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MATRIX SUBORDINATORS AND RELATED UPSILON TRANSFORMATIONS*

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Abstract. A class of upsilon transformations of Lévy measures for matrix subordinators is introduced. Some regularizing properties of these transformations are derived, such as absolute continuity and complete monotonicity. The class of Lévy measures with completely monotone matrix densities is characterized. Examples of infinitely divisible nonnegative definite random matrices are constructed using an upsilon transformation.

Key words. infinite divisibility, random matrices, Lévy measures, cone valued random variables, completely monotone matrix functions

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1. Introduction.

Let $X = \{X_t\}_{t \geqq 0}$ be a Lévy process, denote by ρ the Lévy measure of X_1, and write $\mathfrak{L}(\mathbf{R})$ for the class of all Lévy measures on $\mathbf{R}$. The upsilon transformation Υ_0, introduced in [7] and studied further in [8], [1], and [2], is defined as the mapping on $\mathfrak{L}(\mathbf{R})$ into $\mathfrak{L}(\mathbf{R})$, given by

$$\widetilde{\rho}(\mathrm{d}x) = \int_0^\infty \rho(\xi^{-1}\,\mathrm{d}x)\,e^{-\xi}\,\mathrm{d}\xi. \tag{1.1}$$

This mapping is one-to-one, smooth, and strongly regularizing, and $\Upsilon_0(\mathfrak{L}(\mathbf{R}))$ is a proper subset of $\mathfrak{L}(\mathbf{R})$. Furthermore, Υ_0 has a stochastic representation as follows. Let Y be the random variable defined by the stochastic integral

$$Y = \int_0^1 \bigl|\log(1-s)\bigr|\,\mathrm{d}X_s. \tag{1.2}$$

Then Y has Lévy measure $\widetilde{\rho}$.

In [2] the mapping is extended to a mapping on the class $\mathfrak{L}(\mathbf{R}^d)$ of all d-dimensional Lévy measures, via the direct extension of (1.2) to higher dimension, with X now a d-dimensional Lévy process. Relations to self-decomposability and to distributions of Steutel–Goldie–Bondesson type and Thorin type are discussed in the above-mentioned papers.

In the present paper we extend some of these results to the case where X is a matrix subordinator, with the mapping Υ_0 then being from $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$ into $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$, where $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$ is the class of Lévy measures on the cone $\overline{\mathbf{M}}_m^+$ of symmetric nonnegative definite $m\times m$ matrices. This extension, from $\mathbf{R}_+$ to $\overline{\mathbf{M}}_m^+$, is rather different in nature than the one from $\mathbf{R}$ to $\mathbf{R}^d$ studied in [2]. Our approach is based on analytic concepts, such as Laplace transforms, and not on stochastic integral representation tools.

This approach leads to the construction of concrete and new examples of infinitely divisible distributions on $\overline{\mathbf{M}}_m^+$ and their associated matrix subordinators. Such matrix-valued Lévy processes are important in the study of subordination of cone-parameter

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Lévy processes; see [21], [22]. Furthermore, random matrices of this type are useful in constructing other infinitely divisible random vectors and matrices, which are covariance mixtures of Gaussian vectors and matrices, respectively; cf. [4], [6]. We also note that instances of infinitely divisible random matrices in the cone $\overline{\mathbf{M}}_m^+$ arise naturally as quadratic covariation matrices of vector-valued Lévy processes [19].

It is relevant to mention that the most common examples in statistics of laws of positive definite random matrices, such as the Wishart and gamma matrix distributions, are not infinitely divisible [18], [13]. Constructions of matrix distributions analogous to one-dimensional distributions have traditionally been done by fairly direct generalization of the one-dimensional probability densities (see, for example, [14]). There is no reason to believe that such an approach will lead to infinitely divisible matrix laws. In this paper we use the framework of Lévy measures and the mapping Υ_0 to construct infinitely divisible matrix versions of one-dimensional distributions, including the gamma distribution and simple cases of tempered stable distributions. A different approach, using Lévy copulas and the one-dimensional Υ_0 mapping, is discussed in [1].

Section 2 of this paper sets out some, mostly well-known, results from matrix theory that are needed for defining and establishing properties of the Υ_0-type mapping for matrix subordinators. The class of matrix subordinators is introduced in section 3. A family of upsilon transformations Υ_q for matrix subordinators is defined and studied in section 4, where we establish their key properties of uniqueness, smoothness, and regularization. We investigate, in particular, the image set $\mathfrak{B} = \Upsilon_0(\mathfrak{L}(\mathbf{M}_m^+))$, where $\mathbf{M}_m^+$ is the open subcone of $\overline{\mathbf{M}}_m^+$. The class $\mathfrak{B}$ constitutes a matrix generalization of the Steutel–Goldie–Bondesson class of infinitely divisible laws on $\mathbf{R}_+$. Section 5 considers several subclasses of matrix subordinators. Specifically, we characterize the stable matrix distributions that belong to $\mathfrak{B}$, introduce infinitely divisible matrix extensions of the one-dimensional gamma distribution and of some simple tempered stable laws, and study their images under Υ_0. As a by-product, our construction of Lévy measures yields examples of completely monotone matrix functions, including Bessel-type functions, different from those commonly used in the classical matrix distribution theory. Finally, in section 6 we show a relation of the transformation $\Upsilon_{m/2}$ to the so-called mat G random matrices.

2. Some matrix theory. In this section we establish matrix notation and recall factorization results and integration over the cone of positive definite matrices. We also review completely monotone functions of matrices and the gamma and bi-gamma matrix functions.

2.1. Notation. First, we introduce the following standard notation. Let $\mathbf{M}_{m\times m}$ be the linear space of $m\times m$ real matrices, $\mathbf{M}_m$ be the linear subspace of symmetric matrices, $\overline{\mathbf{M}}_m^+$ the closed cone of nonnegative definite matrices in $\mathbf{M}_m$, $\mathbf{M}_m^+$ or $\{X > 0\}$ be the open cone of positive definite matrices in $\mathbf{M}_m$, and let $\mathbf{U}_m^+$ (respectively, $\mathbf{L}_m^+$) be the open cone of upper (respectively, lower) triangular matrices with positive diagonal elements.

For $X \in \mathbf{M}_{m\times m}$, $X^\top$ is the transpose of X and $\mathrm{tr}(X)$ is the trace of X. For X in $\overline{\mathbf{M}}_m^+$, $X^{1/2}$ is a unique symmetric matrix in $\overline{\mathbf{M}}_m^+$ such that $X = X^{1/2}X^{1/2}$. Given a nonsingular matrix X in $\mathbf{M}_{m\times m}$, X^{-1} denotes its inverse, $|X|$ its determinant, and $X^{-\top}$ the inverse of its transpose. The eigenvalues of X in $\mathbf{M}_m$, arranged in increasing order, are denoted by $x_1, \ldots, x_n$. When X is in $\mathbf{M}_m^+$ we simply write $X > 0$.

For a matrix $X = (X_{ij})$ in $\mathbf{M}_m$, the upper triangular matrix $(X_{kj})_{1\leqq k\leqq j\leqq m}$

with $[m]$ entries determines X, where we use consistently the notation $[m] = m(m+1)/2$ and $\langle m\rangle = (m+1)/2$. In this way we identify $\mathbf{M}_m^+$ with a subset of $\mathbf{R}^{[m]}$, considering $(X_{kj})_{1\leqq k\leqq j\leqq m}$ as a column vector in $\mathbf{R}^{[m]}$. The identity matrix in $\mathbf{M}_m$ is denoted by I_m.

When dealing with matrix subordinators, a useful matrix norm is the *trace norm* defined for $X \in \mathbf{M}_{m\times m}$ as

$$\|X\| = \mathrm{tr}\left(\left\{XX^\top\right\}^{1/2}\right). \tag{2.1}$$

We denote by $\mathbf{S}_m = \{U \in \mathbf{M}_m\colon \|U\| = 1\}$ the unit disk of $\mathbf{M}_{m\times m}$ and write $\overline{\mathbf{S}}_m^+ = \mathbf{S}_m \cap \overline{\mathbf{M}}_m^+$. Whenever the product of two matrices X and Y makes sense, one has

$$\mathrm{tr}(XY) \leqq \|X\|\,\|Y\|. \tag{2.2}$$

For X in $\overline{\mathbf{M}}_m^+$, $\|X\| = \mathrm{tr}(X)$, and, in particular, if $U \in \mathbf{S}_m^+$, $\mathrm{tr}(U) = \|U\| = 1$.

For $X > 0$, $Y > 0$ one has the trace inequalities (see [16], [17])

$$x_1\,\mathrm{tr}(Y) \leqq \mathrm{tr}(XY) \leqq x_n\,\mathrm{tr}(Y) \tag{2.3}$$

and also

$$|X| \leqq x_m^m \leqq \left(\mathrm{tr}(X)\right)^m. \tag{2.4}$$

An *axis* $E^{ij} = (e_{kl}^{ij})$ is a matrix in $\mathbf{M}_{m\times m}$ which has zero elements everywhere except for a one in the ith row and jth column. For any $i = 1,\ldots,m$, $j = 1,\ldots,n$, E^{ij} is such that $\|E^{ij}\| = 1$. The set of *matrix axes* is the canonical basis for $\mathbf{M}_{m\times m}$ and the only ones belonging to $\overline{\mathbf{M}}_m^+$ are the diagonals E^{ii}, $i = 1,\ldots,m$.

For a general random matrix M in $\mathbf{M}_{m\times m}$ its Fourier transform is defined as

$$\phi_M(\Theta) = \mathbf{E}\exp\left(i\,\mathrm{tr}\left(M\Theta^\top\right)\right), \qquad \Theta \in \mathbf{M}_{m\times m}.$$

2.2. Factorization of matrices, disintegration, and Jacobians. A useful tool in the study of probability measures in $\mathbf{M}_m^+$ is the so-called LU-decomposition (lower-upper decomposition) of matrices. For $X > 0$ we denote by $\overline{X}$ a unique matrix in $\mathbf{U}_m^+$ such that $X = \overline{X}^\top\overline{X}$. Sometimes it is useful to write $X = \underline{X}^\top\underline{X}$, where $\underline{X} = \overline{X}^\top$ is in $\mathbf{L}_m^+$. One trivially has $|X| = \prod_{j=1}^m \overline{X}_{jj}^2 = \prod_{j=1}^m \underline{X}_{jj}^2$.

An important role in this work is played by the *anti-matrix* of X (abbreviated as *anti-*X) defined for $X > 0$ as a unique matrix $\mathbf{X}$ in $\mathbf{M}_m^+$ such that

$$\mathbf{X} = \overline{X}\,\overline{X}^\top. \tag{2.5}$$

Sometimes anti-X is called the disguised matrix (see [14]). We observe that $\mathrm{tr}(\mathbf{X}) = \mathrm{tr}(X)$ and $|\mathbf{X}| = |X|$.

Given a function $h\colon \mathbf{M}_m^+ \to [0,\infty)$ we write

$$\int_{X>0} h(X)\,\mathrm{d}X = \int_{X>0} h(X)\,\mathrm{d}X_{11}\,\mathrm{d}X_{12}\cdots\mathrm{d}X_{mm}, \tag{2.6}$$

where the right-hand side is the Lebesgue integral over the cone $\mathbf{M}_m^+$ considered as a subset of $\mathbf{R}^{[m]}$. We recall that the Lebesgue measure of $\overline{\mathbf{M}}_m^+\backslash\mathbf{M}_m^+$ is zero (see [30, Lemma 4.73]) and that the measure

$$\vartheta(\mathrm{d}X) = \frac{\mathrm{d}X}{|X|^{\langle m\rangle}} \tag{2.7}$$

is invariant under the transformation $X \to AXA^\top$, for $X \in \mathbf{M}_m^+$ and any nonsingular $m \times m$ matrix A (see [12, Example 6.19]).

There are two alternative representations of integral (2.6) which are useful in this work. First, using the facts that $\mathrm{d}X = 2^m \bar{J}(\overline{X})\,\mathrm{d}\overline{X}$ and $\mathrm{d}X = 2^m \underline{J}(\underline{X})\,\mathrm{d}\underline{X}$ (see [20, Theorem 1.28]), we have

$$\int_{X>0} h(X)\,\mathrm{d}X = 2^m \int_{\mathbf{U}_m^+} h\big(\overline{X}^\top \overline{X}\big)\,\bar{J}(\overline{X})\,\mathrm{d}\overline{X} \tag{2.8}$$

$$= 2^m \int_{\mathbf{L}_m^+} h\big(\underline{X}^\top \underline{X}\big)\,\underline{J}(\underline{X})\,\mathrm{d}\underline{X}, \tag{2.9}$$

where $\mathrm{d}\overline{X} = \prod_{1\leqq i \leqq j \leqq m} \mathrm{d}\overline{X}_{ij}$, $\mathrm{d}\underline{X} = \prod_{1\leqq i \leqq j \leqq m} \mathrm{d}\underline{X}_{ij}$ and where, for a triangular matrix T with nonzero diagonal elements, we write

$$\bar{J}(T) = \prod_{j=1}^m T_{jj}^{m+1-j} = \big|TT^\top\big|^{\langle m\rangle}\,\underline{J}(T^{-1}), \tag{2.10}$$

where

$$\underline{J}(T) = \prod_{j=1}^m T_{jj}^{j}. \tag{2.11}$$

The function $\bar{J}(T)$ is $[m]$-homogeneous, that is, for each $t > 0$

$$\bar{J}(tT) = t^{[m]}\bar{J}(T). \tag{2.12}$$

On the other hand, making the change of variable $X = rU$, $r = \mathrm{tr}(X)$, for which $\mathrm{tr}(U) = 1$ and $\mathrm{d}X = r^{[m]-1}\,\mathrm{d}r\,\mathrm{d}U$ (see [20, p. 111]), we also have

$$\int_{X>0} h(X)\,\mathrm{d}X = \int_{\mathbf{S}_m^+}\int_0^\infty h(rU)\,r^{[m]-1}\,\mathrm{d}r\,\mathrm{d}U. \tag{2.13}$$

In addition, the following transformations and Jacobians are used several times in this work. Given $X > 0$, and A and B upper triangular matrices with positive diagonals, for $\overline{V} = A\overline{X}B$ we have (see [20, Theorem 1.16])

$$\mathrm{d}\overline{V} = \bar{J}(A)\,\underline{J}(B)\,\mathrm{d}\overline{X}. \tag{2.14}$$

For a symmetric matrix $X > 0$ and a nonsingular matrix $C > 0$, if $Y = CXC^\top$, we have (see [20, Theorem 1.20])

$$\mathrm{d}Y = |C|^{m+1}\,\mathrm{d}X. \tag{2.15}$$

If $V > 0$, $Y > 0$, and $\overline{V} = \overline{Y}^{-1}$, we have (see [20, Theorem 1.27])

$$\mathrm{d}\overline{V} = |\overline{Y}|^{-(m+1)}\mathrm{d}\overline{Y}. \tag{2.16}$$

2.3. Completely monotone matrix functions. Given a Radon measure Q on $\mathbf{M}_m^+$, its Laplace transform is defined as

$$\mathcal{L}Q(\Theta) = \int_{\overline{\mathbf{M}}_m^+} \exp\big(-\mathrm{tr}(\Theta X)\big)\,Q(\mathrm{d}X), \tag{2.17}$$

where in the trace operation the matrix parameter Θ is interpreted as $\Theta = (\frac{1}{2}(1+\delta_{ij})\Theta_{ij})$, in which case

$$\operatorname{tr}(X\Theta) = \operatorname{tr}(\Theta X) = \sum_{1 \leqq k \leqq j \leqq m} \Theta_{jk} X_{jk}. \tag{2.18}$$

Sometimes we use the notation $\operatorname{etr}(-X\Theta) = \exp(-\operatorname{tr}(\Theta X))$.

As in the one-dimensional case, there is a one-to-one correspondence between Laplace transforms and completely monotone functions of matrices.

For a function $h\colon \mathbf{M}_m^+ \to \mathbf{R}$ and $Y \in \mathbf{M}_m^+$, define the operator $\Delta_Y h\colon \mathbf{M}_m^+ \to \mathbf{R}$ by $\Delta_Y h(X) = h(X+Y) - h(X)$ and let $\nabla_X := -\Delta_X$. We say that h is a completely monotone function of X if it is nonnegative and if for all finite sets $\{X_1, \ldots, X_n\} \subset \mathbf{M}_m^+$ and $Y \in \mathbf{M}_m^+$, $\nabla_{X_1} \cdots \nabla_{X_n} h(Y) \geqq 0$.

From [25] and [32] we have the following analogue of Bernstein's theorem.

Lemma 2.1. *For any function $\mathbf{M}_m^+ \to \mathbf{R}_+$ completely monotone in X there exists a unique nonnegative Radon measure Q on $\mathbf{M}_m^+$ such that*

$$h(X) = \int_{\overline{\mathbf{M}}_m^+} \operatorname{etr}(-XY)\, Q(\mathrm{d}Y), \qquad X \in \overline{\mathbf{M}}_m^+, \tag{2.19}$$

and conversely.

A completely monotone function on $\mathbf{M}_m^+$ is called *regular* [6] if Q is concentrated on the open cone $\mathbf{M}_m^+$ (and in particular $Q(\{0\}) = 0$). In this work we very often deal with functions that are completely monotone with respect to the anti-matrix $\mathbf{X}$ rather than with respect to X. We observe in what follows that there are functions which are completely monotone in both X and $\mathbf{X}$, but this is not the case in general.

There are two matrix functions which play a similar role, as does $1/x$ for real valued functions. First, for $a > \frac{1}{2}(m-1)$, consider the Laplace transform (see [14, equation 1.4.6])

$$|X|^{-a} = \frac{1}{\Gamma_m(a)} \int_{Y>0} \operatorname{etr}(-YX)|Y|^{a-\langle m\rangle}\, \mathrm{d}Y. \tag{2.20}$$

Then the function $h(X) = |X|^{-a}$ is completely monotone in X, and since $|X| = |\mathbf{X}|$, the function $|X|^{-a}$ is also completely monotone in $\mathbf{X}$ for $a > \frac{1}{2}(m-1)$.

Another important class of functions completely monotone in X is given by inverse powers of the trace function. Specifically, for $p > 0$ and a fixed $Y_0 > 0$, define the Radon measure

$$Q_{Y_0}(A) = \frac{1}{\Gamma(p)\big(\operatorname{tr}(Y_0)\big)^p} \int_0^\infty \int_{\mathbf{S}_m^+} 1_A(r\omega)\, r^{p-1}\, \delta_{\{\operatorname{tr}(Y_0)^{-1} Y_0\}}(\mathrm{d}\omega)\, \mathrm{d}r. \tag{2.21}$$

Then

$$\big(\operatorname{tr}(XY_0)\big)^{-p} = \int_{\overline{\mathbf{M}}_m^+} \operatorname{etr}(-XY)\, Q_{Y_0}(\mathrm{d}Y) \tag{2.22}$$

is a function completely monotone in X for all $p > 0$ and $Y_0 > 0$. However, this function is not completely monotone in $\mathbf{X}$ unless $Y_0 = \mathrm{I}_m$.

Section 5 presents additional concrete examples of functions completely monotone in $\mathbf{X}$.

2.4. The matrix gamma and bi-gamma functions. The *matrix gamma function* $\Gamma_m(a)$ is defined for $\operatorname{Re} a > \frac{1}{2}(m-1)$ as

$$\Gamma_m(a) = \int_{X>0} \exp\big(-\operatorname{tr}(X)\big)\,|X|^{a-\langle m\rangle}\,\mathrm{d}X. \tag{2.23}$$

We denote by $\Gamma(X; a, m)$ the corresponding matrix distribution on $\mathbf{M}_m^+$ whose density is given by

$$\gamma(X; a, m) = \Gamma_m(a)^{-1} \exp\big(-\operatorname{tr}(X)\big)\,|X|^{a-\langle m\rangle}, \qquad X > 0. \tag{2.24}$$

When $a = \langle m\rangle$, using (2.23) and (2.13) and recalling that $\Gamma_m(\langle m\rangle) = \pi^{m/2}$, one can prove that

$$c_m = \int_{\mathbf{S}_m^+} \mathrm{d}U = \frac{\pi^{m/2}}{([m]-1)!}. \tag{2.25}$$

Another important matrix function that appears in this work is the *bi-gamma matrix function*, defined, for $\operatorname{Re} a > \frac{1}{2}(m-1)$ and Θ_1, Θ_2 in $\mathbf{M}_m^+$, as

$$\mathbf{\Gamma}_m(a, \Theta_1, \Theta_2) = \int_{X>0} \exp\big(-\operatorname{tr}(X\Theta_1 + \mathbf{X}\Theta_2)\big)\,|X|^{a-\langle m\rangle}\,\mathrm{d}X. \tag{2.26}$$

One can easily prove that for a nonnegative constant b

$$\mathbf{\Gamma}_m(a, 0, \Theta) = |\Theta|^{-a-\langle m\rangle}\,\underline{J}(\overline{\Theta}\,)^2\,\Gamma_m(a), \tag{2.27}$$

$$\mathbf{\Gamma}_m(a, b\mathrm{I}_m, \Theta) = \mathbf{\Gamma}_m(a, 0, b\mathrm{I}_m + \Theta). \tag{2.28}$$

3. Infinite divisibility in the cone $\mathbf{M}_m^+$. We now review several facts about infinitely divisible matrices with values in the cone $\overline{\mathbf{M}}_m^+$. The study of infinitely divisible random elements in closed cones was initiated in [10], [31] and recently considered in [3], [21], [22], [23], [24].

A random matrix M is *infinitely divisible in* $\overline{\mathbf{M}}_m^+$ if and only if for each integer $p \geqq 1$ there exist p independent identically distributed random matrices $M_1, \ldots, M_p$ in $\overline{\mathbf{M}}_m^+$ such that $M \overset{\text{law}}{=} M_1 + \cdots + M_p$.

3.1. Lévy–Khinchin representation. As for general cones, the Lévy–Khinchin representation of infinitely divisible random matrices in $\overline{\mathbf{M}}_m^+$ has the following special form. We refer to [31].

PROPOSITION 3.1. *A random matrix M is infinitely divisible in $\overline{\mathbf{M}}_m^+$ if and only if its cumulant transform is of the form*

$$\mathcal{C}(\Theta; M) = i\operatorname{tr}(\Psi^0\Theta) + \int_{\overline{\mathbf{M}}_m^+} \big(e^{i\operatorname{tr}(X\Theta)} - 1\big)\,\rho(\mathrm{d}X), \qquad \Theta \in \mathbf{M}_m^+, \tag{3.1}$$

where $\Psi^0 \in \overline{\mathbf{M}}_m^+$ is called the drift and the Lévy measure ρ is such that $\rho(\mathbf{M}_m \backslash \overline{\mathbf{M}}_m^+) = 0$ and ρ has order of singularity

$$\int_{\overline{\mathbf{M}}_m^+} \min\big(1, \operatorname{tr}(X)\big)\,\rho(\mathrm{d}X) < \infty. \tag{3.2}$$

Moreover, the Laplace transform of M is given by

$$\mathcal{L}_M(\Theta) = \exp\big\{ -\mathcal{K}(\Theta; M)\big\}, \qquad \Theta \in \mathbf{M}_m^+, \tag{3.3}$$

where $\mathcal{K}$ is the cumulant transform or Laplace exponent

$$\mathcal{K}(\Theta; M) = \operatorname{tr}(\Psi^0\Theta) + \int_{\overline{\mathbf{M}}_m^+} \big(1 - e^{-\operatorname{tr}(X\Theta)}\big)\, \rho(\mathrm{d}X). \tag{3.4}$$

We denote by $\mathrm{ID}(\overline{\mathbf{M}}_m^+)$ the set of infinitely divisible matrix distributions with cumulant transform function (3.1). By $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$ we denote the cone of Lévy measures on $\overline{\mathbf{M}}_m^+$, i.e., the measures on $\overline{\mathbf{M}}_m^+$ with order of singularity (3.2). The subclass of Lévy measures concentrated on the open cone $\mathbf{M}_m^+$ is denoted by $\mathfrak{L}(\mathbf{M}_m^+)$. We use the notation $M \in \mathrm{ID}(\overline{\mathbf{M}}_m^+)$ to indicate that the law of M belongs to $\mathrm{ID}(\overline{\mathbf{M}}_m^+)$.

Remark 3.1. (a) A random matrix M in $\mathrm{ID}(\overline{\mathbf{M}}_m^+)$ has independent components if and only if it is diagonal. This follows from the Lévy–Khinchin representation (3.1) and the fact that the only matrix axes E^{ij} in $\overline{\mathbf{M}}_m^+$ are the diagonal ones. In this case the Lévy measure ρ is concentrated in the diagonal matrix axes E^{ii}, $i = 1, \dots, m$.

(b) The above fact proves that there are infinitely divisible matrices with values in $\mathbf{M}_m^+$ whose Lévy measures are concentrated on the singular matrices $\overline{\mathbf{M}}_m^+ \setminus \mathbf{M}_m^+$.

3.2. Matrix subordinators and the Lévy–Itô decomposition. If M is an infinitely divisible matrix in $\overline{\mathbf{M}}_m^+$ with cumulant transform $\mathcal{K}$ given by (3.4), there is a matrix-valued Lévy process $\{M_t\}_{t \geqq 0}$ such that

$$\mathcal{L}_{M_t}(\Theta) = \exp\big(-t\mathcal{K}(\Theta;\, M)\big).$$

We call this Lévy process a *matrix subordinator*, in view of the following result.

A matrix process $\{M_t\}$ is said to be $\overline{\mathbf{M}}_m^+$-increasing if for all $0 \leqq s < t$, $M_t - M_s \in \overline{\mathbf{M}}_m^+$ with probability one. The following result is obtained from [24, Theorem 83] and [23], where it was proved for more general closed cones.

LEMMA 3.1. *Let $\{M_t\}$ be a matrix subordinator. Then*

(a) *the norm process $\{m_t = \|M_t\|;\ t \geqq 0\}$ is a one-dimensional subordinator*;

(b) $\mathbf{P}\{M_t \in \overline{\mathbf{M}}_m^+,\ 0 \leqq t < \infty\} = 1$;

(c) $\mathbf{P}\{M_t$ *is* $\overline{\mathbf{M}}_m^+$*-increasing*$\} = 1$;

(d) $\{M_t\}$ *is of bounded variation with respect to the trace norm* $\|\cdot\|$;

(e) $\mathbf{P}\{\lim_{t\to\infty} \|t^{-1}M_t - \Psi^0\| = 0\} = 1$.

If $\{M_t\}$ is a matrix subordinator with Lévy–Khinchin representation (3.1), it has a Lévy–Itô decomposition

$$M_t = t\Psi^0 + \int_{\overline{\mathbf{M}}_m^+} x\, N_t(\mathrm{d}x), \tag{3.5}$$

where $N_t(\mathrm{d}x)$ is a Poisson random measure on $\overline{\mathbf{M}}_m^+$ with

$$\mathbf{E}\big\{N_t(\mathrm{d}x)\big\} = \rho(\mathrm{d}x)\, t.$$

4. Upsilon transformations for matrix subordinators. In this section we define the upsilon transformations for Lévy measures of matrix subordinators and establish their key properties of uniqueness, smoothness, and regularization. As a tool, a Laplace-type transformation of such measures is introduced.

We recall that $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$ denotes the set of Lévy measures concentrated on the closed cone $\overline{\mathbf{M}}_m^+$, while $\mathfrak{L}(\mathbf{M}_m^+)$ is the subset of Lévy measures concentrated in the open cone $\mathbf{M}_m^+$.

4.1. Definition and first properties. For ρ in $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$ and real q consider the mapping $\Upsilon_q\colon\ \rho \to \widetilde{\rho}_q$ given by

$$\widetilde{\rho}_q(\mathrm{d}Z) = \int_{X>0} \rho\big(\overline{X}^{-\top} \mathrm{d}Z\, \overline{X}^{-1}\big)|X|^q e^{-\operatorname{tr}(X)}\,\mathrm{d}X. \tag{4.1}$$

Remark 4.1. (a) A useful equivalent representation of the measure $\widetilde{\rho}$ is obtained using (2.8)

$$\widetilde{\rho}_q(\mathrm{d}Z) = 2^m \int_{\mathbf{U}_m^+} \rho\big(\overline{X}^{-\top} \mathrm{d}Z\overline{X}^{-1}\big)\, |\overline{X}|^{2q}\, \bar{J}(\overline{X}\,)\, e^{-\operatorname{tr}(\overline{X}^\top\overline{X})}\,\mathrm{d}\overline{X}. \tag{4.2}$$

(b) Using (2.13) the mapping $\widetilde{\rho}$ is also expressible as

$$\widetilde{\rho}_q(\mathrm{d}Z) = \int_0^\infty \int_{\mathbf{S}_m^+} r^{[m]+mq-1}\, e^{-u} \rho\big(r^{-1}\overline{U}^{-\top} \mathrm{d}Z\,\overline{U}^{-1}\big)\, |U|^q\,\mathrm{d}U\,\mathrm{d}r. \tag{4.3}$$

For $q = 0$, $m = 1$ the definition (4.1) specializes to the mapping Υ_0 on $\mathfrak{L}(\mathbf{R}_+)$ given by (1.1). As proved in [7], [8], [9], and [2] this mapping, although arising as establishing a connection between classical and free infinite divisibility (see [7]), has turned out to possess a number of other properties of interest purely in classical infinite divisibility. In particular, it was shown that Υ_0 maps the class of self-decomposable laws on $\mathbf{R}_+$ into the Thorin class (the class of generalized gamma convolutions), and that the class of all infinitely divisible laws on $\mathbf{R}_+$ is mapped into the Steutel–Goldie–Bondesson class (the class of mixtures of nonnegative exponential distributions). Further, there are extensions of these results to the infinitely divisible distributions on $\mathbf{R}^d$ (see [2]).

In the matrix case it is possible to prove that $\widetilde{\rho}_q = \Upsilon_q(\rho)$ is a Lévy measure on $\overline{\mathbf{M}}_m^+$ for $q > -1$. However, in the present paper we deal only with the case $q = 0$, except in section 6, where we point out a relation of the map $\Upsilon_{m/2}$ to mat G random matrices. The case $q = -1$, $m = 1$ was considered in [2] and [29]. The case $q > -2$, $m = 1$ and its relation to Rosiński's tempered stable laws [27] will be considered elsewhere.

In what follows we write $\widetilde{\rho}$ for $\widetilde{\rho}_0$. We first prove that $\widetilde{\rho} = \Upsilon(\rho)$ is a Lévy measure on $\overline{\mathbf{M}}_m^+$. If μ is a probability measure in $\mathrm{ID}(\overline{\mathbf{M}}_m^+)$ with Lévy measure ρ, we shall also use the notation $\widetilde{\mu} = \Upsilon(\mu)$ to indicate that $\widetilde{\mu}$ is the infinitely divisible probability measure associated to the transformed Lévy measure $\widetilde{\rho}$.

Lemma 4.1. *Υ_0 is a well-defined linear mapping from $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$ into $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$.*

Proof. Since for any $Z \in \overline{\mathbf{M}}_m^+$, $V \in \overline{\mathbf{M}}_m^+$ one has that $V^{-\top} Z V^{-1} \in \overline{\mathbf{M}}_m^+$, it follows that $\widetilde{\rho}$ is a measure on $\overline{\mathbf{M}}_m^+$. The linearity is obvious. Further we have to prove that $\widetilde{\rho}$ is a Lévy measure, i.e.,

$$\int_{\overline{\mathbf{M}}_m^+} \min\big(1, \operatorname{tr}(Z)\big)\, \widetilde{\rho}(\mathrm{d}Z) < \infty,$$

under the assumption that

$$\int_{\overline{\mathbf{M}}_m^+} \min\big(1, \operatorname{tr}(X)\big)\, \rho(\mathrm{d}X) < \infty.$$

In what follows we use several times the fact that for nonnegative a, b, c such that $a \leqq c$, we have $\min(a,b) \leqq \min(c,b)$.

Using (2.13) we find

$$\begin{aligned}
\int_{\overline{\mathbf{M}}_m^+} \min\big(1, \operatorname{tr}(Z)\big)\, \widetilde{\rho}(\mathrm{d}Z) &= \int_{\overline{\mathbf{M}}_m^+}\int_{X>0} e^{-\operatorname{tr}(X)} \min\big(1, \operatorname{tr}(Z)\big)\, \rho\big(\overline{X}^{-\top} \mathrm{d}Z \overline{X}^{-1}\big)\, \mathrm{d}X \\
&= \int_{X>0} e^{-\operatorname{tr}(X)} \int_{\overline{\mathbf{M}}_m^+} \min\big(1, \operatorname{tr}\big(\overline{X}^{\top} Y \overline{X}\big)\big)\, \rho(\mathrm{d}Y)\, \mathrm{d}X \\
&= \int_{\mathbf{S}_m^+} \mathrm{d}U \int_0^\infty r^{[m]-1} e^{-r} \int_{\overline{\mathbf{M}}_m^+} \min\big(1, r \operatorname{tr}\big(\overline{U}^{\top} Y \overline{U}\big)\big)\, \rho(\mathrm{d}Y)\, \mathrm{d}r. \qquad (4.4)
\end{aligned}$$

We notice that $\operatorname{tr}(\overline{U}^{\top} Y \overline{U}) \leqq \operatorname{tr}(U)\, \operatorname{tr}(Y) = \operatorname{tr}(Y)$. Assuming that $0 < r < 1$, we have

(i) $\min(1, r \operatorname{tr}(\overline{U}^{\top} Y \overline{U})) \leqq \min(1, \operatorname{tr}(Y))$ since $r \operatorname{tr}(\overline{U}^{\top} Y \overline{U}) \leqq \operatorname{tr}(Y)$,

(ii) $\int_0^1 r^{[m]-1}\, \mathrm{d}r < \infty$ since $[m] > 0$.

Then, splitting the last expression in (4.4) into two parts and using also the fact that ρ is a Lévy measure, we obtain that

$$\begin{aligned}
I_1 &= \int_{\mathbf{S}_m^+} \mathrm{d}U \int_0^1 r^{[m]-1} e^{-r} \int_{\overline{\mathbf{M}}_m^+} \min\big(1, r \operatorname{tr}\big(\overline{U}^{\top} Y \overline{U}\big)\big)\, \rho(\mathrm{d}Y)\, \mathrm{d}r \\
&\leqq c_m \int_0^1 r^{[m]-1}\, \mathrm{d}r \int_{\overline{\mathbf{M}}_m^+} \min\big(1, \operatorname{tr}(Y)\big)\, \rho(\mathrm{d}Y) < \infty,
\end{aligned}$$

where c_m is given by (2.25).

Next when $r > 1$, we have

(i) $\min(1, r \operatorname{tr}(\overline{U}^{\top} Y \overline{U})) \leqq \min(1, r \operatorname{tr}(Y)) \leqq \min(r, r \operatorname{tr}(Y)) = r \min(1, \operatorname{tr}(Y))$,

(ii) $\int_1^\infty r^{[m]} e^{-r}\, \mathrm{d}r < \infty$ since $[m] > -1$.

Then

$$\begin{aligned}
I_2 &= \int_{\mathbf{S}_m^+} \mathrm{d}U \int_1^\infty r^{[m]-1} e^{-r} \int_{\overline{\mathbf{M}}_m^+} \min\big(1, r \operatorname{tr}\big(\overline{U}^{\top} Y \overline{U}\big)\big)\, \rho(\mathrm{d}Y)\, \mathrm{d}r \\
&\leqq c_m \int_1^\infty r^{[m]} e^{-r}\, \mathrm{d}r \int_{\overline{\mathbf{M}}_m^+} \min\big(1, \operatorname{tr}(Y)\big)\, \rho(\mathrm{d}Y) < \infty.
\end{aligned}$$

Hence, $\int_{\overline{\mathbf{M}}_m^+} \min(1, \operatorname{tr}(Z))\, \widetilde{\rho}(\mathrm{d}Z) = I_1 + I_2 < \infty$. Lemma 4.1 is proved.

Several properties and characteristics of ρ are transferred immediately to $\widetilde{\rho}$. For example, finiteness of $\widetilde{\rho}$ is determined by that of ρ.

Lemma 4.2. *$\Upsilon_0(\rho)$ is finite if and only if ρ is finite.*

Proof. It follows since

$$\int_{\overline{\mathbf{M}}_m^+} \widetilde{\rho}(\mathrm{d}Z) = \int_{\overline{\mathbf{M}}_m^+} \int_{X>0} \rho\big(\overline{X}^{-\top} \mathrm{d}Z \overline{X}^{-1}\big)\, e^{-\operatorname{tr}(X)}\, \mathrm{d}X = \int_{X>0} e^{-\operatorname{tr}(X)}\, \mathrm{d}X \int_{\overline{\mathbf{M}}_m^+} \rho(\mathrm{d}Y)$$

and $\int_{X>0} e^{-\operatorname{tr}(X)}\, \mathrm{d}X = \Gamma_m(\langle m \rangle a) < \infty$ by (2.23).

We next show how the cumulant transform of $\widetilde{\rho}$ is computed from the one for ρ. For a Lévy measure $\nu \in \mathfrak{L}(\overline{\mathbf{M}}_m^+)$, we denote by $\mathcal{K}_\nu(\Theta)$ its cumulant transform given by (3.4) with $\Psi^0 = 0$.

LEMMA 4.3. *The cumulant transform of $\Upsilon_0(\rho)$ is given by*

$$\mathcal{K}_{\Upsilon_0(\rho)}(\Theta) = \int_{X>0} e^{-\operatorname{tr}(X)} \mathcal{K}_\rho\big(\overline{X}\Theta\overline{X}^\top\big)\, \mathrm{d}X.$$

Proof. Using (3.4) and the change of variable $Y = \overline{X}^{-\top} Z \overline{X}^{-1}$ we have

$$\begin{aligned}
\mathcal{K}_{\widetilde{\rho}}(\Theta) &= \int_{\overline{\mathbf{M}}_m^+} \big(1 - \operatorname{etr}\big(-\overline{X}^\top Y \overline{X} Z\Theta\big)\big)\, \widetilde{\rho}(\mathrm{d}Z)\\
&= \int_{\overline{\mathbf{M}}_m^+} \int_{X>0} \big(1 - \operatorname{etr}(-Z\Theta)\big)\, \rho\big(\overline{X}^{-\top} \mathrm{d}Z \overline{X}^{-1}\big)\, e^{-\operatorname{tr}(X)}\, \mathrm{d}X\\
&= \int_{X>0} e^{-\operatorname{tr}(X)}\, \mathrm{d}X \int_{\overline{\mathbf{M}}_m^+} \big(1 - \operatorname{etr}\big(-\overline{X}^\top Y \overline{X}\Theta\big)\big)\, \rho(\mathrm{d}Y)\\
&= \int_{X>0} e^{-\operatorname{tr}(X)} \mathcal{K}_\rho\big(\overline{X}\Theta\overline{X}^\top\big)\, \mathrm{d}X
\end{aligned}$$

as claimed. Lemma 4.3 is proved.

4.2. Laplace transform and uniqueness. A useful tool is the following Laplace transform of Lévy measures on $\overline{\mathbf{M}}_m^+$.

LEMMA 4.4. *For ρ in $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$, the Laplace transform*

$$\mathcal{L}\rho(\Theta) = \int_{X>0} \operatorname{etr}(-X\Theta)|X|\, \rho(\mathrm{d}X) \tag{4.5}$$

is finite for any $\Theta \in \mathbf{M}_m^+$.

Proof. Recall that $\operatorname{tr}(X\Theta) > 0$ since $X, \Theta \in \mathbf{M}_m^+$. If $\operatorname{tr}(X) \leqq 1$, then $|X| \leqq x_m^m \leqq x_m \leqq \operatorname{tr}(X)$, where x_m is the maximum eigenvalue of X. Hence

$$\begin{aligned}
\int_{\operatorname{tr}(X)\leqq 1} \operatorname{etr}(-X\Theta)|X|\, \rho(\mathrm{d}X) &\leqq \int_{\operatorname{tr}(X)\leqq 1} \operatorname{etr}(-X\Theta)\, \operatorname{tr}(X)\, \rho(\mathrm{d}X)\\
&\leqq \int_{\operatorname{tr}(X)\leqq 1} \operatorname{tr}(X)\, \rho(\mathrm{d}X) < \infty,
\end{aligned}$$

since ρ is a Lévy measure.

Next, if $\theta_1 > 0$ is the smallest eigenvalue of Θ, from (2.3) we have $\operatorname{tr}(X\Theta) \geqq \theta_1 \operatorname{tr}(X) > 0$. Hence, using (2.4) and (2.2)

$$|X| \leqq \big(\operatorname{tr}(X)\big)^m \leqq \theta_1^{-m}\big(\operatorname{tr}(X\Theta)\big)^m \leqq (mp)!\, \theta_1^{-m} \operatorname{etr}(X\Theta)$$

and therefore

$$\int_{\operatorname{tr}(X)\geqq 1} \operatorname{etr}(-X\Theta)|X|\, \rho(\mathrm{d}X) \leqq m!\, \theta_1^{-m} \int_{\operatorname{tr}(X)\geqq 1} \rho(\mathrm{d}X) = m!\, \theta_1^{-m} \rho(\operatorname{tr}(X) \geqq 1) < \infty,$$

since ρ is a Lévy measure. Lemma 4.4 is proved.

The above Laplace transform has the following property which implies that the mapping Υ_0 is one-to-one.

THEOREM 4.1. *For ρ in $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$*

$$\mathcal{L}\widetilde{\rho}(\Theta) = |\Theta|^{-\langle m\rangle -1} \int_{V>0} \mathcal{L}\rho(\mathbf{V})\,|V|\,\mathrm{etr}(-\mathbf{\Theta}^{-1}V)\,\mathrm{d}V, \tag{4.6}$$

for $\Theta \in \mathbf{M}_m^+$.

Proof. Using (4.1) and (4.2), we have

$$\begin{aligned}\mathcal{L}\widetilde{\rho}(\Theta) &= \int_{Z>0} |Z|\,\mathrm{etr}(-Z\Theta)\,\widetilde{\rho}(\mathrm{d}Z)\\ &= 2^m \int_{Z>0} |Z|\,\mathrm{etr}(-Z\Theta) \int_{\mathbf{U}_m^+} \bar{J}(\overline{X}\,)\,\mathrm{etr}\big(-\overline{X}^\top\overline{X}\,\big)\,\rho\big(\overline{X}^{-\top}\mathrm{d}\,Z\overline{X}^{-1}\,\big)\,\mathrm{d}\overline{X}\\ &= 2^m \int_{\mathbf{U}_m^+} |\overline{X}|^2\bar{J}(\overline{X}\,)\,\mathrm{etr}\big(-\overline{X}^\top\overline{X}\,\big) \int_{Y>0} |Y|\,\mathrm{etr}\big(-Y\overline{X}\,\Theta\overline{X}^\top\,\big)\,\rho(\mathrm{d}Y)\,\mathrm{d}\overline{X}\\ &= 2^m \int_{\mathbf{U}_m^+} |\overline{X}|^2\bar{J}(\overline{X}\,)\,\mathrm{etr}\big(-\overline{X}^\top\overline{X}\,\big)\,\mathcal{L}\rho\big(\overline{X}\Theta\overline{X}^\top\,\big)\,\mathrm{d}\overline{X}.\end{aligned}$$

Further we make the change of variable $\overline{V} = \overline{X}\,\underline{\Theta}^\top$ for which $\mathrm{d}\overline{V} = \underline{J}\big(\underline{\Theta}^\top\big)\,\mathrm{d}\overline{X}$ (see (2.14)) and using the facts

(a) $\overline{X} = \overline{V}\underline{\Theta}^{-\top}$,
(b) $\bar{J}(\overline{X}) = \bar{J}(\overline{V})\,\bar{J}(\underline{\Theta})^{-1}$,
(c) $\overline{X}^\top\overline{X} = \underline{\Theta}^{-1}\overline{V}^\top\overline{V}\underline{\Theta}^{-\top}$, $\overline{X}\Theta\overline{X}^\top = \overline{V}\,\overline{V}^\top$,
(d) $\mathrm{tr}(\overline{X}^\top\overline{X}) = \mathrm{tr}(\underline{\Theta}^{-1}\overline{V}^\top\overline{V}\,\underline{\Theta}^{-\top}) = \mathrm{tr}((\underline{\Theta}\,\underline{\Theta}^\top)^{-1}\,\overline{V}^\top\,\overline{V}) = \mathrm{tr}(\mathbf{\Theta}^{-1}V)$,
(e) from (2.10) $\bar{J}(\underline{\Theta})^{-1} = |\Theta|^{-\langle m\rangle}\underline{J}(\underline{\Theta})$,
(f) $|\overline{X}| = |\overline{V}|\,|\underline{\Theta}^{-1}|$

we obtain

$$\begin{aligned}\mathcal{L}\widetilde{\rho}(\Theta) &= \frac{2^m}{|\Theta|}\int_{\mathbf{U}_m^+} \frac{|\overline{V}|^2}{\underline{J}(\underline{\Theta})}\,\bar{J}(\overline{V}\,)\,\bar{J}(\underline{\Theta})^{-1}\,\mathrm{etr}(-\mathbf{\Theta}^{-1}V)\,\mathcal{L}\rho(\mathbf{V})\,\mathrm{d}\overline{V}\\ &= 2^m|\Theta|^{-\langle m\rangle-1}\int_{\mathbf{U}_m^+} |\overline{V}|^2\bar{J}(\overline{V}\,)\,\mathrm{etr}(-\mathbf{\Theta}^{-1}V)\,\mathcal{L}\rho(\mathbf{V})\,\mathrm{d}\overline{V}.\end{aligned}$$

Hence, using (2.8)

$$\mathcal{L}\widetilde{\rho}(\Theta) = |\Theta|^{-\langle m\rangle-1}\int_{V>0}\mathrm{etr}(-\mathbf{\Theta}^{-1}V)\,\mathcal{L}\rho(\mathbf{V})\,|V|\,\mathrm{d}V$$

as claimed. Theorem 4.1 is proved.

COROLLARY 4.1. *The mapping Υ_0 is one-to-one.*

Proof. Making the change of variable $\Sigma = \mathbf{\Theta}^{-1}$ (for which $\overline{\Sigma}^\top\,\overline{\Sigma}^{-1} = (\overline{\Theta}\,\overline{\Theta}^\top)^{-1}$) in (4.6) we have

$$|\Sigma|^{-\langle m\rangle-1}\mathcal{L}\widetilde{\rho}(\mathbf{\Sigma}^{-1}) = \int_{V>0}|V|\,\mathrm{etr}(-\Sigma V)\,\mathcal{L}\rho(\mathbf{V})\,\mathrm{d}V,$$

where the right-hand side is the Laplace transform of $g(V) = |V|\mathcal{L}\rho(\mathbf{V})$ with respect to the Lebesgue measure $\mathrm{d}V$ on $\mathbf{R}^{[m]}$. Then the result follows from the inversion formula for the Laplace transform of matrix functions in [15] and recalling that, given $X > 0$, the anti-matrix $\mathbf{X} = \overline{X}\,\overline{X}^\top$ is uniquely determined. Corollary 4.1 is proved.

4.3. Regularizing properties of Υ_0. The following result gives the existence of the Lévy density of the transformed Lévy measure determined by the mapping $\widetilde{\rho} = \Upsilon_0(\rho)$, when ρ is concentrated on the open cone $\mathbf{M}_m^+$, in which case $\widetilde{\rho}$ is also concentrated on $\mathbf{M}_m^+$. Furthermore, we prove that its Lévy density $\tilde{l}(X)$ is completely monotone in $\mathbf{X}$. That is, there exists a completely monotone function g such that $l(X) = g(\mathbf{X})$.

Given a measure Q on $\mathbf{M}_m^+$, we denote by $\underset{\leftarrow}{Q}$ the measure on $\mathbf{M}_m^+$ induced by the mapping $X \to \mathbf{X}^{-1}$. In particular, when ρ is a Lévy measure, $\underset{\leftarrow}{\rho}$ satisfies

$$\int_{Y>0} \min\left(1, \operatorname{tr}(Y^{-1})\right) \underset{\leftarrow}{\rho}(\mathrm{d}Y) < \infty. \tag{4.7}$$

THEOREM 4.2. *Let ρ be in $\mathfrak{L}(\mathbf{M}_m^+)$. Then the Lévy measure $\widetilde{\rho}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{M}_m^+$, with Lévy density $\tilde{l}$ given by*

$$\tilde{l}(X) = \int_{Y>0} \bar{J}(\overline{Y}\,)^{-2} \operatorname{etr}(-\mathbf{X}Y^{-1})\, \rho(\mathrm{d}Y). \tag{4.8}$$

Moreover, $\tilde{l}(X) = g(\mathbf{X})$, *where*

$$g(\mathbf{X}) = \int_{Y>0} \operatorname{etr}(-\mathbf{X}Y)\, Q(\mathrm{d}Y), \tag{4.9}$$

and $Q(\mathrm{d}Y)$ is a Radon measure concentrated on $\mathbf{M}_m^+$ such that

$$\int_{Y>0} \min\left(1, \operatorname{tr}(Y^{-1})\right) \bar{J}(\overline{Y}\,)^{-2}\, Q(\mathrm{d}Y) < \infty. \tag{4.10}$$

The connection between Q and the Lévy measure ρ is given by

$$Q(\mathrm{d}Y) = \bar{J}(\overline{Y}\,)^{2} \underset{\leftarrow}{\rho}(\mathrm{d}Y). \tag{4.11}$$

Proof. Let $\mathcal{A}$ be a Borel set of $\mathbf{M}_m^+$. Using (2.8) we have

$$\widetilde{\rho}(\mathcal{A}) = 2^m \int_{Y>0} \int_{\mathbf{U}_m^+} 1_{\mathcal{A}}\big(\overline{X}^{\top} Y \overline{X}\,\big) \operatorname{etr}\left(-\overline{X}^{\top}\, \overline{X}\,\right) \bar{J}(\overline{X}\,)\, \mathrm{d}\overline{X}\, \rho(\mathrm{d}Y).$$

For fixed Y, we make the change of variable $\overline{V} = \overline{Y}\,\overline{X}$ for which $\mathrm{d}\overline{X} = \bar{J}(\overline{Y}^{-1})\,\mathrm{d}\overline{V}$ (see (2.14)). We observe that

(a) $\overline{X} = \overline{Y}^{-1}\overline{V}$,

(b) $\bar{J}(\overline{X}) = \bar{J}(\overline{V})\,\bar{J}(\overline{Y}^{-1})$,

(c) $V = \overline{V}^{\top}\,\overline{V} = \overline{X}^{\top}\,\overline{Y}^{\top}\overline{Y}\,\overline{X} = \overline{X}^{\top} Y \overline{X}$,

(d) $\overline{X}^{\top}\overline{X} = \overline{V}^{\top}\overline{Y}^{-\top}\overline{Y}^{-1}\overline{V} = \overline{V}^{\top}\mathbf{Y}^{-1}\overline{V}$,

(e) $\operatorname{tr}(\overline{X}^{\top}\overline{X}) = \operatorname{tr}(\overline{V}^{\top}\mathbf{Y}^{-1}\overline{V}) = \operatorname{tr}(\overline{V}\,\overline{V}^{\top}\mathbf{Y}^{-1}) = \operatorname{tr}(\mathbf{V}Y^{-1})$.

Then we have that for each $Y > 0$

$$\begin{aligned}&\int_{\mathbf{U}_m^+} 1_{\mathcal{A}}\big(\overline{X}^{\top} Y \overline{X}\,\big) \operatorname{etr}\left(-\overline{X}^{\top}\overline{X}\,\right) \bar{J}(\overline{X}\,)\, \mathrm{d}\overline{X} \\ &\quad = 2^m \int_{\mathbf{U}_m^+} 1_{\mathcal{A}}\big(\overline{V}^{\top}\overline{V}\,\big) \operatorname{etr}(-\mathbf{V}Y^{-1})\, \bar{J}(\overline{Y}\,)^{-2}\, \bar{J}(\overline{V}\,)\, \mathrm{d}\overline{V}\end{aligned}$$

and

$$\widetilde{\rho}(\mathcal{A}) = 2^m \int_{Y>0} \int_{\mathbf{U}_m^+} 1_{\mathcal{A}}\big(\overline{V}^\top \overline{V}\big) \operatorname{etr}(-\mathbf{V}Y^{-1})\, \bar{J}(\overline{Y})^{-2}\, \bar{J}(\overline{V})\, \mathrm{d}\overline{V}\, \rho(\mathrm{d}Y)$$
$$= 2^m \int_{Y>0} \bar{J}(\overline{Y})^{-2} \int_{\mathbf{U}_m^+} 1_{\mathcal{A}}\big(\overline{V}^\top \overline{V}\big) \operatorname{etr}(-\mathbf{V}Y^{-1})\, \bar{J}(\overline{V})\, \mathrm{d}\overline{V}\, \rho(\mathrm{d}Y).$$

Finally, using (2.8) and the Fubini theorem we obtain

$$\widetilde{\rho}(\mathcal{A}) = \int_{V>0} 1_{\mathcal{A}}(V) \int_{Y>0} \bar{J}(\overline{Y})^{-2} \operatorname{etr}(-\mathbf{V}Y^{-1})\, \rho(\mathrm{d}Y)\, \mathrm{d}V$$
$$= \int_{V>0} 1_{\mathcal{A}}(V) \int_{Y>0} \overline{J}(\overline{Y})^{2} \operatorname{etr}(-\mathbf{V}Y)\, \underset{\leftarrow}{\rho}(\mathrm{d}Y)\, \mathrm{d}V.$$

Formula (4.10) follows from the first expression, and the rest of the theorem is obtained from the second one, on defining $Q(\mathrm{d}Y)$ by (4.11).

Remark 4.2. Condition (4.10) may be restated as follows: There is a finite measure H concentrated on $\mathbf{M}_m^+$ such that

$$Q(\mathrm{d}Y) = \max\Big(1, \big(\operatorname{tr}(Y^{-1})\big)^{-1}\Big)\, \bar{J}(\overline{Y})^2\, H(\mathrm{d}Y).$$

4.4. The image class $\mathfrak{B} = \Upsilon_0(\mathfrak{L}(\mathrm{M}_m^+))$.

In this section we consider the image of the Lévy measures in $\mathfrak{L}(\overline{\mathbf{M}}_m^+)$ under the mapping Υ_0 and their associated infinitely divisible matrix distributions. In particular, we characterize the image $\Upsilon_0(\mathfrak{L}(\mathbf{M}_m^+))$ showing that it is a natural matrix analogue of the so-called Steutel–Goldie–Bondesson class of infinitely divisible laws in $\mathbf{R}_+$ (see [2], [28, Theorem 51.10]).

DEFINITION 4.1. *A* probability measure μ in the set $\mathrm{ID}(\mathbf{M}_m^+)$ belongs to *class* $\mathfrak{B} = \mathfrak{B}(\mathbf{M}_m^+)$ *if its Lévy measure is of the form* $\widetilde{\rho} = \Upsilon_0(\rho)$ *for some* $\rho \in \mathfrak{L}(\mathbf{M}_m^+)$. *We shall say that a* random matrix M belongs to $\mathfrak{B}$ *or that its Lévy density belongs to* $\mathfrak{B}$ *if its probability law is in* $\mathfrak{B}(\mathbf{M}_m^+)$.

Given an infinitely divisible random matrix with the Lévy density ρ, we also use the notation $\Upsilon_0(M)$ to indicate a nonnegative definite and infinitely divisible random matrix having Lévy measure $\Upsilon_0(\rho)$.

As for the one-dimensional case, there is a simple characterization of the class $\mathfrak{B}$ in terms of the complete monotonicity of Lévy densities. In the matrix case the Lévy density is completely monotone with respect to the anti-matrix $\mathbf{X}$.

THEOREM 4.3. *A probability measure μ in* $\mathrm{ID}(\mathbf{M}_m^+)$ *belongs to* $\mathfrak{B}$ *if and only if its Lévy density* $\tilde{l}(X)$ *exists and is completely monotone in* $\mathbf{X}$. *That is,* $\tilde{l}(X) = g(\mathbf{X})$ *for some completely monotone function g on* $\mathbf{M}_m^+$.

Proof. The "if part" follows from Theorem 4.2. Conversely, suppose that μ is infinitely divisible with Lévy density $\tilde{l}(X) = g(\mathbf{X})$, where g is completely monotone, i.e.,

$$g(\mathbf{X}) = \int_{Y>0} \operatorname{etr}(-\mathbf{X}Y)\, Q(\mathrm{d}Y), \tag{4.12}$$

where Q is a Radon measure concentrated on $\mathbf{M}_m^+$. Observe that since $\tilde{l}$ is a Lévy density the measure Q satisfies

(4.13)
$$\int_{Y>0}\int_{X>0} \min\big(1, \operatorname{tr}(X)\big) \operatorname{etr}(-XY)\, Q(\mathrm{d}Y)\, \mathrm{d}X = \int_{X>0} \min\big(1, \operatorname{tr}(X)\big)\, \tilde{l}(X)\, \mathrm{d}X < \infty.$$

Let

$$\rho(\mathrm{d}Y) = \bar{J}(\overline{Y})^2 \underset{\leftarrow}{Q}(\mathrm{d}Y).$$

Then

$$\begin{aligned}
&\int_{Y>0} \big(1 - \mathrm{etr}(-\mathbf{Y}\Theta)\big)\, \widetilde{\rho}(\mathrm{d}Y) \\
&\quad = \int_{Y>0}\int_{X>0} \big(1 - \mathrm{etr}(-\mathbf{Y}\Theta)\big)\, \rho\big(\overline{X}^{-\top}\mathrm{d}Y\overline{X}^{-1}\big)\, e^{-\operatorname{tr}(X)}\, \mathrm{d}X \\
&\quad = \int_{Y>0}\int_{X>0} \big(1 - \mathrm{etr}(-\mathbf{Y}\Theta)\big)\, e^{-\operatorname{tr}(X)}\, \bar{J}\big(\overline{X}^{-\top}\overline{Y}\big)^2\, \mathrm{d}X\, \underset{\leftarrow}{Q}\big(\overline{X}^{-\top}\mathrm{d}Y\overline{X}^{-1}\big) \\
&\quad = \int_{Z>0}\int_{X>0} \big(1 - \mathrm{etr}\big(-\Theta\overline{X}^{\top} Z\overline{X}\big)\big)\, \bar{J}(\overline{Z})^2 e^{-\operatorname{tr}(X)}\, \mathrm{d}X\, \underset{\leftarrow}{Q}(\mathrm{d}Z)
\end{aligned}$$

and using (2.8)

$$\begin{aligned}
&\int_{Y>0} \big(1 - \mathrm{etr}(-\mathbf{Y}\Theta)\big)\, \widetilde{\rho}(\mathrm{d}Y) \\
&\quad = 2^m \int_{Z>0}\int_{\mathbf{U}_m^+} \big(1 - \mathrm{etr}\big(-\Theta\overline{X}^{\top} Z\overline{X}\big)\big)\, \bar{J}(\overline{Z})^2\, e^{-\operatorname{tr}(\overline{X}^{\top}\overline{X})}\bar{J}(\overline{X})\, \mathrm{d}\overline{X}\, \underset{\leftarrow}{Q}(\mathrm{d}Z).
\end{aligned}$$

For Z fixed, making the change of variable $\overline{V} = \overline{Z}\,\overline{X}$ for which $\mathrm{d}\overline{V} = \bar{J}(\overline{Z})\,\mathrm{d}\overline{X}$ (see (2.14)) and using the fact that $\bar{J}(\overline{X}) = \bar{J}(\overline{V})\,\bar{J}(\overline{Z})^{-1}$ we obtain

$$\begin{aligned}
&\int_{Y>0} \big(1 - \mathrm{etr}(-\Theta Y)\big)\, \widetilde{\rho}(\mathrm{d}Y) \\
&\quad = 2^m \int_{Z>0}\int_{\mathbf{U}_m^+} \big(1 - \mathrm{etr}\big(-\Theta\overline{V}^{\top}\overline{V}\big)\big)\ \mathrm{etr}\big(-V\,\overline{V}^{\top}\,\overline{Z}^{-\top}\overline{Z}^{-1}\big)\, \bar{J}(\overline{V})\, \mathrm{d}\overline{V}\, \underset{\leftarrow}{Q}(\mathrm{d}Z) \\
&\quad = 2^m \int_{Z>0}\int_{\mathbf{U}_m^+} \big(1 - \mathrm{etr}\big(-\Theta\overline{V}^{\top}\,\overline{V}\big)\big)|V|^q\, \mathrm{etr}(-\mathbf{V}\mathbf{Z}^{-1})\, \bar{J}(\overline{V})\, \mathrm{d}\overline{V}\, \underset{\leftarrow}{Q}(\mathrm{d}Z).
\end{aligned}$$

Finally, using again (2.8)

$$\begin{aligned}
\int_{Y>0} \big(1 - \mathrm{etr}(-\Theta Y)\big)\, \widetilde{\rho}(\mathrm{d}Y) &= \int_{Z>0}\int_{V>0} \big(1 - \mathrm{etr}(-\Theta V)\big)\ \mathrm{etr}(-\mathbf{V}\mathbf{Z}^{-1})\, \mathrm{d}V\, \underset{\leftarrow}{Q}(\mathrm{d}Z) \\
&= \int_{V>0} \big(1 - \mathrm{etr}(-\Theta V)\big) \int_{Z>0} \mathrm{etr}(-\mathbf{V}\mathbf{Z}^{-1}) \underset{\leftarrow}{Q}(\mathrm{d}Z)\, \mathrm{d}V \\
&= \int_{V>0} \big(1 - \mathrm{etr}(-\Theta V)\big)\, g(\mathbf{V})\, \mathrm{d}V,
\end{aligned}$$

where

$$g(\mathbf{V}) = \int_{Z>0} \mathrm{etr}(-\mathbf{V}\mathbf{Z}^{-1})\, \underset{\leftarrow}{Q}(\mathrm{d}Z) = \int_{Z>0} \mathrm{etr}(-\mathbf{V}Z)\, Q(\mathrm{d}Z).$$

Hence, by the uniqueness of the Lévy–Khinchin representation, $\widetilde{\rho}$ has a Lévy density $\tilde{l}(X) = g(\mathbf{X})$ and therefore $\tilde{l}$ is the Levy density of the law μ, showing that $\mu \in \mathfrak{B}$. Theorem 4.3 is proved.

When the Lévy measure $\rho \in \mathfrak{L}(\mathbf{M}_m^+)$ has a density l, the image measure $\widetilde{\rho}$ belongs always to the class $\mathfrak{B}$ and its Lévy density is given as follows.

COROLLARY 4.2. *Let ρ be a Lévy measure with Lévy density l. Then $\widetilde{\rho}$ belongs to the class $\mathfrak{B}(\mathbf{M}_m^+)$, with Lévy density given by*

$$\tilde{l}(X) = \int_{Y>0} \operatorname{etr}(-\mathbf{X}Y^{-1})\, l(\mathbf{Y})\, \frac{\mathrm{d}Y}{|Y|^{\langle m\rangle}}. \tag{4.14}$$

Proof. Since ρ has a density, it is concentrated on the open cone $\mathbf{M}_m^+$ and therefore, from the last theorem, $\widetilde{\rho}$ is in $\mathfrak{B}(\mathbf{M}_m^+)$. Moreover, we use (2.8) in (4.8) and make the change of variable $\underline{V} = \overline{Y}^{\top}$ (for which $\mathrm{d}\underline{V} = \mathrm{d}\overline{Y}$) to obtain

$$\begin{aligned}
\tilde{l}(X) &= \int_{Y>0} \bar{J}(\overline{Y}\,)^{-2} \operatorname{etr}(-\mathbf{X}Y^{-1})\, l(Y)\, \mathrm{d}Y \\
&= 2^m \int_{\mathbf{U}_m^+} \bar{J}(\overline{Y}\,)^{-2} \operatorname{etr}\big(-\mathbf{X}\overline{Y}^{-\top}\overline{Y}^{-1}\big)\, l(Y)\, \bar{J}(\overline{Y}\,)\, \mathrm{d}\overline{Y} \\
&= 2^m \int_{\mathbf{L}_m^+} \operatorname{etr}\big(-\mathbf{X}\underline{V}^{-1}\underline{V}^{-\top}\big)\, l(\mathbf{V})\, \bar{J}(\underline{V})^{-1}\, \mathrm{d}\underline{V}.
\end{aligned}$$

Using the fact $\bar{J}(\underline{V}) = |V|^{\langle m\rangle}\, \underline{J}(\underline{V})^{-1}$ (see (2.10)) and (2.9) we finally obtain

$$\tilde{l}(X) = 2^m \int_{\mathbf{L}_m^+} \operatorname{etr}(-\mathbf{X}V^{-1})\, l(\mathbf{V})|V|^{-\langle m\rangle} \underline{J}(\underline{V})\, \mathrm{d}\underline{V} = \int_{V>0} \operatorname{etr}(-\mathbf{X}V^{-1})\, l(\mathbf{V})\, \frac{\mathrm{d}V}{|V|^{\langle m\rangle}}$$

as we wanted to prove. Corollary 4.2 is proved.

From the proof of the above theorem, we obtain that the cumulant transform of a random matrix in the class $\mathfrak{B}$ can be expressed in terms of the measure Q and the bi-gamma function $\mathbf{\Gamma}_m(a,\cdot,\cdot)$ defined by (2.26).

COROLLARY 4.3. *The law of M in* $\mathrm{ID}(\mathbf{M}_m^+)$ *belongs to $\mathfrak{B}$ if and only if its cumulant transform is given by*

$$\mathcal{K}(\Theta; M) = \operatorname{tr}(\Psi^0\Theta) + \int_{X>0} \big\{\mathbf{\Gamma}_m(\langle m\rangle, 0, X) - \mathbf{\Gamma}_m(\langle m\rangle, \Theta, X)\big\}\, Q(\mathrm{d}X), \tag{4.15}$$

where Q is a Radon measure on $\mathbf{M}_m^+$ such that

$$\int_{X>0} \Big(|X|^{-(m+1)}\underline{J}(\overline{X}\,)^2 - |\mathrm{I}_m + X|^{-(m+1)}\underline{J}\big(\overline{\mathrm{I}_m + X}\,\big)\Big)^2 Q(\mathrm{d}X) < \infty, \tag{4.16}$$

and $\Psi^0 \in \overline{\mathbf{M}}_m^+$.

Proof. From the last theorem we have that M is in $\mathfrak{B}(\mathbf{M}_m^+)$ if and only if there exists Q such that (4.12) and (4.14) are satisfied, where $\tilde{l}$ is the density of the Lévy measure of M. Then, using (2.26) we have that for Θ in $\mathbf{M}_m^+$,

$$\begin{aligned}
&\int_{X>0} \big(1 - \operatorname{etr}(-\Theta X)\big)\, \tilde{l}(X)\, \mathrm{d}X \\
&\quad= \int_{X>0} \big(1 - \operatorname{etr}(-\Theta X)\big) \int_{Y>0} \operatorname{etr}\big(-\overline{X}\,\overline{X}^{\top} Y\big)\, Q(\mathrm{d}Y)\, \mathrm{d}X \\
&\quad= \int_{Y>0} \big\{\mathbf{\Gamma}_m(\langle m\rangle, 0, Y) - \mathbf{\Gamma}_m(\langle m\rangle, \Theta, Y)\big\}\, Q(\mathrm{d}Y),
\end{aligned}$$

proving (4.15). When $\Theta = I$, condition (4.14) yields

$$\int_{Y>0} \big\{\mathbf{\Gamma}_m(\langle m\rangle, 0, Y) - \mathbf{\Gamma}_m(\langle m\rangle, \mathrm{I}_m, Y)\big\}\, Q(\mathrm{d}Y) < 0.$$

Then, (4.16) is obtained using (2.27) and (2.28). Corollary 4.3 is proved.

5. Examples. The purpose of this section is to illustrate the usefulness of Theorem 4.3 in constructing infinitely divisible random matrix laws belonging to the class $\mathfrak{B}$. Specifically, we characterize the subclass of stable matrix distributions belonging to $\mathfrak{B}$ and introduce infinitely divisible matrix versions of the gamma distribution. Further examples will be considered elsewhere.

5.1. A subclass of stable matrix distributions. Let $\mathfrak{S}_\alpha = \mathfrak{S}_\alpha(\mathbf{M}_m^+)$ be the class of α-stable matrix distributions on $\mathbf{M}_m^+$. In this subsection we characterize the matrix stable distributions that belong to the class $\mathfrak{B}$ and characterize the image $\Upsilon_0(\mathfrak{S}_\alpha)$.

We recall from [4] that for $0 < \alpha < 1$, the Lévy measure ρ of an α-stable random matrix in $\mathbf{M}_m^+$ has the form

$$\rho(\mathcal{C}) = \int_{\mathbf{S}_m^+} \lambda(\mathrm{d}U) \int_0^\infty 1_{\mathcal{C}}(rU)\,\frac{\mathrm{d}r}{r^{1+\alpha}}, \qquad \mathcal{C} \in \mathfrak{B}_0(\mathbf{M}_m), \tag{5.1}$$

where λ is a finite measure (the *spectral measure*) on $\mathbf{S}_m^+$. Recall also that ρ is the Lévy measure of an α-stable matrix distribution if and only if $\rho(a^{-1}\,\mathrm{d}X) = a^\alpha \rho(\mathrm{d}X)$ for all $a > 0$. In terms of densities, the change of variable $Y = aX$ gives $\mathrm{d}Y = a^{[m]}\,\mathrm{d}X$, and therefore a function $l\colon \mathbf{M}_m^+ \to \mathbf{R}_+$ is the density of an α-stable Lévy measure if and only if $l(a^{-1}X) = a^{\alpha+[m]} l(X)$ for each $a > 0$.

The simplest example of a matrix stable distribution is constructed from the function

$$h(X) = \frac{1}{(\mathrm{tr}(X))^{\alpha+[m]}} \tag{5.2}$$

on $\mathbf{M}_m^+$, for which we have the following properties.

Lemma 5.1. *Let $h\colon \mathbf{M}_m^+ \to \mathbf{R}_+$ be given by* (5.2). *Then*

(a) *h is a Lévy density of a matrix stable law on $\mathbf{M}_m^+$ if and only if* $0 < \alpha < 1$;

(b) *h is a completely monotone function of X and $\mathbf{X}$ whenever $\alpha > 0$.*

Proof. From the polar decomposition (2.13)

$$\begin{aligned}\int_{X>0} \min\big(1, \mathrm{tr}(X)\big)\, h(X)\,\mathrm{d}X &= \int_{\mathbf{S}_m^+}\int_0^\infty \min(1,r)\, r^{-[m]-\alpha} r^{[m]-1}\,\mathrm{d}r\,\mathrm{d}U \\ &= \int_{\mathbf{S}_m^+}\int_0^\infty \min(1,r)\, r^{-1-1}\,\mathrm{d}r\,\mathrm{d}U = \int_0^\infty \min(1,r)\, r^{-1-\alpha}\,\mathrm{d}r < \infty,\end{aligned}$$

since $c_m = \int_{\mathbf{S}_m^+} \mathrm{d}U < \infty$ and $r^{-1-\alpha}$ is the one-dimensional Lévy density of the α-stable distribution. Finally, h is completely monotone in X and $\mathbf{X}$, by (2.22) with $Y_0 = \mathrm{I}_0$. Lemma 5.1 is proved.

Contrary to the one-dimensional case, there are α-stable matrix distributions whose Lévy measures are not absolutely continuous with respect to the Lebesgue measure on $\mathbf{M}_m^+$, and thus they cannot belong to the class $\mathfrak{B}$. For example, let M be a random diagonal matrix with independent diagonal elements, each one with a positive one-dimensional α-stable distribution. The corresponding Lévy measure ρ of M is of the form

$$\rho(\mathcal{C}) = \sum_{i=1}^m a_i \int_0^\infty 1_{\mathcal{C}}(rE^{ii})\,\frac{\mathrm{d}r}{r^{1+\alpha}},$$

where $a_i = \lambda(E^{ii})$, $i = 1,\ldots,m$. Since E^{ii} belongs to $\overline{\mathbf{M}}_m^+ \backslash \mathbf{M}_m^+$, $i = 1,\ldots,m$, ρ cannot have a density with respect to the Lebesgue measure on $\mathbf{M}_m^+$.

A characterization of the matrix stable distributions that belong to the class $\mathfrak{B}$ is given by the following result.

PROPOSITION 5.1. *Let* $0 < \alpha < 1$. *An* α*-stable matrix distribution* μ *in* $\mathbf{M}_m^+$ *belongs to the class* $\mathfrak{B}$ *if and only if its Lévy measure has a density that is representable as*

$$\int_{\mathbf{S}_m^+} \frac{\bar{J}(\overline{U})^{-2}}{[\mathrm{tr}(\mathbf{X}U^{-1})]^{\alpha+[m]}}\,\lambda(\mathrm{d}U) \tag{5.3}$$

for some finite measure λ *on* $\mathbf{S}_m^+$ *concentrated on positive definite matrices.*

Proof. We first prove that, when λ is a finite measure on $\mathbf{S}_m^+$ concentrated on $\mathbf{M}_m^+$, the function given by (5.3) is the Lévy density of an α-stable matrix distribution in $\mathfrak{B}$. Let ρ be the Lévy measure (5.1) of an α-stable matrix law, corresponding to the measure λ. Then $\widetilde{\mu} = \Upsilon_0(\mu)$ belongs to $\mathfrak{B}$ and, using (4.8), its Lévy density is computed as follows:

$$\begin{aligned}
\tilde{l}(X) &= \int_{Y>0} \bar{J}(\overline{Y})^{-2}\ \mathrm{etr}(-\mathbf{X}Y^{-1})\,\rho(\mathrm{d}Y)\\
&= \int_{\mathbf{S}_m^+} \lambda(\mathrm{d}U) \int_0^\infty \bar{J}(r^{1/2}\overline{U})^{-2}\ \mathrm{etr}(-r^{-1}\mathbf{X}U^{-1})\,\frac{\mathrm{d}r}{r^{1+\alpha}}\\
&= \int_{\mathbf{S}_m^+} \lambda(\mathrm{d}U) \int_0^\infty r^{-[m]}\bar{J}(\overline{U})^{-2}\ \mathrm{etr}(-r^{-1}\mathbf{X}U^{-1})\,\frac{\mathrm{d}r}{r^{1+\alpha}},
\end{aligned}$$

where we have used (2.12). Hence, making the change of variable $t = 1/r$, we find

$$\tilde{l}(X) = \int_{\mathbf{S}_m^+} \lambda(\mathrm{d}U)\,\bar{J}(\overline{U})^{-2} \int_0^\infty t^{[m]+\alpha-1}\ \mathrm{etr}(-t\mathbf{X}U^{-1})\,\mathrm{d}t.$$

Using the one-dimensional gamma function, we have that

$$\int_0^\infty t^{[m]+\alpha-1}\ \exp\big(-t\,\mathrm{tr}(\mathbf{X}U^{-1})\big)\,\mathrm{d}t = \Gamma(\alpha+[m])[\mathrm{tr}(\mathbf{X}U^{-1})]^{-\alpha-[m]}.$$

Therefore,

$$\tilde{l}(X) = \Gamma(\alpha+[m]) \int_{\mathbf{S}_m^+} \bar{J}(\overline{U})^{-2}[\mathrm{tr}(\mathbf{X}U^{-1})]^{-\alpha-[m]}\,\lambda(\mathrm{d}U) \tag{5.4}$$

which is of the form (5.3) with λ substituted by $\Gamma(\alpha+[m])\,\lambda$. It is trivial that $\tilde{l}(a^{-1}X) = a^{\alpha+[m]}\tilde{l}(X)$ for each $a > 0$; therefore the measure $\widetilde{\mu}$ is α-stable.

Conversely, suppose that μ_α is an α-stable matrix distribution with completely monotone Lévy density l_α. We have to prove that l_α is of the form (5.3). From Theorem 4.3, there is a Radon measure Q on $\mathbf{M}_m^+$ such that

$$l_\alpha(X) = \int_{Y>0} \mathrm{etr}(-\mathbf{X}Y)\,Q(\mathrm{d}Y),$$

and, moreover, $\rho(\mathrm{d}Y) = \bar{J}(\overline{Y})^2 \underleftarrow{Q}(\mathrm{d}Y)$ is a Lévy measure. Then

$$l_\alpha(X) = \int_{Y>0} \bar{J}(\overline{Y})^{-2}\,\mathrm{etr}(-\mathbf{X}Y^{-1})\,\rho(\mathrm{d}Y).$$

We show that ρ is also α-stable. From the fact that l_α is a stable density, we obtain that $Q(a\,\mathrm{d}Y) = a^{\alpha+[m]}Q(\mathrm{d}Y)$ and $\underleftarrow{Q}(a^{-1}\,\mathrm{d}Y) = a^{\alpha+[m]}\underleftarrow{Q}(\mathrm{d}Y)$ for all $a > 0$. Hence $\rho(a^{-1}\,\mathrm{d}Y) = a^\alpha\rho(\mathrm{d}Y)$ for all $a > 0$, i.e., ρ is the Lévy measure of an α-stable matrix distribution μ. That is, $\mu_\alpha = \Upsilon_0(\mu)$, and by the first part of the proposition, $l_\alpha(X)$ is of the form (5.3), as we had to prove. Proposition 5.1 is proved.

5.2. Gamma-type and simple tempered matrix distributions. In the one-dimensional case, the gamma distribution $G(\eta,\sigma)$ with probability density

$$f(x)=\left(\frac{\sigma^{-\eta}}{\Gamma(\eta)}\right)x^{\eta-1}\exp\left(-\frac{x}{\sigma}\right),\qquad x>0,\quad \sigma,\ \eta>0, \tag{5.5}$$

is an infinitely divisible law with the Lévy density

$$h(x)=\eta x^{-1}\exp(-\sigma^{-1}x). \tag{5.6}$$

As mentioned before, the usual matrix gamma distribution with density (2.24) is not infinitely divisible. Using a matrix analogue of (5.6) we show how to construct a gamma-type matrix distribution which belongs to the class $\mathfrak{B}$ and therefore is infinitely divisible.

We first present a general class of simple Lévy densities whose matrix distributions belong to $\mathfrak{B}$ and are related to a simple case of the tempered stable distributions of Rosiński [26], [27]. For $\beta<1$, define $h_\beta(X)\colon\ \overline{\mathbf{M}}_m^+\to\mathbf{R}_+$ by

$$h_\beta(X)=\frac{1}{(\mathrm{tr}(X))^{[m]+\beta}}\ \mathrm{etr}(-X). \tag{5.7}$$

The following is a straightforward result. We recall that $c_m=\pi^{m/2}/([m]-1)!$ (see (2.25)).

Lemma 5.2. *Let $h_\beta\colon\ \mathbf{M}_m^+\to\mathbf{R}_+$ be given by (5.7). Then*

(a) *h_β is a Lévy density on $\mathbf{M}_m^+$ if and only if $\beta<1$;*

(b) *if $\beta<0$, $\int_{X>0}h_\beta(X)\,\mathrm{d}X=c_m$ and for $\beta\geqq0$, $\int_{X>0}h_\beta(X)\,\mathrm{d}X=+\infty$;*

(c) *if $\beta>-[m]$, h_β is completely monotone in X and in $\mathbf{X}$.*

Proof. Using (2.13) we have

$$\begin{aligned}\int_{X>0}\min\big(1,\mathrm{tr}(X)\big)\,h_\beta(X)\,\mathrm{d}X&=\int_{\mathbf{S}_m^+}\int_0^\infty\min(1,r)\,r^{-[m]-\beta}e^{-r}r^{[m]-1}\,\mathrm{d}r\,\mathrm{d}U\\&=\int_{\mathbf{S}_m^+}\mathrm{d}U\int_0^\infty\min(1,r)\,r^{-\beta-1}e^{-r}\,\mathrm{d}r\\&=c_m\int_0^\infty\min(1,r)\,r^{-\beta-1}e^{-r}\,\mathrm{d}r,\end{aligned}$$

where $c_m<\infty$ (see (2.25)). Hence, (a) follows since in the one-dimensional case $r^{-\beta-1}e^{-r}$ is a Lévy density if and only if $\beta<1$. The first part of (b) is obtained since $\int_0^\infty r^{-\beta-1}e^{-r}\,\mathrm{d}r=\Gamma(-\beta)$ for $\beta<0$ and using (2.25). The second part follows since $\int_0^\infty r^{-\beta-1}e^{-r}\,\mathrm{d}r=+\infty$ if $\beta>0$. Finally, assertion (c) follows from (2.22) with $Y_0=\mathrm{I}_m$. Lemma 5.2 is proved.

The above lemma shows that when $\beta<1$, the function h_β is the Lévy density of an infinitely divisible matrix distribution in the class $\mathfrak{B}$. We write $M\sim G_\beta(\mathrm{I}_m)$ to indicate that the random matrix M has an infinitely divisible distribution with Lévy density (5.7). Any random matrix $M\sim G_\beta(\mathrm{I}_m)$ has an *orthogonal-symmetric* distribution, in the sense that $O^\top MO\overset{\text{law}}{=}M$ for any O in the orthogonal group $\mathcal{O}(m)$.

The matrix distributions $G_\beta(\mathrm{I}_m)$, $\beta<1$, form a building block for constructing more interesting matrix laws.

In the case of linear transformations, for any $\Sigma\in\mathbf{M}_m^+$, the random matrix $R=\Sigma^{1/2}M\Sigma^{1/2}$ is infinitely divisible with Lévy density

$$h_\beta(X,\Sigma)=\frac{|\Sigma|^{-\langle m\rangle}}{(\mathrm{tr}(X\Sigma^{-1}))^{[m]+\beta}}\ \mathrm{etr}(-X\Sigma^{-1}). \tag{5.8}$$

We write $R \sim G_\beta(\Sigma)$ to indicate that the random matrix R has the matrix distribution associated to (5.8).

In particular, the case $\beta = 0$ is a *gamma-type matrix distribution.* The name is suggested by the following facts.

PROPOSITION 5.2. *Let $R \sim G_0(\Sigma)$ and $\Sigma \in \mathbf{M}_m^+$. Then R has cumulant transform*

$$\mathcal{K}(\Theta; R) = \int_{\mathbf{S}_m^+} \log\left(1 + \mathrm{tr}(U\Sigma^{1/2}\Theta\Sigma^{1/2})\right)^{-1} \mathrm{d}U. \tag{5.9}$$

Proof. Using (3.4) (with $\Psi^0 = 0$) and (2.13) we have

$$\begin{aligned}\mathcal{K}(\Theta; \Sigma^{1/2}M\Sigma^{1/2}) &= \int_{X>0} (1 - e^{-\mathrm{tr}(X\Sigma^{1/2}\Theta\Sigma^{1/2})})\, h(X)\, \mathrm{d}X \\ &= \int_{\mathbf{S}_m^+}\int_0^\infty (1 - e^{-r\,\mathrm{tr}(U\Sigma^{1/2}\Theta\Sigma^{1/2})})\, r^{-1}\, e^{-r}\, \mathrm{d}r\, \mathrm{d}U.\end{aligned}$$

From the cumulant transform of the one-dimensional gamma distribution

$$\int_0^\infty (1 - e^{-r\,\mathrm{tr}(U\Sigma^{1/2}\Theta\Sigma^{1/2})})\, r^{-1}e^{-r}\, \mathrm{d}r = \log\left(1 + \mathrm{tr}(U\Sigma^{1/2}\,\Theta\Sigma^{1/2})\right)^{-1},$$

from which (5.9) follows. Next, we recall that $\mathrm{tr}(\mathbf{X}) = \mathrm{tr}(X)$ and therefore $h(\mathbf{X}) = h(X)$. Using the change of variable $Y = \Sigma^{1/2}X\Sigma^{1/2}$ for which $\mathrm{d}Y = |\Sigma|^{\langle m\rangle}\,\mathrm{d}X$ (see (2.15))

$$\begin{aligned}\mathcal{K}(\Theta; R) &= \int_{X>0} (1 - e^{-\mathrm{tr}(X\Sigma^{1/2}\Theta\Sigma^{1/2})})\, h(\mathbf{X})\, \mathrm{d}X \\ &= \int_{Y>0} (1 - e^{-\mathrm{tr}(Y\Theta)})\, h(\Sigma^{-1/2}Y\Sigma^{-1/2})|\Sigma|^{-\langle m\rangle}\, \mathrm{d}Y.\end{aligned}$$

Proposition 5.2 is proved.

If $M \sim G_0(\mathrm{I}_m)$, $\mathrm{tr}(M)$ follows a one-dimensional gamma distribution and, more generally, any marginal distribution $\mathrm{tr}(\Sigma M)$ follows a one-dimensional gamma convolution (i.e., it is the weak limit of finite convolutions of gamma distributions). We write $M \sim G_0(\Sigma)$ to indicate that the random matrix M has a matrix law given by (5.9).

COROLLARY 5.1. (a) *If $M \sim G_0(\Sigma)$ for $\Sigma \in \mathbf{M}_m^+$, $\mathrm{tr}(\Sigma M)$ follows a one-dimensional gamma convolution law.*

(b) *If $M \sim G_0(\mathrm{I}_m)$, $\mathrm{tr}(M)$ has the one-dimensional gamma distribution $G(c_m, 1)$.*

Proof. From (5.9), by taking $\Theta = \theta\Sigma$, $\theta > 0$, we have that the cumulant transform of $\mathrm{tr}(\Sigma M)$ is

$$\log \mathbf{E}e^{-\theta\,\mathrm{tr}(\Sigma M)} = \int_{\mathbf{S}_m^+} \log\left(1 + \theta\,\mathrm{tr}(\Sigma U)\right)^{-1} \mathrm{d}U = \int_0^\infty \log(1 + \theta u)^{-1}\, \nu_\Sigma(\mathrm{d}u),$$

where ν_Σ is the measure on $(0, \infty)$ induced by $\mathrm{d}U$ under the transformation $U \to \mathrm{tr}(\Sigma^{1/2}U\Sigma^{1/2})$. Then $\mathrm{tr}(\Sigma M)$ follows a one-dimensional gamma convolution [11]. When $\Sigma = \mathrm{I}_m$,

$$\int_{\mathbf{S}_m^+} \log\left(1 + \theta\,\mathrm{tr}(U)\right)^{-1} \mathrm{d}U = \log(1 + \theta)^{-c_m},$$

which is the cumulant transform of the one-dimensional gamma distribution $G(c_m, 1)$. Corollary 5.1 is proved.

For $0 < \beta < 1$, the matrix law $G_\beta(\mathrm{I}_m)$ is related to the one-dimensional tempered β-stable distributions recently introduced in [26] and [27]. The relation is in the sense that any one-dimensional marginal has a tempered β-stable distribution.

PROPOSITION 5.3. *Let $0 < \beta < 1$ and let $M \sim G_\beta(\mathrm{I}_m)$. Then*

(a) *M has cumulant transform $\mathcal{K}_\beta(\Theta, M)$*

$$\mathcal{K}_\beta(\Theta, M) = -k_\beta \Big\{ \int_{\mathbf{S}_m^+} \big(1 + \mathrm{tr}(U\Theta)\big)^\beta \,\mathrm{d}U - c_m \Big\}, \tag{5.10}$$

where $k_\beta = \Gamma(1-\beta)/\beta$;

(b) *for any $\Sigma \in \mathbf{M}_m^+$, $\mathrm{tr}(\Sigma M)$ has a one-dimensional tempered β-stable distribution.*

Proof. Using (3.4) (with $\Psi^0 = 0$) and (2.13) we have

$$\begin{aligned}\mathcal{K}_\beta(\Theta; M) &= \int_{X>0} (1 - e^{-\mathrm{tr}(X\Theta)})\, h_\beta(X)\,\mathrm{d}X \\ &= \int_{\mathbf{S}_m^+} \int_0^\infty (1 - e^{-r\,\mathrm{tr}(U\Theta)})\, r^{-1-\beta} e^{-r}\,\mathrm{d}r\,\mathrm{d}U.\end{aligned}$$

For $0 < \beta < 1$ one has

$$\int_0^\infty (1 - e^{-r\,\mathrm{tr}(U\Theta)})\, r^{-1-\beta} e^{-r}\,\mathrm{d}r = \frac{1}{\beta}\,\Gamma(1-\beta)\big[\big(1 + \mathrm{tr}(U\Theta)\big)^\beta - 1\big],$$

from which (a) follows.

From (5.10), taking ν_Σ as the measure on $(0,\infty)$ induced by $\mathrm{d}U$ and the transformation $U \to \mathrm{tr}(\Sigma U)$, we have

$$\begin{aligned}\log \mathbf{E} e^{-\theta\,\mathrm{tr}(\Sigma M)} &= -k_\beta \Big\{ \int_{\mathbf{S}_m^+} \big(1 + \theta\,\mathrm{tr}(\Sigma U)\big)^\beta \,\mathrm{d}U - c_m \Big\} \\ &= -k_\beta \Big\{ \int_0^\infty (1 + \theta u)^\beta \nu_\Sigma(\mathrm{d}u) - c_m \Big\},\end{aligned}$$

which is the cumulant transform of a tempered β-stable distribution, $0 < \beta < 1$ [26, Proposition 2.2]. Proposition 5.3 is proved.

5.3. The class $\Upsilon_0(G_\beta)$ of matrix distributions. In this subsection we study the matrix distributions which are the image of G_β under the upsilon transformation Υ_0. More specifically, an infinitely divisible random matrix R in $\mathbf{M}_m^+$ will have a *matrix distribution* $\Upsilon_0(G_\beta(\Sigma))$ if its Lévy measure is of the form $\Upsilon_0(\rho)$, where ρ is the Lévy measure with Lévy density $h_\beta(X; \Sigma)$ given by (5.8), for $\Sigma > 0$ and $\beta < 1$. In this case we write $R \sim \Upsilon_0(G_\beta(\Sigma))$.

The Lévy density of an $\Upsilon_0(G_\beta(\Sigma))$ law is obtained from Corollary 4.2. This construction yields a matrix extension of the one-dimensional Bessel function, different from the one defined in [15], commonly used in the classical multivariate statistical literature [14].

For $\beta < 1$ and $\Sigma > 0$ fixed, define *the matrix Bessel-type function* $\mathbb{K}\colon \mathbf{M}_m^+ \to \mathbf{R}_+$ by

$$\mathbb{K}(X; \beta, \Sigma) = |\Sigma|^{-\langle m\rangle} \int_{Y>0} \mathrm{etr}\big\{ -(\mathbf{X}Y^{-1} + \Sigma^{-1}\mathbf{Y})\big\}\big(\mathrm{tr}(\mathbf{Y}\Sigma^{-1})\big)^{-[m]-\beta} \frac{\mathrm{d}Y}{|Y|^{\langle m\rangle}}. \tag{5.11}$$

The following result is based on infinite divisibility theory and shows that the function $\mathbb{K}$ is well defined and completely monotone in $\mathbf{X}$.

Proposition 5.4. *Let $\beta < 1$ and $\Sigma > 0$. Then $\mathbb{K}(X; \beta, \Sigma)$ is the Lévy density of the matrix distribution $\Upsilon_0(G_\beta(\Sigma))$.*

Proof. From (4.14) in Corollary 4.2, the Lévy density $\tilde{l}$ of $\Upsilon_0(R)$ is computed as follows:

$$\begin{aligned}\tilde{l}(X) &= \int_{Y>0} \operatorname{etr}(-\mathbf{X}Y^{-1})\, h_\beta(\mathbf{Y}; \Sigma)\, \frac{\mathrm{d}Y}{|Y|^{\langle m\rangle}} \\ &= |\Sigma|^{-\langle m\rangle} \int_{Y>0} \operatorname{etr}(-\mathbf{X}Y^{-1}) \big(\operatorname{tr}(\mathbf{Y}\Sigma^{-1})\big)^{-[m]-\beta} \operatorname{etr}(-\mathbf{Y}\Sigma^{-1})\, \frac{\mathrm{d}Y}{|Y|^{\langle m\rangle}} \\ &= |\Sigma|^{-\langle m\rangle} \int_{Y>0} \operatorname{etr}\big\{-(\mathbf{X}Y^{-1} + \Sigma^{-1}\mathbf{Y})\big\} \big(\operatorname{tr}(\mathbf{Y}\Sigma^{-1})\big)^{-[m]-\beta} \frac{\mathrm{d}Y}{|Y|^{\langle m\rangle}} \\ &= \mathbb{K}(X; \beta, \Sigma).\end{aligned}$$

Proposition 5.4 is proved.

As a by-product of the above result we obtain the following analytical property of $\mathbb{K}$.

Corollary 5.2. *The integral $\int_{X>0} \mathbb{K}(X; \beta, \Sigma)\, \mathrm{d}X$ is finite if and only if $\beta < 0$.*

Proof. From Lemma 5.2 we have that $\int_X h_\beta(X; \Sigma)\, \mathrm{d}X$ is finite if and only if $\beta < 0$. Let ρ and $\widetilde{\rho}$ be the Lévy measures given by h_β and $\mathbb{K}(X; \beta, \Sigma)$, respectively. Since $\widetilde{\rho} = \Upsilon_0(\rho)$, by Lemma 4.2, $\widetilde{\rho}$ is a finite measure if and only if ρ is finite. Corollary 5.2 is proved.

Remark 5.1. (a) When $m = 1$ the Lévy density (5.12) takes the form

$$\mathbb{K}(x; \beta, \tau) = \tau^\beta \int_0^\infty \exp\big\{-(xy + \tau^{-1}y^{-1})\big\}\, y^\beta\, \mathrm{d}y,$$

which is related to the one-dimensional modified Bessel function of the third kind and with index $\kappa = \beta + 1$. In particular, when $\beta = 0$, we obtain the image class of one-dimensional distributions under the mapping Υ_0.

(b) The image class $\Upsilon_q(G_\beta(\Sigma))$ for $\beta < 1$ and $q < 0$ will be studied elsewhere.

6. A relation of the mapping $\boldsymbol{\Upsilon_{m/2}}$ to $\operatorname{mat} G$ random matrices. In this section we point out a connection with $\operatorname{mat} G$ random matrices when considering the mapping $\widetilde{\rho}_{m/2} = \Upsilon_{m/2}(\rho)$ of a Lévy measure ρ in $\mathfrak{L}(\mathbf{M}_m^+)$.

Following [5], we say that a random matrix G_R in $\mathbf{M}_{n\times m}$ is $\operatorname{mat} G$ if $G_R \overset{\text{law}}{=} NR$, where R is a random matrix in $\mathbf{M}_{p\times m}$ such that $R^\top R$ is in $\mathrm{ID}(\overline{\mathbf{M}}_m^+)$ and N is a standard normal random matrix $N_{np}(0; \mathrm{I}_m \otimes \mathrm{I}_p)$ in $\mathbf{M}_{n\times p}$ independent of R.

For $\Delta \in \mathbf{M}_n^+, \Sigma \in \mathbf{M}_m^+$, denote by $\varphi_{nm}(Z; \Delta \otimes \Sigma)$ the density function of a zero mean normal matrix distribution with covariance $\Delta \otimes \Sigma$. That is,

$$\varphi_{nm}(Z; \Delta \otimes \Sigma) = \frac{|\Delta|^{-n/2}|\Sigma|^{-m/2}}{(2\pi)^{mn/2}} \operatorname{etr}\bigg(-\frac{1}{2}\big(\Delta^{-1}Z\Delta^{-1}Z^\top\big)\bigg).$$

Let ρ be the Lévy measure of $R^\top R$ and assume it is concentrated on $\mathbf{M}_m^+$. For simplicity let $m = p$. It is proved in [6] that G_R is an infinitely divisible random matrix whose Lévy measure ν_ρ is absolutely continuous with Lévy density

$$u_v(Z) = \int_{\Sigma>0} \varphi_{nm}(Z; \mathrm{I}_n \otimes \Sigma)\, \rho(\mathrm{d}\Sigma), \qquad Z \in \mathbf{M}_{nm}.$$

For $\rho \in \mathfrak{L}(\mathbf{M}_m^+)$, let $\widetilde{\rho}_{m/2}$ be the Lévy measure given by (4.1), and let X, Y be in $\mathrm{ID}(\overline{\mathbf{M}}_m^+)$ with Lévy measures ρ and $\widetilde{\rho}_{m/2}$, and let $N \sim N_{nm}(0; \mathrm{I}_n \otimes \mathrm{I}_m)$ be independent of X and Y. Then $G_Y = N\overline{Y}$ has Lévy density given by

$$
\begin{aligned}
u_{\widetilde{\rho}_{m/2}}(Z) &= \int_{\Sigma>0} \varphi_{nm}(Z; \mathrm{I}_n \otimes \Sigma)\, \widetilde{\rho}_{m/2}(\mathrm{d}\Sigma) \\
&= \int_{\Sigma>0} \int_{X>0} \varphi_{nm}(Z; \mathrm{I}_n \otimes \Sigma)\, \rho\big(\overline{X}^{-\top} \mathrm{d}\Sigma \overline{X}^{-1}\big) |X|^{m/2} e^{-\operatorname{tr}(X)}\, \mathrm{d}X \\
&= \int_{\Sigma>0} \int_{X>0} \varphi_{nm}\big(Z; \mathrm{I}_n \otimes \overline{X}\Sigma\overline{X}^{\top}\big)\, \rho(\mathrm{d}\Sigma) |X|^{m/2} e^{-\operatorname{tr}(X)}\, \mathrm{d}X \\
&= \int_{\Sigma>0} \int_{X>0} \varphi_{nm}(Z\overline{X}^{-1}; \mathrm{I}_n \otimes \Delta)\, \rho(\mathrm{d}\Sigma)\, e^{-\operatorname{tr}(X)}\, \mathrm{d}X \\
&= \int_{X>0} e^{-\operatorname{tr}(X)} \int_{\Sigma>0} \varphi_{nm}(Z\overline{X}^{-1}; \mathrm{I}_n \otimes \Sigma)\, \rho(\mathrm{d}\Sigma)\, \mathrm{d}X \\
&= \int_{X>0} e^{-\operatorname{tr}(X)} \nu_\rho(Z\overline{X}^{-1})\, \mathrm{d}X.
\end{aligned}
$$

Thus, we have proved the following result.

PROPOSITION 6.1. *For $\rho \in \mathfrak{L}(\mathbf{M}_m^+)$, let X, Y be infinitely divisible random matrices in $\mathbf{M}_m^+$ with Lévy measures ρ and $\widetilde{\rho}_{m/2} = \Upsilon_{m/2}(\rho)$, respectively. Then, the Lévy measure $\nu_{\widetilde{\rho}_{m/2}}$ of the* mat G *random matrix G_Y is given by the mapping*

$$
\nu_{\widetilde{\rho}_{m/2}} = \Upsilon_{m/2}(\nu_\rho), \tag{6.1}
$$

where ν_ρ is the Lévy measure of G_X.

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