 : L_2(X_1, \mu) \longrightarrow \mathcal{A} \subset H$$

is an isometric isomorphism of $L_2(X_1, \mu)$ onto the subspace $\mathcal{A}$ of H.

Furthermore

$$RR^* = I \quad : \quad L_2(X_1, \mu) \longrightarrow L_2(X_1, \mu),$$
$$R^* R = P \quad : \quad H \longrightarrow \mathcal{A},$$

where P is the orthogonal projection of H onto $\mathcal{A}$.

In the subsequent five sections we will use this scheme in five different cases defining and calculating explicitly in each case all the necessary data. The non-trivial and essential part of the job is to find the measurable spaces X, X_1, and Y and to construct the corresponding unitary operator U, appropriate for each specific case.

The key idea here, as well as in [12, 13, 14, 15], is to make use of appropriate commutative subgroups of biholomorphisms of the domain under consideration, and doing the corresponding group Fourier transform, to reduce the system of partial differential equations, which define the Bergman space, to the ordinary ones, depending on certain parameters. Solving the last ordinary differential equations we come to the independent "real analysis type" description of the Bergman space.

In what follows we will use the five different types of commutative subgroups, described in Section 3, and we will name each of the five subsequent sections according to the above subgroups.

5. Quasi-elliptic case

The results for this case are already known, see [10]. For the sake of completeness we present them here.

Denote by $\tau(\mathbb{B}^n)$ the base of the unit ball $\mathbb{B}^n$, considered as a Reinhard domain, i.e.,

$$\tau(\mathbb{B}^n) = \{r = (r_1, \ldots, r_n) = (|z_1|, \ldots, |z_n|) \, : \, r^2 = r_1^2 + \cdots + r_n^2 \in [0, 1)\},$$

which belongs to $\mathbb{R}_+^n = \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$. Introduce in $\mathbb{C}^n$ the polar coordinates $z_k = t_k r_k$, $r_k \in \mathbb{R}_+$, where $t_k \in \mathbb{T} = S^1$, $k = 1, \ldots, n$. Then under the identification

$$z = (z_1, \ldots, z_n) = (t_1 r_1, \ldots, t_n r_n) = (t, r),$$

where $t = (t_1, \ldots, t_n) \in \mathbb{T}^n = \mathbb{T} \times \cdots \times \mathbb{T}$, $r = (r_1, \ldots, r_n) \in \tau(\mathbb{B}^n)$, we have $\mathbb{B}^n = \mathbb{T}^n \times \tau(\mathbb{B}^n)$ and

$$L_2(\mathbb{B}^n, \mu_\lambda) = L_2(\mathbb{T}^n) \otimes L_2(\tau(\mathbb{B}^n), \mu),$$

where

$$L_2(\mathbb{T}^n) = \bigotimes_{k=1}^{n} L_2(\mathbb{T}, \frac{dt_k}{it_k})$$

and the measure $d\mu$ in $L_2(\tau(\mathbb{B}^n), \mu)$ is given by

$$d\mu = \mu_\lambda(r) \prod_{k=1}^{n} r_k dr_k = c_\lambda (1 - r^2)^\lambda \prod_{k=1}^{n} r_k dr_k.$$

We define the discrete Fourier transform $\mathcal{F} : L_2(\mathbb{T}) \to l_2 = l_2(\mathbb{Z})$ by

$$\mathcal{F} : f \longmapsto c_n = \frac{1}{\sqrt{2\pi}} \int_{S^1} f(t) \, t^{-n} \, \frac{dt}{it}, \quad n \in \mathbb{Z}. \tag{5.1}$$

The operator $\mathcal{F}$ is unitary and

$$\mathcal{F}^{-1} = \mathcal{F}^* : \{c_n\}_{n \in \mathbb{Z}} \longmapsto f = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} c_n \, t^n.$$

In terms of the scheme of Section 4 we have here

$$\begin{aligned} X &= \mathbb{Z}^n, & L_2(X, \mu) &= l_2(\mathbb{Z}^n), \\ X_1 &= \mathbb{Z}_+^n, & L_2(X_1, \mu) &= l_2(\mathbb{Z}_+^n), \\ Y &= \tau(\mathbb{B}^n), & L_2(Y, \eta) &= L_2(\tau(\mathbb{B}^n), \end{aligned}$$

the unitary operator U is defined as follows,

$$U = \mathcal{F}_{(n)} \otimes I \, : \, L_2(\mathbb{T}^n) \otimes L_2(\tau(\mathbb{B}^n), \mu) \longrightarrow l_2(\mathbb{Z}^n) \otimes L_2(\tau(\mathbb{B}^n), \mu),$$

where $\mathcal{F}_{(n)} = \mathcal{F} \otimes \cdots \otimes \mathcal{F}$, and the function g_0 (sequence in this case) has the form

$$g_0(r) = \left\{ \left(\frac{(2\pi)^n \, \Gamma(n + |p| + \lambda + 1)}{p! \, \Gamma(n + \lambda + 1)} \right)^{1/2} r^p \right\}_{p \in \mathbb{Z}_+^n}, \quad r \in \tau(\mathbb{B}^n),$$

and $\mathbb{Z}_+^n = \mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+$ with $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$.

Introduce the isometric embedding

$$R_0 \; : \; l_2(\mathbb{Z}_+^n) \longrightarrow l_2(\mathbb{Z}^n) \otimes L_2(\tau(\mathbb{B}^n), \mu)$$

where

$$R_0 \; : \; \{c_p\}_{p \in \mathbb{Z}_+^n} \longmapsto c_p(r) = \left\{ \begin{array}{ll} \left(\frac{(2\pi)^n \, \Gamma(n+|p|+\lambda+1)}{p! \, \Gamma(n+\lambda+1)} \right)^{1/2} c_p r^p, & p \in \mathbb{Z}_+^n \\ 0, & p \in \mathbb{Z}^n \setminus \mathbb{Z}_+^n \end{array} \right. .$$

Then the adjoint operator $R_0^* : l_2(\mathbb{Z}^n) \otimes L_2(\tau(\mathbb{B}^n), \mu) \longrightarrow l_2(\mathbb{Z}^n)$ is defined by

$$R_0^* \; : \; \{f_p(r)\}_{p \in \mathbb{Z}^n} \longmapsto$$
$$\left\{ \left(\frac{(2\pi)^n \, \Gamma(n + |p| + \lambda + 1)}{p! \, \Gamma(n + \lambda + 1)} \right)^{1/2} \int_{\tau(\mathbb{B}^n)} r^p \, f_p(r) \, c_\lambda (1 - r^2)^\lambda \prod_{k=1}^n r_k dr_k \right\}_{p \in \mathbb{Z}_+^n} .$$

Theorem 5.1. *The operator $R = R_0 U$ maps $L_2(\mathbb{B}^n, \mu_\lambda)$ onto $l_2(\mathbb{Z}_+^n)$, and the restriction*

$$R|_{\mathcal{A}_\lambda^2(\mathbb{B}^n)} \; : \; \mathcal{A}_\lambda^2(\mathbb{B}^n) \longrightarrow l_2(\mathbb{Z}_+^n)$$

is an isometric isomorphism.

The adjoint operator

$$R^* = U^* R_0 \; : \; l_2(\mathbb{Z}_+^n) \longrightarrow \mathcal{A}_\lambda^2(\mathbb{B}^n) \subset L_2(\mathbb{B}^n, \mu_\lambda)$$

is the isometric isomorphism of $l_2(\mathbb{Z}_+^n)$ onto the subspace $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ of $L_2(\mathbb{B}^n, \mu_\lambda)$.

Furthermore

$$RR^* = I \quad : \quad l_2(\mathbb{Z}_+^n) \longrightarrow l_2(\mathbb{Z}_+^n),$$
$$R^* R = B_{\mathbb{B}^n, \lambda} \quad : \quad L_2(\mathbb{B}^n, \mu_\lambda) \longrightarrow \mathcal{A}_\lambda^2(\mathbb{B}^n),$$

where $B_{\mathbb{B}^n, \lambda}$ is the Bergman projection of $L_2(\mathbb{B}^n, \mu_\lambda)$ onto $\mathcal{A}_\lambda^2(\mathbb{B}^n)$.

Theorem 5.2. *The isometric isomorphism*

$$R^* = U^* R_0 : l_2(\mathbb{Z}_+^n) \longrightarrow \mathcal{A}_\lambda^2(\mathbb{B}^n)$$

is given by

$$R^* \; : \; \{c_p\}_{p \in \mathbb{Z}_+^n} \longmapsto (2\pi)^{-\frac{n}{2}} \sum_{p \in \mathbb{Z}_+^n} \left(\frac{(2\pi)^n \, \Gamma(n + |p| + \lambda + 1)}{p! \, \Gamma(n + \lambda + 1)} \right)^{1/2} c_p \, z^p. \qquad (5.2)$$

Corollary 5.3. *The inverse isomorphism*

$$R : \mathcal{A}_\lambda^2(\mathbb{B}^n) \longrightarrow l_2(\mathbb{Z}_+^n)$$

is given by

$$R : \; \varphi(z) \; \longmapsto \tag{5.3}$$

$$\left\{ (2\pi)^{-\frac{n}{2}} \left(\frac{(2\pi)^n \, \Gamma(n + |p| + \lambda + 1)}{p! \, \Gamma(n + \lambda + 1)} \right)^{1/2} \int_D \varphi(z) \, \overline{z}^p \, \mu_\lambda(z) \, dv(z) \right\}_{p \in \mathbb{Z}_+^n}.$$

6. Quasi-parabolic case

We represent the space $L_2(\mathcal{D}, \eta_\lambda)$ as the following tensor product,

$$L_2(\mathcal{D}, \eta_\lambda) = L_2(\mathbb{C}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda),$$

and consider the unitary operator $U_1 = I \otimes F \otimes I$ acting on it. Here

$$(Ff)(\xi) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} f(u)e^{-i\xi u} du \tag{6.1}$$

is the standard Fourier transform on $L_2(\mathbb{R})$.

For the operators in (2.10) we have obviously

$$U_1 \frac{1}{2} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) U_1^{-1} \;\; = \;\; \frac{i}{2} \left(\xi + \frac{\partial}{\partial v} \right),$$

$$U_1 \left(\frac{\partial}{\partial \overline{z}_k} - i \frac{\partial}{\partial u} z_k \right) U_1^{-1} \;\; = \;\; \frac{\partial}{\partial \overline{z}_k} + \xi z_k, \quad k = 1, \dots, n-1.$$

Thus the image $\mathcal{A}_1(\mathcal{D}) = U_1(\mathcal{A}_0(\mathcal{D}))$ is the subspace of $L_2(\mathcal{D}, \eta_\lambda)$ which consists of all functions satisfying the equations

$$\frac{i}{2} \left(\xi + \frac{\partial}{\partial v} \right) \varphi = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial \overline{z}_k} + \xi z_k \right) \varphi = 0, \quad k = 1, \dots, n-1. \tag{6.2}$$

The first equation is easy to solve, and its general solution has the form

$$\varphi(z', \xi, v) = \psi_0(z', \xi)e^{-\xi v}.$$

The function φ has to be in $L_2(\mathcal{D}, \eta_\lambda)$, which implies that its support on the variable ξ has to be in $\mathbb{R}_+$. That is,

$$\varphi(z', \xi, v) = \chi_{\mathbb{R}_+}(\xi)\psi(z', \xi)e^{-\xi v}. \tag{6.3}$$

Further, the function φ has to satisfy the second equations in (6.2), that is,

$$\left(\frac{\partial}{\partial \overline{z}_k} + \xi z_k \right) \psi(z', \xi) = 0, \quad k = 1, \dots, n-1.$$

Introduce in $\mathbb{C}^{n-1}$ the polar coordinates, $z_k = r_k t_k$, where $r_k \in \mathbb{R}_+$, $t_k \in S^1 = \mathbb{T}$, $k = 1, \ldots, n-1$. Then, it is easy to see that

$$\frac{\partial}{\partial \overline{z}_k} + \xi z_k = \frac{t_k}{2}\left(\frac{\partial}{\partial r_k} - \frac{t_k}{r_k}\frac{\partial}{\partial t_k} + 2\xi r_k\right), \quad k = 1, \ldots, n-1.$$

Represent now

$$\begin{aligned} L_2(\mathcal{D}, \eta_\lambda) &= L_2(\mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+, \eta_\lambda) \\ &= L_2(\mathbb{R}_+^{n-1}, rdr) \otimes L_2(\mathbb{T}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda), \end{aligned}$$

where

$$rdr = \prod_{k=1}^{n-1} r_k dr_k, \quad L_2(\mathbb{T}^{n-1}) = \bigotimes_{k=1}^{n-1} L_2\left(S^1, \frac{dt_k}{it_k}\right).$$

Introduce the unitary operator $U_2 = I \otimes \mathcal{F}_{(n-1)} \otimes I \otimes I$ acting from

$$L_2(\mathbb{R}_+^{n-1}, rdr) \otimes L_2(\mathbb{T}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)$$

onto

$$\begin{aligned} & L_2(\mathbb{R}_+^{n-1}, rdr) \otimes l_2(\mathbb{Z}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda) \\ =\ & l_2(\mathbb{Z}^{n-1}, L_2(\mathbb{R}_+^{n-1}, rdr) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)), \end{aligned}$$

where $\mathcal{F}_{(n-1)} = \mathcal{F} \otimes \cdots \otimes \mathcal{F}$, and each $\mathcal{F}$ is the one-dimensional discrete Fourier transform (5.1).

For φ of the form (6.3) we have

$$U_2\varphi = \chi_{\mathbb{R}_+}(\xi)e^{-\xi v}(\mathcal{F} \otimes I)\psi = \chi_{\mathbb{R}_+}(\xi)e^{-\xi v}\{c_p(r,\xi)\}_{p\in\mathbb{Z}^{n-1}}.$$

Furthere, the sequence $\{d_p\}_{p\in\mathbb{Z}^{n-1}} = \chi_{\mathbb{R}_+}(\xi)e^{-\xi v}\{c_p(r,\xi)\}_{p\in\mathbb{Z}^{n-1}}$ has to satisfy the equations

$$U_2\frac{t_k}{2}\left(\frac{\partial}{\partial r_k} - \frac{t_k}{r_k}\frac{\partial}{\partial t_k} + 2\xi r_k\right)U_2^{-1}\{d_p\}_{p\in\mathbb{Z}^{n-1}} = 0, \quad k = 1, \ldots, n-1,$$

or equivalently

$$U_2\frac{t_k}{2}\left(\frac{\partial}{\partial r_k} - \frac{t_k}{r_k}\frac{\partial}{\partial t_k} + 2\xi r_k\right)U_2^{-1}\{c_p\}_{p\in\mathbb{Z}^{n-1}} = 0, \quad k = 1, \ldots, n-1.$$

The operator

$$\delta = U_2\frac{t_k}{2}\left(\frac{\partial}{\partial r_k} - \frac{t_k}{r_k}\frac{\partial}{\partial t_k} + 2\xi r_k\right)U_2^{-1}$$

acts as follows,

$$
\delta \{c_p\}_{p \in \mathbb{Z}^{n-1}} = U_2 \frac{t_k}{2} \left(\frac{\partial}{\partial r_k} - \frac{t_k}{r_k} \frac{\partial}{\partial t_k} + 2\xi r_k \right) (2\pi)^{-\frac{n-1}{2}} \sum_{p \in \mathbb{Z}^{n-1}} c_p \, t^p
$$

$$
= U_2 \left((2\pi)^{-\frac{n-1}{2}} \sum_{p \in \mathbb{Z}^{n-1}} \hat{t}_k^p \frac{t_k}{2} \left(\frac{\partial c_p}{\partial r_k} t_k^{p_k} - \frac{t_k}{r_k} p_k c_p t_k^{P_k-1} + 2\xi r_k t_k^{p_k} c_p \right) \right)
$$

$$
= U_2 \left((2\pi)^{-\frac{n-1}{2}} \sum_{p \in \mathbb{Z}^{n-1}} \hat{t}_k^p t_k^{p_k+1} \frac{1}{2} \left(\frac{\partial}{\partial r_k} - \frac{p_k}{r_k} + 2\xi r_k \right) c_p \right)
$$

$$
= U_2 \left((2\pi)^{-\frac{n-1}{2}} \sum_{p \in \mathbb{Z}^{n-1}} t^p \frac{1}{2} \left(\frac{\partial}{\partial r_k} - \frac{p_k-1}{r_k} + 2\xi r_k \right) c_{p-e_k} \right)
$$

$$
= \left\{ \frac{1}{2} \left(\frac{\partial}{\partial r_k} - \frac{p_k-1}{r_k} + 2\xi r_k \right) c_{p-e_k} \right\}_{p \in \mathbb{Z}^{n-1}},
$$

where $p = (p_1, \ldots, p_{n-1})$, $e_k = (0, \ldots, 0, 1, 0, \ldots 0)$, $t^p = t_1^{p_1} \cdots t_{n-1}^{p_{n-1}}$, and $\hat{t}_k^p$ is t^p with the multiple $t_k^{p_k}$ omitted. That is,

$$
U_2 \frac{t_k}{2} \left(\frac{\partial}{\partial r_k} - \frac{t_k}{r_k} \frac{\partial}{\partial t_k} + 2\xi r_k \right) U_2^{-1} \{c_p\}_{p \in \mathbb{Z}^{n-1}}
$$

$$
= \left\{ \frac{1}{2} \left(\frac{\partial}{\partial r_k} - \frac{p_k-1}{r_k} + 2\xi r_k \right) c_{p-e_k} \right\}_{p \in \mathbb{Z}^{n-1}}.
$$

Now the space $\mathcal{A}_2(\mathcal{D}) = U_2(\mathcal{A}_1(\mathcal{D}))$ consists of all sequences $\{d_p\}_{p \in \mathbb{Z}^{n-1}}$, where

$$
d_p = \chi_{\mathbb{R}_+}(\xi) e^{-\xi v} c_p(r, \xi), \qquad p \in \mathbb{Z}^{n-1},
$$

which satisfy the equations

$$
\frac{1}{2} \left(\frac{\partial}{\partial r_k} - \frac{p_k}{r_k} + 2\xi r_k \right) c_p = 0, \qquad k = 1, \ldots, n-1.
$$

These equations are easy to solve and their general solution has the form

$$
c_p' = g_p(\xi) \, r^p \, e^{-\xi |r|^2}, \qquad p \in \mathbb{Z}^{n-1},
$$

where $|r|^2 = r_1^2 + \cdots + r_{n-1}^2$.

Further, each function

$$
d_p = \chi_{\mathbb{R}_+}(\xi) \, g_p(\xi) \, r^p \, e^{-\xi(|r|^2+v)}
$$

has to be in $L_2(\mathbb{R}_+^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)$, and moreover we need that

$$
\{d_p\}_{p \in \mathbb{Z}^{n-1}} \in l_2(\mathbb{Z}^{n-1}, L_2(\mathbb{R}_+^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)).
$$

In particular, this implies that $d_p \equiv 0$ for all $p \in \mathbb{Z}^{n-1} \setminus \mathbb{Z}_+^{n-1}$.

We set

$$\alpha_p(\xi) = c_p(\xi) \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}},$$

where $c_p(\xi) \in L_2(\mathbb{R}_+)$ and is prolonged by zero to the negative half-axis, the constant c_λ is given by (2.2), $|p| = p_1 + \cdots + p_{n-1}$, $p! = p_1! \cdots p_{n-1}!$, and $p \in \mathbb{Z}_+^{n-1}$. Then it is easy to see that

$$\|d_p\|_{L_2(\mathbb{R}_+^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)} = \|c_p\|_{L_2(\mathbb{R}_+)},$$

and

$$\|\{d_p\}_{p \in Z_+^{n-1}}\|_{l_2(\mathbb{Z}_+^{n-1},\, L_2(\mathbb{R}_+^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda))} = \|c_p\|_{l_2(\mathbb{Z}_+^{n-1},\, L_2(\mathbb{R}_+))}.$$

Indeed,

$$\begin{aligned}
\|d_p\|^2 &= \int_{\mathbb{R}_+^{n+1}} |c_p(\xi)|^2 \frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p!\,\Gamma(\lambda+1)} r^{2p} e^{-2\xi(|r|^2+v)} \frac{c_\lambda}{4} v^\lambda\, r\,dr\,d\xi\,dv \\
&= \int_{\mathbb{R}_+^{n+1}} |c_p(\xi/2)|^2 \frac{\xi^{|p|+\lambda+n}}{2p!\,\Gamma(\lambda+1)} r^p e^{-\xi(r_1+\cdots+r_{n-1}+v)} v^\lambda\, dr\,d\xi\,dv \\
&= \int_{\mathbb{R}_+} |c_p(\xi/2)|^2 \frac{\xi^{|p|+\lambda+n}}{2p!\,\Gamma(\lambda+1)}\, d\xi \int_{\mathbb{R}_+} v^\lambda e^{-\xi v}\, dv \\
&\quad \cdot \int_{\mathbb{R}_+^{n-1}} r^p e^{-\xi(r_1+\cdots+r_{n-1})} dr.
\end{aligned}$$

By [4], formula 3.351.3, we have

$$\int_{\mathbb{R}_+^{n-1}} r^p e^{-\xi(r_1+\cdots+r_{n-1})} dr = \frac{p!}{\xi^{|p|+n-1}},$$

and by [4], formula 3.381.4, we have

$$\int_{\mathbb{R}_+} v^\lambda e^{-\xi v}\, dv = \frac{\Gamma(\lambda+1)}{\xi^{\lambda+1}}.$$

Thus

$$\|d_p\|^2_{L_2(\mathbb{R}_+^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)} = \int_{\mathbb{R}_+} |c_p(\xi/2)|^2 \frac{d\xi}{2} = \int_{\mathbb{R}_+} |c_p(\xi)|^2 d\xi = \|c_p\|^2_{L_2(\mathbb{R}_+)}.$$

Note that in terms of the scheme of Section 4 we have here

$$\begin{aligned}
X &= \mathbb{Z}^{n-1} \times \mathbb{R}, & L_2(X,\mu) &= l_2(\mathbb{Z}^{n-1}) \otimes L_2(\mathbb{R}), \\
X_1 &= \mathbb{Z}_+^{n-1} \times \mathbb{R}_+, & L_2(X_1,\mu) &= l_2(\mathbb{Z}_+^{n-1}) \otimes L_2(\mathbb{R}_+), \\
Y &= \mathbb{R}_+^{n-1} \times \mathbb{R}_+, & L_2(Y,\eta) &= L_2(\mathbb{R}_+^{n-1}, r\,dr) \otimes L_2(\mathbb{R}_+, \eta_\lambda),
\end{aligned}$$

the unitary operator U is defined as follows,

$$\begin{aligned}
U &= U_2 U_1 U_0 : L_2(D_n, \widetilde{\mu}_\lambda) \longrightarrow \\
&\quad l_2(\mathbb{Z}^{n-1}) \otimes L_2(\mathbb{R}_+^{n-1}, r\,dr) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)) \\
&= l_2(\mathbb{Z}^{n-1}, L_2(\mathbb{R}_+^{n-1}, r\,dr) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)),
\end{aligned}$$

and the function g_0 (function-sequence in this case) has the form

$$g_0(p, r, \xi, v) = \left\{ \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p! \, \Gamma(\lambda+1)} \right)^{\frac{1}{2}} r^p \, e^{-\xi(|r|^2+v)} \right\}_{p \in \mathbb{Z}_+^{n-1}},$$

where $(r, \xi, v) \in \mathbb{R}_+^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$.

Summarizing the above we come to the following statement.

Lemma 6.1. *The unitary operator $U = U_2 U_1 U_0$ maps the Bergman space $\mathcal{A}_\lambda^2(D_n)$ onto the space $\mathcal{A}_2(\mathcal{D}) = g_0 \, l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$ which is the closed subspace of $l_2(\mathbb{Z}^{n-1}, L_2(\mathbb{R}_+^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda))$ and consisits of all sequences $\{d_p(r, \xi, v)\}_{p \in \mathbb{Z}_+^{n-1}}$, where the functions $d_p(r, \xi, v)$, $p \in \mathbb{Z}_+^{n-1}$, have the form*

$$d_p(r, \xi, v) = \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p! \, \Gamma(\lambda+1)} \right)^{\frac{1}{2}} r^p \, e^{-\xi(|r|^2+v)} \, c_p(\xi), \qquad \xi \in \mathbb{R}_+,$$

with $c_p \in L_2(\mathbb{R}_+)$.

Introduce now the isometric imbedding

$$R_0 : l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \longrightarrow l_2(\mathbb{Z}^{n-1}, L_2(\mathbb{R}_+^{n-1}, r dr) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda))$$

by the rule

$$R_0 : \{c_p(\xi)\}_{p \in \mathbb{Z}_+^{n-1}} \longmapsto$$

$$\left\{ \chi_{\mathbb{Z}_+^{n-1}}(p) \chi_{\mathbb{R}_+}(\xi) \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p! \, \Gamma(\lambda+1)} \right)^{\frac{1}{2}} r^p \, e^{-\xi(|r|^2+v)} \, c_p(\xi) \right\}_{p \in \mathbb{Z}^{n-1}},$$

where the function $c_p(\xi)$ is extended by zero for $\xi \in \mathbb{R} \setminus \mathbb{R}_+$ for each $p \in \mathbb{Z}_+^{n-1}$.

The adjoint operator

$$R_0^* : l_2(\mathbb{Z}^{n-1}, L_2(\mathbb{R}_+^{n-1}, r dr) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$$

has obviously the form

$$R_0^* : \{d_p(r, \xi, v)\}_{p \in \mathbb{Z}^{n-1}} \longmapsto$$

$$\left\{ \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p! \, \Gamma(\lambda+1)} \right)^{\frac{1}{2}} \int_{\mathbb{R}_+^n} r^p \, e^{-\xi(|r|^2+v)} \, d_p(r, \xi, v) \, r dr \, \frac{c_\lambda v^\lambda}{4} \, dv \right\}_{p \in \mathbb{Z}_+^{n-1}}.$$

Then we have

$$R_0^* R_0 = I \quad : \quad l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)),$$
$$R_0 R_0^* = P_2 \quad : \quad l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+^{n-1}, r dr) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)) \longrightarrow \mathcal{A}_2(\mathcal{D}),$$

where P_2 is the orthogonal projection of $l_2(\mathbb{Z}^{n-1}, L_2(\mathbb{R}_+^{n-1}, r dr) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda))$ onto $\mathcal{A}_2(\mathcal{D})$.

Thus finally we have

Theorem 6.2. *The operator $R = R_0^* U$ maps $L_2(D_n, \widetilde{\mu}_\lambda)$ onto $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$, and the restriction*

$$R|_{\mathcal{A}_\lambda^2(D_n)} \; : \; \mathcal{A}_\lambda^2(D_n) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$$

is an isometric isomorphism.

   The adjoint operator

$$R^* = U^* R_0 \; : \; l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \longrightarrow \mathcal{A}_\lambda^2(D_n) \subset L_2(D_n, \widetilde{\mu}_\lambda)$$

is the isometric isomorphism of $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$ onto the subspace $\mathcal{A}_\lambda^2(D_n)$ of $L_2(D_n, \widetilde{\mu}_\lambda)$.

   Furthermore

$$RR^* = I \quad : \quad l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)),$$
$$R^* R = B_{D_n, \lambda} \quad : \quad L_2(D_n, \widetilde{\mu}_\lambda) \longrightarrow \mathcal{A}_\lambda^2(D_n),$$

where $B_{D_n, \lambda}$ is the Bergman projection of $L_2(D_n, \widetilde{\mu}_\lambda)$ onto $\mathcal{A}_\lambda^2(D_n)$.

Theorem 6.3. *The isometric isomorphism*

$$R^* = U^* R_0 : l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+)) \longrightarrow \mathcal{A}_\lambda^2(D_n)$$

is given by

$$R^* \; : \; \{c_p(\xi)\}_{p \in \mathbb{Z}_+^{n-1}} \; \longmapsto$$

$$(2\pi)^{-\frac{n}{2}} \sum_{p \in \mathbb{Z}_+^{n-1}} \int_{\mathbb{R}_+} \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}} c_p(\xi)\,(z')^p\,e^{i\xi z_n}\,d\xi. \qquad (6.4)$$

Proof. Calculate

$$R^* = U^* R_0 : \quad \{c_p(\xi)\}_{p \in \mathbb{Z}_+^{n-1}}$$

$$\longmapsto U^* \left\{ \chi_{\mathbb{R}_+}(\xi) \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}} r^p\,e^{-\xi(|r|^2+v)}\,c_p(\xi) \right\}_{p \in \mathbb{Z}_+^{n-1}}$$

$$\longmapsto U_0^* U_1^* \left(\chi_{\mathbb{R}_+}(\xi)\,(2\pi)^{-\frac{n-1}{2}} \sum_{p \in \mathbb{Z}_+^{n-1}} \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}} e^{-\xi(|r|^2+v)}\,c_p(\xi)\,(z')^p \right)$$

$$\longmapsto U_0^* \left((2\pi)^{-\frac{n}{2}} \sum_{p \in \mathbb{Z}_+^{n-1}} \int_{\mathbb{R}_+} \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}} e^{-\xi(|r|^2+v)}\,c_p(\xi)\,(z')^p e^{-iu\xi} d\xi \right)$$

$$= (2\pi)^{-\frac{n}{2}} \sum_{p \in \mathbb{Z}_+^{n-1}} \int_{\mathbb{R}_+} \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}} c_p(\xi)\,(z')^p\,e^{i\xi z_n}\,d\xi.$$

$\square$

Corollary 6.4. *The inverse isomorphism*

$$R : \mathcal{A}_\lambda^2(D_n) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$$

is given by

$$R : \varphi(z) \longmapsto \left\{ (2\pi)^{-\frac{n}{2}} \left(2^{n-3} c_\lambda \frac{(2\xi)^{|p|+\lambda+n}}{p!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}} \right.$$
$$\left. \cdot \int_{D_n} \varphi(z)\, (\overline{z'})^p \, e^{-i\xi \overline{z}_n} \, (\mathrm{Im}\, z_n - |z'|^2)^\lambda \, dv(z) \right\}_{p \in \mathbb{Z}_+^{n-1}}. \quad (6.5)$$

7. Nilpotent case

The first step here will be the same as in the quasi-parabolic case. We consider the space

$$L_2(\mathcal{D}, \eta_\lambda) = L_2(\mathbb{C}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda),$$

and the unitary operator $U_1 = I \otimes F \otimes I$ acting on it, where F is the Fourier transform on $L_2(\mathbb{R})$, see (6.1).

Then the image $\mathcal{A}_1(\mathcal{D}) = U_1(\mathcal{A}_0(\mathcal{D}))$ is the subspace of $L_2(\mathcal{D}, \eta_\lambda)$ which consists of all functions satisfying the equations (6.2). The $L_2(\mathcal{D}, \eta_\lambda)$-solution of the first equation in (6.2) has the form

$$\varphi(z', \xi, v) = \chi_{\mathbb{R}_+}(\xi)\psi(z', \xi)e^{-\xi v}, \quad (7.1)$$

and the function ψ has to satisfy the equations

$$\left(\frac{\partial}{\partial \overline{z}_k} + \xi z_k \right) \psi(z', \xi) = 0, \quad k = 1, \dots, n-1.$$

Using the standard Cartesian coordinates $x' = (x_1, \dots, x_{n-1})$ and $y' = (y_1, \dots, y_{n-1})$, where $z_k = x_k + iy_k$, in $\mathbb{C}^{n-1} = \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, we have

$$L_2(\mathcal{D}, \eta_\lambda) = L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda).$$

Consider the unitary operator $U_2 = F_{(n-1)} \otimes I \otimes I \otimes I$, where $F_{(n-1)} = F \otimes \cdots \otimes F$ is $(n-1)$-dimensional Fourier transform, acting on above tensor decomposition.

We have

$$U_2 \left[\frac{1}{2} \left(\frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k} \right) + \xi \, (x_k + iy_k) \right] U_2^{-1} = \frac{i}{2} \left(\xi_k + \frac{\partial}{\partial y_k} \right) + i\xi \left(\frac{\partial}{\partial \xi_k} + y_k \right).$$

Thus the image $\mathcal{A}_2(\mathcal{D}) = U_2(\mathcal{A}_1(\mathcal{D}))$ is the subspace of $L_2(\mathcal{D}, \eta_\lambda)$ which consists of all functions

$$\varphi(\xi', y', \xi, v) = \chi_{\mathbb{R}_+}(\xi)\psi(\xi', y', \xi)e^{-\xi v},$$

where the function ψ has to satisfy the equations

$$i \left[\frac{1}{2} \left(\xi_k + \frac{\partial}{\partial y_k} \right) + \xi \left(\frac{\partial}{\partial \xi_k} + y_k \right) \right] \psi(\xi', y', \xi) = 0, \quad k = 1, \dots, n-1.$$

Introduce the following change of variables,

$$u_k = \frac{1}{2\sqrt{\xi}}\xi_k - \sqrt{\xi}y_k, \quad v_k = \frac{1}{2\sqrt{\xi}}\xi_k + \sqrt{\xi}y_k, \quad k = 1, \ldots, n-1,$$

or

$$\xi_k = \sqrt{\xi}\left(u_k + v_k\right), \quad y_k = \frac{1}{2\sqrt{\xi}}\left(-u_k + v_k\right), \quad k = 1, \ldots, n-1,$$

and the corresponding unitary operator U_3 acting on $L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)$ by the rule

$$(U_3\varphi)(u', v', \xi, v) = \varphi\left(\sqrt{\xi}\left(u' + v'\right), \frac{1}{2\sqrt{\xi}}\left(-u' + v'\right), \xi, v\right),$$

where $u' = (u_1, \ldots, u_{n-1})$ and $v' = (v_1, \ldots, v_{n-1})$.

We have obviously

$$U_3\, i\left[\frac{1}{2}\left(\xi_k + \frac{\partial}{\partial y_k}\right) + \xi\left(\frac{\partial}{\partial \xi_k} + y_k\right)\right]U_3^{-1} = i\sqrt{\xi}\left(\frac{\partial}{\partial v_k} + v_k\right).$$

Thus the image $\mathcal{A}_3(\mathcal{D}) = U_3(\mathcal{A}_2(\mathcal{D}))$ is the subspace of $L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)$ which consists of all functions

$$\varphi(u', v', \xi, v) = \chi_{\mathbb{R}_+}(\xi)\widetilde{\psi}(u', v', \xi)e^{-\xi v},$$

satisfying the equations

$$i\sqrt{\xi}\left(\frac{\partial}{\partial v_k} + v_k\right)\varphi(u', v', \xi, v) = 0, \quad k = 1, \ldots, n-1,$$

or the corresponding functions function $\widetilde{\psi}(u', v', \xi)$ have to satisfy the equations

$$\left(\frac{\partial}{\partial v_k} + v_k\right)\widetilde{\psi}(u', v', \xi) = 0, \quad k = 1, \ldots, n-1.$$

These equations are easy to solve and their general solution has the form

$$\widetilde{\psi}(u', v', \xi) = \pi^{-\frac{n-1}{4}}\, e^{-\frac{|v'|^2}{2}}\, \chi_{\mathbb{R}_+}(\xi)\, e^{-\xi v}\left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda\Gamma(\lambda+1)}\right)^{\frac{1}{2}}\psi(u', \xi),$$

where $\psi(u', \xi) \in L_2(\mathbb{R}^{n-1} \times \mathbb{R})$. Moreover, we have that

$$
\begin{aligned}
\|\varphi\|_{L_2(\mathbb{R}^{n-1})\otimes L_2(\mathbb{R}^{n-1})\otimes L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+, \eta_\lambda)} &= \|\chi_{\mathbb{R}_+}(\xi)\psi(u', \xi)\|_{L_2(\mathbb{R}^{n-1}\times\mathbb{R})} \\
&= \|\psi(u', \xi)\|_{L_2(\mathbb{R}^{n-1}\times\mathbb{R}_+)}.
\end{aligned}
$$

Indeed,

$$
\begin{aligned}
\|\varphi\|^2 &= \int_{\mathbb{R}^{2n-1}\times\mathbb{R}_+} \pi^{-\frac{n-1}{2}} e^{-|v'|^2} \chi_{\mathbb{R}_+}(\xi)\, e^{-2\xi v} \frac{4(2\xi)^{\lambda+1}}{c_\lambda\Gamma(\lambda+1)}\, |\psi(u',\xi)|^2 \\
&\qquad \cdot \frac{c_\lambda}{4} v^\lambda du'dv'd\xi dv \\
&= \pi^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1}} e^{-|v'|^2}\, dv' \int_{\mathbb{R}^{n-1}\times\mathbb{R}_+} |\psi(u',\xi)|^2 \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)}\, du'd\xi \\
&\qquad \cdot \int_{\mathbb{R}_+} v^\lambda e^{-2\xi v} dv \\
&= \int_{\mathbb{R}^{n-1}\times\mathbb{R}_+} |\psi(u',\xi)|^2\, du'd\xi = \|\psi(u',\xi)\|^2_{L_2(\mathbb{R}^{n-1}\times\mathbb{R}_+)}.
\end{aligned}
$$

Calculating the integral over $v \in \mathbb{R}_+$ we have used formula 3.381.4 from [4].

Note that in terms of the scheme of Section 4 we have here

$$
\begin{aligned}
X &= \mathbb{R}^{n-1}\times\mathbb{R}, & L_2(X,\mu) &= L_2(\mathbb{Z}^{n-1})\otimes L_2(\mathbb{R}), \\
X_1 &= \mathbb{R}^{n-1}\times\mathbb{R}_+, & L_2(X_1,\mu) &= L_2(\mathbb{R}^{n-1})\otimes L_2(\mathbb{R}_+), \\
Y &= \mathbb{R}^{n-1}\times\mathbb{R}_+, & L_2(Y,\eta) &= L_2(\mathbb{R}^{n-1})\otimes L_2(\mathbb{R}_+,\eta_\lambda),
\end{aligned}
$$

the unitary operator U is defined as follows,

$$
U = U_3 U_2 U_1 U_0 : L_2(D_n,\widetilde{\mu}_\lambda) \to L_2(\mathbb{R}^{n-1})\otimes L_2(\mathbb{R}^{n-1})\otimes L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+,\eta_\lambda),
$$

and the function g_0 has the form

$$
g_0(v',\xi,v) = \pi^{-\frac{n-1}{4}} e^{-\xi v - \frac{|v'|^2}{2}} \left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda\Gamma(\lambda+1)}\right)^{\frac{1}{2}}, \quad (v',\xi,v) \in \mathbb{R}^{n-1}\times\mathbb{R}_+\times\mathbb{R}_+.
$$

Summarizing the above we come to the following statement.

Lemma 7.1. *The unitary operator* $U = U_3 U_2 U_1 U_0$ *maps the Bergman space* $\mathcal{A}^2_\lambda(D_n)$ *onto the space* $\mathcal{A}_3(\mathcal{D}) = g_0 L_2(\mathbb{R}^{n-1}\times\mathbb{R}_+)$ *which is the closed subspace of* $L_2(\mathbb{R}^{n-1})\otimes L_2(\mathbb{R}^{n-1})\otimes L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+,\eta_\lambda)$ *and consisits of all functions of the form*

$$
\varphi(u',v',\xi,v) = \pi^{-\frac{n-1}{4}} e^{-\xi v - \frac{|v'|^2}{2}} \left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda\Gamma(\lambda+1)}\right)^{\frac{1}{2}} \psi(u',\xi),
$$

where $\psi(u',\xi) \in L_2(\mathbb{R}^{n-1}\times\mathbb{R}_+)$.

Introduce now the isometric imbedding

$$
R_0 : L_2(\mathbb{R}^{n-1}\times\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}^{n-1})\otimes L_2(\mathbb{R}^{n-1})\otimes L_2(\mathbb{R})\otimes L_2(\mathbb{R}_+,\eta_\lambda)
$$

by the rule

$$
R_0 : \psi(u',\xi) \longmapsto \pi^{-\frac{n-1}{4}} e^{-\xi v - \frac{|v'|^2}{2}} \chi_{\mathbb{R}_+}(\xi) \left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda\Gamma(\lambda+1)}\right)^{\frac{1}{2}} \psi(u',\xi),
$$

where the function $\psi(u',\xi)$ is extended by zero for $\xi \in \mathbb{R}\setminus\mathbb{R}_+$ for each $u' \in \mathbb{R}^{n-1}$.

The adjoint operator

$$R_0^* : L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda) \longrightarrow L_2(\mathbb{R}^{n-1} \times \mathbb{R}_+)$$

has obviously the form

$$R_0^* \ : \ \varphi(u', v', \xi, v) \ \longmapsto$$

$$\pi^{-\frac{n-1}{4}} \int_{\mathbb{R}^{n-1} \times \mathbb{R}_+} e^{-\xi v - \frac{|v'|^2}{2}} \left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda \Gamma(\lambda+1)} \right)^{\frac{1}{2}} f(u', v', \xi, v) \, dv' \frac{c_\lambda}{4} \, v^\lambda dv.$$

Then we have

$$R_0^* R_0 = I \quad : \quad L_2(\mathbb{R}^{n-1} \times \mathbb{R}_+) \longrightarrow L_2(\mathbb{R}^{n-1} \times \mathbb{R}_+),$$
$$R_0 R_0^* = P_3 \quad : \quad L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda) \longrightarrow \mathcal{A}_3(\mathcal{D}),$$

where P_3 is the orthogonal projection of $L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)$ onto $\mathcal{A}_3(\mathcal{D})$.

Thus finally we have

Theorem 7.2. *The operator $R = R_0^* U$ maps $L_2(D_n, \widetilde{\mu}_\lambda)$ onto $L_2(\mathbb{R}^{n-1} \times \mathbb{R}_+)$, and the restriction*

$$R|_{\mathcal{A}_\lambda^2(D_n)} \ : \ \mathcal{A}_\lambda^2(D_n) \longrightarrow L_2(\mathbb{R}^{n-1} \times \mathbb{R}_+)$$

is an isometric isomorphism.

The adjoint operator

$$R^* = U^* R_0 \ : \ L_2(\mathbb{R}^{n-1} \times \mathbb{R}_+) \longrightarrow \mathcal{A}_\lambda^2(D_n) \subset L_2(D_n, \widetilde{\mu}_\lambda)$$

is the isometric isomorphism of $L_2(\mathbb{R}^{n-1} \times \mathbb{R}_+)$ onto the subspace $\mathcal{A}_\lambda^2(D_n)$ of $L_2(D_n, \widetilde{\mu}_\lambda)$.

Furthermore

$$RR^* = I \quad : \quad L_2(\mathbb{R}^{n-1} \times \mathbb{R}_+) \longrightarrow L_2(\mathbb{R}^{n-1} \times \mathbb{R}_+),$$
$$R^* R = B_{D_n, \lambda} \quad : \quad L_2(D_n, \widetilde{\mu}_\lambda) \longrightarrow \mathcal{A}_\lambda^2(D_n),$$

where $B_{D_n, \lambda}$ is the Bergman projection of $L_2(D_n, \widetilde{\mu}_\lambda)$ onto $\mathcal{A}_\lambda^2(D_n)$.

For this and the two remaining cases we will not give exact formulas for the operators R and R^*. If needed, these formulas can be easily obtained by direct though rather lengthy calculations.

8. Quasi-nilpotent case

This case is just a mixture of the two previous cases, quasi-parabolic and nilpotent.

Given an integer $1 \le k \le n-2$, we will write the points of D_n as $z = (z', w', z_n)$, where $z' \in \mathbb{C}^k$ and $w' \in \mathbb{C}^{n-k-1}$, and the points of $\mathcal{D}$ as (z', w', ζ), respectively.

According to this notation we represent

$$L_2(\mathcal{D}, \eta_\lambda) = L_2(\mathbb{C}^k) \otimes L_2(\mathbb{C}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda).$$

Applying, as in previous two cases, the unitary operator $U_1 = I \otimes I \otimes F \otimes I$, we have that the image $\mathcal{A}_1(\mathcal{D}) = U_1(\mathcal{A}_0(\mathcal{D}))$ consists of all L_2-functions of the form

$$\varphi(z', w', \xi, v) = \chi_{\mathbb{R}_+}(\xi)\psi(z', w', \xi)\, e^{-\xi v},$$

which satisfy the equations

$$\begin{aligned}
\left(\frac{\partial}{\partial \overline{z}_l} + \xi z_l\right)\psi(z', w', \xi) &= 0, \quad l = 1, \ldots, k, \\
\left(\frac{\partial}{\partial \overline{w}_m} + \xi w_m\right)\psi(z', w', \xi) &= 0, \quad m = 1, \ldots, n-k-1.
\end{aligned}$$

Now passing to the polar coordinates in $\mathbb{C}^k$, $z_l = r_l t_l$, where $r_l \in \mathbb{R}_+$, $t_l \in S^1 = \mathbb{T}$, $l = 1, \ldots, k$, and Cartesian coordinates in $\mathbb{C}^{n-k-1}$, $x' = (x_1, \ldots, x_{n-k-1})$, $y' = (y_1, \ldots, y_{n-k-1})$, where $w_m = x_m + iy_m$, $m = 1, \ldots, n-k-1$, we have that the space $L_2(\mathcal{D}, \eta_\lambda)$ can be represented in the form

$$L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{T}^k) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda).$$

Introduce the unitary operator $U_2 = I \otimes \mathcal{F}_{(k)} \otimes F_{(n-k-1)} \otimes I \otimes I \otimes I$ acting from $L_2(\mathcal{D}, \eta_\lambda)$ onto

$$\begin{aligned}
&L_2(\mathbb{R}_+^k, rdr) \otimes l_2(\mathbb{Z}^k \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda) \\
&= \; l_2(\mathbb{Z}^k, L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)),
\end{aligned}$$

where $\mathcal{F}_{(k)} = \mathcal{F} \otimes \cdots \otimes \mathcal{F}$ is the k-dimensional discrete Fourier transform and $F_{(n-k-1)} = F \otimes \cdots \otimes F$ is the $(n-k-1)$-dimensional Fourier transform.

Then, by the results of the previous two sections, the image $\mathcal{A}_2(\mathcal{D}) = U_2(\mathcal{A}_1(\mathcal{D}))$ consists of all sequences $\{d_p(r, \xi', y', \xi, v)\}_{p \in \mathbb{Z}_+^k}$, where the functions

$$d_p(r, \xi', y', \xi, v) = \left(\frac{2^{k+2}}{c_\lambda}\frac{(2\xi)^{|p|+\lambda+k+1}}{p!\,\Gamma(\lambda+1)}\right)^{\frac{1}{2}} r^p\, e^{-\xi(|r|^2+v)}\, \widetilde{d}_p(\xi', y', \xi),$$

with $(\xi', y', \xi) \in \mathbb{R}^{n-k-1} \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+$, belong to the space $L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)$. Moreover, the corresponding functions $\widetilde{d}_p(\xi', y', \xi)$ have to satisfy the equations

$$i\left[\frac{1}{2}\left(\xi_m + \frac{\partial}{\partial y_m}\right) + \xi\left(\frac{\partial}{\partial \xi_m} + y_m\right)\right]\widetilde{d}_p(\xi', y', \xi) = 0, \quad m = 1, \ldots, n-k-1.$$

Introduce the following change of variables

$$u_m = \frac{1}{2\sqrt{\xi}}\xi_m - \sqrt{\xi}y_m, \quad v_m = \frac{1}{2\sqrt{\xi}}\xi_m + \sqrt{\xi}y_m, \quad m = 1, \ldots, n-k-1,$$

or

$$\xi_m = \sqrt{\xi}\,(u_m + v_m), \quad y_m = \frac{1}{2\sqrt{\xi}}\,(-u_m + v_m), \quad m = 1, \ldots, n-k-1,$$

and the corresponding unitary operator U_3 acting on

$$l_2(\mathbb{Z}^k, L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda))$$

by the rule

$$U_3 : \{d_p(r, \xi', y', \xi, v)\}_{p \in \mathbb{Z}^k} \mapsto \left\{ d_p \left(r, \sqrt{\xi} \, (u' + v'), \frac{1}{2\sqrt{\xi}} (-u' + v'), \xi, v \right) \right\}_{p \in \mathbb{Z}^k},$$

where $u' = (u_1, \ldots, u_{n-k-1})$ and $v' = (v_1, \ldots, v_{n-k-1})$.

Combining the results of the previous two sections we have that in terms of the scheme of Section 4 our data now are as follows,

$$
\begin{aligned}
X &= \mathbb{Z}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}, & L_2(X, \mu) &= l_2(\mathbb{Z}^k) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}), \\
X_1 &= \mathbb{Z}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+, & L_2(X_1, \mu) &= l_2(\mathbb{Z}_+^k) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}_+), \\
Y &= \mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+, & L_2(Y, \eta) &= L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}_+, \eta_\lambda),
\end{aligned}
$$

the unitary operator U is defined as follows,

$$
\begin{aligned}
U &= U_3 U_2 U_1 U_0 \ : \ L_2(D_n, \widetilde{\mu}_\lambda) \longrightarrow \\
&\quad l_2(\mathbb{Z}^k, L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)),
\end{aligned}
$$

and the function g_0 (function-sequence in this case) has the form

$$g_0(p, r, v', \xi, v) = \pi^{-\frac{n-k-1}{4}} \left(\frac{2^{k+2}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+k+1}}{p! \, \Gamma(\lambda+1)} \right)^{\frac{1}{2}} r^p \, e^{-\xi(|r|^2+v) - \frac{|v'|^2}{2}},$$

where $(p, r, v', \xi, v) \in \mathbb{Z}_+^k \times \mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+ \times \mathbb{R}_+$.

Summarizing the above we come to the following statement.

Lemma 8.1. *The unitary operator $U = U_3 U_2 U_1 U_0$ maps the Bergman space $\mathcal{A}_\lambda^2(D_n)$ onto the space $\mathcal{A}_3(\mathcal{D}) = U_3(\mathcal{A}_2(\mathcal{D})) = g_0 \, l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))$ which is the closed subspace of $l_2(\mathbb{Z}^k, L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda))$ and consisits of all sequences $\{d_p(r, u', v', \xi, v)\}_{p \in \mathbb{Z}_+^k}$, where the functions $d_p(r, u', v', \xi, v)$, $p \in \mathbb{Z}_+^k$, have the form*

$$d_p(r, u', v', \xi, v) = \pi^{-\frac{n-k-1}{4}} \left(\frac{2^{k+2}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+k+1}}{p! \, \Gamma(\lambda+1)} \right)^{\frac{1}{2}} r^p \, e^{-\xi(|r|^2+v) - \frac{|v'|^2}{2}} \, c_p(u', \xi),$$

with $c_p(u', \xi) \in L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)$.

Moreover,

$$\|\{d_p\}_{p \in \mathbb{Z}_+^k}\| = \|\{c_p\}_{p \in \mathbb{Z}_+^k}\|_{l_2(\mathbb{Z}_+^k, \, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))}.$$

We check now the above norm equality. Obviously it is sufficient to check only that

$$\|d_p\|_{L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda)} = \|c_p\|_{L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)}.$$

Calculate

$$
\begin{aligned}
\|d_p\|^2 &= \pi^{-\frac{n-k-1}{2}} \int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}^{n-k-1} \times \mathbb{R} \times \mathbb{R}_+} \frac{2^{k+2}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+k+1}}{p!\,\Gamma(\lambda+1)} \\
&\quad \cdot\ r^{2p}\, e^{-2\xi(|r|^2+v)-|v'|^2}\, |c_p(u',\xi)|^2\, r dr\, du'\, dv'\, d\xi\, \frac{c_\lambda}{4}\, v^\lambda dv \\
&= \int_{\mathbb{R}^{n-k-1}\times\mathbb{R}_+} |c_p(u',\xi)|^2\, du' d\xi\, \frac{2^k (2\xi)^{|p|+k}}{p!} \int_{\mathbb{R}_+^k} r^{2k} e^{-2\xi|r|^2} r dr \\
&\quad \cdot\ \pi^{-\frac{n-k-1}{2}} \int_{\mathbb{R}^{n-k-1}} e^{-|v'|^2} dv'\, \frac{(2\xi)^{\lambda+1}}{\Gamma(\lambda+1)} \int_{\mathbb{R}_+} v^\lambda e^{-2\xi v} dv.
\end{aligned}
$$

By [4], formulas 3.351.3, 3.321.3, and 3.381.4, each of the last three integrals (with the corresponding multiple) is equal to 1, thus

$$
\|d_p\|^2 = \int_{\mathbb{R}^{n-k-1}\times\mathbb{R}_+} |c_p(u',\xi)|^2\, du' d\xi = \|c_p\|^2.
$$

Introduce the isometric imbedding R_0 of the space $l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+))$ into

$$
l_2(\mathbb{Z}^k, (L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda))
$$

which maps the sequence $\{c_p(u',\xi)\}_{p\in\mathbb{Z}_+^k}$ to

$$
\left\{ \pi^{-\frac{n-k-1}{4}} \chi_{\mathbb{Z}_+^k}(p) \chi_{\mathbb{R}_+}(\xi) \left(\frac{2^{k+2}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+k+1}}{p!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}} r^p\, e^{-\xi(|r|^2+v)-\frac{|v'|^2}{2}} c_p(u',\xi) \right\}_{p\in\mathbb{Z}^k},
$$

where the functions $c_p(u',\xi)$ is extended by zero for $\xi \in \mathbb{R}\setminus\mathbb{R}_+$ for each $u' \in \mathbb{R}^{n-k-1}$ and each $p \in \mathbb{Z}^k$.

The adjoint operator R_0^* acts from

$$
l_2(\mathbb{Z}^k, L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda))
$$

onto $l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+))$ as follows,

$$
\begin{aligned}
R_0^* \ : \ & \{d_p(r,u',v',\xi,v)\}_{p\in\mathbb{Z}^k} \longmapsto \left\{ \pi^{-\frac{n-k-1}{4}} \left(\frac{2^{k+2}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+k+1}}{p!\,\Gamma(\lambda+1)} \right)^{\frac{1}{2}} \right. \\
& \left. \cdot \int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} r^p\, e^{-\xi(|r|^2+v)-\frac{|v'|^2}{2}}\, d_p(r,u',v',\xi,v)\, r dr\, dv'\, \frac{c_\lambda v^\lambda}{4}\, dv \right\}_{p\in\mathbb{Z}_+^k}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
R_0^* R_0 &= I : l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)) \longrightarrow l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1}\times\mathbb{R}_+)) \\
R_0 R_0^* &= P_3,
\end{aligned}
$$

where P_3 is the orthogonal projection of

$$
l_2(\mathbb{Z}^k, L_2(\mathbb{R}_+^k, rdr) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}^{n-k-1}) \otimes L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+, \eta_\lambda))
$$

onto $\mathcal{A}_3(\mathcal{D})$.

Thus finally we have

Theorem 8.2. *The operator $R = R_0^* U$ maps $L_2(D_n, \widetilde{\mu}_\lambda)$ onto $l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))$, and the restriction*

$$R|_{\mathcal{A}_\lambda^2(D_n)} : \mathcal{A}_\lambda^2(D_n) \longrightarrow l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))$$

is an isometric isomorphism.
    The adjoint operator

$$R^* = U^* R_0 : l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)) \longrightarrow \mathcal{A}_\lambda^2(D_n) \subset L_2(D_n, \widetilde{\mu}_\lambda)$$

is the isometric isomorphism of $l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+))$ onto the subspace $\mathcal{A}_\lambda^2(D_n)$ of $L_2(D_n, \widetilde{\mu}_\lambda)$.
    Furthermore

$$RR^* = I : l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)) \longrightarrow l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)),$$

$$R^* R = B_{D_n, \lambda} : L_2(D_n, \widetilde{\mu}_\lambda) \longrightarrow \mathcal{A}_\lambda^2(D_n),$$

where $B_{D_n, \lambda}$ is the Bergman projection of $L_2(D_n, \widetilde{\mu}_\lambda)$ onto $\mathcal{A}_\lambda^2(D_n)$.

9. Quasi-hyperbolic case

We represent $D = \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$ in the form $\mathbb{C}^{n-1} \times \Pi$, where Π is the upper half-plane, and introduce in D the "non-isotropic" upper semi-sphere

$$\Omega = \{(z', \zeta) \in \mathbb{C}^{n-1} \times \Pi : |z'|^2 + |\zeta| = 1\}.$$

The points of Ω admit the natural parameterization

$$z_k = s_k t_k, \qquad \text{where } s_k \in [0, 1), \ t_k \in S^1, \ k = 1, \ldots, n-1,$$
$$\zeta = \rho e^{i\theta}, \qquad \text{where } \rho \in (0, 1], \ \theta \in (0, \pi),$$

and

$$\sum_{k=1}^{n-1} s_k^2 + \rho = 1,$$

which in turn induces the following representation of the points $(z', \zeta) \in D = \mathbb{C}^{n-1} \times \Pi$

$$z_k = r^{\frac{1}{2}} s_k t_k, \quad k = 1, \ldots, n-1, \qquad \zeta = r\rho e^{i\theta},$$

where $r \in \mathbb{R}_+$.

    We represent now $D = \tau(\mathbb{B}^{n-1}) \times \mathbb{T}^{n-1} \times \mathbb{R}_+ \times (0, \pi)$, where $\tau(\mathbb{B}^{n-1}) = \{s = (s_1, \ldots, s_{n-1}) \in \mathbb{R}_+^{n-1} : \sum_{k=1}^{n-1} s_k^2 < 1\}$ is the base (in the sense of a Reinhardt domain) of the unit ball $\mathbb{B}^{n-1}$, and $\mathbb{T}^{n-1} = S^1 \times \cdots \times S^1$ is the $n-1$ dimensional torus. Introduce the new coordinate system (s, t, r, θ) in D, where $s = (s_1, \ldots, s_{n-1}) \in \tau(\mathbb{B}^{n-1})$, $t = (t_1, \ldots, t_{n-1}) \in \mathbb{T}^{n-1}$, $r \in \mathbb{R}_+$, and $\theta \in (0, \pi)$, which is connected with the old one (z', ζ) by the formulas

$$s_k = \frac{|z_k|}{\sqrt{|z'|^2 + |\rho|}}, \quad t_k = \frac{z_k}{|z_k|}, \quad r = |z'|^2 + |\rho|, \quad \theta = \arg \zeta, \qquad (9.1)$$

or

$$z_k = r^{\frac{1}{2}} s_k t_k, \qquad \zeta = r(1 - |s|^2)e^{i\theta},$$

where $k = 1, \ldots, n-1$.

We pass now the operators in equations (2.10) to the new coordinate system. For a function $f = f(s_1, \ldots, s_{n-1}, t_1, \ldots, t_{n-1}, r, \theta)$, consider

$$
\begin{aligned}
\frac{\partial}{\partial \overline{\zeta}} f &= \frac{\cos\theta + i\sin\theta}{2} \left(\frac{\partial}{\partial |\zeta|} + i\frac{1}{|\zeta|}\frac{\partial}{\partial\theta} \right) \\
&\quad \cdot f\left(\frac{|z_1|}{\sqrt{|z'|^2 + |\rho|}}, \ldots, \frac{|z_{n-1}|}{\sqrt{|z'|^2 + |\rho|}}, t_1, \ldots, t_{n-1}, r, \theta \right) \\
&= \frac{\cos\theta + i\sin\theta}{2}\left(\frac{\partial f}{\partial r} - \frac{1}{2}\sum_{l=1}^{n-1}\frac{\partial f}{\partial s_l}\frac{|z_l|}{(|z'|^2 + |\zeta|)^{\frac{3}{2}}} + i\frac{1}{|\zeta|}\frac{\partial f}{\partial\theta} \right) \\
&= \frac{\cos\theta + i\sin\theta}{2}\left(\frac{\partial f}{\partial r} - \frac{1}{2}\sum_{l=1}^{n-1}\frac{s_l}{r}\frac{\partial f}{\partial s_l} + i\frac{1}{r(1-|s|^2)}\frac{\partial f}{\partial\theta} \right) \\
&= \frac{\cos\theta + i\sin\theta}{2r}\left(r\frac{\partial f}{\partial r} - \frac{1}{2}\sum_{l=1}^{n-1}s_l\frac{\partial f}{\partial s_l} + i\frac{1}{1-|s|^2}\frac{\partial f}{\partial\theta} \right).
\end{aligned}
$$

Further,

$$
\begin{aligned}
\frac{\partial f}{\partial \overline{z}_k} - iz_k\frac{\partial f}{\partial u} &= \frac{t_k}{2}\left(\frac{\partial f}{\partial |z_k|} - \frac{t_k}{|z_k|}\frac{\partial f}{\partial t_k} \right) - iz_k\left(\cos\theta\frac{\partial f}{\partial |\zeta|} - \frac{\sin\theta}{|\zeta|}\frac{\partial f}{\partial\theta} \right) \\
&= \frac{t_k}{2}\left(2|z_k|\frac{\partial f}{\partial r} - \frac{1}{2}\sum_{l=1}^{n-1}\frac{\partial f}{\partial s_l}\frac{2|z_l||z_k|}{(|z'|^2+|\zeta|)^{\frac{3}{2}}} + \frac{\partial f}{\partial s_k}\frac{1}{\sqrt{|z'|^2+|\zeta|}} - \frac{t_k}{|z_k|}\frac{\partial f}{\partial t_k} \right) \\
&\quad - iz_k\left[\cos\theta\left(\frac{\partial f}{\partial r} - \frac{1}{2}\sum_{l=1}^{n-1}\frac{\partial f}{\partial s_l}\frac{|z_l|}{(|z'|^2+|\zeta|)^{\frac{3}{2}}} \right) - \frac{\sin\theta}{|\zeta|}\frac{\partial f}{\partial\theta} \right] \\
&= t_k|z_k|\left(\frac{\partial f}{\partial r} - \frac{1}{2}\sum_{l=1}^{n-1}\frac{\partial f}{\partial s_l}\frac{|z_l|}{(|z'|^2+|\zeta|)^{\frac{3}{2}}} \right) + \frac{t_k}{2\sqrt{|z'|^2+|\zeta|}}\frac{\partial f}{\partial s_k} - \frac{t_k^2}{2|z_k|}\frac{\partial f}{\partial t_k} \\
&\quad - it_k|z_k|\left[\cos\theta\left(\frac{\partial f}{\partial r} - \frac{1}{2}\sum_{l=1}^{n-1}\frac{\partial f}{\partial s_l}\frac{|z_l|}{(|z'|^2+|\zeta|)^{\frac{3}{2}}} \right) - \frac{\sin\theta}{|\zeta|}\frac{\partial f}{\partial\theta} \right] \\
&= t_k r^{\frac{1}{2}} s_k\left(\frac{\partial f}{\partial r} - \frac{1}{2}\sum_{l=1}^{n-1}\frac{s_l}{r}\frac{\partial f}{\partial s_l} \right) + \frac{t_k}{2r^{\frac{1}{2}}}\frac{\partial f}{\partial s_k} - \frac{t_k^2}{2r^{\frac{1}{2}}s_k}\frac{\partial f}{\partial t_k} \\
&\quad - it_k r^{\frac{1}{2}} s_k\left[\cos\theta\left(\frac{\partial f}{\partial r} - \frac{1}{2}\sum_{l=1}^{n-1}\frac{s_l}{r}\frac{\partial f}{\partial s_l} \right) - \frac{\sin\theta}{r(1-|s|^2)}\frac{\partial f}{\partial\theta} \right]
\end{aligned}
$$

$$
= \quad \frac{t_k}{r^{\frac{1}{2}} s_k} \left\{ s_k^2 \left(r \frac{\partial f}{\partial r} - \frac{1}{2} \sum_{l=1}^{n-1} s_l \frac{\partial f}{\partial s_l} \right) + \frac{s_k}{2} \frac{\partial f}{\partial s_k} - \frac{t_k}{2} \frac{\partial f}{\partial t_k} \right.
$$
$$
\left. - i s_k^2 \left[\cos\theta \left(r \frac{\partial f}{\partial r} - \frac{1}{2} \sum_{l=1}^{n-1} s_l \frac{\partial f}{\partial s_l} \right) - \frac{\sin\theta}{1 - |s|^2} \frac{\partial f}{\partial \theta} \right] \right\}.
$$

That is, the equations (2.10) are equivalent to

$$
\left(r \frac{\partial}{\partial r} - \frac{1}{2} \sum_{l=1}^{n-1} s_l \frac{\partial}{\partial s_l} + i \frac{1}{1 - |s|^2} \frac{\partial}{\partial \theta} \right) f = 0 \tag{9.2}
$$

$$
\left\{ s_k^2 \left(r \frac{\partial f}{\partial r} - \frac{1}{2} \sum_{l=1}^{n-1} s_l \frac{\partial f}{\partial s_l} \right) + \frac{s_k}{2} \frac{\partial f}{\partial s_k} - \frac{t_k}{2} \frac{\partial f}{\partial t_k} \right.
$$
$$
\left. - i s_k^2 \left[\cos\theta \left(r \frac{\partial f}{\partial r} - \frac{1}{2} \sum_{l=1}^{n-1} s_l \frac{\partial f}{\partial s_l} \right) - \frac{\sin\theta}{1 - |s|^2} \frac{\partial f}{\partial \theta} \right] \right\} f = 0,
$$

where $k = 1, \ldots, n - 1$.

From the first of these equations we have

$$
r \frac{\partial f}{\partial r} - \frac{1}{2} \sum_{l=1}^{n-1} s_l \frac{\partial f}{\partial s_l} = -i \frac{1}{1 - |s|^2} \frac{\partial f}{\partial \theta},
$$

substituting into the second we obtain

$$
-i \frac{s_k^2}{1 - |s|^2} \frac{\partial f}{\partial \theta} + \frac{s_k}{2} \frac{\partial f}{\partial s_k} - \frac{t_k}{2} \frac{\partial f}{\partial t_k} - \frac{s_k^2 \cos\theta}{1 - |s|^2} \frac{\partial f}{\partial \theta} + i \frac{s_k^2 \sin\theta}{1 - |s|^2} \frac{\partial f}{\partial \theta} \quad =
$$
$$
\frac{s_k}{2} \frac{\partial f}{\partial s_k} - \frac{t_k}{2} \frac{\partial f}{\partial t_k} + i \frac{s_k^2}{1 - |s|^2} (\sin\theta + i \cos\theta - 1) \frac{\partial f}{\partial \theta} \quad = \quad 0.
$$

From the last equation, for each $k = 1, \ldots, n - 1$, we have

$$
\frac{s_k}{2} \frac{\partial f}{\partial s_k} = \frac{t_k}{2} \frac{\partial f}{\partial t_k} - i \frac{s_k^2}{1 - |s|^2} (\sin\theta + i \cos\theta - 1) \frac{\partial f}{\partial \theta}
$$

Summing up these equations for $k = 1, \ldots, n - 1$ and substituting to (9.2), we have

$$
r \frac{\partial f}{\partial r} - \frac{1}{2} \sum_{l=1}^{n-1} t_l \frac{\partial f}{\partial t_l} + i \left[1 + \frac{|s|^2}{1 - |s|^2} (\sin\theta + i \cos\theta) \right] \frac{\partial f}{\partial \theta} = 0.
$$

That is, finally the equations (2.10) are equivalent to

$$
r \frac{\partial f}{\partial r} - \frac{1}{2} \sum_{l=1}^{n-1} t_l \frac{\partial f}{\partial t_l} + i \left[1 + \frac{|s|^2}{1 - |s|^2} (\sin\theta + i \cos\theta) \right] \frac{\partial f}{\partial \theta} \quad = \quad 0
$$

$$
s_k \frac{\partial f}{\partial s_k} - t_k \frac{\partial f}{\partial t_k} + i \frac{2 s_k^2}{1 - |s|^2} (\sin\theta + i \cos\theta - 1) \frac{\partial f}{\partial \theta} \quad = \quad 0,
$$

where $k = 1, \ldots, n - 1$.

The direct calculation shows that under the change of variables (9.1) we have

$$dv(z', \zeta) = r^n (1 - |s|^2) \prod_{k=1}^{n-1} s_k ds_k \prod_{k=1}^{n-1} \frac{dt_k}{it_k} \, dr d\theta,$$

and

$$\eta_\lambda = \frac{c_\lambda}{4} r^\lambda (1 - |s|^2)^\lambda \frac{c_\lambda}{4} \sin^\lambda \theta.$$

The intermediate result obtained we formulate in the following lemma.

Lemma 9.1. *The space $\mathcal{A}_0(\mathcal{D}) = U_0(\mathcal{A}_\lambda^2(D_n))$ consists of all functions $f = f(s, t, r, \theta)$ which satisfy the equations*

$$r \frac{\partial f}{\partial r} - \frac{1}{2} \sum_{l=1}^{n-1} t_l \frac{\partial f}{\partial t_l} + i \left[1 + \frac{|s|^2}{1 - |s|^2} (\sin\theta + i\cos\theta) \right] \frac{\partial f}{\partial \theta} \quad = \quad 0 \qquad (9.3)$$

$$s_k \frac{\partial f}{\partial s_k} - t_k \frac{\partial f}{\partial t_k} + i \frac{2s_k^2}{1 - |s|^2} (\sin\theta + i\cos\theta - 1) \frac{\partial f}{\partial \theta} \quad = \quad 0, \qquad (9.4)$$

where $k = 1, \ldots, n-1$, and belong to the space

$$\begin{aligned}
L_2(\mathcal{D}, \eta_\lambda) \quad &= \quad L_2(\tau(\mathbb{B}^{n-1}), (1 - |s|^2)^{\lambda+1} s ds) \otimes L_2(\mathbb{T}^{n-1}) \\
&\quad \otimes L_2(\mathbb{R}_+, r^{\lambda+n} dr) \otimes L_2((0, \pi), \frac{c_\lambda}{4} \sin^\lambda \theta d\theta).
\end{aligned}$$

Introduce the unitary operator $U_1 = I \otimes \mathcal{F}_{(n-1)} \otimes M \otimes I$ which acts from the space

$$L_2(\tau(\mathbb{B}^{n-1}), (1 - |s|^2)^{\lambda+1} s ds) \otimes L_2(\mathbb{T}^{n-1}) \otimes L_2(\mathbb{R}_+, r^{\lambda+n} dr) \otimes L_2((0, \pi), \frac{c_\lambda}{4} \sin^\lambda \theta d\theta)$$

onto the space

$$\begin{aligned}
&L_2(\tau(\mathbb{B}^{n-1}), (1 - |s|^2)^{\lambda+1} s ds) \otimes l_2(\mathbb{Z}^{n-1}) \otimes L_2(\mathbb{R}) \otimes L_2((0, \pi), \frac{c_\lambda}{4} \sin^\lambda \theta d\theta) \\
= \quad &l_2(\mathbb{Z}^{n-1}, L_2(\tau(\mathbb{B}^{n-1}), (1 - |s|^2)^{\lambda+1} s ds) \otimes L_2(\mathbb{R}) \otimes L_2((0, \pi), \frac{c_\lambda}{4} \sin^\lambda \theta d\theta)),
\end{aligned}$$

where the Mellin transform $M : L_2(\mathbb{R}_+, r^{\lambda+n} dr) \longrightarrow L_2(\mathbb{R})$ is given by

$$(M\psi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} r^{-i\xi + \frac{\lambda+n-1}{2}} \psi(r) dr,$$

and $\mathcal{F}_{(n-1)} = \mathcal{F} \otimes \cdots \otimes \mathcal{F}$ is the $(n-1)$-dimensional discrete Fourier transform and each $\mathcal{F}$ is given by (5.1).

We note that

$$(M^{-1}\psi)(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} r^{-i\xi - \frac{\lambda+n+1}{2}} \psi(\xi) d\xi,$$

and

$$M r \frac{\partial}{\partial r} M^{-1} \psi \quad = \quad i \left(\xi + i \frac{\lambda + n + 1}{2} \right) \psi,$$

$$\mathcal{F}_k t_k \frac{\partial}{\partial t_k} \mathcal{F}_k^{-1} d_{p_k} \quad = \quad p_k d_{p_k}, \qquad p_k \in \mathbb{Z}.$$

Now the image $\mathcal{A}_1(\mathcal{D}) = U_1(\mathcal{A}_0(\mathcal{D}))$ consists of all sequences $d = \{d_p\}_{p \in \mathbb{Z}^{n-1}}$ the components

$$d_p = d_p(s, \xi, \theta) \in L_2(\tau(\mathbb{B}^{n-1}), (1 - |s|^2)^{\lambda+1} s\,ds) \otimes L_2(\mathbb{R}) \otimes L_2\left((0, \pi), \frac{c_\lambda}{4} \sin^\lambda \theta\, d\theta\right)$$

of which satisfy the equations

$$U_1 \left(r\frac{\partial}{\partial r} - \frac{1}{2} \sum_{l=1}^{n-1} t_l \frac{\partial}{\partial t_l} + i\left[1 + \frac{|s|^2}{1-|s|^2}(\sin\theta + i\cos\theta) \right] \frac{\partial}{\partial\theta} \right) U_1^{-1} d_p$$

$$= \left(\xi + i\frac{\lambda + n + 1}{2} \right) d_p + i\frac{|p|}{2} d_p + \left[1 + \frac{|s|^2}{1-|s|^2}(\sin\theta + i\cos\theta) \right] \frac{\partial d_p}{\partial\theta}$$

$$= 0, \tag{9.5}$$

where $|p| = p_1 + \cdots + p_{n-1}$, and

$$U_1 \left(s_k \frac{\partial}{\partial s_k} - t_k \frac{\partial}{\partial t_k} + i\frac{2s_k^2}{1-|s|^2}(\sin\theta + i\cos\theta - 1)\frac{\partial}{\partial\theta} \right) U_1^{-1} d_p =$$

$$s_k \frac{\partial d_p}{\partial s_k} - p_k d_p + i\frac{2s_k^2}{1-|s|^2}(\sin\theta + i\cos\theta - 1)\frac{\partial d_p}{\partial\theta} = 0, \tag{9.6}$$

where $k = 1, \dots, n-1$, $p = (p_1, \dots, p_{n-1}) \in \mathbb{Z}^{n-1}$.

Equation (9.5) is easy to solve. Using [4], formula 2.558.4, we have

$$d_p(s, \xi, \theta) = \widetilde{d}_p(s, \xi)\, e^{-2\left(\xi + i\frac{\lambda+n+|p|+1}{2}\right) \arctan\left[\left(1 - i\frac{|s|^2}{1-|s|^2}\right)\tan\frac{\theta}{2} + \frac{|s|^2}{1-|s|^2}\right]}.$$

Introduce the temporary notations

$$E = e^{-2\left(\xi + i\frac{\lambda+n+|p|+1}{2}\right) \arctan\left[\left(1 - i\frac{|s|^2}{1-|s|^2}\right)\tan\frac{\theta}{2} + \frac{|s|^2}{1-|s|^2}\right]},$$

$$\alpha = \frac{|s|^2}{1 - |s|^2}.$$

Then $d_p = \widetilde{d}_p E$. By (9.5) we have

$$\frac{\partial d_p}{\partial\theta} = -\frac{\xi + i\frac{\lambda+n+|p|+1}{2}}{1 + \alpha(\sin\theta + i\cos\theta)}\, \widetilde{d}_p E.$$

Calculate

$$\frac{\partial d_p}{\partial s_k} = \frac{\partial \widetilde{d}_p}{\partial s_k} E + \widetilde{d}_p \frac{\partial E}{\partial s_k}$$

$$= \frac{\partial \widetilde{d}_p}{\partial s_k} E - \widetilde{d}_p E\, 2 \left(\xi + i\frac{\lambda + n + |p| + 1}{2} \right) \frac{1 - i\tan\frac{\theta}{2}}{1 + \left[(1 - i\alpha)\tan\frac{\theta}{2} + \alpha\right]^2} \frac{\partial\alpha}{\partial s_k},$$

where

$$\frac{\partial\alpha}{\partial s_k} = \frac{(1 - |s|^2)2s_k + |s|^2 2s_k}{(1 - |s|^2)^2} = 2s_k(1 + \alpha)^2.$$

Thus

$$\frac{\partial d_p}{\partial s_k} = E\frac{\partial \widetilde{d}_p}{\partial s_k} - E\left(\xi + i\frac{\lambda + n + |p| + 1}{2}\right) 2s_k(1+\alpha)^2 A\, \widetilde{d}_p,$$

where

$$
\begin{aligned}
A &= \frac{2(1 - i\tan\frac{\theta}{2})}{1 + \left[(1 - i\alpha)\tan\frac{\theta}{2} + \alpha\right]^2} \\
&= \frac{2\cos^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + (1 - 2i\alpha - \alpha^2)\sin^2\frac{\theta}{2} + 2\alpha(1 - i\alpha)\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \alpha^2\cos^2\frac{\theta}{2}} \\
&= \frac{1 + \cos\theta - i\sin\theta}{1 + \alpha^2\cos\theta + \alpha(1 - i\alpha)\sin\theta - i\alpha(1 - \cos\theta)} \\
&= \frac{1 + \cos\theta - i\sin\theta}{(1 - i\alpha)[1 + \alpha(\sin\theta + i\cos\theta)]}.
\end{aligned}
$$

Thus finally

$$\frac{\partial d_p}{\partial s_k} = E\frac{\partial \widetilde{d}_p}{\partial s_k} - E\left(\xi + i\frac{\lambda + n + |p| + 1}{2}\right)\frac{2s_k(1+\alpha)^2(1 + \cos\theta - i\sin\theta)}{(1 - i\alpha)[1 + \alpha(\sin\theta + i\cos\theta)]}\, \widetilde{d}_p.$$

Substituting the above in (9.6) and canceling out E, we have

$$
\begin{aligned}
&s_k\frac{\partial \widetilde{d}_p}{\partial s_k} - \left(\xi + i\frac{\lambda + n + |p| + 1}{2}\right)\frac{2s_k^2(1+\alpha)^2(1 + \cos\theta - i\sin\theta)}{(1 - i\alpha)[1 + \alpha(\sin\theta + i\cos\theta)]}\, \widetilde{d}_p \\
&\quad - p_k\widetilde{d}_p - i\left(\xi + i\frac{\lambda + n + |p| + 1}{2}\right)\frac{2s_k^2(1+\alpha)(\sin\theta + i\cos\theta - 1)}{1 + \alpha(\sin\theta + i\cos\theta)}\, \widetilde{d}_p \\
&= s_k\frac{\partial \widetilde{d}_p}{\partial s_k} - p_k\widetilde{d}_p - \left(\xi + i\frac{\lambda + n + |p| + 1}{2}\right)\frac{2s_k^2(1+\alpha)}{(1 - i\alpha)[1 + \alpha(\sin\theta + i\cos\theta)]}\, B\, \widetilde{d}_p \\
&= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
B &= (1 + \alpha)(1 + \cos\theta - i\sin\theta) + (i + \alpha)(\sin\theta + i\cos\theta - 1) \\
&= (1 - i)[1 + \alpha(\sin\theta + i\cos\theta)].
\end{aligned}
$$

That is, we have

$$s_k\frac{\partial \widetilde{d}_p}{\partial s_k} - p_k\widetilde{d}_p - \left(\xi + i\frac{\lambda + n + |p| + 1}{2}\right)\frac{2s_k^2(1 - i)(1 + \alpha)}{1 - i\alpha}\, \widetilde{d}_p = 0,$$

where

$$\frac{1 + \alpha}{1 - i\alpha} = \frac{1}{(1 - |s|^2)\left(1 - \frac{i|s|^2}{1 - |s|^2}\right)} = \frac{1}{1 - (1 + i)|s|^2}.$$

Thus finally the equation (9.6) is reduced to

$$s_k\frac{\partial \widetilde{d}_p}{\partial s_k} - p_k\widetilde{d}_p - \left(\xi + i\frac{\lambda + n + |p| + 1}{2}\right)\frac{2(1 - i)s_k^2}{1 - (1 + i)|s|^2}\, \widetilde{d}_p = 0, \qquad (9.7)$$

where $k = 1, \ldots, n - 1$, $p = (p_1, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1}$.

The common general solution of (9.7) for $k = 1, \ldots, n-1$ is given by

$$\widetilde{d}_p = \widetilde{c}_p(\xi)\, s^p [1 - (1+i)|s|^2]^{-\frac{\lambda+n+|p|+1}{2}+i\xi}.$$

Thus the general solution of the equations (9.5) and (9.6) has the form

$$
\begin{aligned}
d_p(s,\xi,\theta) \;=\; & \widetilde{c}_p(\xi)\, s^p [1 - (1+i)|s|^2]^{-\frac{\lambda+n+|p|+1}{2}+i\xi} \\
& \cdot e^{-2\left(\xi + i\frac{\lambda+n+|p|+1}{2}\right)\arctan\left[\left(1 - i\frac{|s|^2}{1-|s|^2}\right)\tan\frac{\theta}{2} + \frac{|s|^2}{1-|s|^2}\right]}.
\end{aligned}
$$

But, for each $p \in \mathbb{Z}^{n-1}$, the function d_p has to be in $L_2(\tau(\mathbb{B}^{n-1}), (1-|s|^2)^{\lambda+1}sds) \otimes L_2(\mathbb{R}) \otimes L_2((0,\pi), \frac{c_\lambda}{4}\sin^\lambda\theta d\theta)$. This implies first that $d_p \equiv 0$ for all $p \in \mathbb{Z}^{n-1}\backslash\mathbb{Z}_+^{n-1}$. Second, introduce

$$
\begin{aligned}
\alpha_p(\xi) = \Bigg(& \int_{\tau(\mathbb{B}^{n-1})\times(0,\pi)} s^{2p}(1-|s|^2)^{\lambda+1}|1-(1-i)|s|^2|^{-(\lambda+n+|p|+1)+2i\xi} \\
& \cdot e^{-4\left(\xi + i\frac{\lambda+n+|p|+1}{2}\right)\arctan\left[\left(1 - i\frac{|s|^2}{1-|s|^2}\right)\tan\frac{\theta}{2} + \frac{|s|^2}{1-|s|^2}\right]} \frac{c_\lambda}{4}\sin^\lambda\theta\, sdsd\theta \Bigg)^{-\frac{1}{2}}. \quad (9.8)
\end{aligned}
$$

Then, setting $\widetilde{c}_p(\xi) = \alpha_p(\xi)\, c_p(\xi)$ with $c_p \in L_2(\mathbb{R})$, we have that for each $p \in \mathbb{Z}_+^{n-1}$

$$\|d_p\|_{L_2(\tau(\mathbb{B}^{n-1}),(1-|s|^2)^{\lambda+1}sds)\otimes L_2(\mathbb{R})\otimes L_2((0,\pi),\frac{c_\lambda}{4}\sin^\lambda\theta d\theta)} = \|c_p\|_{L_2(\mathbb{R})}.$$

Note that in terms of the scheme of Section 4 we have here

$$
\begin{aligned}
X &= \mathbb{Z}^{n-1}\times\mathbb{R}, & L_2(X,\mu) &= l_2(\mathbb{Z}^{n-1})\otimes L_2(\mathbb{R}), \\
X_1 &= \mathbb{Z}_+^{n-1}\times\mathbb{R}, & L_2(X_1,\mu) &= l_2(\mathbb{Z}_+^{n-1})\otimes L_2(\mathbb{R}_+), \\
Y &= \tau(\mathbb{B}^{n-1})\times(0,\pi), & L_2(Y,\eta) &= L_2(\tau(\mathbb{B}^{n-1}),(1-|s|^2)^{\lambda+1}sds) \\
& & & \otimes L_2((0,\pi), \tfrac{c_\lambda}{4}\sin^\lambda\theta d\theta),
\end{aligned}
$$

the unitary operator U is defined as follows,

$$
\begin{aligned}
U = U_1 U_0 \;&:\; L_2(D_n,\widetilde{\mu}_\lambda) \longrightarrow \\
& l_2\left(\mathbb{Z}_+^{n-1}, L_2(\tau(\mathbb{B}^{n-1}),(1-|s|^2)^{\lambda+1}sds)\otimes L_2(\mathbb{R})\otimes L_2((0,\pi), \tfrac{c_\lambda}{4}\sin^\lambda\theta d\theta)\right),
\end{aligned}
$$

and the function g_0 (function-sequence in this case) has the form

$$g_0(s,\xi,\theta) = \{g_0(p,s,\xi,\theta)\}_{p\in\mathbb{Z}_+^{n-1}},$$

where

$$
\begin{aligned}
g_0(p,s,\xi,\theta) \;=\; & \alpha_p(\xi)\, s^p [1 - (1+i)|s|^2]^{-\frac{\lambda+n+|p|+1}{2}+i\xi} \\
& \cdot e^{-2\left(\xi + i\frac{\lambda+n+|p|+1}{2}\right)\arctan\left[\left(1 - i\frac{|s|^2}{1-|s|^2}\right)\tan\frac{\theta}{2} + \frac{|s|^2}{1-|s|^2}\right]},
\end{aligned}
$$

here $p \in \mathbb{Z}_+^{n-1}$ and $(s,\xi,\theta) \in \tau(\mathbb{B}^{n-1})\times\mathbb{R}\times(0,\pi)$.

Summarizing the above we come to the following statement.

Lemma 9.2. *The unitary operator* $U = U_1 U_0$ *maps the Bergman space* $\mathcal{A}_\lambda^2(D_n)$ *onto the space* $\mathcal{A}_1(\mathcal{D}) = g_0 \, l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}))$ *which is the closed subspace of*

$$l_2(\mathbb{Z}_+^{n-1}, L_2(\tau(\mathbb{B}^{n-1}), (1-|s|^2)^{\lambda+1} s ds) \otimes L_2(\mathbb{R}) \otimes L_2((0,\pi), \frac{c_\lambda}{4}\sin^\lambda \theta d\theta))$$

and consisits of all sequences $\{d_p(s,\xi,\theta)\}_{p\in\mathbb{Z}_+^{n-1}}$, *where the functions* $d_p = d_p(s,\xi,\theta)$, $p \in \mathbb{Z}_+^{n-1}$, *have the form*

$$\begin{aligned}
d_p &= c_p(\xi)\,\alpha_p(\xi)\, s^p [1-(1+i)|s|^2]^{-\frac{\lambda+n+|p|+1}{2}+i\xi} \\
&\quad \cdot e^{-2\left(\xi+i\frac{\lambda+n+|p|+1}{2}\right)\arctan\left[\left(1-i\frac{|s|^2}{1-|s|^2}\right)\tan\frac{\theta}{2}+\frac{|s|^2}{1-|s|^2}\right]},
\end{aligned}$$

with $c_p \in L_2(\mathbb{R})$ *and* α_p *given by (9.8).*

 Moreover,

$$\begin{aligned}
&\|\{d_p\}\|_{l_2(\mathbb{Z}_+^{n-1}, L_2(\tau(\mathbb{B}^{n-1}),(1-|s|^2)^{\lambda+1}sds)\otimes L_2(\mathbb{R})\otimes L_2((0,\pi),\frac{c_\lambda}{4}\sin^\lambda\theta d\theta))} \\
&= \|\{c_p\}\|_{l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}))}.
\end{aligned}$$

Introduce the isometric imbedding R_0 of the space $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}))$ into the space

$$l_2(\mathbb{Z}_+^{n-1}, L_2(\tau(\mathbb{B}^{n-1}), (1-|s|^2)^{\lambda+1} s ds) \otimes L_2(\mathbb{R}) \otimes L_2((0,\pi), \frac{c_\lambda}{4}\sin^\lambda \theta d\theta))$$

by the rule

$$R_0 : \{c_p(\xi)\}_{p\in\mathbb{Z}_+^{n-1}} \longmapsto \{c_p(\xi)\,\alpha_p(\xi)\,\beta_p(s,\xi,\theta)\}_{p\in\mathbb{Z}^{n-1}},$$

where the functions $\beta_p = \beta_p(s,\xi,\theta)$ are given by

$$\beta_p = s^p[1-(1+i)|s|^2]^{-\frac{\lambda+n+|p|+1}{2}+i\xi} e^{-2\left(\xi+i\frac{\lambda+n+|p|+1}{2}\right)\arctan\left[\left(1-i\frac{|s|^2}{1-|s|^2}\right)\tan\frac{\theta}{2}+\frac{|s|^2}{1-|s|^2}\right]}. \tag{9.9}$$

We note that

$$\alpha_p(\xi) = \left(\int_{\tau(\mathbb{B}^{n-1})\times(0,\pi)} |\beta_p(s,\xi,\theta)|^2 (1-|s|^2)^{\lambda+1}\frac{c_\lambda}{4}\sin^\lambda\theta\, sdsd\theta\right)^{-\frac{1}{2}}.$$

The adjoint operator R_0^* which acts from

$$l_2(\mathbb{Z}_+^{n-1}, L_2(\tau(\mathbb{B}^{n-1}), (1-|s|^2)^{\lambda+1} s ds) \otimes L_2(\mathbb{R}) \otimes L_2((0,\pi), \frac{c_\lambda}{4}\sin^\lambda \theta d\theta))$$

onto the space $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}))$ has obviously the form

$$R_0^* \; : \; \{d_p(s,\xi,\theta)\}_{p\in\mathbb{Z}^{n-1}} \; \longmapsto$$

$$\left\{\alpha_p(\xi)\int_{\tau(\mathbb{B}^{n-1})\times(0,\pi)} \overline{\beta_p(s,\xi,\theta)}\, d_p(s,\xi,\theta)\,(1-|s|^2)^{\lambda+1}\frac{c_\lambda}{4}\sin^\lambda\theta\, sdsd\theta\right\}_{p\in\mathbb{Z}_+^{n-1}}.$$

Then we have

$$\begin{aligned} R_0^* R_0 &= I : \; l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R})) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R})),\\ R_0 R_0^* &= P_1, \end{aligned}$$

where P_1 is the orthogonal projection of $l_2(\mathbb{Z}_+^{n-1}, L_2(\tau(\mathbb{B}^{n-1}), (1-|s|^2)^{\lambda+1}sds)\otimes L_2(\mathbb{R})\otimes L_2((0,\pi), \frac{c_\lambda}{4}\sin^\lambda\theta d\theta))$ onto $\mathcal{A}_1(\mathcal{D})$.

Then finally we have

Theorem 9.3. *The operator $R = R_0^* U$ maps $L_2(D_n, \widetilde{\mu}_\lambda)$ onto $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}))$, and the restriction*

$$R|_{\mathcal{A}_\lambda^2(D_n)} \; : \; \mathcal{A}_\lambda^2(D_n) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}))$$

is an isometric isomorphism.

The adjoint operator

$$R^* = U^* R_0 \; : \; l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R})) \longrightarrow \mathcal{A}_\lambda^2(D_n) \subset L_2(D_n, \widetilde{\mu}_\lambda)$$

is the isometric isomorphism of $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}))$ onto the subspace $\mathcal{A}_\lambda^2(D_n)$ of $L_2(D_n, \widetilde{\mu}_\lambda)$.

Furthermore

$$\begin{aligned} RR^* = I &: \; l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R})) \longrightarrow l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R})),\\ R^* R = B_{D_n,\lambda} &: \; L_2(D_n, \widetilde{\mu}_\lambda) \longrightarrow \mathcal{A}_\lambda^2(D_n), \end{aligned}$$

where $B_{D_n,\lambda}$ is the Bergman projection of $L_2(D_n, \widetilde{\mu}_\lambda)$ onto $\mathcal{A}_\lambda^2(D_n)$.

10. Toeplitz operators with special symbols

In this section we show that in each case of the previous five sections there exists a class of bounded measurable symbols a, such that the corresponding Toeplitz operators T_a are unitary equivalent to certain multiplication operators $\gamma_a I$. In each case the symbols are invariant with respect to the action of the corresponding commutative subgroup of Section 3. The specific form of γ_a and the space in which this multiplication operator acts depend essentially on the case under consideration. This fact implies an important joint feature, in each case *the C^*-algebra generated by corresponding Toeplitz operators is commutative.* Furthermore, being unitary equivalent to a multiplication operator $\gamma_a I$ such a Toeplitz operator thus admits a spectral type representation, which gives an easy access to its important properties: boundedness, compactness, spectral properties, invariant subspaces, etc.

10.1. Quasi-elliptic case

We will call a function $a(z)$, $z \in D_n$, *quasi-elliptic* if it is separately radial, i.e., $a(z) = a(r) = a(r_1, \ldots, r_n)$, or equivalently if a is invariant under the action of the quasi-elliptic group. The following result has been proved in [10].

Theorem 10.1. *Let $a = a(r)$ be a bounded measurable quasi-elliptic function. Then the Toeplitz operator T_a acting on $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ is unitary equivalent to the multiplication operator $\gamma_a I = R T_a R^*$ acting on $l_2(\mathbb{Z}_+^n)$, where R and R^* are given by (5.3) and (5.2), respectively. The sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(p)\}_{p \in \mathbb{Z}_+^n}$ is given by*

$$
\begin{aligned}
\gamma_{a,\lambda}(p) &= \frac{2^n \, \Gamma(n + |p| + \lambda + 1)}{p! \, \Gamma(\lambda + 1)} \int_{\tau(\mathbb{B}^n)} a(r) \, r^{2p} \, (1 - r^2)^\lambda \prod_{k=1}^n r_k dr_k \\
&= \frac{\Gamma(n + |p| + \lambda + 1)}{p! \, \Gamma(\lambda + 1)} \int_{\Delta(\mathbb{B}^n)} a(\sqrt{r}) \, r^p \, (1 - (r_1 + \cdots + r_n))^\lambda \, dr,
\end{aligned}
$$

where $p \in \mathbb{Z}_+^n$, $\Delta(\mathbb{B}^n) = \{r = (r_1, \ldots, r_n) : r_1 + \cdots + r_n \in [0,1), \ r_k \geq 0, \ k = 1, \ldots, n\}$, $dr = dr_1 \cdots dr_n$, and $\sqrt{r} = (\sqrt{r_1}, \ldots, \sqrt{r_n})$.

10.2. Quasi-parabolic case

We will call a function $a(z)$, $z \in D_n$, *quasi-parabolic* if $a(z) = a(r, y_n) = a(r_1, \ldots, r_{n-1}, \operatorname{Im} z_n)$, i.e., a is invariant under the action of the quasi-parabolic group.

Theorem 10.2. *Let $a = a(r, y_n)$ be a bounded measurable quasi-parabolic function. Then the Toeplitz operator T_a acting on $\mathcal{A}_\lambda^2(D_n)$ is unitary equivalent to the multiplication operator $\gamma_a I = R T_a R^*$ acting on $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}_+))$, where R and R^* are given by (6.5) and (6.4), respectively. The sequence $\gamma_a = \{\gamma_a(p, \xi)\}_{p \in \mathbb{Z}_+^{n-1}}$, $\xi \in \mathbb{R}_+$, is given by*

$$
\gamma_a(p, \xi) = \frac{(2\xi)^{|p| + \lambda + n}}{p! \, \Gamma(\lambda + 1)} \int_{\mathbb{R}_+^n} a(\sqrt{r}, v + r_1 + \cdots + r_{n-1}) \, r^p \, e^{-2\xi(v + r_1 + \cdots + r_{n-1})} \, v^\lambda \, dr dv,
$$

(10.1)

where $\sqrt{r} = (\sqrt{r}_1, \ldots, \sqrt{r}_{n-1})$.

Proof. The operator T_a is obviously unitary equivalent to the operator

$$
\begin{aligned}
R T_a R^* &= R B_{D_n, \lambda} a B_{D_n, \lambda} R^* = R(R^* R) a (R^* R) R^* \\
&= (R R^*) R a R^* (R R^*) = R a R^* \\
&= R_0^* U_2 U_1 U_0 \, a(r, y_n) U_0^{-1} U_1^{-1} U_2^{-1} R_0 \\
&= R_0^* U_2 U_1 \, a(r, v + |r|^2) \, U_1^{-1} U_2^{-1} R_0 \\
&= R_0^* \, a(r, v + |r|^2) R_0 \\
&= T.
\end{aligned}
$$

Now, for $c = \{c_p(\xi)\}_{p \in \mathbb{Z}_+^{n-1}}$, we have

$$
\begin{aligned}
Tc &= R_0^* \left\{ a(r, v + |r|^2) \chi_{\mathbb{R}_+}(\xi) \left(\frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p!\, \Gamma(\lambda+1)} \right)^{\frac{1}{2}} r^p \, e^{-\xi(|r|^2+v)} \, c_p(\xi) \right\}_{p \in \mathbb{Z}^{n-1}} \\
&= \left\{ \frac{2^{n+1}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+n}}{p!\, \Gamma(\lambda+1)} \int_{\mathbb{R}_+^n} a(r, v + |r|^2) \, r^{2p} \, e^{-2\xi(|r|^2+v)} \right. \\
&\qquad \left. \cdot c_p(\xi) \frac{c_\lambda}{4} \, v^\lambda \, rdrdv \right\}_{p \in \mathbb{Z}_+^{n-1}} \\
&= \left\{ \frac{(2\xi)^{|p|+\lambda+n}}{p!\, \Gamma(\lambda+1)} \int_{\mathbb{R}_+^n} a(\sqrt{r}, v + r_1 + \cdots + r_{n-1}) \, r^p \right. \\
&\qquad \left. \cdot e^{-2\xi(v+r_1+\cdots+r_{n-1})} \, c_p(\xi) \, v^\lambda \, drdv \right\}_{p \in \mathbb{Z}_+^{n-1}} \\
&= \left\{ \gamma_a(p, \xi) \cdot c_p(\xi) \right\}_{p \in \mathbb{Z}_+^{n-1}},
\end{aligned}
$$

with

$$
\gamma_a(p, \xi) = \frac{(2\xi)^{|p|+\lambda+n}}{p!\, \Gamma(\lambda+1)} \int_{\mathbb{R}_+^n} a(\sqrt{r}, v + r_1 + \cdots + r_{n-1}) \, r^p \, e^{-2\xi(v+r_1+\cdots+r_{n-1})} \, v^\lambda \, drdv,
$$

where $p \in \mathbb{Z}_+^{n-1}$, $\xi \in \mathbb{R}_+$, and $\sqrt{r} = (\sqrt{r}_1, \ldots, \sqrt{r}_{n-1})$. $\square$

10.3. Nilpotent case

Recall that the nilpotent group $\mathbb{R}^{n-1} \times \mathbb{R}$ acts on D_n as follows. For $(b, h) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$
\tau_{(b,h)} : \ (z', z_n) \longmapsto (z' + b, z_n + h + 2iz' \cdot b + i|b|^2).
$$

We note that both quantities $y' = \operatorname{Im} z'$ and $\operatorname{Im} z_n - |z'|^2$ are invariant under the action of this group.

We will call a function $a(z)$, $z \in D_n$, *nilpotent* if $a(z) = a(y', \operatorname{Im} z_n - |z'|^2)$, i.e., a is invariant under the action of the nilpotent group.

Theorem 10.3. *Let* $a = a(y', \operatorname{Im} z_n - |z'|^2)$ *be a bounded measurable nilpotent function. Then the Toeplitz operator* T_a *acting on* $\mathcal{A}_\lambda^2(D_n)$ *is unitary equivalent to the multiplication operator* $\gamma_a I = R T_a R^*$ *acting on* $L_2(\mathbb{R}^{n-1} \times \mathbb{R}_+)$, *where* R *and* R^* *are given in Section 7. The function* $\gamma_a = \gamma_a(u', \xi)$, *where* $u' \in \mathbb{R}^{n-1}$ *and* $\xi \in \mathbb{R}_+$, *is given by*

$$
\gamma_a(u', \xi) = \frac{(2\xi)^{\lambda+1}}{\pi^{\frac{n-1}{2}} \Gamma(\lambda+1)} \int_{\mathbb{R}^{n-1} \times \mathbb{R}_+} a\left(\frac{1}{2\sqrt{\xi}}(-u' + v'), v\right) e^{-2\xi v - |v'|^2} \, v^\lambda dv' dv.
\tag{10.2}
$$

Proof. The operator T_a is obviously unitary equivalent to the operator

$$
\begin{aligned}
R\, T_a\, R^* &= R\, B_{D_n,\lambda}\, a\, B_{D_n,\lambda}\, R^* = R(R^*R)a(R^*R)R^* \\
&= (RR^*)RaR^*(RR^*) = RaR^* \\
&= R_0^* U_3 U_2 U_1 U_0\, a(y', \operatorname{Im} z_n - |z'|^2) U_0^{-1} U_1^{-1} U_2^{-1} U_3^{-1} R_0 \\
&= R_0^* U_3 U_2 U_1\, a(y', v)\, U_1^{-1} U_2^{-1} U_3^{-1} R_0 \\
&= R_0^* U_3\, a(y', v) U_3^{-1} R_0 \\
&= R_0^*\, a(\frac{1}{2\sqrt{\xi}}(-u' + v'), v) R_0 \\
&= T.
\end{aligned}
$$

Now,

$$
\begin{aligned}
T\psi &= R_0^* \left[a(\frac{1}{2\sqrt{\xi}}(-u' + v'), v)\, \pi^{-\frac{n-1}{4}}\, e^{-\xi v - \frac{|v'|^2}{2}}\, \chi_{\mathbb{R}_+}(\xi) \right. \\
&\qquad\qquad \left. \cdot \left(\frac{4(2\xi)^{\lambda+1}}{c_\lambda \Gamma(\lambda+1)} \right)^{\frac{1}{2}} \psi(u', \xi) \right] \\
&= \pi^{-\frac{n-1}{2}} \int_{\mathbb{R}^{n-1} \times \mathbb{R}_+} e^{-2\xi v - |v'|^2}\, \frac{4(2\xi)^{\lambda+1}}{c_\lambda \Gamma(\lambda+1)}\, a(\frac{1}{2\sqrt{\xi}}(-u' + v'), v) \\
&\qquad\qquad \cdot \psi(u', \xi)\, \frac{c_\lambda}{4}\, v^\lambda dv' dv \\
&= \gamma_a(u', \xi) \cdot \psi(u', \xi),
\end{aligned}
$$

with

$$
\gamma_a(u', \xi) = \frac{(2\xi)^{\lambda+1}}{\pi^{\frac{n-1}{2}} \Gamma(\lambda+1)} \int_{\mathbb{R}^{n-1} \times \mathbb{R}_+} a(\frac{1}{2\sqrt{\xi}}(-u' + v'), v)\, e^{-2\xi v - |v'|^2}\, v^\lambda dv' dv,
$$

where $u' = (u_1, \ldots, u_{n-1}) \in \mathbb{R}_+^{n-1}$ and $\xi \in \mathbb{R}_+$. $\qquad\qquad\qquad\square$

10.4. Quasi-nilpotent case

For an integer $1 \le k \le n-2$, we keep using the notation $z = (z', w', z_n)$ for points of D_n, where $z' \in \mathbb{C}^k$ and $w' \in \mathbb{C}^{n-k-1}$. Recall that the quasi-nilpotent group $\mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$ acts on D_n as follows. For $(t, a, h) \in \mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$,

$$
\tau_{(t,b,h)} : \; (z', w', z_n) \longmapsto (tz', w' + b, z_n + h + 2iw' \cdot b + i|b|^2).
$$

We note that the quantities r, where $r = (r_1, \ldots, r_k)$ with $r_l = |z_l|$, $y' = \operatorname{Im} w'$, and $\operatorname{Im} z_n - |w'|^2$ are invariant under the action of this group.

We will call a function $a(z)$, $z \in D_n$, *quasi-nilpotent* if $a(z) = a(r, y', \operatorname{Im} z_n - |w'|^2)$, i.e., a is invariant under the action of the quasi-nilpotent group, corresponding to the above parameter k.

Theorem 10.4. *Let $a = a(r, y', \operatorname{Im} z_n - |w'|^2)$ be a bounded measurable quasi-nilpotent function. Then the Toeplitz operator T_a acting on $\mathcal{A}_\lambda^2(D_n)$ is unitary equivalent to the multiplication operator $\gamma_a I = R T_a R^*$ acting on*

$$l_2(\mathbb{Z}_+^k, L_2(\mathbb{R}^{n-k-1} \times \mathbb{R}_+)),$$

where R and R^* are given in Section 8. The sequence $\gamma_a = \{\gamma_a(p, u', \xi)\}_{p \in \mathbb{Z}_+^k}$, $(u', \xi) \in \mathbb{R}^{n-k-1} \times \mathbb{R}_+$, is given by

$$
\begin{aligned}
\gamma_a(p, u', \xi) &= \pi^{-\frac{n-k-1}{2}} \frac{(2\xi)^{|p|+\lambda+k+1}}{p! \, \Gamma(\lambda+1)} \\
&\quad \cdot \int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} a(\sqrt{r}, \frac{1}{2\sqrt{\xi}}(-u'+v'), v + r_1 + \cdots + r_k) \\
&\quad \cdot r^p \, e^{-2\xi(v+r_1+\cdots+r_k) - |v'|^2} \, v^\lambda \, dr \, dv' \, dv,
\end{aligned}
\tag{10.3}
$$

where $\sqrt{r} = (\sqrt{r}_1, \ldots, \sqrt{r}_k)$.

Proof. The operator T_a is obviously unitary equivalent to the operator

$$
\begin{aligned}
R T_a R^* &= R B_{D_n, \lambda} a B_{D_n, \lambda} R^* = R(R^*R) a (R^*R) R^* \\
&= (RR^*) R a R^* (RR^*) = R a R^* \\
&= R_0^* U_3 U_2 U_1 U_0 \, a \, U_0^{-1} U_1^{-1} U_2^{-1} U_3^{-1} R_0 \\
&= R_0^* U_3 U_2 U_1 \, a(r, y', v + |r|^2) \, U_1^{-1} U_2^{-1} U_3^{-1} R_0 \\
&= R_0^* U_3 \, a(r, y', v + |r|^2) \, U_3^{-1} R_0 \\
&= R_0^* \, a(r, \frac{1}{2\sqrt{\xi}}(-u'+v'), v + |r|^2) R_0 \\
&= T.
\end{aligned}
$$

We have

$$
\begin{aligned}
T\{c_p(u', \xi)\}_{p \in \mathbb{Z}_+^k} &= R_0^* \left\{ a(r, \frac{1}{2\sqrt{\xi}}(-u'+v'), v + |r|^2) \pi^{-\frac{n-k-1}{4}} \chi_{\mathbb{Z}_+^k}(p) \chi_{\mathbb{R}_+}(\xi) \right. \\
&\quad \left. \cdot \left(\frac{2^{k+2}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+k+1}}{p! \, \Gamma(\lambda+1)} \right)^{\frac{1}{2}} r^p \, e^{-\xi(|r|^2+v) - \frac{|v'|^2}{2}} c_p(u', \xi) \right\}_{p \in \mathbb{Z}^k} \\
&= \left\{ \pi^{-\frac{n-k-1}{2}} \frac{2^{k+2}}{c_\lambda} \frac{(2\xi)^{|p|+\lambda+k+1}}{p! \, \Gamma(\lambda+1)} \right. \\
&\quad \cdot \int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} a(r, \frac{1}{2\sqrt{\xi}}(-u'+v'), v + |r|^2) \\
&\quad \left. \cdot r^{2p} \, e^{-2\xi(|r|^2+v) - |v'|^2} c_p(u', \xi) \, r \, dr \, dv' \, \frac{c_\lambda}{4} v^\lambda \, dv \right\}_{p \in \mathbb{Z}_+^k}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \Bigg\{ \pi^{-\frac{n-k-1}{2}} \frac{(2\xi)^{|p|+\lambda+k+1}}{p!\,\Gamma(\lambda+1)} \\
& \cdot \int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} a\left(\sqrt{r}, \frac{1}{2\sqrt{\xi}}(-u'+v'), v+r_1+\cdots+r_k\right) \\
& \cdot r^p\, e^{-2\xi(v+r_1+\cdots+r_k)-|v'|^2}\, c_p(u',\xi)\, v^\lambda\, dr dv' dv \Bigg\}_{p \in \mathbb{Z}_+^k} \\
= \quad & \left\{ \gamma_a(p,u',\xi) \cdot c_p(u',\xi) \right\}_{p \in \mathbb{Z}_+^k},
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_a(p,u',\xi) \quad = \quad & \pi^{-\frac{n-k-1}{2}} \frac{(2\xi)^{|p|+\lambda+k+1}}{p!\,\Gamma(\lambda+1)} \\
& \cdot \int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}_+} a\left(\sqrt{r}, \frac{1}{2\sqrt{\xi}}(-u'+v'), v+r_1+\cdots+r_k\right) \\
& \cdot r^p\, e^{-2\xi(v+r_1+\cdots+r_k)-|v'|^2}\, v^\lambda\, dr dv' dv
\end{aligned}
$$

where $p \in \mathbb{Z}_+^k$, $u' \in \mathbb{R}_+^{n-k-1}$, $\xi \in \mathbb{R}_+$, and $\sqrt{r} = (\sqrt{r}_1, \ldots, \sqrt{r}_k)$. $\qquad\square$

10.5. Quasi-hyperbolic case

Recall that the quasi-hyperbolic group $\mathbb{T}^{n-1} \times \mathbb{R}_+$ acts on D_n as follows. For $(t,r) \in \mathbb{T}^{n-1} \times \mathbb{R}_+$,

$$
\tau_{(t,r)} : \ (z',z_n) \longmapsto (r^{\frac{1}{2}} t z', r z_n).
$$

We will call a function $a(z)$, $z \in D_n$, *quasi-hyperbolic* if a is invariant under the action of this group.

A convenient in our context way to describe such invariant functions is as follows. Consider the group of non-isotropic dilations $\{\delta_r\}$, $r \in \mathbb{R}_+$, acting on $\mathbb{R}_+^{n-1} \times \Pi$ by the rule

$$
\delta_r : (q_1, \ldots, q_{n-1}, \zeta) \longmapsto (r^{\frac{1}{2}} q_1, \ldots, r^{\frac{1}{2}} q_{n-1}, r\zeta).
$$

Then each function $\tilde{a} = \tilde{a}(q_1, \ldots, q_{n-1}, \zeta)$ with is non-isotropic homogeneous of zero order on $\mathbb{R}_+^{n-1} \times \Pi$ depends only on its values on the non-isotropic upper half sphere

$$
\Omega_+ = \left\{ (q_1, \ldots, q_{n-1}, \zeta) \in \mathbb{R}_+^{n-1} \times \Pi : \sum_{k=1}^{n-1} q_k^2 + |\zeta| = 1 \right\},
$$

and thus, passing to the polar coordinates in the upper half-plane Π, is a function of the form

$$
\tilde{a} = \tilde{a}(q_1, \ldots, q_{n-1}, \rho, \theta) = \tilde{a}\left(\frac{q_1}{\sqrt{|q|^2+\rho}}, \ldots, \frac{q_{n-1}}{\sqrt{|q|^2+\rho}}, \frac{\rho}{|q|^2+\rho}, \theta \right),
$$

where $|q|^2 = \sum_{k=1}^{n-1} q_k^2$, $\rho = |\zeta|$, and $\theta = \arg \zeta$.

Further, we parameterize the points of Ω_+ by points $s = (s_1, \ldots, s_{n-1}, \theta)$ of $\tau(\mathbb{B}^{n-1}) \times (0, \pi)$ as follows,

$$\frac{q_k}{\sqrt{|q|^2 + \rho}} = s_k, \quad k = 1, \ldots, n-1, \quad \frac{\rho}{|q|^2 + \rho} = 1 - |s|^2, \quad \text{and} \quad \theta = \theta.$$

Thus each function on Ω_+ is of the form $a(s, \theta)$, where $s = (s_1, \ldots, s_{n-1}) \in \tau(\mathbb{B}^{n-1})$ and $\theta \in (0, \pi)$.

Now each quasi-hyperbolic function, defined in D_n, can be uniquely represented in the form

$$a = a\left(\frac{|z_1|}{\sqrt{|z'|^2 + |z_n - i|z'|^2|}}, \ldots, \frac{|z_{n-1}|}{\sqrt{|z'|^2 + |z_n - i|z'|^2|}}, \arg(z_n - i|z'|^2)\right),$$
(10.4)

where a is a function, defined in $\tau(\mathbb{B}^{n-1}) \times (0, \pi)$, and this correspondence is one to one.

Theorem 10.5. *Let a be a bounded measurable quasi-hyperbolic function of the form (10.4). Then the Toeplitz operator T_a acting on $\mathcal{A}_\lambda^2(D_n)$ is unitary equivalent to the multiplication operator $\gamma_a I = R T_a R^*$ acting on $l_2(\mathbb{Z}_+^{n-1}, L_2(\mathbb{R}))$, where R and R^* are given in Section 9. The sequence $\gamma_a = \{\gamma_a(p, \xi)\}_{p \in \mathbb{Z}_+^{n-1}}, \xi \in \mathbb{R}$, is given by*

$$\gamma_a(p, \xi) = \alpha_p^2(\xi) \int_{\tau(\mathbb{B}^{n-1}) \times (0, \pi)} a(s, \theta) |\beta_p(s, \xi, \theta)|^2 (1 - |s|^2)^{\lambda+1} \frac{c_\lambda}{4} \sin^\lambda \theta \, ds d\theta,$$
(10.5)

where the functions $\alpha_p(\xi)$ and $\beta_p(s, \xi, \theta)$ are given by (9.8) and (9.9), respectively.

Proof. The operator T_a is obviously unitary equivalent to the operator

$$
\begin{aligned}
R T_a R^* &= R B_{D_n, \lambda} a B_{D_n, \lambda} R^* = R(R^*R)a(R^*R)R^* \\
&= (RR^*)RaR^*(RR^*) = RaR^* \\
&= R_0^* U_1 U_0 \, a \, U_0^{-1} U_1^{-1} R_0 \\
&= R_0^* U_1 \, a(s, \theta) \, U_1^{-1} R_0 \\
&= R_0^* \, a(s, \theta) R_0 \\
&= T.
\end{aligned}
$$

Now,

$$
\begin{aligned}
T\{c_p(\xi)\} &= R_0^* \left\{a(s, \theta)) \, \alpha_p(\xi) \, \beta_p(s, \xi, \theta) \, c_p(\xi)\right\}_{p \in \mathbb{Z}_+^{n-1}} \\
&= \left\{\alpha_p^2(\xi) \int_{\tau(\mathbb{B}^{n-1}) \times (0, \pi)} a(s, \theta) \, |\beta_p(s, \xi, \theta)|^2 \, c_p(\xi) \, (1 - |s|^2)^{\lambda+1}\right. \\
&\qquad \left. \cdot \frac{c_\lambda}{4} \sin^\lambda \theta \, ds d\theta\right\}_{p \in \mathbb{Z}_+^{n-1}} \\
&= \left\{\gamma_a(p, \xi) \cdot c_p(\xi)\right\}_{p \in \mathbb{Z}_+^{n-1}},
\end{aligned}
$$

with

$$\gamma_a(p,\xi) = \alpha_p^2(\xi) \int_{\tau(\mathbb{B}^{n-1})\times(0,\pi)} a(s,\theta) \, |\beta_p(s,\xi,\theta)|^2 \, (1-|s|^2)^{\lambda+1} \frac{c_\lambda}{4} \, \sin^\lambda\theta \, sdsd\theta,$$

where $p = (p_1, \ldots, p_{n-1}) \in \mathbb{Z}_+^{n-1}$, $\xi \in \mathbb{R}$, and the functions $\alpha_p(\xi)$ and $\beta_p(s,\xi,\theta)$ are given by (9.8) and (9.9), respectively. $\qquad\square$

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