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Representation of Infinitely Divisible Distributions on Cones

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Abstract We investigate infinitely divisible distributions on cones in Fréchet spaces. We show that every infinitely divisible distribution concentrated on a normal cone has the regular Lévy–Khintchine representation if and only if the cone is regular. These results are relevant to the study of multidimensional subordination.

Keywords Infinitely divisible distributions on cones · Regular cones · Lévy–Khintchine representation · Multidimensional subordination

1 Introduction

Recall that a subordinator is a Lévy process with nondecreasing trajectories. As a natural generalization of this notion one may consider Lévy processes taking values in cones of Euclidean or more general vector spaces. Such processes are determined by infinitely divisible distributions concentrated on cones. Skorohod [19, Theorem 3.21] showed that an infinitely divisible distribution μ on $\mathbb{R}^d$ is concentrated on a normal closed cone K if and only its Fourier transform admits a representation

$$\widehat{\mu}(y) = \exp\left\{ i\langle y, b_0\rangle + \int_K \left(e^{i\langle y,x\rangle} - 1\right)\nu(dx)\right\}, \quad \text{for all } y \in \mathbb{R}^d, \tag{1}$$

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where $b_0 \in K$ and ν is a Lévy measure concentrated on K such that

$$\int_{K\cap\{\|x\|\le 1\}} \|x\|\nu(dx) < \infty \tag{2}$$

(see Sect. 2 for the relevant definitions; also [16, Theorem 83]). Therefore, μ has the Laplace transform $L_\mu(y)$, defined for y in the dual cone K', of the form

$$L_\mu(y) = \exp\left\{-\langle y, b_0\rangle - \int_K \left(1 - e^{-\langle y,x\rangle}\right)\nu(dx)\right\} \tag{3}$$

(cf. Remark 2 in Sect. 3). Furthermore, if $\{X(t), t \ge 0\}$ is a Lévy process with $\mathcal{L}(X(1)) = \mu$ satisfying (1), then its Lévy-Itô decomposition is of the form

$$X(t) = tb_0 + \lim_{\epsilon\to 0}\sum_{s\le t} \Delta X(s) 1_{\|\Delta X(s)\|>\epsilon} \quad \text{a.s.}, \tag{4}$$

where $\Delta X(s) \in K$ for all $s \ge 0$ a.s. This is a consequence of (1–2). Representations (1) and (3) are fundamental in the study of cone valued Lévy process, multivariate subordination and type G random vectors; see [2, 3, 12, 13].

Dettweiler [8] generalized representation (3) to infinite dimensional vector spaces. Namely, he showed that if μ is an infinitely divisible probability measure concentrated on a normal regular cone K in a locally convex topological vector space E then its Laplace transform $L_\mu(f)$ is given by

$$L_\mu(f) = \exp\left\{-f(b_0) - \int_K \left(1 - e^{-f(x)}\right)\nu(dx)\right\} \quad \text{for all } f \in K', \tag{5}$$

where $b_0 \in K$ and the Lévy measure ν satisfies

$$\int_K (f(x) \wedge 1)\nu(dx) < \infty \quad \text{for all } f \in K'. \tag{6}$$

We will say that μ has the *regular Lévy-Khintchine representation on cone* when (5) holds.

The normality of a cone is a natural assumption precluding pathological situations. However, the question whether the regularity of K is necessary for (5) was left open in [8]. Moreover, no example was known of an infinitely divisible measure on a normal but not regular cone in a Banach space for which (5) would fail. In this paper we show that a normal cone K in a Fréchet space is regular if and only if every infinitely divisible probability measure μ concentrated on K has the regular Lévy-Khintchine representation on cone. This provides the converse to Dettweiler's theorem [8] for measures on Fréchet spaces; we also give a simpler proof of the sufficiency part. In the proof we show that every non regular cone K contains an isomorphic copy of the cone c^+ of nonnegative convergent sequences, we construct an infinitely divisible probability measure on c^+ without the regular Lévy-Khintchine representation on cone, and then we map this measure into K. As far as we know, this result is new for Banach spaces as well. Similarly to the finite dimensional case, the representation (5) is also basic for the study of subordination of infinite dimensional cone parameter Lévy processes. See [14] for a very special Banach space example.

We should also mention extensions of the Lévy-Khintchine representation for infinitely divisible distributions on semigroups (see [4, Theorem 4.3.20]). However, more definitive results for cone semigroups require local compactness of the cones and thus are not applicable here (see Jonasson [10]).

In Sect. 2 we establish notation and recall relevant definitions and facts. We give the main result in Sect. 3. In conclusion of the paper we give two examples. The first one shows that within one Banach space it is possible to have two normal cones such that one is regular and the other is not. The second example shows that the set of completely monotone functions constitutes a normal and regular cone in the Fréchet space $C^\infty(\mathbb{R}_+)$. This is an interesting example of a cone on a Fréchet space for which (5) applies.

2 Preliminaries and Notation

Recall that a Fréchet space is a complete locally convex linear metric space E. Its topology is generated by a nondecreasing sequence $\{\|\cdot\|_i\}_{i\geq 1}$ of semi-norms, see [18, pp. 48–49]. A cone K in E is a closed non-empty set closed under addition and multiplication by nonnegative reals. It is proper if it does not contain a straight line through 0 and $K \neq \{0\}$. A cone K induces a partial order on E by defining $x \leq_K y$ whenever $y - x \in K$. A cone K in E is *normal* if for every $z \in K$ the set $[0, z] := \{x \in K : x \leq_K z\}$ is bounded, see [11, Theorem 3]. It is known that if K is normal then the generating sequence of semi-norms $\{\|\cdot\|_i\}_{i\geq 1}$ can be chosen K*-increasing*, i.e., for every $i \geq 1$ and $x, y \in K$, $\|x\|_i \leq \|x + y\|_i$; note that the converse is obvious. Denote by E' the strong topological dual of E. Given a cone K, the *dual cone* K' is defined by

$$K' = \{f \in E' : f(x) \geq 0 \text{ for all } x \in K\}.$$

An important fact is that

$$K = \{x \in E : f(x) \geq 0 \text{ for all } f \in K'\}$$

see, e.g., [18, Theorem 1.5, p. 126]. In a Fréchet space a cone K is normal if and only if $E' = K' - K'$ (see [18, Corollary 3, p. 220] and [11, Corollary 1]); every normal cone in E is proper [18, Corollary 1, p. 216]. A sequence (x_n) in E is said to be K*-increasing* if $x_n \leq_K x_{n+1}$ for each n; it is K*-majorized* if there exists $x \in K$ with $x_n \leq_K x$ for all $n \geq 1$. A cone K is said to be *regular* if every K-increasing and K-majorized sequence in K is convergent. Denote by c (c_0, respectively) the Banach space of converging (converging to 0, respectively) sequences of real numbers with the supremum norm. Let c^+ and c_0^+ denote the corresponding cones of nonnegative sequences. Clearly both cones are normal and c_0^+ is regular but c^+ is not. In a finite dimensional vector space every proper cone is normal and regular.

For basic facts on infinitely divisible distributions on Banach spaces we refer to Araujo and Giné [1] and to Dettweiler [7] for a general case of locally convex spaces. From Dettweiler [7, Satz 2.5] we infer that μ is an infinitely divisible distribution on a separable Fréchet space E if and only if its Fourier transform $\widehat{\mu}$ has the Lévy-Khintchine representation of the following form. There exist $b \in E$, a positive quadratic form Q on E', and a Borel measure ν on E such that $\nu(\{0\}) = 0$ and $\nu(D^c) < \infty$ for some compact convex symmetric subset D of E such that for all $f \in E'$

$$\widehat{\mu}(f) = \exp\left\{ if(b) - \frac{1}{2}Q(f) + \int_E \big(e^{if(x)} - 1 - if(x)I_D(x)\big)\nu(dx)\right\}. \quad (7)$$

ν is called the Lévy measure of μ and satisfies $\int_E (f(x)^2 \wedge 1)\,\nu(dx) < \infty$ for every $f \in E'$. When E is a Banach space, we usually take as D the unit ball, even though it is not compact when E is infinite dimensional (cf. [1, Theorem 6.2, Chap. 3]).

3 Main Result

Theorem 1 *Let K be a normal cone in a separable Fréchet space E. Then every infinitely divisible probability distribution concentrated on K has the regular Lévy-Khintchine representation on cone if and only if K is regular.*

Proof The sufficiency. Let μ be an infinitely divisible distribution concentrated on a normal regular cone K. Since K is proper, μ has no Gaussian part and by (7) its characteristic function is of the form

$$\widehat{\mu}(f) = \exp\left\{ if(b) + \int_E \big(e^{if(x)} - 1 - if(x)I_D(x)\big)\nu(dx)\right\},$$

where D is a compact convex symmetric subset of E with $\nu(D^c) < \infty$. Since $f(\mu)$ is concentrated on $\mathbb{R}^+$ whenever $f \in K'$, from [17, Theorem 24.11] we infer that $f(\nu)$ is concentrated on $\mathbb{R}^+$, $\int_0^\infty (u \wedge 1)(f(\nu))(du) < \infty$, and

$$f(b) - \int_D f(x)\nu(dx) \geq 0 \quad (8)$$

for every $f \in K'$. Since E' is separable with respect to the weak* topology, there exists a sequence $\{f_n\} \subset K'$ such that $K = \bigcap_n \{x \in K : f_n(x) \geq 0\}$. Hence $\nu(K^c) = 0$, i.e., ν is concentrated on K. From (8) we get

$$f(b) \geq \int_{D\cap K} f(x)\nu(dx) \geq 0 \quad \text{for every } f \in K'. \quad (9)$$

Consequently, $b \in K$ and

$$\widehat{\mu}(f) = \exp\left\{ if(b) + \int_K \big(e^{if(x)} - 1 - if(x)I_D(x)\big)\nu(dx)\right\}.$$

We will now show that for every open neighborhood U of the origin in E

$$\nu(D \cap U^c) < \infty. \quad (10)$$

Indeed, since $0 \notin D \cap U^c$, for every $x \in D \cap U^c$ there exists $f_x \in E'$ such that $f_x(x) > 1$. Hence

$$D \cap U^c \subset \bigcup_{x \in D \cap U^c} \{y \in E : f_x(y) > 1\}.$$

Since $D \cap U^c$ is compact, there exist $x_1, \dots, x_n \in D \cap U^c$ such that

$$D \cap U^c \subset \bigcup_{i=1}^{n} \{y \in E : f_{x_i}(y) > 1\}.$$

Now (10) follows because $\nu(\{y \in E : f(y) > 1\}) \le \int_E (f(x)^2 \wedge 1)\, \nu(dx) < \infty$ for every $f \in E'$.

Fix a decreasing to the origin sequence $\{U_j\} \subset E$ of open neighborhoods of the origin. Since $D \cap U_j^c$ is bounded and $\nu(D \cap U_j^c) < \infty$ for every $j \ge 1$, the integral

$$x_j = \int_{D \cap U_j^c} x \nu(dx)$$

exists in the Pettis sense [9, Theorem 3.1(i)]. (Actually, the integral exists also in the Bochner sense because $D \cap U_j^c$ is compact and hence the identity function on it can be approximated uniformly by simple functions.) We have for every $f \in K'$

$$f(x_j) = \int_{K \cap D \cap U_j^c} f(x) \nu(dx) \ge 0.$$

Hence $x_j \in K$ and the sequence $\{x_j\}$ is K-increasing. By (9) $\{x_j\}$ is K-majorized by $b \in K$ and since K is regular, $x_0 = \lim_{j \to \infty} x_j$ exists. By the Monotone Convergence Theorem for every $f \in K'$

$$f(x_0) = \lim_{j \to \infty} \int_{K \cap D \cap U_j^c} f(x) \nu(dx) = \int_{K \cap D} f(x) \nu(dx).$$

Since $E' = K' - K'$, this equation holds for every $f \in E'$. We conclude that $\widehat{\mu}$ has the form

$$\widehat{\mu}(f) = \exp\left\{ i f(b_0) + \int_K \left(e^{i f(x)} - 1 \right) \nu(dx) \right\}, \tag{11}$$

where $b_0 = b - x_0 \in K$. From Remark 2 given below we infer that L_μ is of the form (5).

The proof of the necessity will be divided into three steps. Assume that K is normal but not regular cone in E. Choose and fix a K-increasing sequence of semi-norms $\{\|\cdot\|_i\}_{i \ge 1}$ generating the topology of E (see Sect. 2).

In the first step we will construct an isomorphic embedding $V : c \to E$ such that $V(c^+) \subset K$. Since K is not regular, there exists $x \in K$ and a K-increasing sequence $\{x_k\} \subset K$ such that $x_k \le_K x$ and $\lim_{k \to \infty} x_k$ does not exist. Therefore, for some $i_0 \ge 1$ and $\epsilon_0 > 0$, there exist $k_j < m_j < k_{j+1}$ such that

$$\|x_{m_j} - x_{k_j}\|_{i_0} > \epsilon_0, \quad j = 1, 2, \dots.$$

Put $y_j = x_{m_j} - x_{k_j}$. Then $y_j \in K$, $\|y_j\|_{i_0} > \epsilon_0$, and for any finite subset $J \subset \mathbb{N}$

$$\sum_{j \in J} y_j \leq_K x. \tag{12}$$

Since K is normal, we have for each $i \geq 1$ and for any finite subset $J \subset \mathbb{N}$

$$\left\| \sum_{j \in J} y_j \right\|_i \leq \|x\|_i.$$

Hence, from [5, Corollary C.1 and 7.2], there exists a subsequence $\{y_{j_n}\}$ such that for $z_n = y_{j_n}$ the map

$$V_0(a) = \sum_{n=1}^{\infty} a_n z_n, \quad a = (a_n)_{n \geq 1} \in c_0,$$

is an isomorphism from c_0 into E. Let $E_0 = V_0(c_0)$. Without loss of generality we may assume that $x \notin E_0$. Indeed, if $x = V_0(a)$ for some $a \in c_0$, then $a_{n_0} \neq 0$ for some $n_0 \geq 1$. We discard $y_{j_{n_0}}$ from $\{y_{j_n}\}$ and define $z_n = y_{j_n}$ if $n < n_0$ and $z_n = y_{j_{n+1}}$ if $n \geq n_0$. The new V_0 is still an isomorphic embedding of c_0 into E and $x \neq E_0$. Define $V : c \to E$ by

$$V(t) = L(t)x + V_0(t - L(t)\mathbf{1}), \tag{13}$$

where $L(t) = \lim_{n \to \infty} t_n$, $t = (t_n) \in c$ and $\mathbf{1} = (1, 1, \ldots)$ is the sequence consisting of 1's. Since $x \notin E_0$, V is one-to-one and

$$E_1 = V(c) = \{rx + y : r \in \mathbb{R}, y \in E_0\}$$

is closed. By the Inverse Mapping Theorem V is an isomorphism between c and E_1. From (12) for every $N \geq 1$ and $t \in c^+$

$$L(t)x + \sum_{n=1}^{N} (t_n - L(t)) z_n = L(t)\left(x - \sum_{n=1}^{N} z_n\right) + \sum_{n=1}^{N} t_n z_n \in K.$$

Letting $N \to \infty$ we get $V(c^+) \subset K$, as claimed.

Now we will construct an infinitely divisible distribution on c^+ which does not have regular Lévy-Khintchine representation on cone. Consider an increasing sequence $0 < x^{(1)} < x^{(2)} < \cdots$ in c^+ given by

$$x_n^{(j)} = \begin{cases} 1 & \text{if } n \leq j, \\ \frac{j}{n} & \text{if } n > j. \end{cases}$$

Let $y^{(1)} = x^{(1)}$ and $y^{(j)} = x^{(j)} - x^{(j-1)}$ when $j \geq 2$. We have

$$y_n^{(j)} = \begin{cases} 0 & \text{if } n < j, \\ \frac{1}{n} & \text{if } n \geq j. \end{cases}$$

Let $\{\xi_j\}$ be a sequence of iid Poisson random variables with parameter 1. We will show that

$$X = \mathbf{1} + \sum_{j=1}^{\infty}(\xi_j - 1)y^{(j)} \tag{14}$$

converges a.s. in c. Consider the partial sums series

$$X^{(k)} = \mathbf{1} + \sum_{j=1}^{k}(\xi_j - 1)y^{(j)}.$$

Since, for any $1 \le k < m$,

$$X_n^{(m)} - X_n^{(k)} = \begin{cases} 0 & \text{if } n \le k, \\ \frac{1}{n}\sum_{j=k+1}^{n\wedge m}(\xi_j - 1) & \text{if } n > k, \end{cases}$$

we get

$$\left\|X^{(m)} - X^{(k)}\right\|_\infty = \sup_{k<n\le m}\left|\frac{1}{n}\sum_{j=k+1}^{n}(\xi_j - 1)\right|.$$

Using Hajek-Renyi-Chow inequality, for any $\epsilon > 0$ (see Theorem 7.4.8 in [6]) we have

$$P\left(\max_{k<n\le m}\left|\frac{1}{n}\sum_{j=k+1}^{n}(\xi_j - 1)\right| \ge \epsilon\right) \le \epsilon^{-2}\sum_{n=k+1}^{m}\frac{1}{n^2} \to 0$$

as $k \to \infty$. Therefore, the series in (14) converges in c in probability. Since it consists of independent random vectors, it converges almost surely as well (by Itô-Nisio Theorem [1, Theorem 2.10, Chap. 3]). Hence X is an infinitely divisible random vector in c with Lévy measure $\nu = \sum_{j=1}^{\infty}\delta_{y^{(j)}}$. Because for every $k \ge 1$

$$X^{(k)} = \mathbf{1} - x^{(k)} + \sum_{j=1}^{k}\xi_j y^{(j)} \in c^+, \tag{15}$$

we have that $P(X \in c^+) = 1$.

We will show that $\mu = \mathcal{L}(X)$ does not admit regular Lévy-Khintchine representation on cone. Suppose to the contrary that

$$L_\mu(f) = \exp\left\{-f(b_0) - \int_{c^+}\left(1 - e^{-f(x)}\right)\nu(dx)\right\}, \quad f \in (c^+)'$$

for some $b_0 \in c^+$. From (15) we infer that for every $f \in (c^+)'$

$$f(b_0) = f(\mathbf{1}) - \lim_{k\to\infty} f\left(x^{(k)}\right).$$

If we take $f(x) = x_n$, the n-th component of $x \in c$, then we get that $f(b_0) = 0$, which implies $b_0 = 0$. However, if we take $f(x) = \lim_{n\to\infty} x_n$ then we get $f(b_0) = 1$, a contradiction.

In the last step of the proof we will show that $\mu_1 = V(\mu)$ does not admit regular Lévy–Khintchine representation, where V is given by (13). Suppose to the contrary that for some $b_1 \in K$

$$L_{\mu_1}(f) = \exp\left\{-f(b_1) - \int_K \left(1 - e^{-f(x)}\right)(V(\nu))(dx)\right\}, \quad f \in K'.$$

By Remark 2 below we have for all $f \in E'$

$$\widehat{\mu_1}(f) = \int_{c^+} e^{if(V(x))}\mu(dx) = \exp\left\{if(b_1) + \int_{c^+} \left(e^{if(V(x))} - 1\right)\nu(dx)\right\}.$$

Since $V(c)$ is a closed subspace of E and E' is separable in the weak* topology, there exists a sequence $\{f_n\} \subset E'$ such that $V(c) = \{x \in E : f_n(x) = 0 \ \forall n \geq 1\}$. Hence for every $t \in \mathbb{R}$ and $n \geq 1$

$$\widehat{\mu_1}(tf_n) = 1 = \exp\{itf_n(b_1)\}.$$

Therefore, $f_n(b_1) = 0$ for every $n \geq 1$, which yields $b_1 \in V(c)$, so that $b_1 = V(b_0)$ for some $b_0 \in c$. Since $V' : E' \to c'$ is onto, for every $g \in c'$ there exists $f \in E'$ such that $V'(f) = g$. Hence

$$\begin{aligned}\widehat{\mu}(g) = \widehat{\mu_1}(f) &= \exp\left\{if(V(b_0)) + \int_{c^+} \left(e^{if(V(x))} - 1\right)\nu(dx)\right\} \\ &= \exp\left\{ig(b_0) + \int_{c^+} \left(e^{ig(x)} - 1\right)\nu(dx)\right\}.\end{aligned}$$

Therefore μ has the representation (11), which is a contradiction with the conclusion of the previous step (cf. Remark 2). The proof is complete. □

Remark 2 The regular Lévy–Khintchine representation on cone (5) is equivalent to the representation (11) when K is normal, so that when one holds then the other does with the same $b_0 \in K$ and ν. This is a standard fact but we will sketch a proof for the sake of completeness. First suppose that (5–6) hold. Since for every $s_1, s_2 < 0$ and $f_1, f_2 \in K^+$, $-s_1 f_1 - s_2 f_2 \in K'$, we have by (5)

$$\begin{aligned}\int_E e^{s_1 f_1(x) + s_2 f_2(x)}\mu(dx) = \exp\Big\{& s_1 f_1(b_0) + s_2 f_2(b_0) \\ &+ \int_K \left(e^{s_1 f_1(x) + s_2 f_2(x)} - 1\right)\nu(dx)\Big\}.\end{aligned}$$

Using the uniqueness of an analytic continuation (see the proof of Theorem 24.11 in [17]) we show that this equation can first be extended to all $s_1 \in \mathbb{C}$ with $\operatorname{Re} s_1 \leq 0$ and then to all $s_2 \in \mathbb{C}$ with $\operatorname{Re} s_2 \leq 0$. Now if we take $s_1 = i$, $s_2 = -i$ and use $K' - K' = E'$, we get (11) for all $f \in E'$. The converse implication is similar but simpler since we may restrict ourself to $f \in K'$ from the beginning.

Example 3 Let $E = c_0$. Since $K_1 := c_0^+$ is normal and regular cone, every infinitely divisible probability measure on K_1 has regular Lévy-Khintchine representation on cone.

Let $\{e_i\}$ denote the standard basis in c_0. Define another cone in c_0 by

$$K_2 = \{(x_n)_{n\geq 1} \in c_0 : x_1 + x_n \geq 0 \text{ for all } n \geq 1\}.$$

If $x, y \in K_2$, $x = (x_n)_{n\geq 1}$, $y = (y_n)_{n\geq 1}$ then

$$\begin{aligned}\|x\| &\leq x_1 + \sup_{n\geq 1} x_n \leq x_1 + \sup_{n\geq 1}(x_n + y_n + y_1)\\ &\leq 2\sup_{n\geq 1}|x_n + y_n| = 2\|x + y\|.\end{aligned}$$

Hence K_2 is normal. Let $x^{(k)} = \sum_{i=2}^{k} e_i$, $k \geq 2$. Clearly, $\{x^{(k)}\}_{k\geq 2} \subset K_2$ is a K_2-increasing sequence and

$$x^{(k)} \leq_{K_2} e_1.$$

Since $\{x^{(k)}\}_{k\geq 2}$ does not converge in c_0, K_2 is not regular. Therefore, there are infinitely divisible probability measures concentrated on K_2 without regular Lévy-Khintchine representation on cone. We conclude that the existence of the regular Lévy-Khintchine representation indeed depends on the cone.

In this context one should mention that, even that c_0^+ is normal and regular, there are c_0^+-valued Lévy processes which are c_0^+-increasing but not of bounded variation, see [15].

Example 4 Recall that a function $x \in C^\infty(\mathbb{R}_+)$ is said to be completely monotone if $(-1)^n D^n x(t) \geq 0$ for all $n \geq 0$ and $t > 0$. The set K of completely monotone functions on $\mathbb{R}_+$ is a cone in $E = C^\infty(\mathbb{R}_+)$. $C^\infty(\mathbb{R}_+)$ is a separable Fréchet space with the topology determined by semi-norms

$$\|x\|_N = \sum_{n=0}^{N} \sup\{|D^n x(t)| : t \in [0, N]\}, \qquad N \geq 1.$$

Since $\|\cdot\|_N$ are K-increasing, K is a normal cone. Let $\{x_k\} \subset K$ be a K-increasing sequence such that $x_k \leq_K x \in K$. Then, for every $n \geq 0$, $k \geq 1$, and $t > 0$, $0 \leq (-1)^n D^n x_k(t) \leq (-1)^n D^n x_{k+1}(t) \leq (-1)^n D^n x(t)$. Let $y^{(n)}(t)$ be a point-wise limit of $D^n x_k(t)$ as $k \to \infty$. By the mean value theorem, for every $N \geq 1$ and s, $t \in [0, N]$, $|y^{(n)}(t) - y^{(n)}(s)| \leq \sup_{u\in[0,N]} |D^{n+1}x(u)||t - s|$. Hence $y^{(n)}$ is continuous and by Dini's theorem, $D^n x_k \to y^{(n)}$ uniformly on each interval $[0, N]$, as $k \to \infty$. We conclude that $y \in C^\infty(\mathbb{R}_+)$ and $y^{(n)} = D^n y$, so that K is regular.

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