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A class of random matrices with infinitely divisible determinants

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Abstract

A class of random matrices whose determinants and some of their powers are infinitely divisible is provided. It includes the right-orthogonally invariant random matrices and mixtures of Wishart matrices.

Keywords: Wishart distribution; Random determinant; Orthogonally invariant matrix distribution; Subordinator; Lévy process

1. Introduction

The purpose of this paper is to show the infinite divisibility of some powers of determinants of a class of random matrices, which includes right-orthogonally invariant random matrices and mixtures of Wishart random matrices.

Let m and n be nonnegative integers. We denote by $\mathbb{M}_{m\times n}$ the linear space of $m\times n$ real matrices, by $\mathbb{M}_m$ the space of symmetric $m\times m$ matrices, by $\overline{\mathbb{S}}_m$ (resp. $\mathbb{S}_m$) the closed (resp. open) cone of nonnegative (resp. positive) definite matrices in $\mathbb{M}_m$ and by $\mathbb{O}(m,n)=\{O\in\mathbb{M}_{m\times n}: OO^\top=\mathrm{I}_m\}$ the Stiefel manifold, where I_m is the $m\times m$ identity matrix. We denote by $\mathrm{tr}(A)$ and $\det(A)$ the trace and the determinant of a matrix A in $\mathbb{M}_m$, respectively.

Recall that a random matrix Z in $\mathbb{M}_{m\times n}$ has a Gaussian distribution $N_{mn}(0;\Sigma\otimes\mathrm{I}_n)$, $\Sigma\in\mathbb{S}_m$, if its density function is given by

$$\varphi_{mn}(x;\Sigma\otimes\mathrm{I}_n)=\{(2\pi)^n\det(\Sigma)\}^{-m/2}\exp\left\{-\tfrac{1}{2}\mathrm{tr}(x^\top\Sigma^{-1}x)\right\},\quad x\in\mathbb{M}_{m\times n}.$$

A random matrix W in $\overline{\mathbb{S}}_m$ is said to have the *Wishart matrix distribution* $W_m(n,\Sigma)$ with *n degrees of freedom*, $n\geqslant m,\Sigma\in\mathbb{S}_m$, if $W\stackrel{d}{=}ZZ^\top$ where $Z\sim N_{mn}(0,\Sigma\otimes\mathrm{I}_n)$ and $\stackrel{d}{=}$ stands for equality in distribution.

This random matrix was introduced in the theory of mathematical statistics by Wishart (1928) and Lévy (1948) who recognized that this matrix distribution is not infinitely divisible.

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The purpose of this paper is to show that some real powers of $\det(W)$ are infinitely divisible and furthermore self-decomposable. We also show that the distribution of $\log(\det(W))$ belongs to the class L_1, a strict subclass of the class of self-decomposable distributions, whose definition will be explained in the next section. We refer to Sato (1999) and Steutel and van Harn (2003) for the study of infinitely divisible distributions.

Furthermore, we prove that any real power $k \geqslant 1$, of the determinant of a covariance mixture of a Gaussian random matrix and of a random rotation of a Wishart random matrix are also infinitely divisible random variables.

2. Main results

The following results provide a class of random matrices for which their determinants and some of their real powers are infinitely divisible. The proofs are given in the next section.

First we present the Wishart case where results are independent of m and n and we can also obtain self-decomposable property.

We recall that a real random variable X is called *self-decomposable* if for any $b \in (0,1)$, there exists Y_b independent of X such that $X \stackrel{d}{=} bX + Y_b$. Any self-decomposable distribution is infinitely divisible. Also the self-decomposable distribution of X is said to belong to the *class* L_1 if Y_b itself is self-decomposable.

Theorem 1. *Let $W \sim W_m(n, \Sigma)$, where $\Sigma \in \mathbb{S}_m$ is nonrandom and $n \geqslant m$.*

(a) *For any real number k with $|k| \geqslant 1$, $(\det(W))^k$ is self-decomposable.*
(b) *The distribution of $\log(\det(W))$ belongs to the class L_1.*

The distribution of a random matrix X in $\mathbb{M}_{m\times n}$ is said to be *MixtG* (*covariance mixture of the normal matrix distribution*) if $X \stackrel{d}{=} S^{1/2}Z$, where Z and S are independent random matrices with S in $\mathbb{S}_m$ and Z having the standard normal matrix distribution $N_{mn}(0; \mathrm{I}_m \otimes \mathrm{I}_n)$. We write X_S to emphasize that the matrix distribution of X depends on that of S. Trivially, any *MixtG* matrix distribution is right-orthogonally invariant, that is, $X_S O \stackrel{d}{=} X_S$ for each $m \times n$ matrix O in $\mathbb{O}(m,n)$.

The distribution of a random matrix W_S in $\mathbb{S}_m$ said to be of *type W*, if $W_S \stackrel{d}{=} X_S X_S^\top$ for some random matrix X_S with *MixtG* distribution with covariance mixture S. We denote such a matrix distribution by $TW_m(n,S)$. From the modelling point of view, W_S has the distribution of the *random rotation* $S^{1/2}WS^{1/2}$, where $W \sim W_m(n, \mathrm{I}_m)$ is a standard Wishart random matrix independent of the random covariance S. For general type W random matrices, the results depend on n and m.

Theorem 2. *Let $W_S \sim TW_m(n,S)$ with S a random matrix in $\mathbb{S}_m$ and $n \geqslant m$.*

(a) *For $n = m, \ldots, m+7$, $\det(W_S)$ is infinitely divisible.*
(b) *When $n = m$, $(\det(W_S))^k$ is infinitely divisible for any nonnegative real $k \geqslant 1$.*

The above results yields the infinite divisibility of the square of the determinant of *MixtG* random matrices.

Corollary 3. (a) *Let Z be a Gaussian random matrix $N_{mm}(0; \Sigma \otimes \mathrm{I}_m)$, where $\Sigma \in \mathbb{S}_m$ is a nonrandom. Then $(\det(Z))^k$ is infinitely divisible for any real $k \geqslant 2$.*

(b) *Let X_S have a MixtG distribution with covariance mixture S in $\mathbb{S}_m$. Then $(\det(X_S))^k$ is infinitely divisible for any real $k \geqslant 2$.*

3. Proofs

We start by collecting useful results on mixtures of chi-square distributions, due to several authors.

Lemma 4. *Let χ_p^2 be a random variable with chi-square distribution with p degrees of freedom and let V be any nonnegative random variable independent of χ_p^2. Then*

(a) *For* $0<p\leqslant 4$, $\chi_p^2 V$ *is infinitely divisible.*
(b) *If* $p=1$, *then* $(\chi_p^2)^k V$ *is infinitely divisible for any nonnegative real* $k\geqslant 1$.

Proof. (a) follows from Steutel and van Harn (2003, p. 344) and recalling that a chi-square random variable χ_p^2 with p degrees of freedom have a gamma distribution $\Gamma(p/2,\frac{1}{2})$ with density $2^{-p/2}\Gamma(\frac{1}{2})^{-1}x^{p/2-1}e^{-x/2}$. (b) is due to Shanbhag and Sreehari (1977), (see also Lemma 5.1 in Maejima and Rosiński, 2001). □

Proof of Theorem 1. (a) It is known (e.g. Gupta and Nagar, 2000, Theorem 3.3.22) that

$$\det(W)\stackrel{d}{=}\det(\Sigma)\prod_{j=1}^{m}\chi_{n-j+1}^2, \tag{1}$$

where $\chi_n^2,\chi_{n-1}^2,\ldots,\chi_{n-m+1}^2$ are independent chi-square random variables with $n,n-1,\ldots,n-m+1$ degrees of freedom, respectively. Then the distributions of $(\det(W))^k$ is self-decomposable, since the product $G_1^{k_1}G_2^{k_2}\cdots G_m^{k_n}$ of independent one-dimensional gamma random variables $G_1,\ldots,G_n$, with real powers $k_j\geqslant 1$, is self-decomposable (e.g. Steutel and van Harn, 2003, Theorem 5.20, Chapter VI).

(b) It is shown in Akita and Maejima (2002) that if Γ is a gamma random variable $\Gamma(t,\gamma)$ and $t\geqslant\frac{1}{2}$, then the distribution of $\log\Gamma$ is in L_1. Also the class L_1 is closed under convolution. Hence, we can conclude that the distribution of $\log(\det(W))=\log(\det(\Sigma))+\sum_{j=1}^{m}\log(\chi_{n-j+1}^2)$ is in the class L_1. □

Proof of Theorem 2. (a) First observe that $\det(W_S)\stackrel{d}{=}\det(S)\det(W)\stackrel{d}{=}\det(S)\prod_{j=1}^{m}\chi_{n-j+1}^2$, where $W\sim W_m(n,\mathrm{I}_m)$. Since we are assuming S is in the open cone $\mathbb{S}_m$, then $P(\det(S)>0)=1$. Also, since $n\geqslant m$, from Theorem 3.2.1 in Gupta and Nagar (2000), $P(\det(W)>0)=1$. Hence $P(\det(W_S)>0)=1$. Next, we write $\det(W_S)\stackrel{d}{=}\det(S)\prod_{i=1}^{m}\chi_{n-i+1}^2=\chi_{n-m+1}^2V$, where $V=\det(S)\prod_{i=1}^{m-1}\chi_{n-i+1}^2$ and where $\det(S)$ is independent of all the above chi-square random variables. The assertion follows from Lemma 4(a), since $(n-m+1)/2\leqslant 4$ for $m\leqslant n\leqslant m+7$.

(b) It follows from Lemma 4(b) by taking $V=(\det(S))^k\prod_{i=1}^{m-1}\chi_{m-i+1}^{2k}$, since $n=m$. □

Proof of Corollary 3. It follows from Theorems 1 and 2, since for the square case $n=m$, $\det(W)=(\det(Z))^2$ and $\det(W_S)=(\det(X_S))^2$. □

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