Centro de Investigación en Matemáticas, A.C.

BLOWUP IN FINITE TIME OF SOME NONLINEAR SYSTEMS OF PDEs AND SPDEs

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Introduction

Systems of coupled partial differential equations (PDEs) and stochastic partial differential equations (SPDEs) can be used as models to describe the distribution of heat in mixtures of components that can burn, turbulence phenomenon, population dynamics, neurophysiology, reaction-diffusion processes, branching diffusions, hydrodynamic limit of particles, among others (see E. Pardoux [37]).

Some of the classical problems arising from the study of PDEs and SPDEs include existence and uniqueness of solutions. Consequently, a natural question to ask is whether a solution exists globally in time or it explodes (or blows up) at some finite time. The study of globability and blowup in finite time of systems of PDEs and SPDEs has been intensively studied during the last decades, starting with the pioneering works of S. Kaplan [28] and H. Fujita [20] (see H. A. Levine [31], K. Deng and H. A. Levine [12], V. A. Galaktionov et al. [21], V. A. Galaktionov [22] and P. Quittner and Ph. Souplet [42] for reviews).

One of the earliest studies of explosion in finite time is due to H. Fujita [20], who considered the parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + u^{1+\beta}, \quad t > 0, \quad x \in D, \qquad (1) \\ u(0,x) &= a(x), \quad x \in D, \\ u(t,x) &= 0, \quad x \in \partial D, \end{aligned}$$

where $D \subseteq \mathbb{R}^d$ is a domain with smooth boundary ∂D, $\beta > 0$ and $a : \bar{D} \to [0,\infty)$ is a given bounded and uniformly continuous function. Solutions of this equation are considered in the mild sense. As is mentioned in [20], in the hole space ($D = \mathbb{R}^d$) the dimension plays an important role in determining the existence of explosive or global positive solutions: if $0 < d\beta < 2$ and $a(x_0) > 0$ for some $x_0 \in \mathbb{R}$, then every nontrivial positive solution of (1) blows up in finite time; if $d\beta > 2$, then (1) admits a global solution for sufficiently small initial value, which means that $0 \leq a(x) \leq \delta e^{-\kappa|x|^2}$ for some constants $\delta, \kappa > 0$.

When D is bounded, in [20] it is considered the smallest eigenvalue $\lambda > 0$ of $-\Delta$, with corresponding eigenfunction $\phi > 0$ in D, namely

$$-\Delta\phi(x) = \lambda\phi(x), \quad x \in D, \tag{2}$$
$$\phi(x) = 0, \quad x \in \partial D.$$

Since $\phi \in L^2(D)$, ϕ can be normalized as $\int_D \phi(x)dx = 1$. In this setting, any positive solution of (1) blows up in finite time if

$$\int_D a(x)\phi(x)dx \geq \lambda^{1/\beta}.$$

In contrast to this result, if we replaced the nonlinear term $u^{1+\beta}$ in equation (1) by any continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $|f(r)| \leq \lambda|r|$ and $|a(x)| \leq M\phi(x)$, for some constants $\lambda, M > 0$, but ϕ normalized as $\max_{x\in D}\phi(x) = 1$, then any solution of (1) is global.

Now consider the coupled system of PDEs

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + v^p, \quad t > 0, \quad x \in \mathbb{R}^d,\\
\frac{\partial v}{\partial t} &= \Delta v + u^q, \quad t > 0, \quad x \in \mathbb{R}^d,\\
u(0,x) &= u_0(x), \quad x \in \mathbb{R}^d,\\
v(0,x) &= v_0(x), \quad x \in \mathbb{R}^d,
\end{aligned} \tag{3}$$

where $p, q > 0$, $d \in \mathbb{N}$ and u_0, v_0 are nonnegative, continuous and bounded functions. M. Escobedo and M. Herrero [18] showed that if $0 < pq \leq 1$, then every solution of (3) is global. Note that this criterion for globability does not depend on the dimension d. However, in the case $pq > 1$ several behaviours can appear. For instance, if $pq > 1$, $\frac{\max\{p,q\}+1}{pq-1} < \frac{d}{2}$, $u_0 \in L^\infty(\mathbb{R}^d) \cap L^{\alpha_1}(\mathbb{R}^d)$ and $v_0 \in L^\infty(\mathbb{R}^d) \cap L^{\alpha_2}(\mathbb{R}^d)$ with $\alpha_1 = (d/2)((pq-1)/(q+1))$, $\alpha_2 = (d/2)((pq-1)/(p+1))$, then there exists $\epsilon > 0$ such that if $\|u_0\|_{L^{\alpha_1}(\mathbb{R}^d)} + \|v_0\|_{L^{\alpha_2}(\mathbb{R}^d)} \leq \epsilon$, then every solution of (3) is global. In contrast, if $pq > 1$, $\frac{\max\{p,q\}+1}{pq-1} < \frac{d}{2}$ and $u_0(x) \geq Ce^{-k|x|^2}$ for some constants $C > 0$ and $k > 0$ (or similarly $v_0(x) \geq Ce^{-k|x|^2}$ for some constants $C > 0$ and $k > 0$), then the solution of (3) blows up in finite time.
Conditions on existence of explosive solutions of (3) change significantly when we restrict the system (3) to a bounded domain $D \subseteq \mathbb{R}^d$, with Dirichlet condition on the boundary. This case was considered by K. Deng [11] and L. Wang [49].

In the case of a bounded domain $D \subseteq \mathbb{R}^d$, if $0 < pq \leq 1$ then every solution of (3) remains global for any bounded initial conditions (as in the case $D = \mathbb{R}^d$), but if $pq > 1$ and for some

$\delta \in (0,1)$, $\Delta u_0 + (1-\delta)v_0^p \geq 0$ and $\Delta v_0 + (1-\delta)u_0^q \geq 0$, then every solution of (3) blows up in finite time.

Consider a random perturbation in a PDE, driven by a real-valued standard Brownian motion $\{W_t; t \geq 0\}$, defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For instance, consider the nonlinear SPDE

$$\begin{aligned} du(t,x) &= \left(\Delta u + u^{1+\beta}(t,x)\right) dt + \kappa u(t,x) dW_t, \quad t > 0, \quad x \in D, \\ u(0,x) &= f(x) \geq 0, \quad x \in D, \\ u(t,x) &= 0, \quad t \geq 0, \quad x \in \partial D, \end{aligned} \tag{4}$$

which is Fujita's equation but with a linear multiplicative noise. We assume that $D \subseteq \mathbb{R}^d$ is a bounded domain, $\beta > 0$, $\kappa \in \mathbb{R}$ and $f : D \to \mathbb{R}_+$ is a $C^2(D)$ function not identically zero. Blowup in finite time for that kind of equations was proved by M. Dozzi and J. A. López-Mimbela [15]. They proved the probabilistic counterpart of Fujita's result: if τ is the explosion time of (4), then there exist random times τ_* and τ^* such that $\tau_* \leq \tau \leq \tau^*$ and, moreover

$$\begin{aligned} \tau_* &= \inf\left\{t \geq 0 : \int_0^t e^{\kappa\beta W_r - \beta(\lambda+\kappa^2/2)r} dr \geq \left(\beta L^\beta \|\phi\|_\infty^\beta\right)^{-1}\right\}, \\ \tau^* &= \inf\left\{t \geq 0 : \int_0^t e^{\kappa\beta W_r - \beta(\lambda+\kappa^2/2)r} dr \geq \left(\beta L^\beta \left(\int_D \phi^2(x) dx\right)^\beta\right)^{-1}\right\}, \end{aligned}$$

where ϕ and λ are as in (2) and the initial condition is taken of the form $f = L\phi$, for some constant $L > 0$. Note that these bounds allow to determine a nontrivial interval for the probability of explosion in finite time of system (4). As it is shown in [15],

$$\begin{aligned} \mathbb{P}(\tau^* < \infty) &= \mathbb{P}\left(\int_0^\infty e^{\kappa\beta W_r - \beta(\lambda+\kappa^2/2)r} dr \geq \beta^{-1}\kappa^{-\beta}\left(\int_D \phi^2(x) dx\right)^{-\beta}\right) \quad \text{and} \\ \mathbb{P}(\tau_* < \infty) &= \mathbb{P}\left(\int_0^\infty e^{\kappa\beta W_r - \beta(\lambda+\kappa^2/2)r} dr \geq \beta^{-1}\kappa^{-\beta}\|\phi\|_\infty^{-\beta}\right), \end{aligned}$$

and therefore $\mathbb{P}(\tau < \infty) \in [\mathbb{P}(\tau^* < \infty), \mathbb{P}(\tau_* < \infty)]$. A random variable of the form

$$Z = \int_0^\infty e^{\sigma W_r + \mu r} dr,$$

where $\sigma, \mu \in \mathbb{R}$, is called a Dufresne's functional, and the distribution of such a random quantity was explicitly obtained by D. Dufresne [16]. He showed by means of weak convergence of random walks that if $\mu \geq 0$ then $Z = \infty$ a.s., and if $\mu < 0$ then $Z^{-1} \overset{d}{=} \Gamma(-2\mu/\sigma^2, \sigma^2/2)$, where

$\Gamma(a,m)$ is a Gamma-distributed variable with density function $f(x)=\Gamma(a)^{-1}m^{-a}x^{a-1}e^{-\frac{x}{m}}$, for $x>0$, and $a,m>0$.

Now consider the coupled system of nonlinear SPDEs

$$\begin{aligned}
du_1(t,x) &= [(\Delta+V_1)u_1(t,x)+u_2^p(t,x)]\,dt+\kappa_1 u_1(t,x)\,dW_t, \quad t>0, \quad x\in D,\\
du_2(t,x) &= [(\Delta+V_2)u_2(t,x)+u_1^q(t,x)]\,dt+\kappa_2 u_2(t,x)\,dW_t, \quad t>0, \quad x\in D, \qquad (5)\\
u_i(0,x) &= f_i(x)\geq 0, \quad x\in D,\\
u_i(t,x) &= 0, \quad t\geq 0, \quad x\in\partial D, \quad i=1,2,
\end{aligned}$$

where $V_i=\lambda+\kappa_i^2/2$, $i=1,2$, $\lambda>0$ is the first eigenvalue of the Laplacian on a smooth bounded domain $D\subseteq\mathbb{R}^d$ and $p\geq q>1$. This kind of systems were studied by M. Dozzi, E. T. Kolkovska and J. A. López-Mimbela [14]. Considering subsolutions and supersolutions of a related system of PDEs, in [14] it is shown that the blowup time τ of the above system is lower and upper bounded by random times ϱ_{**} and ϱ^{**}, respectively, which depend respectively on functionals of the form

$$\int_0^t (e^{aW_r}\vee e^{bW_r})dr \quad \text{and} \quad \int_0^t (e^{aW_r}\wedge e^{bW_r})dr, \qquad (6)$$

for some positive constants a,b. In this case, using the Feynamn-Kac approach given in M. Jeanblanc, J. Pitman and M. Yor [26], in [14] a lower bound for $\mathbb{P}(\tau\geq\theta_k)$ is obtained, where θ_k is an exponential random variable, independent of $\{W_t;t\geq 0\}$, with positive parameter k.

As we can see, there are several conditions for explosion in finite time of different systems of PDEs and SPDEs, and these form the subject of this dissertation. Throughout the present work we consider systems of PDEs and systems of SPDEs, and the main results we present are focused on existence of mild solutions, as well as on finding lower and upper bounds for the explosion times of these systems. We start in Chapter 1 with a brief review of background results that we need to develop this work. Proofs are omitted, but we provide references where they can be found.

The first system of PDEs is analysed in Chapter 2. We consider the semilinear system of partial differential equations

$$\begin{aligned}
\frac{\partial u_1(t,x)}{\partial t} &= \frac{1}{2}\frac{\partial^2 u_1(t,x)}{\partial x^2}+\frac{\varphi_1'(x)}{\varphi_1(x)}\frac{\partial u_1(t,x)}{\partial x}+u_2^{1+\beta_1}(t,x), \quad t>0, \quad x\in\mathbb{R},\\
\frac{\partial u_2(t,x)}{\partial t} &= \frac{1}{2}\frac{\partial^2 u_2(t,x)}{\partial x^2}+\frac{\varphi_2'(x)}{\varphi_2(x)}\frac{\partial u_2(t,x)}{\partial x}+u_1^{1+\beta_2}(t,x), \quad t>0, \quad x\in\mathbb{R}, \qquad (7)
\end{aligned}$$

with initial values of the form $u_i(0,x) = h_i(x)/\varphi_i(x)$, where

$$0 < \varphi_i \in L^2(\mathbb{R}, dx) \cap C^2(\mathbb{R}),$$

and $0 \leq h_i \in L^2(\mathbb{R}, dx)$, $\beta_i > 0$ for $i = 1, 2$. As is mentioned in Section 2.1, operators of the form

$$L^\varphi = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\varphi'(x)}{\varphi(x)}\frac{d}{dx}, \quad x \in \mathbb{R},$$

are infinitesimal generators of one-dimensional recurrent diffusion processes with invariant measure $\mu(dx) = \varphi^2(x)dx$.

The main result on existence and uniqueness of local mild solutions of system (7) is Theorem 2.2.1. To prove an existence and uniqueness theorem we use the classical Banach fixed-point theorem. A criterion for explosion in finite time for that kind of systems is given in Theorem 2.3.1, which we consider is the main result of Chapter 2. We distinguish two cases: if $\beta_1 = \beta_2$, then any non-trivial positive mild solution explodes in finite time, and we give an upper bound T^* for the explosion time. In the case of $\beta_1 > \beta_2$, we found a condition on the sizes of the initial conditions that ensures explosion in finite time of positive nontrivial solutions. In the later case, we were also able to find and upper bound T^* for the blowup time of that system which depends both on the initial values f_1, f_2 and the measures $\mu_i(dx) = \varphi_i^2(x)\,dx$, $i = 1, 2$. Chapter 2 corresponds to the accepted paper [24].

Chapter 3 focuses on the study of existence of positive mild solutions and the explosion in finite time of systems of SPDEs of the form

$$\begin{aligned} du_1(t,x) &= \left[\Delta_\alpha u_1(t,x) + u_2^{1+\beta_1}(t,x)\right]dt + \kappa_1 u_1(t,x)\,dW_t, \quad t>0, \quad x\in D,\\ du_2(t,x) &= \left[\Delta_\alpha u_2(t,x) + u_1^{1+\beta_2}(t,x)\right]dt + \kappa_2 u_2(t,x)\,dW_t, \quad t>0, \quad x\in D, \\ u_i(0,x) &= f_i(x)\geq 0, \quad x\in D,\\ u_i(t,x) &= 0, \quad t\geq 0, \quad x\in\mathbb{R}^d\setminus D, \quad i=1,2, \end{aligned} \tag{8}$$

where Δ_α is the fractional Laplacian and $\beta_1 \geq \beta_2 > 0$. We extend the results given in [30] on existence of solution of system (5), for the case $V_i = 0$ and $\alpha \in (0, 2)$. A criterion for existence of explosive solutions is also given. We start by establishing the equivalence between weak solutions of system (8) and weak solutions of a related system of random PDEs (see Section 1.3, expression (1.6)). This result is based on the Doss-Sussman transformation, which basically works for SPDEs with linear multiplicative noise. We use results due to J. M. Ball [1] and M. Juzyniec [27] on equivalence of weak and mild solutions for the system of random PDEs obtained by means of Doss-Sussman transformation (see Theorem 1.3.3). Assuming the existence of a weak solution of the related system of PDEs, the first difficulty is to find an appropriate explosive subsolution whose explosion time can be explicitly computed. We were able to solve a non-homogeneous random Bernoulli equation by means of a change of variables, and the subsolution we found is given in Theorem 3.3.1, which is one of the main results of Chapter 3. We again distinguish two cases: if $\beta_1 = \beta_2$ we find an upper bound τ^* for the explosion time τ of the system (8); if $\beta_1 > \beta_2$ we find an upper bound τ^{**} for τ. Both random times τ^* and τ^{**} depend on functionals of the form

$$\int_0^t (e^{aW_r} \wedge e^{bW_r}) e^{-\mu r} dr, \tag{9}$$

for some positive constants a, b and μ. Note that functionals of the form (6) are special cases of the functionals in (9). In this sense, our results can be considered as an extension of those results given in [30]. The second challenge is to obtain information on the distribution of the functionals in (9), and on the explosion times of (8) in Theorem 3.3.1 we find random times τ' and τ'' for the cases $\beta_1 = \beta_2$ and $\beta_1 > \beta_2$, respectively, satisfying $\tau \le \tau^* \le \tau'$ and $\tau \le \tau^{**} \le \tau''$. The new random times τ' and τ'' depend on integral functionals of the form

$$\int_0^t e^{-(\sigma W_s - \mu s)} \mathbb{1}_{\{\sigma W_s - \mu s \ge 0\}} ds, \tag{10}$$

where $\sigma > 0$ and $\mu > 0$ are constants. The distribution of the perpetual version of (10) is found by a direct computation of the potential measure of the Brownian motion with drift, $\{\sigma W_t - \mu t; t \ge 0\}$, leading to an integral equation for the function

$$H(x,z) := \mathbb{E}\left[\exp\left(-z \int_0^\infty e^{-(\sigma W_s - \mu s + x)} \mathbb{1}_{\{\sigma W_s - \mu s + x \ge 0\}} ds\right)\right], \quad x \ge 0 \quad z \in \mathbb{C},$$

which is solved by Banach fixed-point iterations, (see expressions (3.21) and (3.22)). Using analytic continuation we finally find the Laplace transform of the functional (10). We invert this Laplace transform using a result due to A. Erdélyi [19]. Our findings are included in Theorem 3.2.6.

To show the existence of a mild solution of the related system of random PDEs (1.6), we use the iterated Galerkin method in Theorem 3.3.3 which allows us to construct a supersolution for this system. This supersolution explodes at a random time τ_*, which is a lower bound for τ. It is not easy to manipulate this bound because it depends on the semigroup of the α-stable process killed in D^c, but if we take the initial values as $f_i = L_i \psi$, $i = 1, 2$, where ψ is the first eigenvalue of $-\Delta_\alpha$ on D, then we can find another lower bound τ_{**} such that

$\tau_{**} \leq \tau_*$. The random time τ_{**} depends on functionals of the form

$$\int_0^t (e^{aW_r} \vee e^{bW_r}) e^{-\mu r} dr, \tag{11}$$

where a, b and μ are positive constants. Even though we were unable to obtain explicitly the distribution of the functional in (11), we found another random time $\tau_{\prime\prime}$ which depends upon Dufresne's functional. In summary, we found an interval for $[\tau_{\prime\prime}, \tau'']$ for τ. Finally, taking the initial conditions of the form $f_i = L_i\psi$, $i = 1, 2$, we construct in Section 3.4 the interval $[\mathbb{P}(\tau'' < \infty), \mathbb{P}(\tau_{\prime\prime} < \infty)]$ for $\mathbb{P}(\tau < \infty)$. Chapter 3 corresponds to the accepted paper [25].

In the final Chapter 4 we consider a system of SPDEs of the form

$$\begin{aligned} du_1(t,x) &= \left[\Delta_{\alpha_1} u_1(t,x) + u_2^{1+\beta_1}(t,x)\right] dt + \kappa_1 u_1(t,x)\, dW_t, \quad t > 0, \quad x \in D, \\ du_2(t,x) &= \left[\Delta_{\alpha_2} u_2(t,x) + u_1^{1+\beta_2}(t,x)\right] dt + \kappa_2 u_2(t,x)\, dW_t, \quad t > 0, \quad x \in D, \\ u_i(0,x) &= f_i(x) \geq 0, \quad x \in D, \\ u_i(t,x) &= 0, \quad t \geq 0, \quad x \in \mathbb{R}^d \setminus D, \quad i = 1, 2, \end{aligned} \tag{12}$$

where $\alpha_1, \alpha_2 \in (0, 2]$ and, in general, $\alpha_1 \neq \alpha_2$. As before, D is a bounded domain. We introduce a different notion of blowup of an SPDE, namely, explosion in the $L^p(D)$-norm, for $p \in [1, \infty)$.

This notion of explosion has been investigated recently by several authors, including [8], [9] and [32], initially for a single equation and with a space-time noise. In the special case of $\alpha_1 = \alpha_2$ we give a condition on the initial values of (12) that ensures finite-time blowup in the $L^p(D)$-norm of any positive solutions of (12), as well as an explicit upper bound for the explosion time in the $L^p(D)$-norm sense (see Theorem 4.3.1).

Next, we deal with the more general case of $\alpha_1 \neq \alpha_2$. Here we have to assume that D is a ball of radius $r > 0$, centred at 0. This assumption allow us to use nice estimates of the first eigenfunction ψ of $-\Delta_\alpha$ on D due to T. Kulczycki (see Bogdan et al. [4]). Such estimates are based on the intrinsic ultracontractive property of Δ_α, and are of the form

$$C_1(r^2 - |x|^2)^{\alpha/2} \leq \psi(x) \leq C_2(r^2 - |x|^2)^{\alpha/2}, \tag{13}$$

where C_i, $i = 1, 2$, is a positive constant depending on the radius r and the fractional power $\alpha \in (0, 2]$. Using (13) we obtain condition implying finite-time blowup of system (12), as well as an upper bound for the explosion time in the $L^p(D)$-norm sense, for any $p \in [1, \infty)$.

Chapter 1

Preliminaries

This chapter presents some background results on different aspects treated throughout the thesis.

Since we consider SPDEs involving the fractional Laplacian Δ_α (see Chapter 3 and 4), in Section 1.1 we show some properties of this operator. In particular we give the classical result about the spectrum of Δ_α.

The main tools from stochastic calculus are presented in Section 1.2. In particular we focused on Itô's formula, integration by parts formula and Polarization's identity. These tools will be needed in Section 1.3.

As it was mentioned in the Introduction, we use a change of variable inspired in the Doss transformation, to go from a system of SPDEs to a system of random PDEs. The results concerning equivalence of mild and weak solutions for the system of random PDEs are shown in Section 1.3.

Finally, in Section 1.4 we present some know formulae for the Laplace transform of some integral functionals of Brownian motion.

We do not give proofs of the quoted results, but we provide references where they can be found.

1.1 The fractional Laplacian

Let $\{Y_t;\ t \geq 0\}$ be a spherically symmetric α-stable Lévy process on $\mathbb{R}^d$, $d \in \mathbb{N}$, where $\alpha \in (0, 2]$. This is a process with independent and stationary increments and characteristic function $\mathbb{E}_0\left[e^{iu\cdot Y_t}\right] = e^{-t|u|^\alpha}$, $u \in \mathbb{R}^d$, $t \geq 0$. We will use $\mathbb{E}_x$ and $\mathbb{P}_x$ to denote respectively the expectation and probability of this process starting at $x \in \mathbb{R}^d$. By $\{P_t; t \geq 0\}$ we denote the semigroup of the process $\{Y_t;\ t \geq 0\}$; that is, for all $f \in L^2(\mathbb{R}^d)$,

$$P_t f(x) = \mathbb{E}_x \left[f(Y_t) \right], \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

The proof of the following result can be found in [7, Theorem 2.1]

Theorem 1.1.1. *The semigroup $\{P_t; t \geq 0\}$ admits an integral kernel $\{p^{(\alpha)}(t,x,y), t > 0,\ x, y \in \mathbb{R}^d\}$ satisfying the following properties:*

1. *$\{p^{(\alpha)}(t,x,y), t > 0,\ x, y \in \mathbb{R}^d\}$ is strictly positive on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$;*
2. *$\{p^{(\alpha)}(t,x,y), t > 0,\ x, y \in \mathbb{R}^d\}$ is jointly continuous on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$;*
3. *for any $t > 0$ and $x, y \in \mathbb{R}^d$,*

$$p^{(\alpha)}(t,x,y) = p^{(\alpha)}(t,y,x) = p^{(\alpha)}(t,0,x-y);$$

4. *for any $t > 0$ and for any $x, y \in \mathbb{R}^d$,*

$$p^{(\alpha)}(t,x,y) = t^{-d/\alpha} p^{(\alpha)}(1, t^{-1/\alpha}x, t^{-1/\alpha}y).$$

When $\alpha = 2$ the process $\{Y_t;\ t \geq 0\}$ is just the Brownian motion in $\mathbb{R}^d$ with variance 2 and

$$p^{(2)}(t,x,y) = \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x-y|^2}{4t}}, \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

When $\alpha = 1$, the process $\{Y_t;\ t \geq 0\}$ is the Cauchy process in $\mathbb{R}^d$ whose transition densities are given by

$$p^{(\alpha)}(t,x,y) = \frac{c_d t}{(t^2 + |x-y|^2)^{(d+1)/2}} \quad t > 0, \quad x, y \in \mathbb{R}^d,$$

where

$$c_d = \Gamma((d+1)/2)/\pi^{(d+1)/2}.$$

In what follows $D \subseteq \mathbb{R}^d$ is a domain. Let $\tau^D := \inf\{t > 0 :\ Y_t \notin D\}$ the first hitting time of D^c and consider the killed process $\left\{Y_t^D,\ t \geq 0\right\}$ given by

$$Y_t^D = \begin{cases} Y_t & \text{if } \ t < \tau^D \\ \partial & \text{if } \ t \geq \tau^D, \end{cases}$$

where ∂ is a cemetery point. By $\{P_t^D; t \geq 0\}$ we denote the semigroup associated to the killed process $\{Y_t^D; t \geq 0\}$. That is, for any $f \in L^2(D)$

$$P_t^D f(x) = \mathbb{E}_x \left[f(Y_t) \mathbb{1}_{\{t < \tau^D\}} \right] \quad t \geq 0, \quad x \in D. \tag{1.1}$$

The following result is [7, Theorem 2.3]. We denote by $C_b^n(D)$ the space of bounded functions with continuous derivatives up to order $n \in \mathbb{N}$. For simplicity we write $C_b^0(D) := C_b(D)$.

Theorem 1.1.2. *For any domain $D \subseteq \mathbb{R}^d$ we have $P_t^D f \in C_b(D)$ for $t > 0$ and $f \in L^\infty(D)$.*

For $t > 0$ and $x, y \in \mathbb{R}^d$, let

$$r^D(t,x,y) = \mathbb{E}_x \left[p^{(\alpha)}(t - \tau_D, Y_{\tau_D}, y) \mathbb{1}_{\{t > \tau_D\}} \right]$$

and

$$p^D(t,x,y) = p^{(\alpha)}(t,x,y) - r^D(t,x,y), \quad t \geq 0, \quad x, y \in \mathbb{R}^d.$$

Note that by the right continuity of the sample paths of $\{Y_t; t \geq 0\}$, we have $p^D(t,x,y) = 0$ for all $x \in \mathbb{R}^d \setminus \overline{D}$. The following result gives the main properties of the family $\{p^D(t,x,y); t \geq 0 \text{ and } x, y \in \mathbb{R}^d\}$ (see [7, Theorem 2.4]).

Theorem 1.1.3. *For any $t > 0$, $x \in \mathbb{R}^d$ and all nonnegative Borel measurable functions f on $\mathbb{R}^d$,*

$$P_t^D f(x) = \int_{\mathbb{R}^d} p^D(t,x,y) f(y) dy.$$

The function $p^D(t,\cdot,\cdot)$ is symmetric on $\mathbb{R}^d \times \mathbb{R}^d$ and strictly positive on $D \times D$. As a function of (t,x,y), p^D is continuous on $(0,\infty) \times (\mathbb{R}^d \setminus \partial D) \times (\mathbb{R}^d \setminus \partial D)$. For any $t, s > 0$ and $x, y \in \mathbb{R}^d$, we have the semigroup property

$$p^D(t+s,x,y) = \int_{\mathbb{R}^d} p^D(t,x,z) p^D(s,z,y) dz.$$

Theorem 1.1.4. *For each $p \in [1,\infty)$, $\{P_t^D; t \geq 0\}$ forms a strongly continuous semigroup in $L^p(D)$. If, in addition, D is bounded, then for each $t > 0$, P_t is a linear bounded operator from $L^p(D)$ to $L^q(D)$ for any $p, q \in [1,\infty)$.*

If $D \subseteq \mathbb{R}^d$ is a bounded domain, then there exists a constant $c_{\alpha,d} > 0$ depending only on α and d such that the kernel $\{p^D(t,x,y); t \geq 0,\ x, y \in \mathbb{R}^d\}$ satisfies the relation

$$p^D(t,x,y) \leq p(t,x,y) \leq c_{\alpha,d} t^{-d/\alpha}, \quad t > 0, \quad x, y \in D,$$

(see [4, Chapter 4]). It follows that

$$\int_D \int_D (p^D(t,x,y))^2 dx dy < \infty, \quad t \geq 0.$$

Therefore $\{P_t^D; t \geq 0\}$ is a family of Hilbert-Schmidt linear operators. Hence for any $t > 0$, P_t^D is a compact operator and we have the result in [7, Theorem 2.4]:

Theorem 1.1.5. *Let $D \subseteq \mathbb{R}^d$ be a bounded domain. Then for any $t > 0$, P_t^D has the same eigenvalues $\{e^{-\lambda_n t}\}_{n\in\mathbb{N}}$ in $L^p(D)$, for all $p \in [1, \infty)$, where*

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

with $\lim_{n\to\infty} \lambda_n = \infty$. In particular there is a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ in $L^2(D)$ and corresponding eigenvalues $\{e^{-\lambda_n t}\}_{n\in\mathbb{N}}$ such that

$$P_t^D \psi_n(x) = e^{-\lambda_n t}\psi_n(x), \quad t > 0, \quad x \in D. \tag{1.2}$$

Moreover $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(D)$, the eigenfunctions ψ_n are continuous and bounded in D and $\psi_n = 0$ on D^c, $n \in \mathbb{N}$. In addition, λ_1 has multiplicity 1 and its corresponding eigenfunction ψ_1 is strictly positive on D (see [4, Chapter 4]).

Now we formulate the above properties in terms of infinitesimal generators. The infinitesimal generator of $\{Y_t;\ t \geq 0\}$ is the fractional Laplacian defined by

$$\Delta_\alpha u(x) := \mathcal{A}(d, \alpha)\text{P.V.} \int_{\mathbb{R}^d} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}} dy, \quad u \in C_b^2(\mathbb{R}^d),$$

where $\mathcal{A}(d, \alpha)$ is a positive constant depending only on d and α. Recall that a point $x_0 \in \partial D$ is called regular for $\{Y_t; t \geq 0\}$ if $\mathbb{P}_{x_0}(\tau_D = 0) = 1$. We say that D is regular for $\{Y_t; t \geq 0\}$ if every point $x \in \partial D$ is regular for $\{Y_t; t \geq 0\}$. It follows that if $D \subseteq \mathbb{R}^d$ is a bounded regular domain for $\{Y_t; t \geq 0\}$, then Δ_α satisfies the eigenvalue problem with Dirichlet condition in D^c:

$$\begin{aligned} \Delta_\alpha \psi_n(x) &= -\lambda_n \psi_n(x), \quad x \in D, \\ \psi_n(x) &= 0, \quad x \in D^c. \end{aligned} \tag{1.3}$$

For example when $D = (0, \pi)$ and $\alpha = 2$, then $\psi_n(x) = \sqrt{2/\pi}\sin(nx)$ and $\lambda_n = n^2$. However, for $\alpha \in (0, 2)$ and a general bounded domain $D \subseteq \mathbb{R}^d$, Δ_α is a pseudodifferential nonlocal operator and it is very difficult to obtain properties of the eigenvalues and eigenfunctions using (1.3) only.

1.2 Some tools from stochastic calculus

This section is devoted to a change of variables formula for stochastic integrals, which makes them easy to handle and thus leads to some explicit computations; in particular, those computations are shown in Chapter 3. The main reference for this section is D. Revuz and M. Yor [43]. We first present the Proposition 3.1 in [43].

Theorem 1.2.1 (**Integration by parts formula**). *If* $X := \{X_t; t \geq 0\}$ *and* $Y := \{Y_t; t \geq 0\}$ *are two real-valued continuous semimartingales, then*

$$X_t Y_t = [X, Y](t) + X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s,$$

where $[X, Y](t)$ *is the covariation process between* X *and* Y*.*

As we will see in Section 1.3, one of the main tools to transform a system of SPDEs to a system of random PDEs is based on Itô's formula. Let $C^n(\mathbb{R})$ the space of the continuous functions with continuous derivatives up to order $n \in \mathbb{N}$.

Theorem 1.2.2 (**Itô's formula**). *Let* $F : \mathbb{R} \to \mathbb{R}$ *be in* $C^2(\mathbb{R})$ *and let* $\{X_t; t \geq 0\}$ *be a real-valued continuous semimartingale. Then* $\{F(X_t); t \geq 0\}$ *is a semimartingale such that*

$$F(X_t) = F(X_0) + \int_0^t F'(X_s) dX_s + \frac{1}{2} \int F''(X_s) d[X, X](s).$$

A useful multivariate version of Itô's formula for two processes is proved in [29, Theorem 4.17].

Theorem 1.2.3. *Let* $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be a function twice differentiable in each component. If* $\{X_t; t \geq 0\}$ *and* $\{Y_t; t \geq 0\}$ *are two real-valued continuous semimartingales, then*

$$\begin{aligned}
&F(X_t, Y_t) \\
&= F(X_0, Y_0) + \int_0^t \frac{\partial F}{\partial x}(X_s, Y_s) dX_s + \int_0^t \frac{\partial F}{\partial y}(X_s, Y_s) dY_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(X_s, Y_s) d[X, X](s) \\
&+ \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial y^2}(X_s, Y_s) d[Y, Y](s) + \int_0^t \frac{\partial^2 F}{\partial x \partial y}(X_s, Y_s) d[X, Y](s).
\end{aligned}$$

Finally, we recall the Polarization's identity (see [43, Theorem 1.9]).

Theorem 1.2.4 (**Polarization's identity**). *If* $\{X_t; t \geq 0\}$ *and* $\{Y_t; t \geq 0\}$ *are two real-valued continuous semimartingales, then*

$$[X, Y](t) = \frac{1}{4} \left([X + Y, X + Y] (t) - [X - Y, X - Y] (t) \right).$$

1.3 Weak and mild solutions and a change of variables

In this section we present the main theorems concerning existence of weak solutions of systems of the form

$$\begin{aligned} du_1(t,x) &= [\Delta_\alpha u_1(t,x) + G_1(u_2(t,x))]\,dt + \kappa_1 u_1(t,x)\,dW_t, \\ du_2(t,x) &= [\Delta_\alpha u_2(t,x) + G_2(u_1(t,x))]\,dt + \kappa_2 u_2(t,x)\,dW_t, \\ u_i(0,x) &= f_i(x) \geq 0, \quad x \in D, \\ u_i(t,x) &= 0, \quad t \geq 0, \quad x \in \mathbb{R}^d \setminus D, \quad i = 1,2. \end{aligned} \tag{1.4}$$

Here, Δ_α is the fractional power of the Laplacian, $\alpha \in (0,2]$, $G_i : \mathbb{R} \to \mathbb{R}_+$ is a locally Lipschitz function, f_i is a bounded measurable function on D, $i = 1,2$, $\{W_t; t \geq 0\}$ is a standard Brownian motion in $\mathbb{R}$, defined in some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t; t \geq 0\}, \mathbb{P})$, and $\beta_i > 0$ and $\kappa_i \in \mathbb{R}$ are constants, $i = 1,2$. We start by recalling the definition of weak solution.

Definition 1.3.1. Let $\tau \in [0,\infty]$ be a stopping time. An $\mathcal{F}_t$-adapted process

$$\{u_i(t,x) : t \geq 0, x \in D\},$$

is a weak solution of (1.4) on $[0,\tau)$ if for all $\phi \in C_b^2(\mathbb{R}^d)$ vanishing on D^c, $t \in [0,\tau)$ and $i = 1,2$,

$$\begin{aligned} \int_D u_i(t,x)\,\phi(x)\,dx = \int_D f_i(x)\,\phi(x)\,dx + \int_0^t \int_D \left(u_i(s,x)\,\Delta_\alpha \phi(x) + G_i(u_{3-i}(s,x))\phi(x)\right) dxds \\ + \kappa_i \int_0^t \int_D u_i(s,x)\,\phi(x) dx dW_s, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{1.5}$$

Let $\{u_i(t,x) : t \geq 0, x \in D\}$ be a weak solution of (1.4) on $[0,\tau)$. Consider the change of variables

$$v_i(t,x) := \exp\{-\kappa_i W_t\}\,u_i(t,x), \quad t \in [0,\tau), \quad x \in D, \quad i = 1,2.$$

This change of variables, inspired in the Doss transformation [13] is useful to transform an SPDE with linear multiplicative noise, into a random PDE , i.e., a family of PDEs parametrized by $\omega \in \Omega$. Proceeding as in [14] one can see that the function $(v_1(t,x), v_2(t,x))$

is a weak solution of the system of random parabolic PDEs

$$\begin{aligned}
\frac{\partial}{\partial t} v_1(t,x) &= \left(\Delta_\alpha v_1(t,x) - \frac{\kappa_1^2}{2} v_1(t,x)\right) + e^{-\kappa_1 W_t} G_1(e^{\kappa_2 W_t} v_2(t,x)),\\
\frac{\partial}{\partial t} v_2(t,x) &= \left(\Delta_\alpha v_2(t,x) - \frac{\kappa_2^2}{2} v_2(t,x)\right) + e^{-\kappa_2 W_t} G_2(e^{\kappa_1 W_t} v_1(t,x)),\\
v_i(0,x) &= f_i(x) \geq 0, \quad x \in D,\\
v_i(t,x) &= 0, \quad t \geq 0, \quad x \in \mathbb{R}^d \setminus D, \quad i = 1,2,
\end{aligned} \tag{1.6}$$

with the same assumptions as in (1.4), i.e., if τ is a stopping time, then for all $\phi \in C_b^2(\mathbb{R}^d)$ vanishing on D^c, $t \in [0,\tau)$ and $i = 1,2$,

$$\begin{aligned}
\int_D v_i(t,x)\phi(x)\,dx = \int_D f_i(x)\phi(x)\,dx &+ \int_0^t \int_D v_i(s,x)\left(\Delta_\alpha \phi(x) - \frac{\kappa_i^2}{2}\phi(x)\right) dxds\\
&+ \int_0^t \int_D e^{-\kappa_i W_s} G_i\left(e^{\kappa_{3-i} W_s} v_{3-i}(s,x)\right)\phi(x)dxds, \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

In fact, let $\{u_i(t,x) : t \geq 0, x \in D\}$ be a weak solution of (1.4) on $[0,\tau)$. Then, expression (1.5) for a fixed $\phi \in C_b^2(\mathbb{R}^d)$ vanishing on D^c, says that the process $\{\int_D u_i(t,x)\phi(x)\,dx, t \in [0,\tau)\}$ is an $\mathcal{F}_t$-semimartingale. Since $x \mapsto e^{-\kappa x}$, $x \in \mathbb{R}$, is twice continuously differentiable, by Itô's formula we get

$$e^{-\kappa W_t} = 1 - \kappa \int_0^t e^{-\kappa W_s} dW_s + \frac{\kappa^2}{2}\int_0^t e^{-\kappa W_s} ds.$$

Recall that the integration by parts formula establishes that

$$Y_t Z_t = [Y,Z](t) + Y_0 Z_0 + \int_0^t Y_s dZ_s + \int_0^t Z_s dY_s,$$

for any two semimartingales $\{Y_t; t \geq 0\}$ and $\{Z_t; t \geq 0\}$. Therefore, taking $Y_{t,i} = e^{-\kappa_i W_t}$ and $Z_{t,i} = \int_D u_i(t,x)\phi(x)\,dx$,

$$\begin{aligned}
\int_D v_i(t,x)\phi(x)\,dx &= \left[e^{-\kappa_i W}, \int_D u_i(\cdot,x)\phi(x)\,dx\right](t) + \int_D f_i(x)\phi(x)\,dx\\
&+ \int_0^t e^{-\kappa_i W_s} d\left[\int_D u_i(s,x)\phi(x)\,dx\right]\\
&+ \int_0^t \int_D u_i(s,x)\phi(x)\,dx \left[-\kappa_i e^{-\kappa_i W_s} dW_s + \frac{\kappa_i^2}{2} e^{-\kappa_i W_s} ds\right]\\
&= \left[e^{-\kappa_i W}, \int_D u_i(\cdot,x)\phi(x)\,dx\right](t) + \int_D f_i(x)\phi(x)\,dx\\
&+ \int_0^t e^{-\kappa_i W_s}\left[\int_D u_i(s,x)\Delta_\alpha\phi(x)\,dx + \int_D G_i(u_{3-i}(s,x))\phi(x)\,dx\right] ds\\
&+ \frac{\kappa_i^2}{2}\int_0^t \int_D e^{-\kappa_i W_s} u_i(s,x)\phi(x)\,dxds.
\end{aligned}$$

By Polarization's identity we see that

$$\left[e^{-\kappa_i W}, \int_D u_i(\cdot, x)\,\phi(x)\,dx\right](t) = -\kappa_i^2 \int_0^t \int_D e^{-\kappa_i W_s} u_i(s,x)\,\phi(x)\,dxds.$$

Therefore

$$\begin{aligned}\int_D v_i(t,x)\,\phi(x)\,dx = &\int_D f_i(x)\,\phi(x)\,dx + \int_0^t \int_D v_i(s,x)\,\Delta_\alpha \phi(x)\,dxds \\ &- \frac{\kappa_i^2}{2} \int_0^t \int_D v_i(s,x)\,\phi(x)\,dxds \\ &+ \int_0^t \int_D e^{-\kappa_i W_s} G_i(e^{\kappa_{3-i} W_s} v_{3-i}(s,x))\phi(x)\,dxds, \quad t \in [0,\tau),\end{aligned}$$

i.e., $(v_1(t,x), v_2(t,x))$ is a weak solution of (1.6) in $[0,\tau)$.

Definition 1.3.2. Let $\tau \in [0,\infty]$ be a stopping time. The vector $(v_1(t,x), v_2(t,x))$ is a mild solution of (1.6) on $[0,\tau)$ if for all $t \in [0,\tau)$, $\mathbb{P}$-c.s. and $i = 1,2$,

$$v_i(t,x) = e^{-\frac{k_i^2}{2}t} P_t^D f_i(x) + \int_0^t e^{-\frac{k_i^2}{2}(t-s) - \kappa_i W_s} P_{t-s}^D G_i(e^{\kappa_{3-i} W_s} v_{3-i}(s,x))ds,$$

where $\{P_t^D; t \geq 0\}$ is the semigroup defined in (1.1).

We have the following theorem due to [27, Theorems 4 and 5] and [1, Theorem page 371]:

Theorem 1.3.3. *The vector $(v_1(t,x), v_2(t,x))$ is a weak solution of system (1.6) on $[0,\tau)$ if and only if $(v_1(t,x), v_2(t,x))$ is a mild solution of (1.6) on $[0,\tau)$.*

Definition 1.3.4. A stopping time $\tau \in [0,\infty]$ is an explosion time of the system (1.4) if

$$\limsup_{t\uparrow\tau} \sup_{x\in D} |u_1(t,x)| = \infty, \ \mathbb{P}\text{-c.s.}, \quad \text{or} \quad \limsup_{t\uparrow\tau} \sup_{x\in D} |u_2(t,x)| = \infty, \ \mathbb{P}\text{-c.s.}$$

The explosion time is defined as the infimum of such τ's being explosion times of system (1.4).

As we will see in Theorem 3.3.3, we can construct a mild solution of (1.6) in some interval $[0,\tau)$, where τ is the explosion time of a suitable supersolution of (1.6).

1.4 A brief review of some exponential functionals of Brownian motion

In this section we give a brief review of some results of M. Yor [50], P. Salminen and M. Yor [44] and A. N. Borodin and P. Salminen [5]. Consider the geometric Brownian motion

$\{e^{W_t+\mu t}; t \geq 0\}$, where $\{W_t; t \geq 0\}$ is a standard Brownian motion in $\mathbb{R}$, defined in some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t; t \geq 0\}, \mathbb{P})$, and $\mu \in \mathbb{R}$ is a constant. The quadratic variation of this geometric Brownian motion is given by

$$A_t := \int_0^t e^{2(W_s+\mu s)} ds, \quad t \geq 0.$$

The functional A_t is called Dufresne's functional and the distribution of its perpetuity, i.e., the distribution of

$$A_\infty := \int_0^\infty e^{2(W_s+\mu s)} ds,$$

was obtained by D. Dufresne [16]. He showed that for $\mu < 0$, $A_\infty \stackrel{d}{=} 1/(2Z_{-\mu})$, where $Z_{-\mu}$ is a Gamma random variable with parameter $-\mu$. This identity has been recovered in paper no. 1 in M. Yor [50], where a relationship with Getoor's studies of last passage times for Bessel processes (see [23]) is used. Basically, M Yor. in paper no. 1 of [50] uses the Lamperti's time change relationship given by

$$e^{W_t+\mu t} = R^{(\mu)}_{\int_0^t e^{2(W_s+\mu s)} ds}, \tag{1.7}$$

where $R^{(\mu)} = \{R^{(\mu)}_t; t \geq 0\}$ is a Bessel process with index μ (or dimension $d = 2(1+\mu)$), i.e., $R^{(\mu)}$ is a diffusion on $\mathbb{R}_+$ having infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{1+2\mu}{2x}\frac{d}{dx}, \quad x \in \mathbb{R}.$$

Note that if $\mu < 0$, we have that $e^{W_t+\mu t} \to 0$, a.s. as $t \to \infty$, and it follows from (1.7) that

$$A_\infty \stackrel{\mathrm{d}}{=} \inf\{u > 0 : R^{(\mu)}_t = 0\}.$$

At this point M. Yor uses Getoor's result to finally get that $\inf\{u > 0 : R^{(\mu)}_t = 0\} \stackrel{\mathrm{d}}{=} 1/(2Z_{-\mu})$.

On the other hand, P. Salminen and M. Yor [44] considered, among other cases, functionals of the form

$$\int_0^\infty e^{2a(W_s+\mu s)} \mathbb{1}_{\{W_s+\mu s<0\}} ds, \quad a > 0, \quad \mu > 0, \tag{1.8}$$

which are called Dufresne's reflected perpetuities. Using again Lamperti's time change together with some results about hitting times, in [44] it is obtained the Laplace transform of the perpetuity. Specifically there holds (see [44, Proposition 3.1, Part (e)]):

Theorem 1.4.1. *If $a, \mu > 0$ then,*

$$\int_0^\infty e^{2a(W_s+\mu s)} \mathbb{1}_{\{W_s+\mu s<0\}} ds \stackrel{d}{=} \int_0^\infty \mathbb{1}_{\{R^{(\mu/a)}_s<1/a\}} ds, \tag{1.9}$$

where $R^{(\mu/a)}_0 = 1/a$.

An important observation is that in A. N. Borodin and P. Salminen [5, Formula 4.1.5.3(1)], it is established that

$$\mathbb{E}_x\left[e^{-\gamma\int_0^\infty \mathbb{1}_{\{R_s^{(\nu)}<1/a\}}ds}\right] = \begin{cases} \frac{2\nu x^{-\nu}I_\nu(x\sqrt{2\gamma})}{\sqrt{2\gamma}r^{1-\nu}I_{\nu-1}(r\sqrt{2\gamma})}, & 0\le x\le r, \\ 1-\frac{x^{-2\nu}I_{\nu+1}(r\sqrt{2\gamma})}{r^{-2\nu}I_{\nu-1}(r\sqrt{2\gamma})}, & 0\le r\le x. \end{cases} \tag{1.10}$$

Therefore, using the scaling property of $\{W_t; t\ge 0\}$ and expressions (1.9) and (1.10), it can be shown that for $\sigma,\mu>0$ and $z>0$,

$$\mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-(\sigma W_t-\mu t)}\mathbb{1}_{\{\sigma W_t-\mu t\ge 0\}}dt\right)\right] = \frac{4\mu I_{\frac{2\mu}{\sigma^2}}\left(\frac{\sqrt{8z}}{\sigma}\right)}{\sigma\sqrt{8z}I_{\frac{2\mu}{\sigma^2}-1}\left(\frac{\sqrt{8z}}{\sigma}\right)},$$

(see expression (3.24)). As we can note, expression (1.10) can be recovered by the arguments used in Section 3.2 of the present work, which are based on the explicit computation of the potential measure of the process $\{\sigma W_t-\mu t; t\ge 0\}$, where $\sigma,\mu>0$, and explicitly solving an integral equation for the Laplace transform of $\int_0^\infty e^{-(\sigma W_t-\mu t)}\mathbb{1}_{\{\sigma W_t-\mu t\ge 0\}}dt$.

Chapter 2

Blowup in finite time of a system of PDEs

2.1 Introduction

In this chapter we present conditions for explosion in finite time of a system of PDEs. To achieve this, we start by considering the semilinear partial differential equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{\varphi'(x)}{\varphi(x)}\frac{\partial u(t,x)}{\partial x} + u^{1+\beta}(t,x), \quad t>0, \quad x\in\mathbb{R}, \tag{2.1}$$

where $\beta > 0$, $\varphi \in C^2(\mathbb{R})$ is a square-integrable, strictly positive function, and the initial value is of the form $u(0,x) = h(x)/\varphi(x)$ with $h \in L^2(\mathbb{R}, dx)$. Setting $\varphi(x) = e^{-x^2/2}$ in (2.1) it becomes

$$\frac{\partial u(t,x)}{\partial t} = L^{\varphi}u(t,x) + u^{1+\beta}(t,x), \quad t>0, \quad x\in\mathbb{R},$$

where $L^{\varphi} := \frac{1}{2}\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x}$ is the infinitesimal generator of the Ornstein-Uhlenbeck semigroup $\{T_t; t\geq 0\}$.

Using essentially Jensen's inequality and the fact that the measure $\mu(dx) = \varphi^2(x)\,dx$ is invariant for $\{T_t; t\geq 0\}$, in [34] it was shown that equation (2.1) exhibits blowup in finite time for any nontrivial initial value of the form $u(0,x) = h(x)/\varphi(x)$, $x\in\mathbb{R}$.

Motivated by this example, in this chapter we provide a criterion for explosion in finite time of positive mild solutions of the 1-dimensional semilinear system

$$\begin{aligned}
\frac{\partial u_1(t,x)}{\partial t} &= \frac{1}{2}\frac{\partial^2 u_1(t,x)}{\partial x^2} + \frac{\varphi_1'(x)}{\varphi_1(x)}\frac{\partial u_1(t,x)}{\partial x} + u_2^{1+\beta_1}(t,x), \quad t>0, \quad x\in\mathbb{R},\\
\frac{\partial u_2(t,x)}{\partial t} &= \frac{1}{2}\frac{\partial^2 u_2(t,x)}{\partial x^2} + \frac{\varphi_2'(x)}{\varphi_2(x)}\frac{\partial u_2(t,x)}{\partial x} + u_1^{1+\beta_2}(t,x), \quad t>0, \quad x\in\mathbb{R},\\
u_i(0,x) &= f_i(x), \quad x\in\mathbb{R}, \quad i=1,2,
\end{aligned} \tag{2.2}$$

where $\beta_1, \beta_2 > 0$ are constants, f_1, f_2 are nonnegative functions and $\varphi_1, \varphi_2 \in C^2(\mathbb{R})\cap L^2(\mathbb{R}, dx)$ are strictly positive. Semilinear systems of this type have been investigated intensively in last years, starting with the pioneering work of V.A. Galaktionov, S.P. Kurdyumov and A.A. Samarskii [21] (see also [22, 18, 35, 14, 33] and the review papers [31, 12]). This kind of systems arise as simplified models of the process of diffusion of heat and burning in a two-component continuous media, where u_1 and u_2 represent the temperatures of the two reactant components.

Recall that a pair (u_1, u_2) of measurable functions is termed *mild solution* of system (2.2) if it solves the system of integral equations

$$u_i(t,x) = T_t^i\left(f_i(x)\right) + \int_0^t T_{t-s}^i\left(u_{3-i}^{1+\beta_i}(s,x)\right)\, ds, \quad t \geq 0, \quad x \in \mathbb{R}, \tag{2.3}$$

where $i = 1, 2$ and $\{T_t^i; t \geq 0\}$ is the semigroup of continuous linear operators on $L^\infty(\mathbb{R}, dx)$ having infinitesimal generator

$$L^{\varphi_i} = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{\varphi_i'}{\varphi_i}\frac{\partial}{\partial x}; \quad i = 1, 2. \tag{2.4}$$

If there exists $T \in (0, \infty)$ such that $\|u_1(t,\cdot)\|_{L^\infty(\mathbb{R},dx)} = \infty$ or $\|u_2(t,\cdot)\|_{L^\infty(\mathbb{R},dx)} = \infty$ for all $t \geq T$, then it is said that (u_1, u_2) *blows up (or explodes) in finite time*, and in this case the infimum of such T's is called the *blowup time* (or the *explosion time*) of (u_1, u_2).

Notice that for any $g \in L^\infty(\mathbb{R}, dx)$ and $i = 1, 2$,

$$T_t^i(g(x)) = \mathbb{E}\left[g\left(X_t^{x,i}\right)\right], \quad t \geq 0, \quad x \in \mathbb{R},$$

where $\{X_t^{x,i}; t \geq 0\}$ is the unique strong solution of the stochastic differential equation

$$Y_t = x + W_t + \int_0^t \frac{\varphi_i'}{\varphi_i}(Y_s)\, ds, \quad t \geq 0, \quad x \in \mathbb{R};$$

here $\{W_t; t \geq 0\}$ is a standard 1-dimensional Brownian motion. It turns out that under our assumptions both processes $\{X_t^{x,i}; t \geq 0\}$, $i = 1, 2$, are recurrent and, moreover, possess corresponding invariant measures

$$\mu_i(dx) = \varphi_i^2(x)\, dx, \quad i = 1, 2. \tag{2.5}$$

The intuitive explanation of the blowup phenomenon in non-linear heat equations of the type

$$\frac{\partial u}{\partial t} = \mathcal{A}u + u^{1+\beta}; \quad u(0) = f \geq 0,$$

where $\beta > 0$ and $\mathcal{A}$ is the generator of a strong Markov process on a locally compact space, is that if the initial value f is "small" then the tendency of the solution to blowup (which it would do if $u^{1+\beta}$ were the only term in the left-hand side of the equation) can be inhibited by the dissipative effect of the migration with generator $\mathcal{A}$; see e.g. [31], [35] or [36]. In view of the ergodicity of the processes $\{X_t^{x,i}; t \geq 0\}$, $i = 1, 2$, the mild solution of (2.2) should therefore blowup in finite time, at least for certain non-trivial positive initial values f_i, $i = 1, 2$.

We are going to give conditions which imply blowup in finite time of system (2.2) under the assumption that φ_1/φ_2 is a strictly positive bounded function such that

$$\inf_{x \in \mathbb{R}} \{\varphi_1(x)/\varphi_2(x)\} > 0,$$

and the initial values are of the form $f_i = h_i/\varphi_i$, where $h_i \in L^2(\mathbb{R}, dx)$, $i = 1, 2$. We distinguish two cases: if $\beta_1 = \beta_2$ we show that any non-trivial positive mild solution of (2.2) blows up in finite time. If $\beta_1 \neq \beta_2$ we prove that a condition on the "sizes" of f_1 and f_2 and on the measures μ_1, μ_2 of the form

$$\int f_1\, d\mu_1 + \int f_2\, d\mu_2 > c_0,$$

(where the constant $c_0 > 0$ is determined by the system parameters) already implies finite time explosion of (2.2); see Theorem 2.3.1 below. Moreover, we find an upper bound T^* for the blowup time of system (2.2) which depends both on the initial values f_1, f_2, and the invariant measures in (2.5). Our setting allows us to consider a wide range of choices for φ_1 and φ_2, for instance

$$\varphi_1(x) = (\sin(x) + 2)\, \varphi_2(x) \ \text{ with } \varphi_2(x) = e^{-x^2/2},$$

or else

$$\varphi_1(x) = \left(e^{-x^2/2} + 1\right) \varphi_2(x) \ \text{ with } \varphi_2(x) = 1/(1 + x^2).$$

In these two cases the functions h_i, $i = 1, 2$, can be chosen of the form

$$h_i(x) = P_i(|x|)/Q_i(|x|),$$

where P_i, Q_i are polynomial functions with non-negative coefficients such that their degrees satisfy $2 \leq \deg(Q_i) - \deg(P_i)$, and $Q_i(0) > 0$.

Another interesting generator of the form (2.4) arises setting $\varphi(x) = x^{\nu + \frac{1}{2}}$, where $\nu > 0$ is a constant. In this case $L^{\varphi} := \frac{1}{2}\frac{\partial^2}{\partial x^2} + \left(\nu + \frac{1}{2}\right)\frac{1}{x}\frac{\partial}{\partial x}$ is the infinitesimal generator of a process $\{R_t; t \geq 0\}$ starting at 1, which is a Bessel process in $\mathbb{R}$ of dimension $2 + 2\nu$. This kind of infinitesimal generators given by L^{φ} arise naturally in the problems involving the determination of the distribution of functionals of the form

$$A_{\mathbf{e}_\lambda} := \int_0^{\mathbf{e}_\lambda} e^{2(W_s+\nu s)} ds,$$

where $\{W_t; t \geq 0\}$ is a standard Brownian motion in $\mathbb{R}$ and $\mathbf{e}_\lambda$ is an exponential random variable with parameter $\lambda > 0$, independent of $\{W_t; t \geq 0\}$. It can be show (see [50]) that

$$\mathbb{P}\left(A_{\mathbf{e}_\lambda} > u\right) = \mathbb{E}^{\lambda}\left[\frac{1}{\psi(R_u)}\right], \quad u > 0,$$

where ψ solves the ordinary differential equation

$$L^{\varphi}\psi(x) = \frac{\lambda\psi(x)}{x^2}, \quad \psi(1) = 1,$$

and is such that $\{\psi(R_t)e^{-\lambda\int_0^t \frac{ds}{R_s^2}}; t \geq 0\}$ is a martingale with respect to $\mathcal{R}_t := \sigma\{R_u : u \leq t\}$, $t \geq 0$, and $\mathbb{E}^{\lambda}$ stands for the expected value with respect to the measure $\mathbb{P}^{\lambda}\big|_{\mathcal{R}_t} = \psi(R_t)e^{-\lambda\int_0^t \frac{ds}{R_s^2}}\mathbb{P}\big|_{\mathcal{R}_t}$.

In the next section we prove existence and uniqueness of local mild solutions of (2.2) using the clasical fixed-point argument, adapted to our context. Our main result in this chapter, Theorem 2.3.1, is stated and proved in Section 2.3.

2.2 Local existence and uniqueness of mild solutions

Our proof of existence, uniqueness and positiveness of mild solutions of system (2.2) is based on [47, Theorem 2.1], (see also [41, Theorem 2.1], [48, Theorem 3], [33, Theorem 2] and [38, Theorem 1]).

For each $\tau \in (0, \infty)$ we define the set

$$E_\tau := \left\{(u_1, u_2)\,|u_1, u_2 : [0, \tau] \to L^{\infty}(\mathbb{R}, dx)\,, |||(u_1, u_2)||| < \infty\right\},$$

where

$$|||(u_1, u_2)||| := \sup_{t\in[0,\tau]}\left\{\|u_1(t,\cdot)\|_{L^{\infty}(\mathbb{R},dx)} + \|u_2(t,\cdot)\|_{L^{\infty}(\mathbb{R},dx)}\right\}.$$

Then $(E_\tau, |||\cdot|||)$ is a Banach space and the sets

$$P_\tau := \{(u_1, u_2) \in E_\tau : u_1 \geq 0, u_2 \geq 0\} \quad \text{and} \quad B_R := \{(u_1, u_2) \in E_\tau : |||(u_1, u_2)||| \leq R\}$$

are closed subsets of E_τ for any $R \in (0, \infty)$. Therefore $(P_\tau \cap B_R, |||\cdot|||)$ is a Banach space for all $\tau, R \in (0, \infty)$.

Theorem 2.2.1. *There exist $\tau, R \in (0,\infty)$ such that system (2.2) has a unique positive mild solution in $P_\tau \cap B_R$.*

Proof. We will prove that the operator $\Psi : P_\tau \cap B_R \to P_\tau \cap B_R$ defined by

$$\begin{aligned}&\Psi\left(\left(u_1(t,x), u_2(t,x)\right)\right)\\ &= \left(T_t^1\left(f_1(x)\right) + \int_0^t T_{t-s}^1\left(u_2^{1+\beta_1}(s,x)\right)ds, T_t^2\left(f_2(x)\right) + \int_0^t T_{t-s}^2\left(u_1^{1+\beta_2}(s,x)\right)ds\right),\end{aligned}$$

is a contraction for certain $\tau, R \in (0,\infty)$. We start by verifying that Ψ is in fact an operator from $P_\tau \cap B_R$ onto $P_\tau \cap B_R$ for suitably chosen $\tau, R \in (0,\infty)$. Let $\tau_0, R_0 \in (0,\infty)$ be such that

$$R_0 > \left(\|f_1\|_{L^\infty(\mathbb{R},dx)} + \|f_2\|_{L^\infty(\mathbb{R},dx)}\right) \text{ and } \tau_0 \le \frac{R_0 - \left(\|f_1\|_{L^\infty(\mathbb{R},dx)} + \|f_2\|_{L^\infty(\mathbb{R},dx)}\right)}{R_0^{1+\beta_1} + R_0^{1+\beta_2}}.$$

If $(u_1, u_2) \in P_{\tau_0} \cap B_{R_0}$ then $\Psi((u_1,u_2))$ has positive components due to the definition of Ψ and the fact that $u_1, u_2 \ge 0$. Hence

$$\begin{aligned}|||\Psi((u_1,u_2))||| &= \sup_{t\in[0,\tau_0]}\left\{\left\|T_t^1\left(f_1(\cdot)\right) + \int_0^t T_{t-s}^1\left(u_2^{1+\beta_1}(s,\cdot)\right)ds\right\|_{L^\infty(\mathbb{R},dx)}\right.\\ &\quad\left. + \left\|T_t^2\left(f_2(\cdot)\right) + \int_0^t T_{t-s}^2\left(u_1^{1+\beta_2}(s,\cdot)\right)ds\right\|_{L^\infty(\mathbb{R},dx)}\right\}\\ &\le \|f_1\|_{L^\infty(\mathbb{R},dx)} + \|f_2\|_{L^\infty(\mathbb{R},dx)} + \tau_0\left(R_0^{1+\beta_1} + R_0^{1+\beta_2}\right),\end{aligned}$$

where we have used the contraction property of the operators T_t^i, $i = 1, 2$, to obtain the last inequality. It follows that $|||\Psi((u_1,u_2))||| \le R_0$, i.e., Ψ is an operator from $P_{\tau_0} \cap B_{R_0}$ onto itself.

In order to prove the contraction property of Ψ we choose τ_0 as above in such a way that

$$\max_{i=1,2}\left\{(1+\beta_i) R_0^{\beta_i}\right\}\tau_0 \in (0,1). \tag{2.6}$$

Let $(u_1,u_2), (\hat{u}_1,\hat{u}_2) \in P_{\tau_0} \cap B_{R_0}$. Using again the contraction property of the operators T_t^i, $i = 1, 2$, and the well-known inequality $|a^p - b^p| \le p(a \vee b)^{p-1}|a-b|$, which holds for all

$a, b > 0$ and $p \geq 1$, we obtain

$$
\begin{aligned}
& |||\Psi((u_1, u_2)) - \Psi((\hat{u}_1, \hat{u}_2))||| \\
= & \sup_{t\in[0,\tau_0]} \left\{ \left\| \int_0^t T^1_{t-s} \left(u_2^{1+\beta_1}(s,\cdot) - \hat{u}_2^{1+\beta_1}(s,\cdot) \right) ds \right\|_{L^\infty(\mathbb{R},dx)} \right. \\
& \left. + \left\| \int_0^t T^2_{t-s} \left(u_1^{1+\beta_2}(s,\cdot) - \hat{u}_1^{1+\beta_2}(s,\cdot) \right) ds \right\|_{L^\infty(\mathbb{R},dx)} \right\} \\
\leq & \sup_{t\in[0,\tau_0]} \int_0^t \left\| u_2^{1+\beta_1}(s,\cdot) - \hat{u}_2^{1+\beta_1}(s,\cdot) \right\|_{L^\infty(\mathbb{R},dx)} ds \\
& + \sup_{t\in[0,\tau_0]} \int_0^t \left\| u_1^{1+\beta_2}(s,\cdot) - \hat{u}_1^{1+\beta_2}(s,\cdot) \right\|_{L^\infty(\mathbb{R},dx)} ds \\
\leq & (1+\beta_1) R_0^{\beta_1} \int_0^{\tau_0} \| u_2(s,\cdot) - \hat{u}_2(s,\cdot) \|_{L^\infty(\mathbb{R},dx)} ds \\
& + (1+\beta_2) R_0^{\beta_2} \int_0^{\tau_0} \| u_1(s,\cdot) - \hat{u}_1(s,\cdot) \|_{L^\infty(\mathbb{R},dx)} ds \\
\leq & \max_{i=1,2} \left\{ (1+\beta_i) R_0^{\beta_i} \right\} \tau_0 \, |||(u_1, u_2) - (\hat{u}_1, \hat{u}_2)||| \, .
\end{aligned}
$$

From the last inequality we conclude, due to (2.6), that Ψ is a contraction in $P_{\tau_0} \cap B_{R_0}$. It follows from the Banach fixed-point theorem that Ψ has a unique fixed point in $P_{\tau_0} \cap B_{R_0}$, which is the unique mild solution of system (2.2). □

2.3 A condition for blowup in finite time

Our main result is the following

Theorem 2.3.1. *Let $\varphi_i \in L^2(\mathbb{R}, dx) \cap C^2(\mathbb{R})$ be a strictly positive function and assume that the initial value f_i admits the representation*

$$f_i(x) := \frac{h_i(x)}{\varphi_i(x)} \geq 0, \quad x \in \mathbb{R}, \tag{2.7}$$

for some positive nontrivial $h_i \in L^2(\mathbb{R}, dx)$, $i = 1, 2$. Suppose in addition that there exist strictly positive constants k_1, k_2 such that

$$k_1 \leq \frac{\varphi_1(x)}{\varphi_2(x)} \leq k_2, \quad x \in \mathbb{R}. \tag{2.8}$$

1. *Assume that $\beta_1 = \beta_2$. Then any non-trivial positive mild solution (u_1, u_2) of system (2.2) blows up in finite time.*

2. Assume that $\beta_1 > \beta_2$. Let $A_0 := \left(\frac{1+\beta_2}{1+\beta_1}\right)^{\frac{1+\beta_2}{\beta_1-\beta_2}} \frac{\beta_1-\beta_2}{1+\beta_1}$ and suppose that

$$\int_{\mathbb{R}} f_1(x)\,\mu_1(dx) + \int_{\mathbb{R}} f_2(x)\,\mu_2(dx) > 2^{\frac{\beta_2}{1+\beta_2}} A_0^{\frac{1}{1+\beta_2}}. \tag{2.9}$$

Then any mild solution (u_1, u_2) of system (2.2) blows up in finite time.

Proof. Let (u_1, u_2) be a mild solution of system (2.2). We denote

$$w_i(t,x) := \varphi_i(x)\,u_i(t,x), \quad t \geq 0, \quad x \in \mathbb{R}.$$

Multiplying both sides of (2.3) by φ_i yields

$$w_i(t,x) = \varphi_i(x)\,T_t^i\left(\frac{h_i}{\varphi_i}(x)\right) + \int_0^t \varphi_i(x)\,T_{t-s}^i\left(w_{3-i}^{1+\beta_i}(s,x)\,\varphi_{3-i}^{-(1+\beta_i)}(x)\right)ds. \tag{2.10}$$

Since the function $g_i(x) := \varphi_i^2(x)$ satisfies the differential equation

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} g_i(x) - \frac{\partial}{\partial x}\left(g_i(x)\,\frac{\varphi_i'(x)}{\varphi_i(x)}\right) = 0, \quad x \in \mathbb{R},$$

it follows that $\mu_i(dx) = \varphi_i^2(x)\,dx$ is invariant for the semigroup $\left\{T_t^i,\ t \geq 0\right\}$. Let us write $\mathbb{E}^i[f] := \int_{\mathbb{R}} f(x)\,\varphi_i(x)\,dx$. Due to (2.10) this implies that

$$\mathbb{E}^i[w_i(t,\cdot)] = \mathbb{E}^i[h_i(\cdot)] + \int_0^t \mathbb{E}^i\left[w_{3-i}^{1+\beta_i}(s,\cdot)\,\varphi_i(\cdot)\,\varphi_{3-i}^{-(1+\beta_i)}(\cdot)\right]ds. \tag{2.11}$$

Define $a := \min\left\{k_1^2, k_2^{-2}\right\}$. From assumption (2.8) we get $\varphi_i^2(x)/\varphi_{3-i}^2(x) \geq a$ for all $x \in \mathbb{R}$ and $i = 1, 2$. Therefore

$$\begin{aligned}
\mathbb{E}^i\left[w_{3-i}^{1+\beta_i}(s,\cdot)\,\varphi_i(\cdot)\,\varphi_{3-i}^{-(1+\beta_i)}(\cdot)\right] &= \int_{\mathbb{R}}\left(\frac{w_{3-i}(s,x)}{\varphi_{3-i}(x)}\right)^{1+\beta_i}\varphi_i^2(x)\,dx \\
&\geq a\,\|\varphi_{3-i}\|_{L^2(\mathbb{R},dx)}^2 \int_{\mathbb{R}}\left(\frac{w_{3-i}(s,x)}{\varphi_{3-i}(x)}\right)^{1+\beta_i}\frac{\varphi_{3-i}^2(x)}{\|\varphi_{3-i}\|_{L^2(\mathbb{R},dx)}^2}dx \\
&\geq a\frac{\|\varphi_{3-i}\|_{L^2(\mathbb{R},dx)}^2}{\|\varphi_{3-i}\|_{L^2(\mathbb{R},dx)}^{2+2\beta_i}}\left(\int_{\mathbb{R}}\frac{w_{3-i}(s,x)}{\varphi_{3-i}(x)}\varphi_{3-i}^2(x)\,dx\right)^{1+\beta_i} \\
&= a\,\|\varphi_{3-i}\|_{L^2(\mathbb{R},dx)}^{-2\beta_i}\left(\mathbb{E}^{3-i}[w_{3-i}(s,\cdot)]\right)^{1+\beta_i},
\end{aligned} \tag{2.12}$$

where we have used Jensen's inequality to obtain the last inequality. Plugging (2.12) into (2.11) renders

$$\mathbb{E}^i[w_i(t,\cdot)] \geq \mathbb{E}^i[h_i(\cdot)] + a\,\|\varphi_{3-i}\|_{L^2(\mathbb{R},dx)}^{-2\beta_i}\int_0^t\left(\mathbb{E}^{3-i}[w_{3-i}(s,\cdot)]\right)^{1+\beta_i}ds. \tag{2.13}$$

Let $y_i(t)$ be the solution of the system

$$\begin{aligned} y_i'(t) &= a\left\|\varphi_{3-i}\right\|_{L^2(\mathbb{R},dx)}^{-2\beta_i} y_{3-i}^{1+\beta_i}(t), \quad t>0, \\ y_i(0) &= \mathbb{E}^i\left[h_i(\cdot)\right], \quad i=1,2. \end{aligned}$$

Putting $b := a\min\left\{\left\|\varphi_1\right\|_{L^2(\mathbb{R},dx)}^{-2\beta_2}, \left\|\varphi_2\right\|_{L^2(\mathbb{R},dx)}^{-2\beta_1}\right\}$ we get the system of differential inequalities

$$\begin{aligned} y_i'(t) &\geq b y_{3-i}^{1+\beta_i}(t), \quad t>0, \\ y_i(0) &= \mathbb{E}^i\left[h_i(\cdot)\right], \quad i=1,2. \end{aligned}$$

Let $(z_1(t), z_2(t))$ be the solution of the system of ordinary differential equations

$$\begin{aligned} z_i'(t) &= b z_{3-i}^{1+\beta_i}(t), \quad t>0, \\ z_i(0) &= \mathbb{E}^i\left[h_i(\cdot)\right], \quad i=1,2. \end{aligned}$$

By the Picard-Lindelöf theorem, this system with $(z_1(0), z_2(0)) = (0,0)$ has a unique local solution $(w_1(t), w_2(t)) \equiv (0,0)$ for all $t \in [0,\tau)$, for some $\tau \in (0,\infty]$. In our case $\mathbb{E}^i\left[h_i(\cdot)\right] \geq 0$. Therefore by a classical comparison theorem, $z_1(t), z_2(t) \geq 0$ for all $t \in [0,\tau)$.

Consider the new function

$$E(t) := z_1(t) + z_2(t), \quad t \geq 0.$$

We deal separately with the two cases in the statement of the theorem:

1. Case $\beta_1 = \beta_2$. Using the fact that

$$x^{1+\beta_1} + y^{1+\beta_1} \geq 2^{-\beta_1}(x+y)^{1+\beta_1}, \quad x\geq 0, \quad y \geq 0, \tag{2.14}$$

we get

$$\begin{aligned} &E'(t) = z_1'(t) + z_2'(t) = b\left(z_1^{1+\beta_1}(t) + z_2^{1+\beta_1}(t)\right) \geq 2^{-\beta_1} b E^{1+\beta_1}(t), \quad t>0, \\ &E(0) = \mathbb{E}^1\left[h_1(\cdot)\right] + \mathbb{E}^2\left[h_2(\cdot)\right]. \end{aligned}$$

Let $I(t)$ be the solution of the ordinary differential equation

$$\begin{aligned} I'(t) &= 2^{-\beta_1} b I^{1+\beta_1}(t), \quad t>0, \\ I(0) &= \mathbb{E}^1\left[h_1(\cdot)\right] + \mathbb{E}^2\left[h_2(\cdot)\right]. \end{aligned}$$

Since I is a subsolution of E (see [46, Lemma 1.2]) and I explodes at time

$$T^* = \frac{2^{\beta_1}}{b\beta_1\left(\mathbb{E}^1\left[h_1(\cdot)\right] + \mathbb{E}^2\left[h_2(\cdot)\right]\right)^{\beta_1}} \in (0,\infty),$$

it follows that E explodes at some time $t_E \le T^*$, and therefore, by a classical comparison theorem we get that

$$\mathbb{E}^1 \left[w_1 \left(t, \cdot \right) \right] = \left\| u_1 \left(t, \cdot \right) \right\|_{L^1(\mathbb{R}, \mu_1)} = \infty \quad \text{ or } \quad \mathbb{E}^2 \left[w_2 \left(t, \cdot \right) \right] = \left\| u_2 \left(t, \cdot \right) \right\|_{L^1(\mathbb{R}, \mu_2)} = \infty$$

for all $t \ge T^*$. Since $\left\| u_i \left(t, \cdot \right) \right\|_{L^1(\mathbb{R}, \mu_i)} \le \left\| u_i \left(t, \cdot \right) \right\|_{L^\infty(\mathbb{R}, dx)} \left\| \varphi_i \right\|^2_{L^2(\mathbb{R}, dx)}$ for all $t \in [0, \infty)$, $i = 1, 2$, we conclude that the mild solution (u_1, u_2) of system (2.2) blows up in finite time.

2. Case $\beta_1 > \beta_2$. Recall that for all $x, y \ge 0$, $\delta > 0$ and $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$ we have Young's inequality

$$xy \le \frac{\delta^{-p} x^p}{p} + \frac{\delta^q y^q}{q}. \tag{2.15}$$

From the definition of A_0 it follows that

$$z_2^{1+\beta_1} \left(t \right) \ge z_2^{1+\beta_2} \left(t \right) - A_0, \quad \text{for all } t \ge 0.$$

In fact, it suffices to choose in (2.15)

$$x = 1, \quad y = z_2^{1+\beta_2} \left(t \right), \quad \delta = \left(\frac{1+\beta_1}{1+\beta_2} \right)^{\frac{1+\beta_2}{1+\beta_1}} \quad \text{and} \quad q = \frac{1+\beta_1}{1+\beta_2}.$$

Therefore we have

$$E' \left(t \right) \ge b \left(z_1^{1+\beta_2} \left(t \right) + z_2^{1+\beta_2} \left(t \right) - A_0 \right).$$

Using again inequality (2.14) we conclude that

$$z_1^{1+\beta_2} \left(t \right) + z_2^{1+\beta_2} \left(t \right) \ge 2^{-\beta_2} E^{1+\beta_2} \left(t \right),$$

hence

$$E' \left(t \right) \ge b \left(2^{-\beta_2} E^{1+\beta_2} \left(t \right) - A_0 \right).$$

Let $I \left(t \right)$ solve the ordinary differential equation

$$I' \left(t \right) = b \left(2^{-\beta_2} I^{1+\beta_2} \left(t \right) - A_0 \right), \quad t > 0,$$
$$I \left(0 \right) = \mathbb{E}^1 \left[h_1 \left(\cdot \right) \right] + \mathbb{E}^2 \left[h_2 \left(\cdot \right) \right].$$

It follows from the same comparison theorem as above that I is a subsolution of E. Using separation of variables we get, for $t \in (0, \infty)$,

$$t = \int_{E(0)}^{I(t)} \frac{dx}{b \left(2^{-\beta_2} x^{1+\beta_2} - A_0 \right)} \le \int_{E(0)}^{\infty} \frac{dx}{b \left(2^{-\beta_2} x^{1+\beta_2} - A_0 \right)} =: T^*. \tag{2.16}$$

But the hypothesis (2.9) implies that $T^* < \infty$. Hence (2.16) cannot hold for sufficiently large t, which yields that I explodes at a finite time $T^{**} \in (0, T^*]$. Therefore E explodes no latter than T^* as well. From here we proceed as in the case $\beta_1 = \beta_2$ to conclude that the mild solution (u_1, u_2) of system (2.2) blows up in finite time also in this case.

□

The following result is an immediate consequence of the previous theorem. Recall that $E(0) = \int_{\mathbb{R}} f_1 \, d\mu_1 + \int_{\mathbb{R}} f_2 \, d\mu_2$ and

$$A_0 = \left(\frac{1+\beta_2}{1+\beta_1}\right)^{\frac{1+\beta_2}{\beta_1-\beta_2}} \frac{\beta_1 - \beta_2}{1+\beta_1}, \quad b = \min\left\{k_1^2, \frac{1}{k_2^2}\right\} \min_{i1,2}\left\{\|\varphi_i\|_{L^2(\mathbb{R},dx)}^{-2\beta_i}\right\}.$$

Corollary 2.3.2. *Under the assumptions of Theorem 2.3.1, if $\beta_1 = \beta_2$ then the explosion time of any non-trivial positive solution of (2.2) is bounded above by*

$$T^* = \frac{2^{\beta_1}}{b\beta_1 (E(0))^{\beta_1}}.$$

If $\beta_1 > \beta_2$ and (2.9) holds, then the time of explosion of (2.2) is bounded above by

$$T^* = \int_{E(0)}^{\infty} \frac{dx}{b\,(2^{-\beta_2} x^{1+\beta_2} - A_0)}.$$

Remark 2.3.3. Theorem 2.3.1 and Corollary 2.3.2 remain valid when $\beta_2 > \beta_1$, with the obvious changes in the correspondent statements.

Chapter 3

Blowup in finite time of a system of SPDEs

3.1 Introduction

Let $D \subseteq \mathbb{R}^d$ be a bounded smooth domain, and let $\kappa_1, \kappa_2 \in \mathbb{R}$, be given constants. Denote by $\{W_t; t \geq 0\}$ a one-dimensional standard Brownian motion defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $f_1, f_2 \in C^2(D)$ be two positive functions. In [14] lower and upper bounds for the explosion time of positive solutions of the semilinear system of SPDEs

$$\begin{aligned} du_1(t,x) &= [(\Delta + V_1)u_1(t,x) + u_2^p(t,x)]\,dt + \kappa_1 u_1(t,x)\,dW_t, \\ du_2(t,x) &= [(\Delta + V_2)u_2(t,x) + u_1^q(t,x)]\,dt + \kappa_2 u_2(t,x)\,dW_t, \\ u_i(0,x) &= f_i(x) \geq 0, \quad x \in D, \\ u_i(t,x) &= 0, \quad t \geq 0, \quad x \in \partial D, \quad i = 1,2, \end{aligned} \tag{3.1}$$

were obtained in the case $V_i = \lambda_1 + \kappa_i^2/2$, $i = 1,2$, where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian on D and $p \geq q > 1$. It was shown that there exist random times ϱ_{**}, ϱ^{**} such that $\varrho_{**} \leq \varrho \leq \varrho^{**}$, where ϱ is the explosion time of (3.1) and the laws of ϱ_{**} and ϱ^{**} are given, respectively, in terms of exponential functionals of the form

$$\int_0^t \left(e^{aW_r} \wedge e^{bW_r}\right) dr \quad \text{and} \quad \int_0^t \left(e^{aW_r} \vee e^{bW_r}\right) dr, \quad t \geq 0, \tag{3.2}$$

for certain real constants a, b. Our aim in this chapter is to obtain lower and upper bounds for the explosion time of positive solutions of the system of SPDEs

$$\begin{aligned} du_1(t,x) &= [\Delta_\alpha u_1(t,x) + G_1(u_2(t,x))]\,dt + \kappa_1 u_1(t,x)\,dW_t, \\ du_2(t,x) &= [\Delta_\alpha u_2(t,x) + G_2(u_1(t,x))]\,dt + \kappa_2 u_2(t,x)\,dW_t, \\ u_i(0,x) &= f_i(x) \geq 0, \quad x \in D, \\ u_i(t,x) &= 0, \quad t \geq 0, \quad x \in \mathbb{R}^d \setminus D, \quad i = 1,2. \end{aligned} \tag{3.3}$$

Here, Δ_α is the fractional power $-(-\Delta)^{\alpha/2}$ of the Laplacian, G_i is a locally Lipschitz positive function such that

$$G_i(z) \geq z^{1+\beta_i}, \quad z \geq 0, \tag{3.4}$$

with $\beta_i > 0$, $i = 1, 2$. We assume (3.4) in Section 3.3.1 only; it is replaced by (3.39) in Section 3.3.2. We refer to Chapter 1 for definitions of blowup times, and for types of solutions of SPDEs. Equations and systems of the above kind arise as mathematical models describing processes of diffusion of heat and burning in two-component continuous media, where the functions u_1, u_2 are treated as temperatures of interacting components in a combustible mixture. Hence, it is natural and relevant to investigate properties of positive solutions of such equations. Since we do not assume G_i to be Lipschitz, $i = 1, 2$, blowup of the solution of (3.3) in finite time cannot be left out. One of the main contributions in this chapter is to show that there are random times τ_{**} and τ^{**} such that $\tau_{**} \leq \tau \leq \tau^{**}$, where τ is the explosion time of (3.3). In this case, the distributions of the random times τ_{**} and τ^{**} are given in terms of functionals of the form

$$\int_0^t \left(e^{aW_s} \wedge e^{bW_s}\right) e^{-Ms} dr \quad \text{and} \quad \int_0^t \left(e^{aW_s} \vee e^{bW_s}\right) e^{-\mu s} ds \tag{3.5}$$

for some positive constants a, b, M and μ, which depend on the parameters of the system (3.3). Notice that the functionals (3.2) are a special case of (3.5), hence the present work can be considered as a generalization and an extension of [14]. Although the laws of the functionals (3.5) are not given explicitly in this work, we find random times $\tau_{\prime\prime}$ and τ'' such that $\tau_{\prime\prime} \leq \tau_{**}$ and $\tau^{**} \leq \tau''$. The random times τ'' and $\tau_{\prime\prime}$ are given in terms of random functionals of the form

$$F_1(t) = \int_0^t e^{-(\sigma W_s - \mu s)} \mathbb{1}_{\{\sigma W_s - \mu s \geq 0\}} dr \quad \text{and} \quad F_2(t) = \int_0^t e^{\sigma W_s - \mu s} ds, \quad t \geq 0,$$

respectively, where σ and μ are certain constants. The function F_2 is known as *Dufresne's functional* and the distribution of its perpetual version $F_2(\infty)$ was computed in [16] for $\mu > 0$. The density function of $F_2(t)$ for $0 \leq t < \infty$ was obtained by M. Yor using techniques based on hitting times of Bessel processes; see [50], [5] and [44]. The function F_1 is known as *one-sided Dufresne's functional*. We believe that the law of its perpetual version could be obtained by the method of hitting times of Bessel processes as in the case of F_2, or else using the method of Pintoux and Privault [40]. In the present work we calculate the probability density function of $F_1(\infty)$ by a straight analytical approach based on the explicit computation of the potential measure of the process $X_t = \sigma W_t - \mu t$, $t \geq 0$. This allows us to obtain a related integral equation for the function

$$H(x,z)=\mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-(X_s+x)}\mathbb{1}_{\{X_s+x\geq 0\}}ds\right)\right],\quad x\geq 0,\quad z>0,$$

(which gives as a special case the Laplace transform of $F_1(\infty)$), and upon solving it we obtain an explicit expression for H. By inverting the transform H we get the distribution of the perpetual functional $F_1(\infty)$ which is needed further to obtain a lower bound for the probability of explosion in finite time. This is the subject of Section 3.2. With the aim of getting suitable sub- and supersolutions of (3.3) –from which we will obtain upper and lower bounds for τ–, in Section 3.3 we transform system (3.3) into a related system of random partial differential equations. This procedure is similar to the one performed in [14] and is inspired in a classical result of Doss [13] (see also Section 1.3). In Section 3.3 we also obtain upper and lower bounds for the explosion time τ. In Section 3.4 we give explicit non-trivial bounds for the probability of explosion in finite time of positive solutions of system (3.3), under the assumptions that $\beta_1=\beta_2$ and the initial values are of the form $f_i(x)=L_i\psi(x)$, $x\in D$, with $L_i>0$, $i=1,2$, where ψ is the eigenfunction corresponding to the first eigenvalue of Δ_α on D. Such bounds depend on the functionals we found in Section 3.3.

3.2 An exponential functional of Brownian motion

Let $\{W_t;t\geq 0\}$ be a one-dimensional standard Brownian motion. Let σ and μ be positive constants. It is well known (see e.g. [16] and Section 1.4) that Dufresne's functional $\int_0^\infty e^{\sigma W_s-\mu s}ds$ has the following distribution for all $c\geq 0$:

$$\mathbb{P}\left(\int_0^\infty e^{\sigma W_s-\mu s}ds>c\right)=\frac{\gamma\left(\frac{2\mu}{\sigma^2},\frac{2}{\sigma^2 c}\right)}{\Gamma\left(\frac{2\mu}{\sigma^2}\right)},\tag{3.6}$$

where $\gamma(a,x)=\int_0^x e^{-s}s^{a-1}ds$ and $\Gamma(a)=\gamma(a,\infty)$ for all $a>0$ and $x\geq 0$.

Let $X_t=\sigma W_t-\mu t$, $t\geq 0$. The motivation of this section is to study, from an analytical point of view, some distributional properties of the exponential functional

$$\int_0^\infty e^{-(X_t+x)}\mathbb{1}_{\{X_t+x\geq 0\}}dt,\quad x\geq 0.$$

This kind of functionals, also named one-sided variants of Dufresne's functional, emerges for instance in the problem of explosion in finite time of systems of SPDEs. In particular we calculate explicitly its Laplace transform and its distribution at $x = 0$. Recall (see [3]) that the potential measure of the process $\{X_t; t \geq 0\}$ is the Borel measure U defined by

$$U(B) = \int_0^\infty \mathbb{P}(X_t \in B)\, dt, \quad B \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ stands for the Borel σ-algebra in $\mathbb{R} = (-\infty, \infty)$.

Lemma 3.2.1. *The measure U is absolutely continuous with respect to the Lebesgue measure, and the density function of U is given by*

$$u(x) = \frac{1}{\mu}\mathbb{1}_{(-\infty,0)}(x) + \frac{1}{\mu}e^{-\frac{2\mu}{\sigma^2}x}\mathbb{1}_{[0,\infty)}(x), \quad x \in \mathbb{R}. \tag{3.7}$$

Proof. First note that the transition probability of $\{X_t; t \geq 0\}$ is given by

$$p(t,x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left[-\frac{(x+\mu t)^2}{2\sigma^2 t}\right], \quad x \in \mathbb{R}, \quad t > 0.$$

From [45, page 242] we know that

$$\begin{aligned} u(x) &= \int_0^\infty p(t,x)\, dt \\ &= \sqrt{\frac{2}{\pi\sigma^2\mu}} \int_0^\infty \exp\left[-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{x}{s} + \sqrt{\frac{\mu}{2\sigma^2}}s\right)^2\right] ds, \quad x \in \mathbb{R}, \end{aligned}$$

where we have used the change of variables $s = \sqrt{\mu t}$ to obtain the second equality. Now we note that for all $s > 0$,

$$\begin{aligned} &\sqrt{\frac{2}{\pi\sigma^2\mu}} e^{-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{x}{s} + \sqrt{\frac{\mu}{2\sigma^2}}s\right)^2} \\ &= \frac{e^{-\frac{\mu}{\sigma^2}(|x|+x)}}{2\mu}\left[-\frac{2}{\sqrt{\pi}}e^{-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s} - \sqrt{\frac{\mu}{2\sigma^2}}s\right)^2}\left(-\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s^2} - \sqrt{\frac{\mu}{2\sigma^2}}\right)\right. \\ &\left. + e^{\frac{2\mu}{\sigma^2}|x|}\frac{2}{\sqrt{\pi}}e^{-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s} + \sqrt{\frac{\mu}{2\sigma^2}}s\right)^2}\left(-\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s^2} + \sqrt{\frac{\mu}{2\sigma^2}}\right)\right]. \end{aligned} \tag{3.8}$$

Integrating both sides of (3.8) with respect to s, we get for all $x \in \mathbb{R}$,

$$\begin{aligned} u(x) = \frac{e^{-\frac{\mu}{\sigma^2}(|x|+x)}}{2\mu} &\left[-\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s} - \sqrt{\frac{\mu}{2\sigma^2}}s\right)^2}\left(-\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s^2} - \sqrt{\frac{\mu}{2\sigma^2}}\right) ds\right. \\ &\left. + e^{\frac{2\mu}{\sigma^2}|x|}\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s} + \sqrt{\frac{\mu}{2\sigma^2}}s\right)^2}\left(-\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s^2} + \sqrt{\frac{\mu}{2\sigma^2}}\right) ds\right]. \end{aligned}$$

Performing the change of variables

$$t = \sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s} - \sqrt{\frac{\mu}{2\sigma^2}}s \quad \text{and} \quad t = \sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s} + \sqrt{\frac{\mu}{2\sigma^2}}s$$

in the integrals of the right hand side renders

$$u(x) = \frac{e^{-\frac{\mu}{\sigma^2}(|x|+x)}}{2\mu}\left[-\mathrm{erf}\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s} - \sqrt{\frac{\mu}{2\sigma^2}}s\right) + e^{\frac{2\mu}{\sigma^2}|x|}\mathrm{erf}\left(\sqrt{\frac{\mu}{2\sigma^2}}\frac{|x|}{s} + \sqrt{\frac{\mu}{2\sigma^2}}s\right)\right]\Bigg|_0^\infty,$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds, \quad z \in \mathbb{R},$$

is the error function. Since $\mathrm{erf}(\infty) = 1$ and $\mathrm{erf}(-\infty) = -1$, it follows that

$$u(x) = \frac{e^{-\frac{2\mu}{\sigma^2}x}}{2\mu}\left[-\mathrm{erf}(-\infty) + e^{\frac{2\mu}{\sigma^2}x}\mathrm{erf}(\infty) + \mathrm{erf}(\infty) - e^{\frac{2\mu}{\sigma^2}x}\mathrm{erf}(\infty)\right] = \frac{1}{\mu}e^{-\frac{2\mu}{\sigma^2}x},$$

for all $x \geq 0$. Similarly, if $x < 0$ we conclude that $u(x) = 1/\mu$ and the result follows. $\square$

Define

$$H(x,z) = \mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-(X_s+x)}\mathbb{1}_{\{X_s+x\geq 0\}}\,ds\right)\right]$$

for all $x \geq 0$, $z \in \mathbb{C}$.

Lemma 3.2.2. *For all $x \geq 0$ and $z \in \mathbb{C}$, $H(x,z)$ satisfies the integral equation*

$$H(x,z) = 1 - \mu^{-1}ze^{\frac{2\mu}{\sigma^2}x}\int_x^\infty e^{-\left(1+\frac{2\mu}{\sigma^2}\right)u}H(u,z)\,du - \mu^{-1}z\int_0^x e^{-u}H(u,z)\,du. \tag{3.9}$$

Proof. For simplicity of notation, for any fixed $z \in \mathbb{C}$, let $f_z(x) = -ze^{-x}\mathbb{1}_{\{x\geq 0\}}$, $x \in \mathbb{R}$. Define the function

$$v_t(x,z) = \mathbb{E}\left[\exp\left(\int_0^t f_z(X_s+x)ds\right)\right], \quad t \geq 0, \quad x \geq 0.$$

Using that

$$\begin{aligned}&\int_0^t \exp\left(\int_s^t f_z(X_u+x)du\right)f_z(X_s+x)ds\\ &= -\int_0^t \frac{d}{ds}\left[\exp\left(\int_s^t f_z(X_u+x)du\right)\right]ds = \exp\left(\int_0^t f_z(X_u+x)du\right) - 1,\end{aligned}$$

from the Dominated Convergence Theorem we get

$$\begin{aligned}v_t(x,z) &= 1 + \mathbb{E}\left[\int_0^t \exp\left(\int_s^t f_z(X_u+x)du\right)f_z(X_s+x)ds\right]\\ &= 1 + \int_0^t \mathbb{E}\left[\exp\left(\int_s^t f_z(X_u+x)du\right)f_z(X_s+x)\right]ds.\end{aligned} \tag{3.10}$$

Since $f_z(X_s+x)$ is measurable with respect to $\sigma(X_r, 0 \leq r \leq s)$, $0 \leq s \leq t$, then

$$\begin{aligned}&\mathbb{E}\left[\exp\left(\int_s^t f_z(X_u+x)du\right)f_z(X_s+x)\right]\\ &= \mathbb{E}\left[f_z(X_s+x)\mathbb{E}\left[\exp\left(\int_s^t f_z(X_u+x)du\right)\Bigg|\,\sigma(X_r, 0 \leq r \leq s)\right]\right].\end{aligned} \tag{3.11}$$

Due to the independence of increments property of $\{X_t; t \geq 0\}$ we get

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\int_s^t f_z(X_u + x)du\right)\middle|\,\sigma(X_r, 0 \leq r \leq s)\right] \\
&= \mathbb{E}\left[\exp\left(\int_s^t f_z(X_u - X_s + X_s + x)du\right)\middle|\,\sigma(X_r, 0 \leq r \leq s)\right] \\
&= h(X_s + x), \qquad (3.12)
\end{aligned}$$

where the function h is defined by

$$h(y) = \mathbb{E}\left[\exp\left(\int_s^t f_z(X_u - X_s + y)du\right)\right].$$

Due to stationarity of increments of $\{X_t; t \geq 0\}$, we obtain that

$$h(y) = \mathbb{E}\left[\exp\left(\int_0^{t-s} f_z(X_u + y)du\right)\right] = v_{t-s}(y, z). \qquad (3.13)$$

Plugging (3.11), (3.12) and (3.13) into (3.10) we finally get

$$\begin{aligned}
&v_t(x, z) \\
&= 1 + \int_0^t \mathbb{E}\left[f_z(X_s + x)v_{t-s}(X_s + x, z)\right] ds = 1 + \mathbb{E}\left[\int_0^t f_z(X_s + x)v_{t-s}(X_s + x, z)ds\right] \\
&= 1 - z\int_{\mathbb{R}} e^{-(x+y)}\mathbb{1}_{[0,\infty)}(x+y)\left(\int_0^t v_{t-s}(x+y, z)\,\mathbb{P}(X_s \in dy)\,ds\right).
\end{aligned}$$

Since the improper integral $\int_0^\infty e^{-(X_s+x)}\mathbb{1}_{\{X_s+x\geq 0\}}\,ds$ is a.s. finite due to [17, Theorem 1.4], using dominated convergence we get

$$v_t(x, z) \to H(x, z) \quad \text{as} \quad t \to \infty.$$

The fact that

$$0 \leq \left|v_{t-s}(x+y, z)\,\mathbb{1}_{[0,t]}(s)\right| \leq 1$$

for all $s \geq 0$ implies, for all $x \geq 0$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}} e^{-(x+y)}\mathbb{1}_{[0,\infty)}(x+y)\left(\int_0^\infty v_{t-s}(x+y, z)\,\mathbb{1}_{[0,t]}(s)\,\mathbb{P}(X_s \in dy)\,ds\right)\right| \\
&\leq \int_{\mathbb{R}} e^{-(x+y)}\mathbb{1}_{[0,\infty)}(x+y)\,U(dy) = \frac{1-e^{-x}}{\mu} + \frac{\sigma^2 e^{-x}}{\mu\sigma^2 + 2\mu^2} \leq \frac{1}{\mu} + \frac{\sigma^2}{\mu\sigma^2 + 2\mu^2}.
\end{aligned}$$

Using again dominated convergence we get that for every $z \in \mathbb{C}$ and every $x \geq 0$ the function $H(\cdot, z)$ satisfies the integral equation

$$H(x, z) = 1 - z\int_{\mathbb{R}} e^{-(x+y)}\mathbb{1}_{[0,\infty)}(x+y)\,H(x+y, z)\,U(dy). \qquad (3.14)$$

From (3.7) and (3.14) it follows that for every $z \in \mathbb{C}$ and every $x \geq 0$

$$\begin{aligned}
H(x,z) &= 1 - z\int_{\mathbb{R}} e^{-(x+y)} \mathbb{1}_{[0,\infty)}(x+y)\, H(x+y,z)\, \mu^{-1} \mathbb{1}_{(-\infty,0)}(y)\, dy \\
&\quad - z\int_{\mathbb{R}} e^{-(x+y)} \mathbb{1}_{[0,\infty)}(x+y)\, H(x+y,z)\, \mu^{-1} e^{-\frac{2\mu}{\sigma^2}y} \mathbb{1}_{[0,\infty)}(y)\, dy \\
&= 1 - \mu^{-1} z \int_{-x}^{0} e^{-(x+y)} H(x+y,z)\, dy - \mu^{-1} z \int_{0}^{\infty} e^{-(x+y)} H(x+y,z)\, e^{-\frac{2\mu}{\sigma^2}y} dy \\
&= 1 - \mu^{-1} z e^{\frac{2\mu}{\sigma^2}x} \int_{x}^{\infty} e^{-\left(1+\frac{2\mu}{\sigma^2}\right)u} H(u,z)\, du - \mu^{-1} z \int_{0}^{x} e^{-u} H(u,z)\, du.
\end{aligned}$$

□

Theorem 3.2.3. *Let $\theta \in \mathbb{C}$ be such that $|\theta| < 1$, and let*

$$I(x,u) = e^{\frac{2\mu}{\sigma^2}x} e^{-\left(1+\frac{2\mu}{\sigma^2}\right)u} \mathbb{1}_{[x,\infty)}(u) + e^{-u} \mathbb{1}_{[0,x)}(u), \quad x \geq 0, \quad u \geq 0.$$

Then, the integral equation

$$g(x) = 1 - \theta \int_{0}^{\infty} I(x,u)\, g(u)\, du \tag{3.15}$$

possesses a unique solution

$$g(x) = \sum_{n \geq 0} (-\theta)^n \psi_n(x) \in C_b\left(\mathbb{R}^+\right),$$

where

$$\psi_0(x) = 1, \quad \psi_{n+1}(x) = \int_{0}^{\infty} I(x,u)\, \psi_n(u)\, du, \quad n \geq 0, \quad x \geq 0.$$

Proof. Consider the Banach space $(C_b(\mathbb{R}^+), \|\cdot\|_\infty)$. We have that

$$\int_{0}^{\infty} I(x,u)\, du = 1 - \frac{2\mu}{\sigma^2 + 2\mu} e^{-x},$$

which implies that the function

$$h(x) := 1 - \theta \int_{0}^{\infty} I(x,u)\, g(u)\, du, \quad x \in \mathbb{R}^+,$$

satisfies $h \in C_b(\mathbb{R}^+)$ for all $g \in C_b(\mathbb{R}^+)$. Now we prove that the operator $T : C_b(\mathbb{R}^+) \to C_b(\mathbb{R}^+)$, defined by $T(g) = h$, is a contraction mapping. In fact, for $g_1, g_2 \in C_b(\mathbb{R}^+)$,

$$\begin{aligned}
\|T(g_1) - T(g_2)\|_\infty &= |\theta| \left\| \int_{0}^{\infty} I(\cdot,u)\, g_1(u)\, du - \int_{0}^{\infty} I(\cdot,u)\, g_2(u)\, du \right\|_\infty \\
&\leq |\theta| \left\| \int_{0}^{\infty} I(\cdot,u)\, du \right\|_\infty \|g_1 - g_2\|_\infty = |\theta| \|g_1 - g_2\|_\infty,
\end{aligned}$$

i.e., T is a contraction mapping. From the Banach fixed point theorem it follows that (3.15) has a unique solution. To prove the power series representation of g first we note that $\|\psi_n\|_\infty \leq 1$ for all $n \geq 0$, which can be easily proved by induction. Then, under the assumption $|\theta| \in [0,1)$, the series

$$\sum_{n\geq 0} (-\theta)^n \psi_n$$

is absolutely and uniformly convergent. By Fubini's theorem we finally get that

$$\begin{aligned} 1 - \theta \int_0^\infty I(x,u) \sum_{n\geq 0} (-\theta)^n \psi_n(u)\, du &= 1 + \sum_{n\geq 0} (-\theta)^{n+1} \int_0^\infty I(x,u)\, \psi_n(u)\, du \\ &= 1 + \sum_{n\geq 0} (-\theta)^{n+1} \psi_{n+1}(x) = \sum_{n\geq 0} (-\theta)^n \psi_n(x). \end{aligned}$$

Therefore

$$g(x) = \sum_{n\geq 0} (-\theta)^n \psi_n(x)$$

for all $x \geq 0$. □

From Lemma 3.2.2 and Theorem 3.2.3 we deduce one of the main results of this section.

Theorem 3.2.4. *For all $x \geq 0$ and all $z \in \mathbb{C}$ such that $|z|\,\mu^{-1} < 1$, the function $H(x,z)$ is the unique solution of the integral equation*

$$F(x,z) = 1 - \mu^{-1} z e^{\frac{2\mu}{\sigma^2}x} \int_x^\infty e^{-\left(1+\frac{2\mu}{\sigma^2}\right)u} F(u,z)\, du - \mu^{-1} z \int_0^x e^{-u} F(u,z)\, du. \tag{3.16}$$

In order to get a closed expression for H, we proceed by induction over $n \geq 0$ to prove that

$$\psi_{n+1}(x) = \sum_{k=1}^{n} B_k \psi_{n+1-k}(x) + B_{n+1}\left(1 - \frac{\frac{2\mu}{\sigma^2}}{n+1+\frac{2\mu}{\sigma^2}} e^{-(n+1)x}\right), \tag{3.17}$$

where

$$B_k := \frac{(-1)^{k-1}\,\Gamma\left(\frac{2\mu}{\sigma^2}\right)\left(\frac{2\mu}{\sigma^2}\right)^k}{k!\,\Gamma\left(k+\frac{2\mu}{\sigma^2}\right)}, \quad k \in \mathbb{N}.$$

For $n = 0$, under the convention $\sum_{k=1}^{0} \equiv 0$ and the fact that $B_1 = 1$, we get

$$\psi_1(x) = \int_0^\infty \left(e^{\frac{2\mu}{\sigma^2}x} e^{-\left(1+\frac{2\mu}{\sigma^2}\right)u} \mathbb{1}_{[x,\infty)}(u) + e^{-u}\mathbb{1}_{[0,x)}(u)\right) du = 1 - \frac{\frac{2\mu}{\sigma^2}}{1+\frac{2\mu}{\sigma^2}} e^{-x},$$

which shows that (3.17) holds for $n=0$. Assume that (3.17) is true for some $n \geq 0$. Then

$$
\begin{aligned}
\psi_{n+1}(x) &= \int_0^\infty I(x,u)\,\psi_n(u)\,du \\
&= \int_0^\infty I(x,u)\left(\sum_{k=1}^{n-1} B_k \psi_{n-k}(u) + B_n\left(1 - \frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}} e^{-nu}\right)\right) du \\
&= \sum_{k=1}^{n-1} B_k \int_0^\infty I(x,u)\,\psi_{n-k}(u)\,du \\
&+ B_n\left(\int_0^\infty I(x,u)\,du - \frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}} \int_0^\infty I(x,u)\,e^{-nu} du\right) \\
&= \sum_{k=1}^{n-1} B_k \psi_{n+1-k}(x) + B_n \psi_1(x) \\
&- \frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}} B_n \left(\frac{1}{n+1} - \frac{\frac{2\mu}{\sigma^2}}{(n+1)\left(n+1+\frac{2\mu}{\sigma^2}\right)} e^{-(n+1)x}\right) \\
&= \sum_{k=1}^{n} B_k \psi_{n+1-k}(x) + B_{n+1}\left(1 - \frac{\frac{2\mu}{\sigma^2}}{n+1+\frac{2\mu}{\sigma^2}} e^{-(n+1)x}\right),
\end{aligned}
$$

where in the second equality we have used the induction hypothesis, the definition of ψ_n for the fourth one and the fact that

$$
B_{n+1} = -\frac{\frac{2\mu}{\sigma^2}}{(n+1)\left(n+\frac{2\mu}{\sigma^2}\right)} B_n
$$

for the last equality. This proves (3.17). Moreover, notice that

$$
\begin{aligned}
&\sum_{n\geq 1} \left(-\mu^{-1} z\right)^n \psi_n(x) \\
&= \sum_{n\geq 1} \left(-\mu^{-1} z\right)^n \left(\sum_{k=1}^{n-1} B_k \psi_{n-k}(x) + B_n\left(1 - \frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}} e^{-nx}\right)\right) \\
&= \sum_{k\geq 1} B_k \sum_{n\geq k+1} \left(-\mu^{-1} z\right)^n \psi_{n-k}(x) + \sum_{n\geq 1} \left(-\mu^{-1} z\right)^n B_n \left(1 - \frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}} e^{-nx}\right) \\
&= \sum_{k\geq 1} B_k \sum_{j\geq 1} \left(-\mu^{-1} z\right)^{j+k} \psi_j(x) + \sum_{n\geq 1} \left(-\mu^{-1} z\right)^n B_n \left(1 - \frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}} e^{-nx}\right) \\
&= \sum_{k\geq 1} B_k \left(-\mu^{-1} z\right)^k \sum_{j\geq 1} \left(-\mu^{-1} z\right)^j \psi_j(x) + \sum_{n\geq 1} \left(-\mu^{-1} z\right)^n B_n \left(1 - \frac{\frac{2\mu}{\sigma^2}}{n+\frac{2\mu}{\sigma^2}} e^{-nx}\right),
\end{aligned}
$$

from which we conclude that

$$\sum_{n\geq 1}\left(-\mu^{-1}z\right)^{n}\psi_{n}\left(x\right) = \frac{\sum_{n\geq 1}\left(-\mu^{-1}z\right)^{n}B_{n}\left(1-\frac{\frac{2\mu}{\sigma^{2}}}{n+\frac{2\mu}{\sigma^{2}}}e^{-nx}\right)}{1-\sum_{n\geq 1}\left(-\mu^{-1}z\right)^{n}B_{n}},$$

and therefore we get

$$H\left(x,z\right)=\frac{1-\sum_{n\geq 1}\left(-\mu^{-1}z\right)^{n}B_{n}\left(\frac{\frac{2\mu}{\sigma^{2}}}{n+\frac{2\mu}{\sigma^{2}}}\right)e^{-nx}}{1-\sum_{n\geq 1}\left(-\mu^{-1}z\right)^{n}B_{n}}. \tag{3.18}$$

From the definition of B_n,

$$\begin{aligned}
&\sum_{n\geq 1}\left(-\mu^{-1}z\right)^{n}B_{n}\left(\frac{\frac{2\mu}{\sigma^{2}}}{n+\frac{2\mu}{\sigma^{2}}}\right)e^{-nx}=\sum_{n\geq 1}\left(-\mu^{-1}ze^{-x}\right)^{n}\frac{\left(-1\right)^{n-1}\Gamma\left(\frac{2\mu}{\sigma^{2}}\right)\left(\frac{2\mu}{\sigma^{2}}\right)^{n}}{n!\Gamma\left(n+\frac{2\mu}{\sigma^{2}}\right)}\left(\frac{\frac{2\mu}{\sigma^{2}}}{n+\frac{2\mu}{\sigma^{2}}}\right)\\
&=1-\frac{2\mu}{\sigma^{2}}\Gamma\left(\frac{2\mu}{\sigma^{2}}\right)\sum_{n\geq 0}\frac{\left(\frac{2z}{\sigma^{2}}e^{-x}\right)^{n}}{n!\Gamma\left(n+1+\frac{2\mu}{\sigma^{2}}\right)}\\
&=1-\frac{2\mu}{\sigma^{2}}\left(\frac{2z}{\sigma^{2}}e^{-x}\right)^{-\frac{\mu}{\sigma^{2}}}\Gamma\left(\frac{2\mu}{\sigma^{2}}\right)\sum_{n\geq 0}\frac{\left(\frac{2\left(\frac{2z}{\sigma^{2}}e^{-x}\right)^{1/2}}{2}\right)^{2n+\frac{2\mu}{\sigma^{2}}}}{n!\Gamma\left(n+1+\frac{2\mu}{\sigma^{2}}\right)}\\
&=1-\frac{2\mu}{\sigma^{2}}\left(\frac{2z}{\sigma^{2}}e^{-x}\right)^{-\frac{\mu}{\sigma^{2}}}\Gamma\left(\frac{2\mu}{\sigma^{2}}\right)I_{\frac{2\mu}{\sigma^{2}}}\left(2\left(\frac{2z}{\sigma^{2}}e^{-x}\right)^{1/2}\right),
\end{aligned} \tag{3.19}$$

where

$$I_{\nu}\left(z\right):=\sum_{k\geq 0}\frac{\left(\frac{z}{2}\right)^{2k+\nu}}{k!\Gamma\left(k+1+\nu\right)},\quad z\in\mathbb{C},$$

is the modified Bessel function of the first kind of order $\nu\in\mathbb{R}$. Similarly, it can be shown that

$$\sum_{n\geq 1}\left(-\mu^{-1}z\right)^{n}B_{n}=1-\Gamma\left(\frac{2\mu}{\sigma^{2}}\right)\left(\frac{2z}{\sigma^{2}}\right)^{\frac{1}{2}-\frac{\mu}{\sigma^{2}}}I_{\frac{2\mu}{\sigma^{2}}-1}\left(2\left(\frac{2z}{\sigma^{2}}\right)^{1/2}\right). \tag{3.20}$$

Plugging (3.19) and (3.20) into (3.18) we get

$$H\left(x,z\right)=2\mu\sigma^{-1}\left(2z\right)^{-1/2}e^{\frac{\mu}{\sigma^{2}}x}\frac{I_{\frac{2\mu}{\sigma^{2}}}\left(2\sigma^{-1}\left(2z\right)^{1/2}e^{-\frac{x}{2}}\right)}{I_{\frac{2\mu}{\sigma^{2}}-1}\left(2\sigma^{-1}\left(2z\right)^{1/2}\right)}, \tag{3.21}$$

for all $x\geq 0$ and $z\in\mathbb{C}$ such that $\left|z\right|\mu^{-1}<1$. In particular we obtain

Theorem 3.2.5. *The equality*

$$\mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-X_t}\mathbb{1}_{\{X_t\geq 0\}}dt\right)\right] = \frac{4\mu I_{\frac{2\mu}{\sigma^2}}\left(\frac{\sqrt{8z}}{\sigma}\right)}{\sigma\sqrt{8z}I_{\frac{2\mu}{\sigma^2}-1}\left(\frac{\sqrt{8z}}{\sigma}\right)} \tag{3.22}$$

holds for every $z\in\mathbb{C}$ *such that* $|z|\,\mu^{-1}<1$.

Let F be the distribution function of the random variable

$$\int_0^\infty e^{-X_t}\mathbb{1}_{\{X_t\geq 0\}}\,dt.$$

Let $\left\{j_{\frac{2\mu}{\sigma^2}-1,n}\right\}_{n\geq 1}$ be the increasing sequence of all positive zeros of the Bessel function of the first kind of order $\frac{2\mu}{\sigma^2}-1>-1$, and let

$$J_{\frac{2\mu}{\sigma^2}-1}(z) := \sum_{m\geq 0}\frac{(-1)^m\left(\frac{z}{2}\right)^{2m+\frac{2\mu}{\sigma^2}-1}}{m!\Gamma\left(m+\frac{2\mu}{\sigma^2}\right)}, \quad z\in\mathbb{C}.$$

From the fact that

$$\frac{J_{\frac{2\mu}{\sigma^2}}(z)}{J_{\frac{2\mu}{\sigma^2}-1}(z)} = -2z\sum_{n\geq 1}\left(z^2-j^2_{\frac{2\mu}{\sigma^2}-1,n}\right)^{-1}, \qquad z\in\mathbb{C}\setminus\left\{\pm j_{\frac{2\mu}{\sigma^2}-1,n}\right\}_{n\geq 1},$$

(see [19, formula 7.9(3)]) and the relation $J_\nu(zi)=i^\nu I_\nu(z)$, which holds for all $\nu,z\in\mathbb{R}$, it follows that

$$z^{-1/2}\frac{I_{\frac{2\mu}{\sigma^2}}\left(z^{1/2}\right)}{I_{\frac{2\mu}{\sigma^2}-1}\left(z^{1/2}\right)} = 2\sum_{n\geq 1}\frac{1}{z+j^2_{\frac{2\mu}{\sigma^2}-1,n}}, \qquad z\in\mathbb{C}\setminus\left\{-j^2_{\frac{2\mu}{\sigma^2}-1,n}\right\}_{n\geq 1}. \tag{3.23}$$

Notice that the function

$$z\mapsto z^{-1/2}\frac{I_{2\mu/\sigma^2}(z^{1/2})}{I_{2\mu/\sigma^2-1}(z^{1/2})}, \quad z\in\mathbb{C},$$

has no poles in the region

$$\{w\in\mathbb{C} : \text{Re } w>0,\ |w|<\mu\}.$$

Using an analytic continuation argument we conclude that

$$\mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-X_t}\mathbb{1}_{\{X_t\geq 0\}}dt\right)\right] = \frac{4\mu I_{\frac{2\mu}{\sigma^2}}\left(\frac{\sqrt{8z}}{\sigma}\right)}{\sigma\sqrt{8z}I_{\frac{2\mu}{\sigma^2}-1}\left(\frac{\sqrt{8z}}{\sigma}\right)}, \tag{3.24}$$

for all $z\in\{w\in\mathbb{C} : \text{Re } w>0\}$. In particular we get that the Laplace transform of the random variable

$$\int_0^\infty e^{-X_t}\mathbb{1}_{\{X_t\geq 0\}}dt$$

is given, for all $z \geq 0$, by

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-z\int_0^\infty e^{-X_t}\mathbb{1}_{\{X_t\geq 0\}}dt\right)\right] &= \frac{8\mu}{\sigma^2}\sum_{n\geq 1}\frac{1}{\frac{8z}{\sigma^2}+j^2_{\frac{2\mu}{\sigma^2}-1,n}} \\
&= \frac{8\mu}{\sigma^2}\sum_{n\geq 1}\int_0^\infty e^{-zy}\left[\frac{\sigma^2}{8}e^{-\left(\frac{\sigma^2}{8}j^2_{\frac{2\mu}{\sigma^2}-1,n}\right)y}\right]dy \\
&= \int_0^\infty e^{-zy}\left[\mu\sum_{n\geq 1}e^{-\left(\frac{\sigma^2}{8}j^2_{\frac{2\mu}{\sigma^2}-1,n}\right)y}\right]dy,
\end{aligned}$$

where we used the fact that

$$\sum_{n\geq 1}\frac{1}{j^2_{\nu,n}} = \frac{1}{4(\nu+1)}$$

for any $\nu > -1$ (see [10, formula (32)]). In this way we have proved the following result.

Theorem 3.2.6. *F is absolutely continuous with respect to the Lebesgue measure. Furthermore, if $y \geq 0$ then*

$$F\left(dy\right) = \mu\left(\sum_{n\geq 1}\exp\left\{-\left(\frac{\sigma^2}{8}j^2_{\frac{2\mu}{\sigma^2}-1,n}\right)y\right\}\right)dy. \tag{3.25}$$

3.3 Bounds for the explosion time

In this section we obtain upper and lower bounds for the explosion time of the semilinear system (3.3). For this, we first construct a suitable subsolution of (3.3) by means of the change of variables

$$v_i\left(t,x\right) := \exp\left\{-\kappa_i W_t\right\}u_i\left(t,x\right), \quad t\geq 0, \quad x\in D, \quad i=1,2,$$

which transforms a weak solution (u_1, u_2) of (3.3) into a weak solution of a system of random parabolic PDEs.

As it was shown in Section 1.3 one can see that the vector $\left(v_1\left(t,x\right), v_2\left(t,x\right)\right)$ is a weak solution of the system of RPDEs

$$\begin{aligned}
&\frac{\partial}{\partial t}v_1(t,x) = \left(\Delta_\alpha v_1(t,x) - \frac{\kappa_1^2}{2}v_1(t,x)\right) + e^{-\kappa_1 W_t}G_1\left(e^{\kappa_2 W_t}v_2(t,x)\right),\\
&\frac{\partial}{\partial t}v_2(t,x) = \left(\Delta_\alpha v_2(t,x) - \frac{\kappa_2^2}{2}v_2(t,x)\right) + e^{-\kappa_2 W_t}G_2\left(e^{\kappa_1 W_t}v_1(t,x)\right),\\
&v_i(0,x) = f_i(x) \geq 0, \quad x \in D,\\
&v_i(t,x) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^d \setminus D, \quad i = 1,2,
\end{aligned} \tag{3.26}$$

with the same assumptions as in (3.3). Notice that $v_i(t,\cdot)$ is non-negative on D for each $t \geq 0$ and $i = 1,2$, which follows from the Feynman-Kac representation of (3.26); see e.g. [2]. Hence

$$u_i(t,\cdot) = \exp\{\kappa_i W_t\}v_i(t,\cdot)$$

is also non-negative on D for each $t \geq 0$ and $i = 1,2$. Moreover, it is clear that if τ is the blowup time of system (3.3), then τ is also the blowup time of system (3.26). Let λ and ψ be, respectively, the first eigenvalue and eigenfunction of Δ_α in D, with ψ normalized so that $\int_D \psi(x)\,dx = 1$.

3.3.1 An upper bound for the explosion time

In order to get an upper bound for the explosion time τ, we first show that the function

$$t \mapsto \int_D v(t,x)\,\psi(x)\,dx, \quad t > 0,$$

satisfies the integral inequality inequality

$$\begin{aligned}
\int_D v_i(t,x)\,\psi(x)\,dx \geq{}& \int_D f_i(x)\,\psi(x)\,dx - \left(\lambda + \frac{\kappa_i^2}{2}\right)\int_0^t \int_D v_i(s,x)\,\psi(x)\,dxds\\
&+ \int_0^t e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_s}\left(\int_D v_{3-i}(s,x)\,\psi(x)\,dx\right)^{1+\beta_i} ds,
\end{aligned} \tag{3.27}$$

for $i = 1,2$ and $t > 0$. In fact, since $v_i(t,x)$ is a weak solution of (3.26) and

$$G_i(z) \geq z^{1+\beta_i}, \quad z \geq 0,$$

then in particular we have

$$\begin{aligned}
\int_D v_i(t,x)\,\psi(x)\,dx \geq{}& \int_D f_i(x)\,\psi(x)\,dx + \int_0^t \int_D v_i(s,x)\,\Delta_\alpha\psi(x)\;dx\,ds\\
&- \frac{\kappa_i^2}{2}\int_0^t \int_D v_i(s,x)\,\psi(x)\;dx\,ds\\
&+ \int_0^t \int_D e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_s} v_{3-i}^{1+\beta_i}(s,x)\,\psi(x)\;dx\,ds.
\end{aligned} \tag{3.28}$$

Since v_i and ψ are non-negative in D, by Hölder's inequality we get that

$$\begin{aligned}\int_D v_{3-i}(s,x)\,\psi(x)\,dx &= \int_D v_{3-i}(s,x)\,\psi^{\frac{1}{1+\beta_i}}(x)\,\psi^{\frac{\beta_i}{1+\beta_i}}(x)\,dx \\ &\leq \left(\int_D v_{3-i}^{1+\beta_i}(s,x)\,\psi(x)\,dx\right)^{\frac{1}{1+\beta_i}}. \end{aligned} \tag{3.29}$$

Using the fact that

$$\Delta_\alpha \psi(x) = -\lambda\psi(x) \quad \text{on} \quad D,$$

we finally obtain expression (3.27). Using now a comparison theorem (see e.g. [46, Lemma 1.2]) and (3.27), we deduce that the function h_i determined by the equation

$$\begin{aligned}\frac{d}{dt}h_i(t) &= -\left(\lambda + \frac{\kappa_i^2}{2}\right)h_i(t) + e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_t}h_{3-i}^{1+\beta_i}(t), \\ h_i(0) &= \int_D f_i(x)\,\psi(x)\,dx,\end{aligned}$$

is a subsolution of v_i, $i = 1, 2$. We define

$$m = \lambda + \max_{i=1,2}\left\{\frac{\kappa_i^2}{2}\right\} \quad \text{and} \quad M_t = \min_{i=1,2}\left\{e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_t}\right\}, \quad t \geq 0,$$

and consider the system of random Ordinary Differential Equations (random ODEs)

$$\frac{d}{dt}z_i(t) = -mz_i(t) + M_t z_{3-i}^{1+\beta_i}(t), \quad z_i(0) = h_i(0), \quad i = 1, 2.$$

Using the transformation

$$y_i(t) := e^{mt}z_i(t), \quad t \geq 0, \quad i = 1, 2,$$

it follows that

$$\frac{d}{dt}y_i(t) = e^{-m\beta_i t}M_t y_{3-i}^{1+\beta_i}(t), \quad y_i(0) = h_i(0), \quad i = 1, 2. \tag{3.30}$$

Using again a comparison argument it follows that

$$h_i(t) \geq z_i(t), \quad t \geq 0, \quad i = 1, 2.$$

For $t \geq 0$ we define

$$E(t) = y_1(t) + y_2(t) \quad \text{with} \quad E(0) = \sum_{i=1}^{2}\int_D f_i(x)\,\psi(x)\,dx.$$

We present the main result of this section, where

$$A := \min_{i=1,2}\left\{(1+\beta_i)\kappa_{3-i} - \kappa_i\right\}.$$

Theorem 3.3.1. *Assume that $A > 0$ and let τ be the blow-up time of system (3.3).*

1. *If $\beta_1 = \beta_2$, then $\tau \le \tau'$, where*

$$\tau' = \inf\left\{t \ge 0 : \int_0^t e^{-(AW_s - m\beta_1 s)} \mathbb{1}_{\{AW_s - m\beta_1 s \ge 0\}} ds \ge 2^{\beta_1}\beta_1^{-1}\left(E\left(0\right)\right)^{-\beta_1}\right\}. \tag{3.31}$$

2. *Suppose $\beta_1 > \beta_2 > 0$. Let*

$$\epsilon_0 = \min\left\{1, \left(\frac{h_2\left(0\right)}{A_0^{1/(1+\beta_2)}}\right)^{\beta_1-\beta_2}, \left(\frac{2^{-(1+\beta_2)}\left(E\left(0\right)\right)^{1+\beta_2}}{A_0}\right)^{\frac{\beta_1-\beta_2}{1+\beta_2}}\right\},$$

with

$$A_0 = \left(\frac{1+\beta_1}{1+\beta_2}\right)^{-\frac{1+\beta_2}{\beta_1-\beta_2}} \frac{\beta_1-\beta_2}{1+\beta_1}.$$

Assume that

$$2^{-\beta_2}\epsilon_0\left(E\left(0\right)\right)^{1+\beta_2} - \epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} A_0 > 0, \tag{3.32}$$

and let

$$C_0 = 2^{-\beta_2}\epsilon_0 - \frac{\epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} A_0}{\left(E\left(0\right)\right)^{1+\beta_2}}.$$

Then $\tau \le \tau''$, where

$$\tau'' = \inf\left\{t \ge 0 : \int_0^t e^{-(AW_s - m\beta_2 s)} \mathbb{1}_{\{AW_s - m\beta_2 s \ge 0\}} ds \ge C_0^{-1}\beta_2^{-1}\left(E\left(0\right)\right)^{-\beta_2}\right\}. \tag{3.33}$$

Proof. Recall that

$$x^{1+\beta_1} + y^{1+\beta_1} \ge 2^{-\beta_1}\left(x+y\right)^{1+\beta_1}$$

for all $x, y \in [0, \infty)$. Therefore, from (3.30) we get

$$\frac{d}{dt}E\left(t\right) \ge 2^{-\beta_1}e^{-m\beta_1 t}M_t E^{1+\beta_1}\left(t\right).$$

Using a comparison argument as before, it is clear that I is a subsolution of E, where

$$\frac{d}{dt}I\left(t\right) = 2^{-\beta_1}e^{-m\beta_1 t}M_t I^{1+\beta_1}\left(t\right), \quad I\left(0\right) = E\left(0\right).$$

The solution of this equation is given by

$$I\left(t\right) = \left(I^{-\beta_1}\left(0\right) - 2^{-\beta_1}\beta_1\int_0^t e^{-m\beta_1 s}M_s ds\right)^{-\frac{1}{\beta_1}}, \quad t \in [0, \tau^*),$$

with

$$\tau^* := \inf\left\{t \ge 0 : \int_0^t e^{-m\beta_1 s}M_s ds \ge 2^{\beta_1}\beta_1^{-1}I^{-\beta_1}\left(0\right)\right\}. \tag{3.34}$$

The inequality $\tau \leq \tau^*$ is clear since I is a subsolution of $v_1 + v_2$. There remains to show the inequality $\tau^* \leq \tau'$, where τ' is defined in (3.31). This follows easily from the fact that

$$e^{-m\beta_1 s} M_s \geq e^{-m\beta_1 s} e^{AW_s} \mathbb{1}_{\{W_s \geq 0\}}$$

and

$$\{AW_s - m\beta_1 \geq 0\} \subseteq \{W_s \geq 0\}$$

for all $s \geq 0$. We conclude that

$$\int_0^t e^{-m\beta_1 s} M_s ds \geq \int_0^t e^{-(AW_s - m\beta_1 s)} \mathbb{1}_{\{AW_s - m\beta_1 s \geq 0\}} ds,$$

and the assertion follows. Therefore $\tau \leq \tau'$.

We now prove part (2) of the theorem. According to Young's inequality,

$$xy \leq \frac{\delta^{-p} x^p}{p} + \frac{\delta^q y^q}{q} \tag{3.35}$$

for all $x, y \in [0, \infty)$, $\delta > 0$ and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Taking A_0 as in the statement and setting

$$x = \epsilon, \quad y = y_2^{1+\beta_2}(t), \quad \delta = \left(\frac{1+\beta_1}{1+\beta_2} \right)^{\frac{1+\beta_2}{1+\beta_1}} \quad \text{and} \quad q = \frac{1+\beta_1}{1+\beta_2}$$

in (3.35), it follows that for all $\epsilon > 0$,

$$y_2^{1+\beta_1}(t) \geq \epsilon y_2^{1+\beta_2}(t) - \epsilon^{\frac{1+\beta_1}{\beta_1 - \beta_2}} A_0, \quad t \geq 0.$$

Using (3.30) we get

$$\frac{d}{dt} E(t) \geq e^{-m\beta_1 t} M_t \left(y_1^{1+\beta_2}(t) + \epsilon y_2^{1+\beta_2}(t) - \epsilon^{\frac{1+\beta_1}{\beta_1 - \beta_2}} A_0 \right). \tag{3.36}$$

Suppose $\epsilon \in (0, 1]$. Using Jensen's inequality we conclude that

$$\begin{aligned} y_1^{1+\beta_2}(t) + \epsilon y_2^{1+\beta_2}(t) &\geq 2^{-\beta_2} \left[y_1(t) + \epsilon^{\frac{1}{1+\beta_2}} y_2(t) \right]^{1+\beta_2} \\ &\geq 2^{-\beta_2} \epsilon \left[y_1(t) + y_2(t) \right]^{1+\beta_2} = 2^{-\beta_2} \epsilon E^{1+\beta_2}(t), \end{aligned}$$

hence

$$\frac{d}{dt} E(t) \geq e^{-m\beta_1 t} M_t \left(2^{-\beta_2} \epsilon E^{1+\beta_2}(t) - \epsilon^{\frac{1+\beta_1}{\beta_1 - \beta_2}} A_0 \right).$$

Take ϵ_0 as in the statement. We claim that

$$E(t) \geq E(0) > 0 \text{ for all } t \geq 0.$$

In fact, let J be the solution of the differential equation

$$J'(t) = e^{-m\beta_1 t} M_t f(J(t)), \quad J(0) = E(0),$$

where

$$f(x) := 2^{-\beta_2}\epsilon_0 x^{1+\beta_2} - \epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} A_0, \quad x \geq 0.$$

By comparison $E(t) \geq J(t)$ for all $t \geq 0$, and therefore it suffices to show that

$$J(t) \geq E(0), \quad t \geq 0.$$

Notice that f is increasing and has only one zero at

$$x_0 = (2^{\beta_2}\epsilon_0^{\frac{1+\beta_2}{\beta_1-\beta_2}} A_0)^{\frac{1}{1+\beta_2}} > 0,$$

with $x_0 < E(0)$ due to (3.32). Let

$$T = \inf\{t > 0 : J(t) < E(0)\}.$$

Then $T > 0$ because J is strictly increasing around 0, and $J(t) \geq E(0)$ for all $t \in (0, T)$. Suppose that $T < \infty$. Being J continuous on $[0, T]$ and differentiable on $(0, T)$, Rolle's theorem yields that $J'(c) = 0$ for some $c \in (0, T)$. Hence $J(c) = x_0$ which implies that $x_0 \geq E(0)$. This contradiction says that $T = \infty$ and

$$E(t) \geq E(0) \text{ for all } t \geq 0,$$

which proves the claim. Therefore,

$$\frac{d}{dt}E(t) \geq e^{-m\beta_1 t} M_t E^{1+\beta_2}(t)\left[2^{-\beta_2}\epsilon_0 - \frac{\epsilon_0^{\frac{1+\beta_1}{\beta_1-\beta_2}} A_0}{(E(0))^{1+\beta_2}}\right].$$

Let C_0 be as in the statement and let I be the solution of the equation

$$\frac{d}{dt}I(t) = e^{-m\beta_1 t} M_t I^{1+\beta_2}(t)\, C_0, \quad t \in [0, \tau^{**}); \quad I(0) = E(0),$$

where τ^{**} will be defined below. Then $I(t) \leq E(t)$. The expression for I is given in this case by

$$I(t) = \left(I^{-\beta_2}(0) - C_0\beta_2 \int_0^t e^{-m\beta_1 s} M_s ds\right)^{-\frac{1}{\beta_2}},$$

for all $t \in [0, \tau^{**})$, with τ^{**} given by

$$\tau^{**} = \inf\left\{t \geq 0 : \int_0^t e^{-m\beta_1 s} M_s ds \geq C_0^{-1}\beta_2^{-1} I^{-\beta_2}(0)\right\}. \tag{3.37}$$

Taking τ'' as in (3.33) and proceeding as in the proof of Part 1, we get $\tau \leq \tau^{**} \leq \tau''$. □

Remark 3.3.2. When $\kappa_1 = \kappa_2 = 0$ and $\beta_1 = \beta_2 > 0$, from the inequality $\tau \le \tau^*$ it follows that

$$\begin{aligned}\mathbb{P}(\tau < \infty) &\ge \mathbb{P}\left(\frac{1}{\lambda} > 2^{\beta_1} I^{-\beta_1}(0)\right) \ge \mathbb{P}\left(\frac{1}{\lambda} > \left(\min_{i=1,2}\left\{\int_D f_i(x)\,\psi(x)\,dx\right\}\right)^{-\beta_1}\right) \\ &= \begin{cases} 1 & \text{if } \lambda^{\frac{1}{\beta_1}} < \min_{i=1,2}\left\{\int_D f_i(x)\,\psi(x)\,dx\right\}, \\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

which is the deterministic result given in [33].

3.3.2 A lower bound for the explosion time

Suppose that $\{Y_t; t \ge 0\}$ is a spherically symmetric α-stable process with infinitesimal generator Δ_α. Let

$$\tau^D := \inf\{t > 0 : \; Y_t \notin D\}$$

and consider the killed process $\left\{Y_t^D, \; t \ge 0\right\}$ given by

$$Y_t^D = \begin{cases} Y_t & \text{if } \; t < \tau^D \\ \partial & \text{if } \; t \ge \tau^D, \end{cases}$$

where ∂ is a cemetery point. Let $T \ge 0$ be a random time. Recall that a pair of $\mathcal{F}_t$-adapted random fields

$$(v_1(t,x), v_2(t,x)), \quad x \in D, \quad t \ge 0,$$

is a mild solution of (3.26) in the interval $[0,T]$ if

$$v_i(t,x) = e^{-\frac{\kappa_i^2}{2}t} P_t^D f_i(x) + \int_0^t e^{-\kappa_i W_r} e^{-\frac{\kappa_i^2}{2}(t-r)} P_{t-r}^D \left[G_i\left(e^{\kappa_{3-i} W_r} v_{3-i}(r,x)\right)\right] dr, \tag{3.38}$$

$\mathbb{P}$-a.s. for all $t \in (0,T]$, $i = 1,2$, where $\left\{P_t^D, \; t \ge 0\right\}$ is the semigroup of the process $\left\{Y_t^D, t \ge 0\right\}$. In what follows we will assume that G_i is a locally Lipschitz positive function such that

$$G_i(z) \le z^{1+\beta_i}, \quad z \ge 0, \quad i = 1,2. \tag{3.39}$$

Moreover, we set

$$A = \min_{i=1,2}\{(1+\beta_i)\kappa_{3-i} - \kappa_i\} \quad \text{and} \quad B = \max_{i=1,2}\{(1+\beta_i)\kappa_{3-i} - \kappa_i\}.$$

Theorem 3.3.3. *Let* $\beta = \max_{i=1,2}\{\beta_i\}$ *and*

$$\phi(t) = e^{-(\kappa_1 \wedge \kappa_2)^2 t/2} \max_{i=1,2}\left\{\sup_{s \in [0,t]} \left\| P_s^D f_i \right\|_\infty\right\}, \quad t \ge 0.$$

Assume that $A > 0$. Then there exists a mild solution (v_1, v_2) of (3.26) such that

$$v_i(t,x) \le \phi(t)B(t),$$

for all $0 \le t < \tau_*$, $x \in D$ and $i = 1, 2$, where

$$B(t) = \left(1 - \beta \int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2} \left\{\phi^{\beta_i}(r)\right\} dr\right)^{-\frac{1}{\beta}}$$

and

$$\tau_* = \inf \left\{ t \ge 0 : \int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2} \left\{\phi^{\beta_i}(r)\right\} dr \ge \frac{1}{\beta} \right\}. \tag{3.40}$$

Proof. Notice that $B(0) = 1$ and

$$\frac{d}{dt} B(t) = \left(e^{AW_t} \vee e^{BW_t}\right) \max_{i=1,2} \left\{\phi^{\beta_i}(t)\right\} B^{1+\beta}(t), \quad t > 0,$$

hence

$$B(t) = 1 + \int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2} \left\{\phi^{\beta_i}(r)\right\} B^{1+\beta}(r)\, dr.$$

Now let $V : [0, \infty) \times D \to \mathbb{R}$ be a non-negative continuous function such that

$$V(t, \cdot) \in C_0(D), \quad t \ge 0,$$

and satisfying

$$V(t,x) \le \phi(t) B(t), \quad t \in [0, \tau_*), \quad x \in D. \tag{3.41}$$

Define the operator F_i by

$$F_i(V(t,x)) := e^{-\frac{\kappa_i^2}{2}t} P_t^D f_i(x) + \int_0^t e^{-\kappa_i W_r} e^{-\frac{\kappa_i^2}{2}(t-r)} P_{t-r}^D \left[G_i\left(e^{\kappa_{3-i} W_r} V(r,x)\right)\right] dr,$$

for $i = 1, 2$. Using (3.39) and that the semigroup $\left\{P_t^D, t \ge 0\right\}$ preserves positivity we get

$$\begin{aligned} F_i(V(t,x)) &\le \phi(t) + \int_0^t e^{((1+\beta_i)\kappa_{3-i} - \kappa_i)W_r} e^{-\frac{(\kappa_1 \wedge \kappa_2)^2}{2}(t-r)} P_{t-r}^D \left[V^{1+\beta_i}(r,x)\right] dr \\ &\le \phi(t) + \int_0^t e^{((1+\beta_i)\kappa_{3-i} - \kappa_i)W_r} e^{-\frac{(\kappa_1 \wedge \kappa_2)^2}{2}(t-r)} \phi^{1+\beta_i}(r)\, B^{1+\beta_i}(r)\, dr, \end{aligned}$$

where we have used (3.41) to obtain the last inequality. Notice that if $t \in [0, \tau_*)$ and $r \in [0, t]$ then

$$e^{-\frac{(\kappa_1 \wedge \kappa_2)^2}{2}(t-r)} \phi(r) \le \phi(t),$$

and since $B(t) \ge 1$,

$$B^{1+\beta_i}(r) \le B^{1+\beta}(r), \quad 0 \le r \le t.$$

Therefore, for all $t \in [0, \tau_*)$ and $x \in D$,

$$\begin{aligned} F_i(V(t,x)) &\leq \phi(t)\left[1 + \int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2}\left\{\phi^{\beta_i}(r)\right\} B^{1+\beta}(r)\, dr\right] \\ &= \phi(t)B(t). \end{aligned}$$

Now we will define increasing sequences which will converge to the mild solution of (3.26). Let

$$v_{1,0}(t,x) = e^{-\frac{\kappa_1^2}{2}t} P_t^D f_1(x), \quad v_{2,0}(t,x) = e^{-\frac{\kappa_2^2}{2}t} P_t^D f_2(x), \quad (t,x) \in [0,\tau_*) \times D,$$

and for any $n \geq 0$ define

$$v_{1,n+1}(t,x) = F_1(v_{2,n}(t,x)), \quad v_{2,n+1}(t,x) = F_2(v_{1,n}(t,x)),$$

for $(t,x) \in [0,\tau_*) \times D$. To prove that $(v_{1,n}(t,x))_{n\geq 0}$ and $(v_{2,n}(t,x))_{n\geq 0}$ are increasing for all $t \in [0,\tau_*)$ and $x \in D$, note that

$$\begin{aligned} v_{i,0}(t,x) &\leq e^{-\frac{\kappa_i^2}{2}t} P_t^D f_i(x) + \int_0^t e^{-\kappa_i W_r} e^{-\frac{\kappa_i^2}{2}(t-r)} P_{t-r}^D \left[G_i\left(e^{\kappa_{3-i}W_r} v_{3-i,0}(r,x)\right)\right] dr \\ &= v_{i,1}(t,x), \quad i = 1,2. \end{aligned}$$

Suppose that, for some $n \geq 0$,

$$v_{i,n} \geq v_{i,n-1}, \quad i = 1,2.$$

Then

$$v_{i,n+1}(t,x) = F_i(v_{3-i,n}(t,x)) \geq F_i(v_{3-i,n-1}(t,x)) = v_{i,n}(t,x)$$

for all $(t,x) \in [0,\tau_*) \times D$, where we have used the monotonicity of F_i, $i = 1,2$. By induction, this shows that both sequences $(v_{1,n}(t,x))_{n\geq 0}$ and $(v_{2,n}(t,x))_{n\geq 0}$ are increasing. Therefore the limits

$$v_1(t,x) := \lim_{n\to\infty} v_{1,n}(t,x) \quad \text{and} \quad v_2(t,x) := \lim_{n\to\infty} v_{2,n}(t,x)$$

exist for all $t \in [0,\tau_*)$ and $x \in D$. From the Monotone Convergence Theorem we conclude that

$$v_i(t,x) = F_i v_{3-i}(t,x), \quad i = 1,2,$$

for all $t \in [0,\tau_*)$ and $x \in D$. Thus, (v_1, v_2) is a mild solution of (3.26). Moreover,

$$v_i(t,x) \leq \phi(t)B(t), \quad i = 1,2,$$

for all $t \in [0,\tau_*)$ and $x \in D$, and the result follows. □

Corollary 3.3.4. *Under the assumptions of Theorem 3.3.3, if*

$$\beta \int_0^\infty \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2} \left\{\phi^{\beta_i}(r)\right\} dr < 1,$$

then the mild solution (v_1, v_2) of (3.26) obtained in Theorem 3.3.3 is global.

3.4 Bounds for the probability of explosion in finite time

Throughout this section we make the following assumptions:

1. $\beta_1 = \beta_2 > 0$,
2. the initial values in (3.3) are of the form

$$f_i(x) = L_i \psi(x), \quad x \in D, \quad i = 1, 2,$$

   where L_1 and L_2 are positive constants,
3. $G(z) = z^{1+\beta_1}$, $z \geq 0$.

As above we denote

$$A = \min_{i=1,2} \left\{(1+\beta_1)\kappa_{3-i} - \kappa_i\right\}, \quad B = \max_{i=1,2} \left\{(1+\beta_1)\kappa_{3-i} - \kappa_i\right\},$$

and assume that $A > 0$. We also abbreviate $\Lambda := \frac{(\kappa_1 \wedge \kappa_2)^2}{2}$.

3.4.1 An upper bound for the probability of blowup in finite time

Consider the random variable τ_{**} defined by

$$\tau_{**} := \inf\left\{t \geq 0 : \int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) e^{-\Lambda\beta_1 r} dr \geq \frac{1}{\beta_1 \|\psi\|_\infty^{\beta_1}} \min_{i=1,2}\left\{\frac{1}{L_i^{\beta_1}}\right\}\right\}. \tag{3.42}$$

It is easy to see that $\tau_{**} \leq \tau_*$. Furthermore, noticing that

$$\begin{aligned}
&\int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) e^{-\Lambda\beta_1 r} dr \\
&= \int_0^t e^{AW_r - \Lambda\beta_1 r} \mathbb{1}_{\{W_r<0\}} dr + \int_0^t e^{BW_r - \Lambda\beta_1 r} \mathbb{1}_{\{W_r \geq 0\}} dr \\
&\leq \int_0^\infty e^{-\Lambda\beta_1 r} dr + \int_0^t e^{BW_r - \Lambda\beta_1 r} dr = \frac{1}{\Lambda\beta_1} + \int_0^t e^{BW_r - \Lambda\beta_1 r} dr,
\end{aligned}$$

it follows that

$$\tau_{\prime\prime} := \inf\left\{t \geq 0 : \frac{1}{\Lambda\beta_1} + \int_0^t e^{BW_r - \Lambda\beta_1 r} dr \geq \frac{1}{\beta_1 \|\psi\|_\infty^{\beta_1}} \min_{i=1,2}\left\{\frac{1}{L_i^{\beta_1}}\right\}\right\} \tag{3.43}$$

satisfies $\tau_{\prime\prime} \leq \tau_{**}$ as long as $A > 0$.

Theorem 3.4.1. *Assume that*

$$\frac{\|\psi\|_\infty^{\beta_1}}{\Lambda} < \min_{i=1,2}\left\{\frac{1}{L_i^{\beta_1}}\right\}.$$

Then

$$\mathbb{P}(\tau < \infty) \leq \frac{\gamma\left(\frac{2\Lambda\beta_1}{B^2}, \frac{2}{B^2\left(\frac{1}{\beta_1\|\psi\|_\infty^{\beta_1}} \min_{i=1,2}\left\{\frac{1}{L_i^{\beta_1}}\right\} - \frac{1}{\Lambda\beta_1}\right)}\right)}{\Gamma\left(\frac{2\Lambda\beta_1}{B^2}\right)}. \tag{3.44}$$

Proof. From the relation $\tau_{\prime\prime} \leq \tau$ and the continuity of paths of Brownian motion, it follows that

$$\begin{aligned}
\mathbb{P}(\tau < \infty) &\leq \mathbb{P}(\tau_{\prime\prime} < \infty) \\
&= 1 - \mathbb{P}\left(\int_0^\infty e^{BW_r - \Lambda\beta_1 r} dr \leq \frac{1}{\beta_1 \|\psi\|_\infty^{\beta_1}} \min_{i=1,2}\left\{\frac{1}{L_i^{\beta_1}}\right\} - \frac{1}{\Lambda\beta_1}\right) \\
&= \mathbb{P}\left(\int_0^\infty e^{BW_r - \Lambda\beta_1 r} dr > \frac{1}{\beta_1 \|\psi\|_\infty^{\beta_1}} \min_{i=1,2}\left\{\frac{1}{L_i^{\beta_1}}\right\} - \frac{1}{\Lambda\beta_1}\right).
\end{aligned}$$

The result follows from (3.6). □

Remark 3.4.2. Notice that $\mathbb{P}(\tau < \infty) < \delta$ for any given $\delta > 0$ provided that the positive constants L_1, L_2 are sufficiently small, i.e., for sufficiently small initial conditions, the system (3.3) explodes in finite time with small probability.

3.4.2 Lower bound for the probability of explosion in finite time

Theorem 3.4.3. *If* $m = \lambda + \frac{1}{2}(\kappa_1 \vee \kappa_2)^2$ *then*

$$\mathbb{P}(\tau < \infty) \geq \frac{8m\beta_1}{A^2} \sum_{n\geq 1} \frac{\exp\left\{-\frac{A^2 2^{\beta_1}}{8\beta_1 (L_1+L_2)^{\beta_1} \|\psi\|_2^{2\beta_1}} j^2_{\frac{2m\beta_1}{A^2}-1,n}\right\}}{j^2_{\frac{2m\beta_1}{A^2}-1,n}}. \tag{3.45}$$

Proof. From the relation $\tau \leq \tau'$, the continuity of paths of Brownian motion and Theorem 3.2.6, it follows that

$$\begin{aligned}
\mathbb{P}(\tau < \infty) &\geq \mathbb{P}\left(\tau' < \infty\right) \\
&= \mathbb{P}\left(\int_0^\infty e^{-(AW_s - m\beta_1 s)} \mathbb{1}_{\{AW_s - m\beta_1 s\}} ds \geq \frac{2^{\beta_1}}{\beta_1 (L_1 + L_2)^{\beta_1} \|\psi\|_2^{2\beta_1}}\right) \\
&= \int_{\frac{2^{\beta_1}}{\beta_1 (L_1+L_2)^{\beta_1} \|\psi\|_2^{2\beta_1}}}^\infty m\beta_1 \sum_{n\geq 1} \exp\left\{-\left(\frac{A^2}{8} j^2_{\frac{2m\beta_1}{A^2}-1,n}\right) y\right\} dy. \\
&= \frac{8m\beta_1}{A^2} \sum_{n\geq 1} \frac{\exp\left\{-\frac{A^2 2^{\beta_1}}{8\beta_1 (L_1+L_2)^{\beta_1} \|\psi\|_2^{2\beta_1}} j^2_{\frac{2m\beta_1}{A^2}-1,n}\right\}}{j^2_{\frac{2m\beta_1}{A^2}-1,n}},
\end{aligned}$$

where we used the Monotone Convergence Theorem to obtain the last equality. □

Remark 3.4.4. Notice that for sufficiently large L_1 and L_2, the relation

$$\frac{8m\beta_1}{A^2} \sum_{n\geq 1} \frac{\exp\left\{-\frac{A^2 2^{\beta_1}}{8\beta_1 (L_1+L_2)^{\beta_1} \|\psi\|_2^{2\beta_1}} j^2_{\frac{2m\beta_1}{A^2}-1,n}\right\}}{j^2_{\frac{2m\beta_1}{A^2}-1,n}} \sim 1 - \frac{\sqrt{8} m \beta_1^{1/2} 2^{\beta_1/2}}{A\pi (L_1 + L_2)^{\beta_1/2} \|\psi\|_2^{\beta_1}}$$

holds; see [10, formula (39)]. Therefore $\mathbb{P}(\tau < \infty) > 1 - \epsilon$ for any given $\epsilon > 0$ provided that the positive constants L_1, L_2 are sufficiently large, i.e., for sufficiently large initial conditions, the solution of system (3.3) explodes in finite time with high probability.

Chapter 4

Blowup in finite time in $L^p(D)$-norm of a system of SPDEs

4.1 Introduction

This chapter addresses the problem of explosion in finite time in $L^p(D)$-norm, of the coupled system of reaction-diffusion SPDEs

$$\begin{aligned} du_1(t,x) &= \left(\Delta_{\alpha_1} u_1(t,x) + u_2^{1+\beta_1}(t,x)\right) dt + \kappa_1 u_1(t,x)\, dW_t, \\ du_2(t,x) &= \left(\Delta_{\alpha_2} u_2(t,x) + u_1^{1+\beta_2}(t,x)\right) dt + \kappa_2 u_2(t,x)\, dW_t, \quad t>0, \quad x \in D, \\ u_i(0,x) &= f_i(x) \geq 0, \quad x \in D, \\ u_i(t,x) &= 0, \quad t \geq 0, \quad x \in \mathbb{R}^d \setminus D, \quad i=1,2, \end{aligned} \tag{4.1}$$

where D is an open domain in $\mathbb{R}^d$, Δ_{α_i} is the fractional power of the Laplacian introduced in Chapter 3, $\beta_i > 0$, $\kappa_i \in \mathbb{R}$ and $\alpha_i \in (0,2)$ are constants, $f_i \in C_b^2(D)$, $i=1,2$, and $\{W_t; t \geq 0\}$ is a Brownian motion defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In this chapter we consider another notion of explosion which was treated in [8] (see also [9] and [32]), where it is proved explosion in finite time in $L^p(D)$-norm of the positive solution of a single SPDE of the form (4.1), with $\alpha = 2$. We say that a function $u : \mathbb{R}_+ \times \Omega \times D \to \mathbb{R}$ explodes in $L^p(D)$-norm if there exists $T_p \in (0,\infty]$ such that

$$\lim_{t \to T_p^-} \mathbb{E}\left[\|u(t,\cdot)\|_{L^p(D)}\right] = \infty. \tag{4.2}$$

When $T_p < \infty$, we say that u explodes in finite time in $L^p(D)$-norm and the infimum of such numbers T_p satisfying (4.2) is called the explosion time of u. We say that a solution of (4.1) explodes in $L^p(D)$-norm if at least one of u_1 or u_2 explodes in the $L^p(D)$-norm. In this case case, $T_p := \min\left\{T_p^1, T_p^2\right\}$ is called the explosion time of system (4.1), where T_p^i is the explosion time of u_i, $i=1,2$. We say that a solution of (4.1) explodes in finite time in $L^p(D)$-norm if at least one of u_1 or u_2 explodes in finite time in $L^p(D)$-norm.

This chapter is organized as follows. In Section 4.2 we recall the notions of weak and mild solutions and we establish their equivalence under our assumptions. The concept of weak solution for the system (4.1) was introduced in Definition 1.3.1 for the case $\alpha_1=\alpha_2$. We need a definition for the general case $\alpha_1,\alpha_2\in(0,2]$, and hence to establish a theorem on existence of weak (or mild) solutions. We make a suitable change of variables in system (4.1) in order to obtain a related system of random PDEs given in (4.4). This change of variables allows to prove, in particular, that there exists a positive weak solution of system (4.1), by proving the existence of a weak solution of the related system (4.4). Moreover, following the results in Section 1.3, we formulate a general result on existence of a local mild solution of the related system (4.4) and we prove that this local mild solution is also a weak solution of the system (4.4). Some of the results on equivalence of solutions of PDEs we use in Section 4.2 are based on [1] and [27].

Further, we find conditions ensuring finite-time blowup of system (4.1) in $L^p(D)$-norm for all $p\in[1,\infty)$. To achieve this, in Section 4.3.1 we adopt the methodology of Chapter 3 to prove the existence of an explosive weak solution of (4.1). We first consider the case $\alpha_1=\alpha_2$. In Theorem 4.3.1 we provide conditions which imply explosion in finite time in $L^p(D)$-norm of system (4.1), where we distinguish the cases $\beta_1=\beta_2$ and $\beta_1>\beta_2$.

The case of $\alpha_1>\alpha_2$ is treated in Section 4.3.2. We follow the same approach as in the case $\alpha_1=\alpha_2$, but first we must obtain suitable upper and lower bounds of the first eigenfunction ψ_i of $-\Delta_{\alpha_i}$, $i=1,2$, with the aim of getting a useful explosive subsolution. The estimations needed for this case are of the form

$$C_{1,i}\mathbb{E}_x[\tau_D^{(i)}]\leq\psi_i(x)\leq C_{2,i}\mathbb{E}_x[\tau_D^{(i)}],\quad x\in D,\quad i=1,2,$$

where $C_{1,i},C_{2,i}$, $i=1,2$, are positive constants depending on the domain D and the fractional power $\alpha_i\in(0,2]$, and $\tau_D^{(i)}$ is the exit time from D of a spherically symmetric α_i-stable process with infinitesimal generator Δ_{α_i}; see [4, Theorem 4.4]. The expression $\mathbb{E}_x[\tau_D^{(i)}]$ can be explicitly calculated when D is an open ball $B(0,r)\subseteq\mathbb{R}^d$ with centre at the origin and an arbitrary radius $r\in(0,\infty)$. For the case of $D=B(0,r)$, we give in Theorem 4.3.2 an upper bound for the explosion time of (4.1). We consider the case $\alpha_1>\alpha_2$, and we distinguish two cases: $\beta_1=\beta_2$ and $\beta_1>\beta_2$. In both cases we must assume that $\beta_1>\frac{\alpha_1-\alpha_2}{\alpha_2}$ and some restrictions on the initial conditions to ensure that the upper bound for the explosion time is finite.

4.2 Weak and mild solutions

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t; t \geq 0\}, \mathbb{P})$ be a filtered probability space.

Definition 4.2.1. Let $\tau \in [0, \infty]$ be a stopping time. A pair of $\mathcal{F}_t$-adapted processes

$$\{u_i(t,x) : t \geq 0,\, x \in D\}, \quad i = 1, 2,$$

is a weak solution of (4.1) on $[0,\tau)$ if for all $\phi_i \in C_b^2(\mathbb{R}^d)$ vanishing on D^c, $t \in [0,\tau)$ and $i = 1, 2$,

$$\begin{aligned}
&\int_D u_i(t,x)\,\phi_i(x)\,dx \\
&= \int_D f_i(x)\,\phi_i(x)\,dx + \int_0^t \int_D \Big(u_i(s,x)\,\Delta_{\alpha_i}\phi_i(x) + u_{3-i}^{1+\beta_i}(s,x)\,\phi_i(x)\Big)\,dxds \\
&+ \kappa_i \int_0^t \int_D u_i(s,x)\,\phi_i(x) dx dW_s, \quad \mathbb{P}\text{-a.s.}
\end{aligned} \tag{4.3}$$

Definition 4.2.2. Let $\tau \in [0, \infty]$ be a stopping time. The vector $(u_1(t,x), u_2(t,x))$ is a mild solution of (4.1) on $[0,\tau)$ if for all $t \in [0,\tau)$, $\mathbb{P}$-c.s. and $i = 1, 2$,

$$u_i(t,x) = P_{t,i}^D f_i(x) + \int_0^t P_{t-r,i}^D \Big(u_{3-i}^{1+\beta_i}(r,x)\Big)\,dr + \kappa_i \int_0^t P_{t-r,i}^D\,(u_i(r,x))\,dWr,$$

where $\{P_{t,i}^D; t \geq 0\}$ is the semigroup defined in (1.1), having infinitesimal generator Δ_{α_i}.

Proceeding as in Section 1.3, it is easy to show that $\{u_i(t,x) : t \geq 0, x \in D\}$, $i = 1, 2$, is a weak solution of (4.1) on $[0,\tau)$ if and only if the vector $(v_1(t,x), v_2(t,x))$ given by

$$v_i(t,x) := \exp\{-\kappa_i W_t\}\,u_i(t,x), \quad t \in [0,\tau), \quad x \in D, \quad i = 1, 2,$$

is a weak solution of the system of random parabolic PDEs

$$\begin{aligned}
\frac{\partial}{\partial t} v_1(t,x) &= \left(\Delta_{\alpha_1} v_1(t,x) - \frac{\kappa_1^2}{2} v_1(t,x)\right) + e^{((1+\beta_1)\kappa_2 - \kappa_1)W_t} v_2^{1+\beta_1}(t,x), \\
\frac{\partial}{\partial t} v_2(t,x) &= \left(\Delta_{\alpha_2} v_2(t,x) - \frac{\kappa_2^2}{2} v_2(t,x)\right) + e^{((1+\beta_2)\kappa_1 - \kappa_2)W_t} v_1^{1+\beta_2}(t,x),
\end{aligned} \tag{4.4}$$

$$v_i(0,x) = f_i(x) \geq 0, \quad x \in D,$$

$$v_i(t,x) = 0, \quad t \geq 0, \quad x \in \mathbb{R}^d \setminus D, \quad i = 1, 2,$$

with the same assumptions as in (4.1), i.e., if τ is a stopping time, then for all $\phi_i \in C_b^2(\mathbb{R}^d)$ vanishing on D^c, $t \in [0,\tau)$ and $i = 1, 2$,

$$\int_D v_i(t,x)\,\phi_i(x)\,dx = \int_D f_i(x)\,\phi_i(x)\,dx + \int_0^t \int_D v_i(s,x)\left(\Delta_{\alpha_i}\phi_i(x) - \frac{\kappa_i^2}{2}\phi_i(x)\right)dxds$$
$$+ \int_0^t \int_D e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_t} v_{3-i}^{1+\beta_i}(t,x)\,\phi_i(x)dxds, \quad \mathbb{P}\text{-a.s.}$$

The following theorem establishes the existence of a local mild solution of system (4.4). Its proof is very similar to that of Theorem 3.3.3, and therefore it is omitted. As in Section 3.3.2, we let $A = \min_{i=1,2}\{(1+\beta_i)\kappa_{3-i} - \kappa_i\}$ and $B = \max_{i=1,2}\{(1+\beta_i)\kappa_{3-i} - \kappa_i\}$. The way of choosing the functions $\phi(t)$ and $B(t)$, $t \geq 0$, in the following theorem, is inspired in the preparation paper by M. J. Ceballos-Lira and A. Pérez [6].

Theorem 4.2.3. *Let $\beta = \max_{i=1,2}\{\beta_i\}$. Assume that $A > 0$ and let*

$$\phi(t) = e^{-(\kappa_1\wedge\kappa_2)^2 t/2} \max_{i=1,2}\left\{\sup_{s\in[0,t]} \left\|P_{s,i}^D f_i\right\|_\infty\right\}, \quad t \geq 0.$$

Then there exists a mild solution $(v_1(t,x), v_2(t,x))$ of system (4.4), such that

$$v_i(t,x) \leq \phi(t)B(t),$$

for all $0 \leq t < \tau$, $x \in D$ and $i = 1, 2$, where

$$B(t) = \left(1 - \beta\int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2}\left\{\phi^{\beta_i}(r)\right\} dr\right)^{-\frac{1}{\beta}},$$

and

$$\tau = \inf\left\{t \geq 0 : \int_0^t \left(e^{AW_r} \vee e^{BW_r}\right) \max_{i=1,2}\left\{\phi^{\beta_i}(r)\right\} dr \geq \frac{1}{\beta}\right\}.$$

Now we establish the equivalence between mild and weak solutions of system (4.4). The proof is an adaptation of [27, Theorems 4 and 5]. First, we prove two auxiliary lemmas. For simplicity of the notation, $S_{t,i} := e^{-\frac{\kappa_i^2}{2}t}P_{t,i}^D$ is the semigroup of $\Delta_{\alpha_i} - \frac{\kappa_i^2}{2}$, which acts on $C_b^2(D)$, for $i = 1, 2$, and the dot product in $L^2(D)$ is denoted by

$$\langle f, g\rangle := \int_D f(x)g(x)dx, \quad f, g \in L^2(D).$$

Lemma 4.2.4. *For all $t \geq 0$ and $\phi \in C_b^2(D)$,*

$$\langle S_{t,i}f_i, \phi\rangle = \langle f_i, \phi\rangle + \int_0^t \left\langle S_{r,i}f_i, \left(\Delta_{\alpha_i} - \frac{\kappa_i^2}{2}\right)\phi\right\rangle dr, \quad i = 1, 2.$$

Proof. Let $\phi \in C_b^2(D)$. Then for all $h>0$ we have that

$$\begin{aligned} &h^{-1}\left[\langle S_{t+h,i}f_i,\phi\rangle - \langle S_{t,i}f_i,\phi\rangle\right] \\ &= h^{-1}\left[\langle S_{h,i}\left(S_{t,i}f_i\right),\phi\rangle - \langle S_{t,i}f_i,\phi\rangle\right] \\ &= h^{-1}\left[\langle S_{t,i}f_i, S_{h,i}\phi-\phi\rangle\right] \to \left\langle S_{t,i}f_i, \left(\Delta_{\alpha_i}-\frac{\kappa_i^2}{2}\right)\phi\right\rangle, \quad \text{as } h\to 0, \end{aligned}$$

where the limit is taken in $L^2(D)$. Therefore for $i=1,2$

$$\frac{d}{dt}\langle S_{t,i}f_i,\phi\rangle = \left\langle S_{t,i}f_i, \left(\Delta_{\alpha_i}-\frac{\kappa_i^2}{2}\right)\phi\right\rangle,$$

which implies that, for $i=1,2$,

$$\langle S_{t,i}f_i,\phi\rangle - \langle f_i,\phi\rangle = \int_0^t \left\langle S_{r,i}f_i, \left(\Delta_{\alpha_i}-\frac{\kappa_i^2}{2}\right)\phi\right\rangle dr.$$

□

Lemma 4.2.5. *For all $t\geq 0$ and $\phi\in C_b^2(D)$,*

$$\begin{aligned} &\int_0^t \left\langle S_{t-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right),\phi\right\rangle dr \\ &= \int_0^t \left\langle e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot),\phi\right\rangle dr \\ &+ \int_0^t\int_0^r \left\langle S_{r-s,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_s} v_{3-i}^{1+\beta_i}(s,\cdot)\right), \left(\Delta_{\alpha_i}-\frac{\kappa_i^2}{2}\right)\phi\right\rangle ds dr, \quad i=1,2. \end{aligned}$$

Proof. Let $\phi\in C_b^2(D)$. Then for all $h>0$ and $i=1,2$,

$$\begin{aligned} &h^{-1}\left[\int_0^{t+h} \left\langle S_{t+h-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right),\phi\right\rangle dr\right. \\ &\left.- \int_0^t \left\langle S_{t-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right),\phi\right\rangle dr\right] \\ &= h^{-1}\left[\int_t^{t+h} \left\langle S_{t+h-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right),\phi\right\rangle dr\right. \\ &+ \int_0^t \left\langle S_{t-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right),S_{h,i}\phi\right\rangle dr \\ &\left.- \int_0^t \left\langle S_{t-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right),\phi\right\rangle dr\right] \\ &= h^{-1}\int_t^{t+h} \left\langle S_{t+h-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right),\phi\right\rangle dr \\ &+ \int_0^t \left\langle S_{t-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right),h^{-1}\left(S_{h,i}\phi-\phi\right)\right\rangle dr \\ &\to \left\langle e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_t} v_{3-i}^{1+\beta_i}(t,\cdot),\phi\right\rangle \\ &+ \int_0^t \left\langle S_{t-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right), \left(\Delta_{\alpha_i}-\frac{\kappa_i^2}{2}\right)\phi\right\rangle dr, \quad \text{as } h\to 0, \end{aligned}$$

which implies that

$$\frac{d}{dt}\int_0^t \left\langle S_{t-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right), \phi\right\rangle dr$$
$$= \left\langle e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_t} v_{3-i}^{1+\beta_i}(t,\cdot), \phi\right\rangle$$
$$+ \int_0^t \left\langle S_{t-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right), \left(\Delta_{\alpha_i} - \frac{\kappa_i^2}{2}\right)\phi\right\rangle dr, \quad i = 1,2,$$

and the result follows as in the proof of the previous lemma. □

Theorem 4.2.6. *If a vector $(v_1(t,x), v_2(t,x))$ is a mild solution of (4.4) in some interval $(0,\tau)$, then $(v_1(t,x), v_2(t,x))$ is a weak solution of (4.4) in $(0,\tau)$.*

Proof. Assume that $(v_1(t,x), v_2(t,x))$ is a mild solution on $(0,\tau)$. Then for all $t \in (0,\tau)$,

$$v_i(t,x) = S_{t,i}f_i(x) + \int_0^t S_{t-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,x)\right) dr.$$

Let $\phi_i \in C_b^2(D)$, $i = 1,2$. Taking the dot product in $L^2(D)$ we obtain

$$\langle v_i(t,\cdot), \phi_i\rangle = \langle S_{t,i}f_i, \phi_i\rangle + \int_0^t \left\langle S_{t-r,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot)\right), \phi_i\right\rangle dr,$$

and using Lemma 4.2.4 and Lemma 4.2.5 we obtain for all $t \in (0,\tau)$,

$$\langle v_i(t,\cdot), \phi_i\rangle$$
$$= \langle f_i, \phi_i\rangle + \int_0^t \left\langle e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot), \phi_i\right\rangle dr + \int_0^t \left\langle S_{r,i}f_i, \left(\Delta_{\alpha_i} - \frac{\kappa_i^2}{2}\right)\phi_i\right\rangle dr$$
$$+ \int_0^t \int_0^r \left\langle S_{r-s,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_s} v_{3-i}^{1+\beta_i}(s,\cdot)\right), \left(\Delta_{\alpha_i} - \frac{\kappa_i^2}{2}\right)\phi_i\right\rangle ds dr$$
$$= \langle f_i, \phi_i\rangle + \int_0^t \left\langle e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot), \phi_i\right\rangle dr$$
$$+ \int_0^t \left\langle S_{r,i}f_i + \int_0^r S_{r-s,i}\left(e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_s} v_{3-i}^{1+\beta_i}(s,\cdot)\right) ds, \left(\Delta_{\alpha_i} - \frac{\kappa_i^2}{2}\right)\phi_i\right\rangle dr$$
$$= \langle f_i, \phi_i\rangle + \int_0^t \left\langle e^{((1+\beta_i)\kappa_{3-i}-\kappa_i)W_r} v_{3-i}^{1+\beta_i}(r,\cdot), \phi_i\right\rangle dr + \int_0^t \left\langle v_i(r,\cdot), \left(\Delta_{\alpha_i} - \frac{\kappa_i^2}{2}\right)\phi_i\right\rangle dr,$$

which implies that $(v_1(t,x), v_2(t,x))$ is a weak solution on $(0,\tau)$. □

4.3 Explosion in finite time

From Theorem 4.2.3 and Theorem 4.2.6 it follows that system (4.1) has a weak solution $\{u_i(t,x); t \geq 0, x \in D\}$. In fact, a mild solution $(v_1(t,x), v_2(t,x))$ of system (4.4) can be constructed from Theorem 4.2.3, which is also a weak solution of (4.4) by Theorem 4.2.6. Using now the change of variables

$$u_i(t,x) = \exp\{\kappa_i W_t\} v_i(t,x),$$

we obtain that $\{u_i(t,x); t \geq 0,\, x \in D\}$ is a weak solution of system (4.1).

Let $u_i\,(t,\phi) := \int_D u_i\,(t,x)\,\phi\,(x)\,dx,\ i = 1,2$. Then, for all $\phi_i \in C_b^2(D)$ the weak solution of system (4.1) can be written as

$$u_i\,(t,\phi_i) = u_i\,(0,\phi_i) + \int_0^t u_i\,(s,\Delta_{\alpha_i}\phi_i)\,ds + \int_0^t u_{3-i}^{1+\beta_i}\,(s,\phi_i)\,ds + \int_0^t \kappa_i u_i\,(s,\phi_i)\,dW_s. \quad (4.5)$$

Let λ_i denote the first positive eigenvalue of $-\Delta_{\alpha_i}$ on D, with respective eigenfunction ψ_i, $i = 1,2$, normalized so that $\int_D \psi_i(x)dx = 1$. We write $\mu_i(t) = \mathbb{E}\left[u_i(t,\psi_i)\right]$.

4.3.1 A criterion for explosion in finite time. Case $\alpha_1 = \alpha_2$.

In this case we write $\alpha := \alpha_1 = \alpha_2$, $\lambda := \lambda_1 = \lambda_2$ and $\psi := \psi_1 = \psi_2$. From Jensen's inequality and (4.5) we get that, for all $t \geq 0$ and $i = 1,2$,

$$\begin{aligned} \mu_i\,(t) &= \mu_i\,(0) - \lambda \int_0^t \mu_i\,(s)\,ds + \int_0^t \mathbb{E}\left[u_{3-i}^{1+\beta_i}\,(s,\psi)\right] ds \\ &\geq \mu_i\,(0) - \lambda \int_0^t \mu_i\,(s)\,ds + \int_0^t \mu_{3-i}^{1+\beta_i}(s)ds. \end{aligned}$$

Now we consider the system of ODEs

$$\begin{aligned} \frac{d}{dt} h_i\,(t) &= -\lambda h_i\,(t) + h_{3-i}^{1+\beta_i}\,(t)\,, \quad t > 0, \\ h_i\,(0) &= \mu_i\,(0)\,, \quad i = 1,2, \end{aligned}$$

and define $E\,(t) = h_1\,(t) + h_2\,(t)$, $t \geq 0$. Let $T_p = \min\{T_p^1, T_p^2\}$ be the explosion time of system (4.1), where T_p^i is the explosion time of u_i, $i = 1,2$. We have the following theorem.

Theorem 4.3.1. *1. Assume that $\beta_1 = \beta_2 > 0$ and $E\,(0) > 2\lambda^{1/\beta_1}$. Then $T_p \leq T^*$ for all $p \in [1,\infty)$, where*

$$T^* = \int_{E(0)}^{\infty} \frac{1}{-\lambda u + 2^{-\beta_1} u^{1+\beta_1}} du < \infty.$$

2. Let $\beta_1 > \beta_2 > 0$ and $A_0 = \left(\frac{1+\beta_1}{1+\beta_2}\right)^{-\frac{1+\beta_2}{\beta_1-\beta_2}} \frac{\beta_1-\beta_2}{1+\beta_1}$, and suppose that there exists $\epsilon_0 \in (0,1]$ such that $f_0 := 2^{-\beta_2}\epsilon_0 E^{1+\beta_2}\,(0) - \epsilon_0^{\frac{1+\beta_2}{\beta_1-\beta_2}} A_0 > \lambda E\,(0)$. Then $T_p \leq T^*$ for all $p \in [1,\infty)$, where

$$T^* = \int_{E(0)}^{\infty} \frac{1}{-\lambda u + (f_0/E^{1+\beta_2}(0)) u^{1+\beta_2}} du < \infty.$$

Proof. 1. We have that

$$\frac{d}{dt}E\left(t\right)=-\lambda E\left(t\right)+\left(h_{1}^{1+\beta_{1}}\left(t\right)+h_{2}^{1+\beta_{1}}\left(t\right)\right),\quad t>0.$$

Using Young's inequality we obtain, as in the proof of Theorem (2.3.1), that

$$\frac{d}{dt}E\left(t\right)\geq-\lambda E\left(t\right)+2^{-\beta_{1}}E^{1+\beta_{1}}\left(t\right),\quad t>0. \tag{4.6}$$

Let $I\left(t\right)$ be the solution of

$$\begin{aligned}&\frac{d}{dt}I\left(t\right)=-\lambda I\left(t\right)+2^{-\beta_{1}}I^{1+\beta_{1}}\left(t\right),\quad t>0,\\&I\left(0\right)=E\left(0\right).\end{aligned}$$

Notice that $I\left(0\right)>2\lambda^{1/\beta_{1}}$ implies $I\left(t\right)\geq I\left(0\right)$, for all $t>0$. Hence, for all $T>0$,

$$\begin{aligned}T&=\int_{I(0)}^{I(T)}\frac{1}{-\lambda u+2^{-\beta_{1}}u^{1+\beta_{1}}}du\\&\leq\int_{I(0)}^{\infty}\frac{1}{-\lambda u+2^{-\beta_{1}}u^{1+\beta_{1}}}du=:T^{*},\end{aligned}$$

where $T^{*}<\infty$ under the assumption $E\left(0\right)>2\lambda^{1/\beta_{1}}$. Therefore the last inequality cannot hold for all sufficiently large T. This means that I must explode in a finite time $T_{I}\leq T^{*}$. By a comparison argument we get that the explosion time T_{E} of E satisfies that $T_{E}\leq T_{I}$. In a similar way we can prove that $T_{\mu_{i}}\leq T_{h_{i}}$, where $T_{\mu_{i}}$ and $T_{h_{i}}$ are the explosion times of μ_{i} and h_{i}, respectively, $i=1,2$. Since $T_{E}=\min\left\{T_{h_{1}},T_{h_{2}}\right\}$, we obtain that

$$\min\left\{T_{\mu_{1}},T_{\mu_{2}}\right\}\leq T^{*}.$$

Using Hölder's inequality we have, for each $i=1,2$ and $p\in[1,\infty)$,

$$\mu_{i}\left(t\right)\leq C_{p,i}\mathbb{E}\left[\left\|u_{i}\left(t,\cdot\right)\right\|_{L^{p}(D)}\right]$$

for some constant $C_{p,i}>0$. Therefore, $T_{p}\leq T^{*}$.

2. Using again Young's inequality we get

$$\frac{d}{dt}E\left(t\right)\geq-\lambda E\left(t\right)+2^{-\beta_{2}}\epsilon_{0}E^{1+\beta_{2}}\left(t\right)-\epsilon_{0}^{\frac{1+\beta_{1}}{\beta_{1}-\beta_{2}}}A_{0}. \tag{4.7}$$

Now we prove that $E(t)\geq E(0)$, for all $t\geq0$. In fact, consider the function

$$f(x)=-\lambda x+2^{-\beta_{2}}\epsilon_{0}x^{1+\beta_{2}}-\epsilon_{0}^{\frac{1+\beta_{1}}{\beta_{1}-\beta_{2}}}A_{0},\quad x\in\mathbb{R}.$$

Let J be the solution of the differential equation

$$J'(t) = f(J(t)), \quad t > 0,$$
$$J(0) = E(0).$$

By comparison $E(t) \geq J(t)$ for all $t \geq 0$. Therefore it suffices to show that $J(t) \geq E(0)$, for $t \geq 0$. It is easy to see that f has only one zero at some point $x_0 < E(0)$. Let $T := \inf\{t > 0 : J(t) < E(0)\}$. Then $T > 0$ since J is strictly increasing around 0, and $J(t) \geq E(0)$ for all $t \in (0, T)$. Being J continuous on $[0, T]$ and differentiable on $(0, T)$, Rolle's theorem yields that $J'(c) = 0$ for some $c \in (0, T)$. Hence $J(c) = x_0$ which implies $x_0 \geq E(0)$. This contradiction says that $T = \infty$ and

$$E(t) \geq E(0), \quad t \geq 0.$$

Therefore

$$\frac{d}{dt}E(t) \geq -\lambda E(t) + E^{1+\beta_2}(t)\left[2^{-\beta_2}\epsilon_0 - \frac{\epsilon_0^{\frac{1+\beta_2}{\beta_1-\beta_2}} A_0}{E^{1+\beta_2}(0)}\right] = -\lambda E(t) + \frac{f_0}{(E^{1+\beta_2}(0))}E^{1+\beta_2}(t).$$

Solving the equation

$$\begin{aligned} \frac{d}{dt}I(t) &= -\lambda I(t) + \frac{f_0}{(E^{1+\beta_2}(0))}I^{1+\beta_2}(t), \quad t > 0, \\ I(0) &= E(0), \end{aligned}$$

taking $T^* := \int_{I(0)}^{\infty} \dfrac{1}{-\lambda u + (f_0/E^{1+\beta_2}(0))u^{1+\beta_2}}du$ (which is finite under the assumption over f_0) and following the argument used in the case $\beta_1 = \beta_2$, we get the result.

□

4.3.2 A criterion for explosion in finite time. Case $\alpha_1 > \alpha_2$ and $D = B(0, r)$.

From [4, Theorem 4.4] we know that there exist positive constants $C_{1,i}$ and $C_{2,i}$, depending on D and α_i, $i = 1, 2$ such that

$$C_{1,i}\mathbb{E}_x[\tau_D^{(i)}] \leq \psi_i(x) \leq C_{2,i}\mathbb{E}_x[\tau_D^{(i)}], \quad x \in D, \tag{4.8}$$

where ψ_i is the first eigenfunction of Δ_{α_i} in D, normalized so that $\int_D \psi_i(x)dx = 1$, and $\tau_D^{(i)}$ is the exit time from D of an spherically symmetric α_i-stable process with infinitesimal generator Δ_{α_i}, $i = 1, 2$.

In the case $D = B(0,r)$, where $B(0,r)$ is the open ball in $\mathbb{R}^d$ with centre at the origin and radius $r \in (0,\infty)$, it is known that

$$\mathbb{E}_x[\tau_D^{(i)}] = c_{\alpha_i,d}\left(r^2 - |x|^2\right)^{\alpha_i/2},$$

where $c_{\alpha_i,d} > 0$ is a constant depending only on α_i and d, $i = 1,2$ (see [4, Theorem 4.4]). Therefore, if $D = B(0,r)$, then there exist positive constants $C_{1,i} := C_1(r,\alpha_i,d)$ and $C_{2,i} := C_2(r,\alpha_i,d)$ such that

$$C_{1,i}\left(r^2 - |x|^2\right)^{\alpha_i/2} \leq \psi_i(x) \leq C_{2,i}\left(r^2 - |x|^2\right)^{\alpha_i/2}. \tag{4.9}$$

From Hölder's inequality we get that

$$\int_D u_{3-i}(s,x)\psi_{3-i}(x)dx \leq \left(\int_D u_{3-i}^{1+\beta_i}(s,x)\psi_i(x)dx\right)^{\frac{1}{1+\beta_i}}\left(\int_D \left(\frac{\psi_{3-i}(x)}{\psi_i(x)}\right)^{\frac{1}{\beta_i}}\psi_{3-i}(x)dx\right)^{\frac{\beta_i}{1+\beta_i}}.$$

Due to (4.9) we have that

$$\begin{aligned}
&\max_{i=1,2}\left\{\left(\int_D \left(\frac{\psi_{3-i}(x)}{\psi_i(x)}\right)^{\frac{1}{\beta_i}}\psi_{3-i}(x)dx\right)^{\beta_i}\right\} \\
&\leq \max_{i=1,2}\left\{\frac{C_{2,3-i}^{1+\beta_i}}{C_{1,i}}\left(\int_D \left(r^2 - |x|^2\right)^{\frac{\alpha_{3-i}-\alpha_i}{2}\frac{1}{\beta_i}+\frac{\alpha_{3-i}}{2}}dx\right)^{\beta_i}\right\} =: C^{-1},
\end{aligned}$$

and $C^{-1} < \infty$ provided that $\beta_i > \frac{\alpha_i - \alpha_{3-i}}{\alpha_{3-i}}$ for each $i = 1,2$. Notice that if $\alpha_1 > \alpha_2$, then the condition $\beta_1 > \frac{\alpha_1-\alpha_2}{\alpha_2}$ implies $\beta_2 > \frac{\alpha_2-\alpha_1}{\alpha_1}$ because $\beta_2 > 0$. Similarly if $\alpha_2 > \alpha_1$, the condition $\beta_2 > \frac{\alpha_2-\alpha_1}{\alpha_1}$ implies $\beta_1 > \frac{\alpha_1-\alpha_2}{\alpha_2}$. Therefore we can suppose, without lost of generality, that $\alpha_1 > \alpha_2$ and $\beta_1 > \frac{\alpha_1-\alpha_2}{\alpha_2}$ to ensure that C is positive and finite. Hence,

$$\left(\int_D u_{3-i}(s,x)\psi_{3-i}(x)dx\right)^{1+\beta_i} \leq C^{-1}\int_D u_{3-i}^{1+\beta_i}(s,x)\psi_i(x)dx, \quad s \in [0,t]. \tag{4.10}$$

We write again $\mu_i(t) = \mathbb{E}\left[u_i(t,\psi_i)\right]$ and denote by $\lambda_{i,r}$ the corresponding positive eigenvalue of ψ_i in $B(0,r)$, $i = 1,2$. Using again Jensen's inequality we get from (4.5) and (4.10) that for all $t \geq 0$,

$$\begin{aligned}
\mu_i(t) &= \mu_i(0) - \lambda_{i,r}\int_0^t \mu_i(s)\,ds + \int_0^t \mathbb{E}\left[u_{3-i}^{1+\beta_i}(s,\psi_i)\right]ds \\
&\geq \mu_i(0) - \lambda_{i,r}\int_0^t \mu_i(s)\,ds + C\int_0^t \mathbb{E}\left[\left(u_{3-i}(s,\psi_{3-i})\right)^{1+\beta_i}\right]ds \\
&\geq \mu_i(0) - \lambda_{i,r}\int_0^t \mu_i(s)\,ds + C\int_0^t \left(\mathbb{E}\left[u_{3-i}(s,\psi_{3-i})\right]\right)^{1+\beta_i}ds \\
&= \mu_i(0) - \lambda_{i,r}\int_0^t \mu_i(s)\,ds + C\int_0^t \mu_{3-i}^{1+\beta_i}(s)ds.
\end{aligned}$$

Now consider the system of ODEs

$$\frac{d}{dt}h_i(t) = -\lambda_r h_i(t) + Ch_{3-i}^{1+\beta_i}(t), \quad t > 0,$$
$$h_i(0) = \mu_i(0), \quad i = 1, 2,$$

with $\lambda_r = \max\{\lambda_{1,r}, \lambda_{2,r}\}$. The proof of the following result is similar to the proof of Theorem 4.3.1.

Theorem 4.3.2. *Assume that $D = B(0, r)$, $\alpha_1 > \alpha_2$, $\beta_1 > \frac{\alpha_1-\alpha_2}{\alpha_2}$ and C as in (4.10).*

1. *If $\beta_1 = \beta_2 > 0$ and $E(0) > 2(\lambda_r/C)^{1/\beta_1}$, then there exists $T^* \in \mathbb{R}^+$ such that for all $p \in [1, \infty)$, $T_p \le T^*$, where*

$$T^* = \int_{E(0)}^{\infty} \frac{1}{-\lambda_r u + 2^{-\beta_1} C u^{1+\beta_1}} du.$$

2. *Let $\beta_1 > \beta_2 > 0$ and let $A_0 = \left(\frac{1+\beta_1}{1+\beta_2}\right)^{-\frac{1+\beta_2}{\beta_1-\beta_2}} \frac{\beta_1-\beta_2}{1+\beta_1}$. Suppose that there exists $\epsilon_0 \in (0, 1]$ such that $f_0 := 2^{-\beta_2} C\epsilon_0 E^{1+\beta_2}(0) - \epsilon_0^{\frac{1+\beta_2}{\beta_1-\beta_2}} CA_0 > \lambda_r E(0)$. Then there exists $T^* \in \mathbb{R}^+$ such that for all $p \in [1, \infty)$, $T_p \le T^*$, where*

$$T^* = \int_{E(0)}^{\infty} \frac{1}{-\lambda u + (f_0/E^{1+\beta_2}(0))u^{1+\beta_2}} du.$$

If $\alpha_2 > \alpha_1$, we must assume $\beta_2 > \frac{\alpha_2-\alpha_1}{\alpha_1}$ in order to ensure $C \in (0, \infty)$, and in the case $\alpha_1 > \alpha_2$, we must assume $\beta_1 > \frac{\alpha_1-\alpha_2}{\alpha_2}$ to ensure $C \in (0, \infty)$. In both cases $\alpha_1 > \alpha_2$ and $\alpha_2 > \alpha_1$, Part 2. of Theorem 4.3.2 remains valid when $\beta_2 > \beta_1$, with the obvious changes in the correspondent statements.

Remark 4.3.3. Conditions on ϵ_0 and f_0 in Part 2. of Theorem 4.3.1 and Theorem 4.3.2 can be satisfied because functions of the form $f(x) = ax^{1+\beta} - bx - c$, $x \ge 0$, increases to infinity, for all constants $\beta, a, b, c > 0$.

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