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φ-Dialgebras and a Class of Matrix "Coquecigrues"

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Abstract. Starting with the Leibniz algebra defined by a φ-dialgebra, we construct examples of "coquecigrues," in the sense of Loday, that is to say, manifolds whose tangent structure at a distinguished point coincides with that of the Leibniz algebra. We discuss some possible implications and generalizations of this construction.

1 Introduction

Around 1990 J. L. Loday introduced the notion of a Leibniz algebra [L1], a generalization of a Lie algebra where the skew-symmetry of the bracket is suppressed. Although his initial (and main) motivation was the homology theory that can be defined on them, it was soon realized that Leibniz algebras were useful in a variety of contexts.

More to the point, Loday also posed "Lie's third problem for Leibniz algebras". That is, given a (say) finite dimensional Leibniz algebra, find a manifold with an algebraic operation, whose tangent structure at some distinguished point would inherit the structure of the Leibniz algebra. Since no such objects (besides the "trivial" case of Lie groups) were explicitly known, they were jokingly dubbed by him "coquecigrues" (which translates to something like "nonsense"). Their construction has proven to be quite an elusive task indeed [KW].

On the other hand, a few years later, Loday also introduced the notion of *dialgebra* [L2], which is in turn a generalization of associative algebra, but possessing two operations, and showed the existence of a functor relating Leibniz algebras to dialgebras, analogous to the functor existing between Lie algebras and associative algebras. Quite recently, Loday's definition of dialgebra was taken as a basis to define *digroups* [F, K, L], where the key elements are the introduction of an appropriate notion of neutral element and inversion.

Following this line of reasoning, in this paper I will construct some very explicit examples of manifolds with the algebraic structure of a digroup that are not Lie groups, but that have the essential properties required for a "coquecigrue." Moreover, we shall see that they have a rather nice geometrical structure.

Certainly, I do not claim to have solved the general problem posed by Loday. My point is rather that the Leibniz algebra structure by itself might not in general

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uniquely determine the type of integral manifold. Hence, most likely there are different classes of "coquecigrues" (and so the ones discussed here would be just one of them). I hope nevertheless that the construction given here sheds some additional light on a possible general structure of these intriguing objects.

2 φ-Dialgebras and Coquecigrues

Let us begin by recalling that a *dialgebra* is a vector space V together with two bilinear associative operations, $\dashv$ and $\vdash$, satisfying the relations

$$x \dashv (y \dashv z) = x \dashv (y \vdash z),$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z),$$
$$(x \dashv y) \vdash z = (x \vdash y) \vdash z.$$

And a well-known fact is that a dialgebra canonically defines a Leibniz algebra, with bracket $[x, y] = x \dashv y - y \vdash x$.

Also, recall that a non-trivial *bar unit* in a dialgebra is an element e satisfying

$$e \vdash x = x = x \dashv e \quad \forall x \in V.$$

Here "non-trivial" just means that the corresponding relations from the pointer side, *i.e.,* $e \dashv x = x = x \vdash e$, do not necessarily hold (in which case it is also well known that the two operations coincide, and the dialgebra is simply an associative algebra with unit).

The set of bar units is called the *halo* of the dialgebra, and shall be denoted here by $\mathrm{hl}(V)$. When it exists, it is an affine subspace of the dialgebra. Indeed, since by bilinearity the operations in V satisfy $0 \vdash x = x \vdash 0 = 0 \dashv x = x \dashv 0 = 0$, if we set

$$N_{\vdash} = \{x | x \vdash y = 0 \quad \forall y\} \quad \text{and} \quad {}_{\dashv}N = \{x | y \dashv x = 0 \quad \forall y\},$$

and e is a non-trivial bar unit, then the halo is the affine space modelled after the subspace $N_{\vdash} \cap {}_{\dashv}N$ and passing through e.

The important example for us, considered also in [F], is the following: Let V be any vector space and fix $\varphi \in V'$ (the algebraic dual). Then one can define a dialgebra structure on V by setting $x \dashv y = \varphi(y)x$ and $x \vdash y = \varphi(x)y$. Verification of the dialgebra axioms is straightforward. For instance, to check the first axiom, we have for the left-hand side $x \dashv (y \dashv z) = x \dashv (\varphi(z)y) = x\varphi(z)\varphi(y)$, while for the right-hand side we have $(x \dashv y) \dashv z = (x\varphi(y)) \dashv z = x\varphi(y)\varphi(z)$, etc. We shall call such a dialgebra a φ-dialgebra, and sometimes denote it by V_φ.

However, the main reason why φ-dialgebras are of interest to us is that it is easy to exhibit their non-trivial bar units. More precisely, we have the following lemma.

Lemma 2.1 *Let V be any vector space, and fix $\varphi \in V'$, $\varphi \neq 0$. Then V_φ is a dialgebra with non-trivial bar units. Moreover, its halo is an affine space modelled after the subspace* $\ker \varphi$.

Proof Since $\varphi \neq 0$, from the equation $x \dashv e = x$, for all $x \in V_\varphi$, we get that e is a bar unit in V_φ if and only if $\varphi(e) = 1$. So, if x_0 is any element in V_φ such that $\varphi(x_0) \neq 0$, $x_0/\varphi(x_0)$ is a bar unit.

Moreover, if e is any fixed bar unit, it is clear that another element e' will be a bar unit if and only if $\varphi(e - e') = 0$. In other words, $N_\vdash = {}_\dashv N = \ker \varphi$ in this case, and hence the bar units in V_φ form an affine hyperplane modelled after $\ker \varphi$, as stated. ∎

The φ-dialgebras give rise to trivial Leibniz algebras, because, as one easily checks, the bracket vanishes identically. In particular, these Leibniz algebras are Lie algebras. Nevertheless, from the point of view of "integration of the linear structure," they are not really trivial. Indeed, and mostly as a motivation for what follows, let us discuss the following example:

Let $V = \mathbb{R}^n$ ($\cong V^*$) denote Euclidean n-space, with the usual interior (dot) product, and fix $e \in S^{n-1}$. Putting $\varphi(x) = e \cdot x$ defines a φ-dialgebra structure in V, hence also a Leibniz algebra structure (abelian in this case). Now, by Lemma 2.1, the fixed element e is a bar-unit for this dialgebra, and, since $e \in S^{n-1}$, projection along the subspace $\ker \varphi$ is orthogonal. (Thus, e is somewhat special, being the bar unit of minimal length, but this is not essential).

Now fix this bar unit e and say that $x \in V$ is (pointer) *invertible* (relative to e)[1] if there exists a unique $y \in V$ such that $y \dashv x = x \vdash y = e$. For a φ-dialgebra this simply means $\varphi(x)y = e$. Hence, applying φ to this equation we see that this is the same as $\varphi(x)\varphi(y) = 1$, and so x is invertible if and only if $x \notin \ker \varphi$, with inverse

$$x^{-1} = \frac{1}{\varphi(x)} e.$$

The set of invertible elements is therefore the open subset $V^\times = V \setminus \ker \varphi$.

Thus, inversion is a well-defined operation for those $x \notin \ker \varphi$, and actually it has a nice geometrical interpretation: to invert an element x we take its projection $\varphi(x)e$ onto the space spanned by e, and then we take the inverse on this line of $\varphi(x)e$ in the sense of classical geometry (see Figure 1).

On the other hand, notice that inversion is certainly not an involution in $V^\times$, since x and y will have the same inverse if and only if $x - y \in \ker \varphi$. In fact, $y = x^{-1} \Rightarrow x = y^{-1}$ if and only if $x = \varphi(x)e$. Thus, the set of invertible elements is not a group, but the subset consisting of the line spanned by e is actually a group isomorphic to the non-zero real numbers.

The key step is now the observation that we can make sense of conjugation by elements of $V^\times$ by considering the action $(x, y) \mapsto x \vdash y \dashv x^{-1}$. (One *a priori* reason why this is the right combination of the dialgebra operations to define an *adjoint* action comes from the second axiom for Leibniz algebras, which guarantees that the left and right *translations* so defined commute.)

[1] When abstracted, this notion of invertibility leads precisely to the definition of digroup, in the sense of [L, F, K].

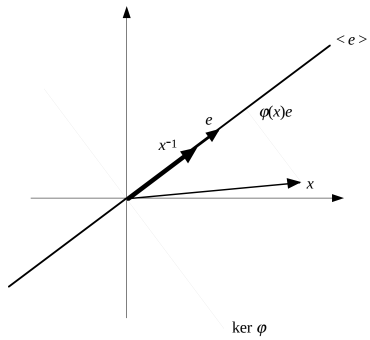

Figure 1: Inversion of an element in $V^{\times}$

Actually, for this particular example this action is trivial, since

$$x \vdash y \dashv x^{-1} = \varphi(x)\varphi\Big(\frac{e}{\varphi(x)}\Big)\, y = y.$$

The point, however, is the following. Take an element $a \in V$, and consider a curve $x(t)$ in $V^{\times}$ such that $x(0) = e$ and $x'(0) = a$, and let $y \in V$ be any other vector. Then we can compute:

$$\begin{aligned}
\frac{d}{dt}|_{t=0}\big(x(t) \vdash y \dashv x(t)^{-1}\big) &= \big(\varphi(x(t))' y\varphi(x^{-1}(t)) + \varphi(x(t))y(\varphi(x^{-1}(t))'\big)\,|_{t=0} \\
&= \varphi(a)y\varphi((e) + \varphi(e)y\varphi\Big(\frac{\varphi(a)e}{\varphi(e)^2}\Big) \\
&= \varphi(a)y - y\varphi(a) \\
&= y \dashv a - a \vdash y = [y, a].
\end{aligned}$$

This might not seem very interesting at first, since, y being fixed, the derivative is of course zero. This is all right, because the Leibniz algebra V is also abelian. Thus, what we have seen is that the tangent space to $V^{\times}$ at e can be identified with the Leibniz algebra V_{φ}, as required by Loday's program, and so following his suggestion we might call $V^{\times}$ a φ-coquecigrue.

Remark This example already shows that the Leibniz algebra structure alone is not enough to determine the coquecigrue, even locally, since in this case the Leibniz algebra is a Lie algebra, but the coquecigrue just constructed is not a Lie group.

Obviously, by Riesz's theorem, what was said applies equally well, for instance, to any Hilbert space. To avoid the technical difficulties involved in the definition of vector fields, tensor products, etc., in infinite dimensions, I will nevertheless stay in the finite dimensional case.

3 The Matrix Case

Using the example of the previous section as a guide, let us now construct a more general and interesting class of coquecigrues.

For this, we recall first that if V is a dialgebra, it is well known that the space $M_k = \mathrm{Mat}(k, V)$ of square $k \times k$ matrices with entries in V is also a dialgebra with the operations defined entry-wise and again denoted $\dashv, \vdash$ [L2]. And if V has a non-trivial bar unit e, M_k also has a non-trivial bar unit, namely

$$E = \begin{pmatrix} e & 0 & \dots & 0 \\ 0 & e & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & e \end{pmatrix}.$$

Moreover, since as a vector space $M_k \cong \mathrm{Mat}(k, \mathbb{R}) \otimes V \cong V \otimes \mathrm{Mat}(k, \mathbb{R})$, M_k is a $\mathrm{Mat}(k, \mathbb{R})$-bimodule, *i.e.,* the product of scalars with vectors in V extends to give actions of the algebra $\mathrm{Mat}(k, \mathbb{R})$ on M_k, both on the right and the left, in an obvious way:

$$(X, A) \mapsto X \otimes A, \text{ where } (X \otimes A)_{ij} = \sum_k x_{ik} a_{kj}, \quad A \in Mat(k, \mathbb{R}), X \in M_k,$$

and similarly for the product on the left side. (We shall usually omit the symbol $\otimes$).

Now, fix a φ-dialgebra V_φ, which we could assume is given as in the example of the previous section, and an integer k, and consider the corresponding space M_k. As noted, M_k is also a dialgebra with the distinguished bar unit E, and moreover, the linear functional φ defines a linear map $M_k \to Mat(k, \mathbb{R})$, which we still denote φ, sending $X = (x_{ij})$ to $\varphi(X) = (\varphi(x_{ij}))$.

Let us state a few properties of this space:

Lemma 3.1 *Let V_φ be a φ-dialgebra, and M_k the associated dialgebra of square $k \times k$ matrices. The following properties hold*

(i) $\varphi(E) = \mathrm{Id} \in \mathrm{Mat}(k, \mathbb{R})$*;*
(ii) $AE = EA$*, for all* $A \in \mathrm{Mat}(k, \mathbb{R})$*;*
(iii) $\varphi(\varphi(X)Y) = \varphi(X)\varphi(Y) = \varphi(X\varphi(Y))$ *for all* $X, Y \in M_k$*;*
(iv) $X \dashv Y = X\varphi(Y)$*,* $X \vdash Y = \varphi(X)Y$ *for all* $X, Y \in M_k$*.*

Proof This is again quite straightforward, so let us just verify (iii). If $X = (x_{ij})$ and $Y = (y_{ij})$, then $(\varphi(X)Y)_{ij} = \sum_k \varphi(x_{ik}) y_{kj}$. Therefore,

$$(\varphi(\varphi(X)Y))_{ij} = \varphi\Big(\sum_k \varphi(x_{ik}) y_{kj}\Big) = \sum_k \varphi(x_{ik})\varphi(y_{kj}) = (\varphi(X)\varphi(Y))_{ij}$$

as desired. ■

(iv) gives us a somewhat more usable description of the dialgebra operations in M_k. And, since matrix multiplication is not commutative, this immediately shows that the resulting Leibniz algebra structure in M_k is certainly non-abelian. Indeed, it is not even a Lie algebra, because

$$[X,Y] = X\varphi(Y) - \varphi(Y)X \neq -(Y\varphi(X) - \varphi(X)Y) = -[Y,X].$$

Nevertheless we can repeat the constructions of the one-dimensional case:

Definition 3.2 Given $X \in M_k$ we say that it has a *pointer inverse relative to E* if there is a unique $Y \in M_k$ such that $Y \dashv X = E, X \vdash Y = E$.

For simplicity we shall simply say that such an X is *invertible.* As in the example, we have the following explicit characterization of inverses:

Lemma 3.3 *$X \in M_k$ is invertible if and only if $\varphi(X) \in \mathrm{GL}(k,\mathbb{R})$, and its inverse is $\varphi(X)^{-1}E$.*

Proof By (iv) in Lemma 3.1 the condition for invertibility becomes $Y\varphi(X) = E$ and $\varphi(X)Y = E$, and applying φ to these equalities we get $\varphi(Y)\varphi(X) = \varphi(X)\varphi(Y) = \mathrm{Id}$ as a necessary condition for X to be invertible. It follows in particular that $\varphi(X) \in \mathrm{GL}(k,\mathbb{R})$ and that we must choose Y to equal both $\varphi(X)^{-1}E$, and $E\varphi(X)^{-1}$. Both quantities coincide however, because of (ii) in Lemma 3.1, and therefore, in the open subset $M_k^\times = \varphi^{-1}(GL(k,\mathbb{R}))$ of M_k, pointer inversion is well defined. ■

Also, notice that Lemmas 3.1 and 3.3 imply that $\varphi(X^{-1}) = \varphi(X)^{-1}$.

Thus $M_k^\times$ is a digroup, and again, $M_k^\times$ acts on the dialgebra M_k by an adjoint action:

$$(X,Y) \mapsto \mathrm{Ad}_X Y = X \vdash Y \dashv X^{-1} = \varphi(X)Y\varphi(X)^{-1}, \quad X \in M_k^\times, Y \in M_k.$$

Lemma 3.4 *The adjoint action defined above is a left $\vdash$ action, in the sense that*

$$\mathrm{Ad}_X(\mathrm{Ad}_Y Z) = \mathrm{Ad}_{X\vdash Y} Z = \mathrm{Ad}_{\varphi(X)Y} Z, \quad X,Y \in M_k^\times, Z \in M_k.$$

Proof It suffices to verify that $(X \vdash Y)^{-1} = Y^{-1} \dashv X^{-1}$, which is rather clear from the characterisation of inverses given in the previous lemma. But it also follows from the dialgebra axioms and the properties of pointer inverses. On one side this is direct:

$$\begin{aligned}(X \vdash Y) \vdash (Y^{-1} \dashv X^{-1}) &= X \vdash (Y \vdash (Y^{-1} \vdash X^{-1})) \\ &= X \vdash ((Y \vdash Y^{-1}) \vdash X^{-1}) \\ &= X \vdash (E \vdash X^{-1}) = E,\end{aligned}$$

while on the other we need to use the first axiom for dialgebras once:

$$\begin{aligned}(Y^{-1} \dashv X^{-1}) \dashv (X \vdash Y) &= Y^{-1} \dashv (X^{-1} \dashv (X \vdash Y)) \\ &= Y^{-1} \dashv (X^{-1} \dashv (X \dashv Y)) \\ &= Y^{-1} \dashv (E \dashv Y) \\ &= Y^{-1} \dashv Y = E.\end{aligned}$$

■

Now, since M_k is a finite dimensional space, we can directly compute derivatives to see that

$$(X^{-1}(t))' = -\varphi(X(t))^{-1}\varphi(X'(t))E\varphi(X(t))^{-1}.$$

(Recall from Lemma 3.1 that E commutes with scalar matrices.)

Thus, if $Y \in M_k$ and $X(t)$ is a curve in $M_k^{\times}$ such that $X(0) = E$ and $X'(0) = A \in M_k$, we have

$$\begin{aligned}\frac{d}{dt}|_{t=0}(X(t) \vdash Y \dashv X(t)^{-1}) &= \big(\varphi(X(t))'Y\varphi(X^{-1}(t)) + \varphi(X(t))Y(\varphi(X^{-1}(t))')\big)|_{t=0} \\ &= \varphi(A)Y\varphi(E) + \varphi(E)Y\varphi\big(\varphi(E)\varphi(A)E\varphi(E)\big) \\ &= \varphi(A)Y - Y\varphi(A) \\ &= Y \dashv A - A \vdash Y = [Y, A].\end{aligned}$$

This time the result of taking derivatives is certainly not trivial, and we have indeed proved the following result.

Theorem 3.5 *The tangent space to $M_k^{\times}$ at the point E is endowed with the Leibniz algebra structure induced from the dialgebra structure of M_k. Thus, the digroup $M_k^{\times}$ is a coquecigrue in the sense of Loday.*

We shall sometimes call it a matrix φ-coquecigrue. Let us now analyze its geometric structure.

First, consider the set

$$\mathrm{GL}(k, \mathbb{R}) \otimes \{E\} = \{AE\,;\, A \in \mathrm{GL}(k, \mathbb{R})\}.$$

It is actually a homomorphic copy of the Lie group $\mathrm{GL}(k, \mathbb{R})$ included in the coquecigrue. Obviously, φ restricted to this subset is a diffeomorphism onto $\mathrm{GL}(k, \mathbb{R})$. But for $A, B \in \mathrm{GL}(k, \mathbb{R})$ an easy computation shows that

$$(AE) \dashv (BE) = (AE) \vdash (BE) = ABE,$$

so that φ restricted to this subset is also a group isomorphism, regardless of which digroup operation we choose in the dialgebra.

Moreover, $X^{-1} = Y^{-1}$ if and only if $\varphi(X - Y) = 0$. Therefore, again we see that the matrix φ-coquecigrue fibers over $\mathrm{GL}(k, \mathbb{R}) \otimes \{E\}$, with fiber $\ker\varphi \subset M_k$. Actually we can say more.

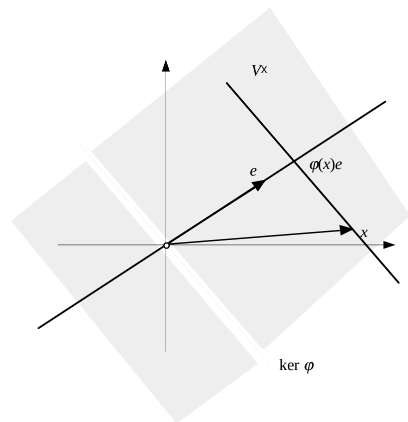

Figure 2: Fibering of $M_k^\times$ over $\mathrm{GL}(k,\mathbb{R})$.

Proposition 3.6 *M_k has the structure of a trivial vector bundle over* $\mathrm{GL}(k,\mathbb{R})$*, with fiber* $\ker\varphi$.

Proof We have already shown that $M_k^\times$ is a vector bundle, but actually we can explicitly set up a global diffeomorphism $M_k^\times \to \mathrm{GL}(k,\mathbb{R}) \times \ker\varphi$:

$$X \mapsto (\varphi(X), X - \varphi(X)E),$$

which gives a global trivialization, since it is linear on the fibers. ■

This gives a rather neat description of the geometric structure of the coquecigrue, illustrated in Figure 2 for the case $k = 1$.

But let us now have a closer look at the relationship to the algebraic structure of the dialgebra. The key point here is that multiplication of X and X^{-1}, in the "reverse" order, gives a "transverse structure" to the fibered structure given in Proposition 3.6, namely.

Lemma 3.7 *Both $X \dashv X^{-1} = X\varphi(X)^{-1}$ and $X^{-1} \vdash X = \varphi(X)^{-1}X$ belong to* $\mathrm{hl}(M_k)$*. In particular, they are transverse to* $\ker\varphi$.

Proof This is a quite straightforward computation, so let us just check one of the conditions for $X \dashv X^{-1}$ to be a bar unit:

$$(X \dashv X^{-1}) \vdash Y = (X \vdash X^{-1}) \vdash Y = \varphi(X\varphi(X^{-1})Y = \mathrm{Id}\, Y = Y,$$

by Lemma 3.1. ■

Notice that in general $X \dashv X^{-1} \neq X^{-1} \vdash X$. We could therefore choose any of them to define the transverse structure (and the results will be essentially the same). For reasons that will be clear soon, we choose the former. In any case, the important consequence of the existence of this transverse structure is that $M_k^\times$ should not be viewed as vector bundle; rather, we have the following:

Proposition 3.8 *The map* $R_A(X) = X + \varphi(X)A$, $X \in M_k^\times$, $A \in \ker\varphi$, *is a right action of* $\ker\varphi$, *as an abelian group, on* $M_k^\times$, *that turns it into a* $\ker\varphi$*-principal bundle over* $\mathrm{GL}(k,\mathbb{R})$.

Proof Notice that the projection is simply the map φ, so the action obviously preserves the fibers.

To see that this is indeed a right action, we compute

$$\begin{aligned} R_B(R_A(X)) &= X + \varphi(X)A + \varphi(X + \varphi(X)A)B \\ &= X + \varphi(X)(A + B) + \varphi(\varphi(X)A)B \\ &= X + \varphi(X)(A + B) = R_{A+B}(X). \end{aligned}$$

Finally, to check equivariance, we first notice that since $A \in \ker\varphi$, we have $(R_A(X))^{-1} = X^{-1}$, and then modify the trivialization of Proposition 3.6, defining $\psi(X) = \varphi(X)^{-1}X - E$. Then, although ψ is no longer linear in X,

$$\begin{aligned} \psi(R_A(X)) &= \varphi((R_A(X)^{-1})R_A(X) - E \\ &= \varphi(X)^{-1}(X + \varphi(X)A) - E \\ &= \varphi(X)^{-1}X - E + A = \psi(X) + A, \end{aligned}$$

proving the equivariance of the action in the global trivialization $(\varphi(X), \psi(X))$. ■

We now combine this proposition and the transverse structure of Lemma 3.7 to obtain our main result:

Theorem 3.9 *For each* $X \in M_k^\times$ *consider the space* $H_X = \{XA, A \in \mathrm{Mat}(k,\mathbb{R})\}$, *regarded as a vector subspace of* $T_X M_k^\times$ *in the natural way. Then* $X \mapsto H_X$ *defines an equivariant horizontal distribution for the action R defined in Proposition* 3.8, i.e., *a connection.*

The horizontal component of a tangent vector $Y \in T_X M_k^\times$ *is given by* $h(Y) = X\varphi(X)^{-1}\varphi(Y)$. *Therefore, the associated* $\ker\varphi$*-valued connection* 1*-form is given by* $\omega = dX - X\varphi(X)^{-1}\varphi(dX)$.

Before proving the theorem, it is perhaps convenient to clarify what we mean by dX and $\varphi(dX)$ in the last expression. At any given point $Z \in M_k$, by dX_Z we simply mean the M_k-valued linear form on $T_Z M_k$ that to a tangent vector $Y \in T_Z M_k \cong M_k$ associates Y itself; $\varphi(dX)$ is then the $\mathrm{Mat}(k,\mathbb{R})$-valued form that associates to this tangent vector the matrix $\varphi(Y)$ (recall that M_k is a $\mathrm{Mat}(k,\mathbb{R})$-module).

Proof of Theorem 3.9 The equivariance of the distribution H under the action R means that $R_{B*}H_X = H_{R_BX}$. Now, if $XA \in H_X$, since R_B is linear in X we have, $R_{B*}(XA) = (R_BX)A$. Thus, H is equivariant.

We still have to show that $H_X \oplus \ker\varphi \cong T_X M_k^\times \cong M_k$. But if $Y \in M_k$ is any vector, we obviously have

$$Y = (X\varphi(X)^{-1}\varphi(Y)) + (Y - X\varphi(X)^{-1}\varphi(Y)).$$

Now, by construction, the first summand on the right-hand side belongs to H_X, while the second satisfies

$$\varphi\big(Y - X\varphi(X)^{-1}\varphi(Y)\big) = \varphi(Y) - \varphi(X)\varphi(X)^{-1}\varphi(Y) = 0,$$

so it belongs to $\ker\varphi$.

Using Lemma 3.7, this proves the desired direct sum decomposition of $T_X M_k^\times$, and also explicitly exhibits the horizontal part of a vector as $h(Y) = X\varphi(X)^{-1}\varphi(Y)$, proving the second assertion.

The expression for the connection form is now almost immediate. Since $\ker\varphi$ is an abelian group, its Lie algebra is identified to itself. Therefore, from the definition of dX and $\varphi(dX)$

$$\omega(Y) = Y - X\varphi(X)^{-1}\varphi(Y),$$

and so, from what has just been shown, ω takes its values on the Lie algebra of the structure group of the principal bundle. Hence, all that would remain to be shown is that $H_X = \ker\omega_X$, which again is clear from the definitions of H_X and ω. ∎

Remark Theorem 3.9 shows that these matrix coquecigrues have a geometrical structure reminiscent of, but not exactly identical to, the one discussed in [KW].

Also, the expression for $h(Y)$ given in the theorem shows exactly how $X\varphi(X)^{-1}$ determines the transverse structure to the fibers.

And again, one has a nice picture of the connection in the case $k = 1$. The horizontal subspace at a point X is the space complementary to $\ker\varphi$ and passing through X (see Figure 3).

Finally, we can also compute the curvature of ω.

Theorem 3.10 *The connection ω of Theorem 3.9 is flat.*

Proof By definition, the curvature of the connection ω is

$$D\omega(Y,Z) = d\omega(h(Y),h(Z)).$$

Now, since $d(\varphi(X)^{-1}) = -\varphi(X)^{-1}d\varphi(X)\varphi(X)^{-1} = -\varphi(X)^{-1}\varphi(dX)\varphi(X)^{-1}$, we have

$$\begin{aligned} d\omega &= -dX \wedge \varphi(X)^{-1}\varphi(dX) - Xd(\varphi(X)^{-1}) \wedge \varphi(dX) \\ &= -dX \wedge \varphi(X)^{-1}\varphi(dX) + X\varphi(X)^{-1}\varphi(dX)\varphi(X)^{-1} \wedge \varphi(dX). \end{aligned}$$

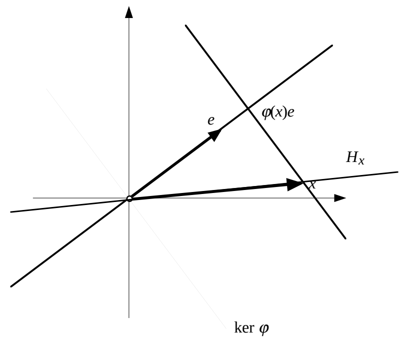

Figure 3: Horizontal space of the connection at a point X

Hence,

$$\begin{aligned} D\omega(Y,Z) &= -X\varphi(X)^{-1}\varphi(X)\varphi(Y)\varphi(X)^{-1}\varphi(X)\varphi(Z) \\ &\quad + X\varphi(X)\varphi(Z)\varphi(X)^{-1}\varphi(X)\varphi(X)^{-1}\varphi(Y) \\ &\quad + X\varphi(X)^{-1}\varphi(X)\varphi(X)^{-1}\varphi(Y)\varphi(X)^{-1}\varphi(X)\varphi(X)^{-1}\varphi(Z) \\ &\quad - X\varphi(X)^{-1}\varphi(X)\varphi(X)^{-1}\varphi(Z)\varphi(X)^{-1}\varphi(X)\varphi(X)^{-1}\varphi(Y) \\ &= -X\varphi(X)^{-1}\varphi(Y)\varphi(X)\varphi(Z) + X\varphi(X)^{-1}\varphi(Z)\varphi(X)\varphi(Y) \\ &\quad + X\varphi(X)^{-1}\varphi(Y)\varphi(X)\varphi(Z) - X\varphi(X)^{-1}\varphi(Z)\varphi(X)\varphi(Y) \\ &= 0. \end{aligned}$$

■

4 Some Final Remarks

Although I have considered here only matrix φ-dialgebras, this is mostly because of the ease with which bar units and inverses can be handled, and it is clear that at least some parts of the previous constructions can be generalized. For this purpose, it is convenient to rewrite the relevant conditions in terms exclusively of the dialgebra operations, as was done in proving Lemma 3.4.

Thus the action of $\ker\varphi$ on $M_k^\times$ is given by $R_A X = X + X \vdash A$. Similarly, the horizontal projection of a vector is $h(Y) = (X \dashv X^{-1}) \dashv Y$, and hence the connection form can be written as $\omega = dX - (X \dashv X^{-1}) \dashv dX$, etc.

Nevertheless, the results proved here do not in general carry over to the abstract dialgebra context without some additional hypotheses that moreover usually depend on the specific point at hand (*e.g.*, Theorem 3.5, where it is necessary that the invertible elements form an open subset). So this generalization is not straightforward. The full extent of this possibility will be discussed in a forthcoming work.

On the other hand, the structure of these matrix coquecigrues raises the question of constructing the algebraic operations of a dialgebra (or a digroup) starting from the geometric structure, namely: is some kind of converse of Theorem 3.9 true? As already seen in the analysis done in [KW], this is probably not an easy matter either.

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