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Spin^q manifolds admitting parallel and Killing spinors

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Abstract

We classify the simply-connected Spin^q manifolds admitting either a non-zero parallel or a non-zero real Killing spinor.

Keywords: Spin^q structure; Quaternionic spinor; Parallel spinor; Killing spinor; Riemannian holonomy

1. Introduction

Among oriented Riemannian manifolds, Spin manifolds are distinguished by the existence of a double cover of their orthonormal frame bundle P_{SO} by a bundle P_{Spin} with fiber a Spin group. The existence of such a bundle depends on a topological condition, namely the vanishing of the second Stiefel–Whitney class. In such a case, the standard representation of Spin(n) gives rise to the spinor vector bundle whose sections are called spinors. Various operators can be defined, such as a covariant derivative and the Dirac operator. Such operators lead to natural spinorial equations whose solutions are linked to the geometry of the manifold. We refer the reader to Hitchin's seminal paper [5], which prompted a great development of Spin and Spin^c geometries.

There are many manifolds which do not admit Spin structures (such as the complex projective plane $\mathbb{CP}^2$) but admit either Spin^c or Spin^q structures. Such structures are defined in a similar way to Spin structures with the Spin group replaced by the Spin^c and Spin^q groups respectively. They also give rise to bundles of spinors and operators whose properties are related to the manifold's geometry. For instance, E. Witten used Spin^c structures to define a new gauge theory (Seiberg–Witten theory) which has yielded many powerful results in smooth 4-manifold theory [15,7]. Shortly after, Okonek and Teleman started to develop the analogous gauge theory for Spin^q structures [11], which it was hoped would be a bridge between Donaldson's theory and Seiberg–Witten theory.

In [14], Wang classified the irreducible simply-connected Spin manifolds admitting non-zero parallel spinors, which turn out to be Ricci-flat. In [8], Moroianu classified the simply-connected Spin^c manifolds admitting non-zero parallel spinors showing that such a manifold must be the product of a Ricci-flat Spin manifold and a Kähler manifold. In this paper, we show that a Spin^q manifold admitting a parallel quaternionic spinor must be the product of a Ricci-flat Spin manifold and a Kähler manifold with its canonical Spin^q structure.

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In [2], Bär classified the simply-connected Spin manifolds admitting real Killing spinors, and in [8], Moroianu studied complete simply-connected Spin^c manifolds admitting real Killing spinors. Here, we study complete simply-connected Spin^q manifolds admitting real Killing spinors.

The note is organized as follows. In Section 2 we recall preliminaries of Spin^q structures. In Section 3 we study Spin^q manifolds admitting a non-zero parallel spinor. In Section 4 we study Spin^q manifolds which admit a real Killing spinor.

2. Preliminaries on structures of spin-type

Let SO(n) denote the special orthogonal group and Spin(n) its universal double-cover. By using the unit complex numbers $U(1)$ or the unit quaternions Sp(1), the Spin group can be "twisted" as follows

$$\mathrm{Spin}^c(n) = (\mathrm{Spin}(n) \times U(1))/\{\pm(1,1)\} = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1),$$
$$\mathrm{Spin}^q(n) = (\mathrm{Spin}(n) \times Sp(1))/\{\pm(1,1)\} = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} Sp(1).$$

These give rise to the following short exact sequences

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \times U(1) \longrightarrow 1,$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^q(n) \longrightarrow \mathrm{SO}(n) \times \mathrm{SO}(3) \longrightarrow 1,$$

respectively.

2.1. Spin structures on oriented Riemannian vector bundles

Let E be an oriented Riemannian vector bundle over a smooth manifold M, with $r = \mathrm{rank}(E) \geq 3$. Let $P_{\mathrm{SO}(E)}$ denote the orthonormal frame bundle of E. A Spin structure on E is a principal Spin(r)-bundle $P_{\mathrm{Spin}(E)}$ together with a 2 sheeted covering

$$\xi : P_{\mathrm{Spin}(E)} \longrightarrow P_{\mathrm{SO}(E)},$$

such that $\xi(pg) = \xi(p)\xi_0(g)$ for all $p \in P_{\mathrm{Spin}(E)}$, and all $g \in \mathrm{Spin}(r)$, where $\xi_0 : \mathrm{Spin}(r) \longrightarrow \mathrm{SO}(r)$ denotes the universal covering map.

The case when $r = \mathrm{rank}(E) = 2$ has to be dealt with differently since the universal cover of SO(2) is non-compact. In this case we set $\xi_0 : \mathrm{SO}(2) \longrightarrow \mathrm{SO}(2)$ to be the connected 2-fold covering of SO(2). When $r = 1$ a Spin structure is only a 2-fold covering of the base manifold M.

Given a Spin structure $P_{\mathrm{Spin}(E)}$ one can associate a spinor bundle

$$\Delta(E) = P_{\mathrm{Spin}(E)} \times_{\mathrm{Spin}(r)} \Delta_r,$$

where Δ_r denotes the standard complex representation of Spin(r).

2.2. Covariant derivatives on oriented Riemannian vector bundles and spin bundles

Let us recall the description of the covariant derivative on spin bundles from [6]. Let E be an oriented Riemannian vector bundle over a manifold M. A covariant derivative on E is a linear map

$$\nabla : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$$

such that

$$\nabla(fe) = df \otimes e + f\nabla e$$

for all $f \in C^\infty(M)$ and all $e \in \Gamma(E)$. Furthermore, if $\{e_i\}$ is a local orthonormal basis of E and the covariant derivative ∇ is induced from a connection on $P_{\mathrm{SO}(E)}$, ∇ can be expressed by the rule

$$\nabla e_i = \sum_{j=1}^{r} \omega_{ji} \otimes e_j, \tag{1}$$

for a collection of 1-forms ω_{ji}. It is also compatible with the inner product, $\langle\cdot,\cdot\rangle$ on E,

$$X\langle e, e'\rangle = \langle\nabla_X e, e'\rangle + \langle e, \nabla_X e'\rangle,$$

for every $X \in TM$ and $e, e' \in \Gamma(E)$. The covariant derivative can be extended to

$$\nabla : \Gamma(T^*M \otimes E) \longrightarrow \Gamma\left(\bigwedge\nolimits^2 T^*M \otimes E\right)$$

by the rule

$$\nabla(\alpha \otimes e) = d\alpha \otimes e - \alpha \wedge \nabla e,$$

and we set $R = \nabla \circ \nabla$. It follows that

$$\nabla(\nabla e_i) = \nabla\left(\sum_{j=1}^{r} \omega_{ji} \otimes e_j\right) = \sum_{j=1}^{r} \Omega_{ji} \otimes e_j,$$

where

$$\Omega_{ji} = d\omega_{ji} + \sum_{k=1}^{r} \omega_{jk} \wedge \omega_{ki}.$$

For $X, Y \in TM$ we get the curvature transformation

$$R_{X,Y}e = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})e = \sum_{j=1}^{r} e_j \Omega_{ji}(X, Y).$$

If E admits a spin structure, the connection on $P_{\mathrm{SO}(E)}$ lifts to $P_{\mathrm{Spin}(E)}$ and there is an induced covariant derivative ∇^s on any spinor bundle $S(E)$ defined as follows,

$$\nabla^s \sigma_\alpha = \frac{1}{2}\sum_{i<j} \omega_{ji} \otimes e_i e_j \cdot \sigma_\alpha,$$

where $\{\sigma_\alpha\}$ is a local orthonormal frame of $S(E)$, and the dot "·" means Clifford multiplication. For a general section $\sigma \in \Gamma(S(E))$ we have that

$$\nabla^s \sigma = d\sigma + \frac{1}{2}\sum_{i<j} \omega_{ji} \otimes e_i e_j \cdot \sigma.$$

The curvature of this connection is

$$R^s \sigma = \frac{1}{2}\sum_{i<j} \Omega_{ji} \otimes e_i e_j \cdot \sigma.$$

In particular, for any tangent vectors $X, Y \in T_xM$,

$$R^s_{X,Y}(\sigma) = \frac{1}{2}\sum_{i<j} \langle R_{X,Y}(e_i), e_j\rangle e_i e_j \cdot \sigma.$$

Remark. Even if the bundle E is not spin, these calculations are valid locally.

2.3. Spin-type structures on the tangent bundle

Let us now consider the tangent bundle TM of an oriented n-dimensional Riemannian manifold with a fixed metric, which is an oriented Riemannian vector bundle. All the previous considerations apply, and we say that a manifold is spin if $w_2(TM) = 0$.

If the manifold is not spin, it means that the spinor bundle Δ is not globally defined due to a $\mathbb{Z}_2$ ambiguity. In order to get around this problem and to be able to carry out certain aspects of geometric analysis as in the case of spin manifolds, other spin-like structures have been defined. This is done by coupling the locally defined spin structure with another locally defined spin structure of an auxiliary bundle so that they cancel each other's $\mathbb{Z}_2$ ambiguities.

Definition 2.1. Let M be an oriented Riemannian manifold with a fixed metric and let $P_{\mathrm{SO}(n)}(M)$ denote its bundle of oriented orthonormal frames.

- M is called Spin^c if it admits a Spin^c structure consisting of a $U(1)$-principal bundle $P_{U(1)}(M)$, a principal Spin$^c(n)$ bundle $P_{\mathrm{Spin}^c(n)}(M)$ and a Spin^c equivariant projection

$$\xi\colon P_{\mathrm{Spin}^c(n)}(M) \longrightarrow P_{\mathrm{SO}(n)}(M) \times P_{U(1)}(M)$$

- M is called Spin^q if it admits a Spin^q structure consisting of a SO(3)-principal bundle $P_{\mathrm{SO}(3)}(M)$, a principal Spin$^q(n)$ bundle $P_{\mathrm{Spin}^q(n)}(M)$ and a Spin^q equivariant projection

$$\xi\colon P_{\mathrm{Spin}^q(n)}(M) \longrightarrow P_{\mathrm{SO}(n)}(M) \times P_{\mathrm{SO}(3)}(M).$$

Denote by ξ_1 and ξ_2 the compositions of ξ with the projection onto $P_{\mathrm{SO}(n)}(M)$ and $P_{\mathrm{SO}(3)}(M)$, respectively.

The existence of a Spin^c structure is equivalent to the second Stiefel–Whitney class being the mod 2 reduction of the first Chern class of $P_{U(1)}(M)$, $w_2(M) = c_1(P_{U(1)}(M))$ (mod 2). On the other hand, M admits a Spin^q structure if and only if $w_2(M) = w_2(P_{\mathrm{SO}(3)}(M))$. We refer the reader to [6] for the theory of Spin and Spin^c structures, and to [9] for Spin^q structures.

In terms of vector bundles, the auxiliary bundle $P_{U(1)}$ of a Spin^c structure has an associated complex line bundle L. Let $\Delta(M)$ denote the locally defined complex spinor bundle of M. Thus, the Spin^c structure has an associated globally defined vector bundle by the complex tensor product $\Delta^c = \Delta(M) \otimes L^{1/2}$, whose sections are called (complex) spinors. Similarly, a Spin^q structure has an associated globally defined (quaternionic) spinor bundle

$$\Delta^q = \Delta(M) \otimes \Delta(E)$$

where $\Delta(E)$ denotes the locally defined complex spinor bundle of the rank 3 oriented Riemannian vector bundle E associated to the auxiliary bundle $P_{\mathrm{SO}(3)}$ of the Spin^q structure.

Example. Let us examine the standard Spin representation on a *non-Spin* quaternion-Kähler 12-manifold M with non-zero scalar curvature. Recall that the complexified tangent bundle factors as

$$TM_c = E \otimes H,$$

where E and H are only locally defined bundles corresponding to the standard complex representations of the holonomy factors $Sp(3)$ and Sp(1) respectively [12]. Notice that the bundle S^2H is non-trivial since the manifold is non-Ricci-flat. Thus, the locally defined spin bundle splits as follows

$$\Delta_{12} = \bigwedge\nolimits_0^3 E \oplus \bigwedge\nolimits_0^2 E \otimes H \oplus E \otimes S^2H \oplus S^3H \tag{2}$$

where $\bigwedge_0^p E$ denotes the (locally defined) bundle corresponding to the primitive subspace in $\bigwedge^p E$ defined as the Hermitian complement to $\varepsilon \wedge \bigwedge^{p-2} E$ with ε a symplectic form invariant by $Sp(3)$, and S^pH denotes the p-th symmetric power of H. If the manifold M is not Spin, the bundle H defines a non-trivial Spin^q structure on M

$$\begin{aligned}\Delta_{12} \otimes H &= \left(\bigwedge\nolimits_0^3 E \oplus \bigwedge\nolimits_0^2 E \otimes H \oplus E \otimes S^2H \oplus S^3H\right) \otimes H\\ &= \bigwedge\nolimits_0^3 E \otimes H \oplus \bigwedge\nolimits_0^2 E \otimes (S^2H + 1) \oplus E \otimes (S^3H + H) \oplus (S^4H \oplus S^2H)\end{aligned}$$

since all of the bundles in the last line are now *globally defined*.

Remarks. 1. A Spin manifold admits trivial Spin^c and Spin^q structures

2. A Spin^c manifold canonically admits a Spin^q structure. If M is not spin, the Spin^c bundle is $\Delta^c = \Delta(M) \otimes L^{1/2}$. Therefore, the direct sum bundle $(\Delta \otimes L^{1/2}) \oplus (\Delta \otimes L^{-1/2})$ defines a Spin^q structure whose SO(3) bundle is the underlying real vector bundle of $S^2(L^{1/2} \oplus L^{-1/2}) = L + \mathbb{C} + L^{-1}$. We shall call this structure the *canonical* Spin^q structure of a Spin^c manifold.

3. A Spin^c manifold is not necessarily Spin.
4. A Spin^q manifold may be neither Spin nor Spin^c.

Examples. The following manifolds show the various possibilities described in the previous remark.

1. The quaternionic projective space $\mathbb{HP}^3$ is Spin and also admits a non-trivial Spin^q structure.
2. The complex Grassmannian $\mathbb{G}r_2(\mathbb{C}^5)$ is not Spin, but can be considered as a Kähler manifold and a quaternion-Kähler manifold for the same metric. The canonical Spin^c structure given by the canonical bundle (viewed as a Kähler manifold) is different from the Spin^q structure given by its quaternionic-Kähler structure (see the example after Proposition 3.1).
3. The real Grassmannian $\mathbb{G}r_4(\mathbb{R}^7)$ admits neither Spin nor Spin^c structures, while it naturally admits a Spin^q structure as a quaternion-Kähler manifold.
4. Any almost-quaternionic manifold admits a Spin^q structure [9].

2.4. *Connections on Spin^q manifolds*

Let M be an n-dimensional oriented Riemannian manifold admitting a Spin^q structure $P_{\mathrm{Spin}^q(n)}(M)$. The Levi-Civita connection ω on M together with a chosen fixed connection θ on $P_{\mathrm{SO}(3)}$ define a connection on $P_{\mathrm{Spin}^q(n)}(M)$ denoted by ∇^q. We shall denote by $\langle\cdot,\cdot\rangle$ all the metrics involved since the arguments determine which metric is being used.

Following definition (1) of a covariant derivative, the Levi-Civita connection induces the covariant derivative $\nabla : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$. More precisely, let $\{v_1, \dots, v_n\}$ denote a local orthonormal frame of TM, then $\nabla v_i = \sum_{j=1}^n \omega_{ji} \otimes v_j$, for the collection of 1-forms $\omega_{ji} = \langle \nabla v_j, v_i\rangle$. The covariant derivative ∇ is compatible with the inner product and if we let $R = \nabla \circ \nabla$, $X, Y \in TM$

$$R_{X,Y} v_i = \sum_{j=1}^{r} v_j \Omega_{ji}(X, Y),$$

where $\Omega_{ji} = d\omega_{ji} + \sum_{k=1}^n \omega_{jk} \wedge \omega_{ki}$.

Similarly, for the rank 3 oriented Riemannian auxiliary vector bundle E associated to $P_{\mathrm{SO}(3)}(M)$, let $\{e_1, e_2, e_3\}$ be a local orthonormal frame so that the covariant derivative induced by the connection θ is $\nabla^E : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E)$, $\nabla^E e_i = \sum_{j=1}^3 \theta_{ji} \otimes e_j$, form a collection of 1-forms θ_{ji}. ∇^E is also compatible with the corresponding metric. Let $R^E = \nabla \circ \nabla$, $X, Y \in TM$, so that

$$R^E_{X,Y} e_i = \sum_{j=1}^{3} e_j \Theta_{ji}(X, Y),$$

where $\Theta_{ji} = d\theta_{ji} + \sum_{k=1}^3 \theta_{jk} \wedge \theta_{ki}$.

Let $\Delta(M)$ and $\Delta(E)$ denote the locally defined spinor bundles of TM and E respectively, where $\Delta(E_x) \cong \mathbb{H} \cong \mathbb{C}^2$ is isomorphic to the quaternions. The quaternionic spinor bundle $\Delta^q(M) = \Delta(M) \otimes \Delta(E)$ is globally defined and inherits the following covariant derivative, let $\psi \in \Gamma(\Delta^q(M))$ then

$$\nabla^q \psi = d\psi + \frac{1}{2}\sum_{i<j} \omega_{ji} v_i \cdot v_j \cdot \psi + \frac{1}{2}\sum_{k<l} \theta_{lk} e_l \cdot e_k \cdot \psi, \tag{3}$$

which is compatible with the induced metric. Moreover

$$\nabla^q(\nabla^q \psi) = \frac{1}{2}\sum_{i<j} \Omega_{ij} v_i \cdot v_j \cdot \psi + \frac{1}{2}\sum_{k<l} \Theta_{kl} e_k \cdot e_l \cdot \psi.$$

Let us now recall that the complexified algebra generated by $\{e_1, e_2, e_3\}$ with the relations $e_k \cdot e_l + e_l \cdot e_k = -2\langle e_k, e_l\rangle$ is isomorphic to the quaternions so that (3) can be rewritten as

$$\nabla^q \psi = d\psi + \frac{1}{2}\sum_{i<j} \omega_{ji} v_i \cdot v_j \cdot \psi + \frac{1}{2}(i\theta_{23} + j\theta_{31} + k\theta_{12}) \cdot \psi$$

for $\psi \in \Gamma(\Delta^q(M))$, and

$$\nabla^q(\nabla^q\psi) = \frac{1}{2}\sum_{i<j}\Omega_{ij}v_i \cdot v_j \cdot \psi + \frac{1}{2}(i\Theta_{23} + j\Theta_{31} + k\Theta_{12}) \cdot \psi, \tag{4}$$

where i, j, k now denote, according to the context, the standard basis of unit quaternions. Now if $\{\varphi^i\}$ is a frame dual to $\{v_i\}$, we can rewrite (4) as

$$\nabla^q(\nabla^q\psi) = \frac{1}{4}\sum_{i<j}\left(\sum_{k,l}R_{ijkl}\varphi^k \wedge \varphi^l\right)v_i \cdot v_j \cdot \psi + \frac{1}{2}(i\Theta_{23} + j\Theta_{31} + k\Theta_{12}) \cdot \psi.$$

Remark. A Spin^q structure on a simply-connected manifold M whose $P_{\mathrm{SO}(3)}$ bundle is trivial with a flat connection is canonically identified with a Spin structure and the covariant derivative ∇^q is the same as ∇ on spinor bundles.

Remark. Since $\Delta(E_x)$ can be identified with the quaternions, we also have an action by the unit quaternions $Sp(1) \subset \mathbb{H}$ as follows. If $q \in Sp(1)$ and $h \in \Delta(E_x)$, then $h \cdot q = hq^{-1}$, and ∇^q commutes with this action.

3. Parallel spinors

Let M be a Spin^q manifold, X a vector field and ψ a quaternionic spinor. Following [3], define the form

$$H^q_\psi(X) = \sum_{\alpha=1}^{n} v_\alpha \cdot (\nabla^q\nabla^q\psi)(X, v_\alpha).$$

Lemma 3.1.

$$H^q_\psi(X) = -\frac{1}{2}\mathrm{Ric}(X) \cdot \psi + \frac{1}{2}(X \lrcorner \Theta) \cdot \psi,$$

where $\Theta = i\Theta_{23} + j\Theta_{31} + k\Theta_{12}$.

The proof of the proposition on page 64 of [3] can be easily modified to our case. □

Definition 3.1. Let M be a simply-connected Riemannian Spin^q manifold. A quaternionic spinor $\psi \in \Gamma(\Delta^q)$ is *parallel* if

$$\nabla^q_X\psi = 0$$

for every vector field X.

Lemma 3.2. *Let X be a vector field and ψ a non-zero parallel spinor. Then*

$$\mathrm{Ric}(X) \cdot \psi = (X \lrcorner \Theta) \cdot \psi, \tag{5}$$

where Ric *denotes the Ricci tensor as a type* $(1, 1)$ *tensor.*

Proof. This follows from the previous lemma since $\nabla^q\psi = 0$ implies $H^q_\psi(X) = 0$ for all X. □

Theorem 3.1. *Let M be a simply-connected Riemannian* Spin^q *manifold. Assume M admits a non-zero parallel spinor ψ. Then, M is isometric to the Riemannian product of a Ricci-flat Spin manifold and a Kähler manifold.*

Proof. Without loss of generality, we can assume that the manifold M is irreducible. More precisely, if the simply-connected manifold M is a Riemannian product $M_1 \times M_2$, then

$$H^2(M, \mathbb{Z}_2) = H^2(M_1, \mathbb{Z}_2) \oplus H^2(M_2, \mathbb{Z}_2),$$
$$w_2(M_1) \equiv w_2(P_{\mathrm{SO}(3)}|_{M_1}) \quad \text{and} \quad w_2(M_2) \equiv w_2(P_{\mathrm{SO}(3)}|_{M_2}),$$

so that both M_1 and M_2 admit Spin^q structures, and also inherit connections. Restricting the bundle $\Delta(M) \otimes \Delta(E)$ to M_1 gives

$$(\Delta(M) \otimes \Delta(E))|_{M_1} = 2^{[\dim(M_2)/2]} \Delta(M_1) \otimes \Delta(E)|_{M_1},$$

and analogously for M_2

$$(\Delta(M) \otimes \Delta(E))|_{M_2} = 2^{[\dim(M_1)/2]} \Delta(M_2) \otimes \Delta(E)|_{M_2}.$$

Since the Spin holonomy of the connection must have a fixed point (subspace) in the representation $\Delta(M) \otimes \Delta(E)$, i.e. a trivial summand, the above restrictions must also have at least one trivial summand. Now, the Spin holonomy of M_1 is the same for each copy of $\Delta(M_1) \otimes \Delta(E)|_{M_1}$ so that by Schur's lemma $\Delta(M_1) \otimes \Delta(E)|_{M_1}$ has a trivial summand and therefore M_1 endowed with this Spin^q structure and connection admits a non-zero parallel spinor. Similarly for M_2.

Now, if ψ is a non-trivial parallel spinor, then $X\|\psi\|^2 = \langle \nabla^q_X \psi, \psi\rangle + \langle \nabla^q_X \psi, \psi\rangle = 0$, so it has constant length and no zeroes. Furthermore, the tangent bundle TM can be considered as a sub-bundle of Δ^q. Consider the isomorphism $\Phi: TM \longrightarrow TM \cdot \psi \subset \Delta^q(M)$, given by $\Phi(X) = X \cdot \psi$. If $X \cdot \psi = 0$ then

$$X \cdot X \cdot \psi = -|X|^2 \psi = 0,$$

which implies $X = 0$, so that Φ is an isomorphism.

Since ψ is parallel, the sub-bundle $TM \cdot \psi$ is a parallel sub-bundle. Indeed,

$$\nabla^q_Y (X \cdot \psi) = (\nabla_Y X) \cdot \psi + X \cdot \nabla^q_Y \psi = (\nabla_Y X) \cdot \psi,$$

for every vector field Y.

Step 1. Define the following distribution

$$\mathcal{D} = \Phi^{-1}((TM \cdot \psi) \cap (iTM \cdot \psi + jTM \cdot \psi + kTM \cdot \psi)),$$

whose fiber at $x \in M$

$$\mathcal{D}_x = \{X \in T_xM \mid \exists Y_1, Y_2, Y_3 \in T_xM, X \cdot \psi = iY_1 \cdot \psi + jY_2 \cdot \psi + kY_3 \cdot \psi\}.$$

The distribution $\mathcal{D}$ and its orthogonal complement $\mathcal{D}^\perp$ are parallel as we show next. First notice

$$\begin{aligned}
\nabla^q (i\psi) &= i d\psi + i\frac{1}{2}\sum_{i<j} \omega_{ji} v_i \cdot v_j \cdot \psi + \frac{1}{2}(i\theta_{23} + j\theta_{31} + k\theta_{12}) i\psi \\
&= i d\psi + i\frac{1}{2}\sum_{i<j} \omega_{ji} v_i \cdot v_j \cdot \psi + i\frac{1}{2}(i\theta_{23} - j\theta_{31} - k\theta_{12}) \psi \\
&= i\nabla^q \psi + (-k\theta_{31} + j\theta_{12})\psi \\
&= (-k\theta_{31} + j\theta_{12})\psi,
\end{aligned}$$

and similarly

$$\begin{aligned}
\nabla^q (j\psi) &= (k\theta_{23} - i\theta_{12})\psi, \\
\nabla^q (k\psi) &= (-j\theta_{23} + i\theta_{31})\psi.
\end{aligned}$$

Let $X \in \Gamma(\mathcal{D})$ and Z be a vector field. Thus

$$\begin{aligned}
\nabla_Z X \cdot \psi &= \nabla_Z X \cdot \psi + X \cdot \nabla^q_Z \psi \\
&= \nabla^q_Z (X \cdot \psi) \\
&= \nabla^q_Z (Y_1 \cdot i\psi + Y_2 \cdot j\psi + Y_3 \cdot k\psi) \\
&= (i\nabla_Z Y_1 \cdot \psi + Y_1 \cdot \nabla^q_Z (i\psi)) + (j\nabla_Z Y_2 \cdot \psi + Y_2 \cdot \nabla^q_Z (j\psi)) + (k\nabla_Z Y_3 \cdot \psi + Y_3 \cdot \nabla^q_Z (k\psi)) \\
&= i(\nabla_Z Y_1 - \theta_{12}(Z) Y_2 + \theta_{31}(Z) Y_3) \cdot \psi + j(\nabla_Z Y_2 + \theta_{12}(Z) Y_1 - \theta_{23}(Z) Y_3) \cdot \psi \\
&\quad + k(\nabla_Z Y_3 - \theta_{31}(Z) Y_1 + \theta_{23}(Z) Y_2) \cdot \psi,
\end{aligned}$$

so $\nabla_Z X \in \Gamma(\mathcal{D})$. On the other hand, if $W \in \Gamma(\mathcal{D}^\perp)$, then $\langle W, X\rangle = 0$ for all $X \in \Gamma(\mathcal{D})$, and $\langle \nabla_Z W, X\rangle = Z\langle W, X\rangle - \langle W, \nabla_Z X\rangle = 0$, for any vector field Z, so that $\nabla_Z W \in \Gamma(\mathcal{D}^\perp)$.

Since M is irreducible, either $\mathcal{D} = TM$ or $\mathcal{D}^\perp = TM$. If $\mathcal{D}^\perp = TM$ then by Lemma 3.2,

$$\mathrm{span}\{\mathrm{Ric}(X) | X \text{ vector field}\} \subset \mathcal{D} = \{0\},$$

so that Θ vanishes identically, the connection of $P_{\mathrm{SO}(3)}(M)$ is flat, and M is Spin (classified in [14]).

If $\mathcal{D} = TM$, we proceed as follows.

Step 2. Let us assume there exists a quaternion $q_0 = ai + bj + ck$ with $a^2 + b^2 + c^2 = 1$, $q_0^2 = -1$, such that the following distribution is non-trivial

$$\mathcal{E} = \Phi^{-1}((TM \cdot \psi) \cap (TM \cdot \psi \cdot q_0)),$$

with fiber at $x \in M$

$$\mathcal{E}_x = \{X \in T_x M \mid \exists Y \in T_x M, X \cdot \psi = Y \cdot \psi \cdot q_0\}.$$

The bundle $\mathcal{E}$ and its orthogonal complement $\mathcal{E}^\perp$ in TM are parallel. Namely, let $X \in \Gamma(\mathcal{E})$ and Z be a vector field

$$\begin{aligned}(\nabla_Z X) \cdot \psi &= (\nabla_Z X) \cdot \psi + X \cdot (\nabla^q_Z \psi) = \nabla^q_Z (X \cdot \psi)\\ &= ((\nabla_Z Y) \cdot \psi + Y \cdot (\nabla^q_Z \psi)) \cdot q_0 = (\nabla_Z Y) \cdot \psi \cdot q_0,\end{aligned}$$

so that $\nabla_Y X \in \Gamma(\mathcal{E})$. Now, let $Z \in \Gamma(\mathcal{E}^\perp)$ and Y be a vector field. By definition $\langle Z, X\rangle = 0$ for all $X \in \Gamma(\mathcal{E})$, and $\langle \nabla_Y Z, X\rangle = Y\langle Z, X\rangle - \langle Z, \nabla_Y X\rangle = 0$, for any vector field Y, so that $\nabla_Y Z \in \Gamma(\mathcal{E}^\perp)$.

Since M is irreducible, either $\mathcal{E} = TM$ or $\mathcal{E}^\perp = TM$. If $\mathcal{E} = TM$, we can define a parallel complex structure on M as follows. For any vector field X, define the almost complex structure J_0 by the equation

$$X \cdot \psi = J_0(X) \cdot \psi \cdot q_0, \tag{6}$$

since by this definition, $J_0(J_0(X)) = -X$.

To see that it is orthogonal multiply (6) by X on the left

$$X \cdot X \cdot \psi = X \cdot J_0(X) \cdot \psi \cdot q_0,$$

$$-|X|^2 \psi = X \cdot J_0(X) \cdot \psi \cdot q_0. \tag{7}$$

Multiply (6) by $J_0(X)$ on the left

$$\begin{aligned}J_0(X) \cdot X \cdot \psi &= J_0(X) \cdot J_0(X) \cdot \psi \cdot q_0,\\ J_0(X) \cdot X \cdot \psi &= -|J_0(X)|^2 \psi \cdot q_0.\end{aligned}$$

Multiply the last equation by $-q_0$ on the right

$$-|J_0(X)|^2 \psi = -J_0(X) \cdot X \cdot \psi \cdot q_0 = (X \cdot J_0(X) + 2\langle X, J_0(X)\rangle) \cdot \psi \cdot q_0. \tag{8}$$

Subtract (8) from (7) to get

$$\psi((-|X|^2 + |J_0(X)|^2) + 2\langle X, J_0(X)\rangle q_0^{-1}) = 0,$$

which is essentially multiplication by a "complex number". Therefore,

$$(-|X|^2 + |J_0(X)|^2) + 2\langle X, J_0(X)\rangle q_0^{-1} = 0,$$

i.e.

$$|X| = |J_0(X)| \quad \text{and} \quad \langle X, J_0(X)\rangle = 0.$$

Now, taking the covariant derivative of (6)

$$\begin{aligned}\nabla^q_Y (X \cdot \psi) &= (\nabla_Y X) \cdot \psi + X \cdot (\nabla^q_Y \psi) = (\nabla_Y X) \cdot \psi\\ &= (\nabla_Y (J_0(X)) \cdot \psi + J_0(X) \cdot (\nabla^q_Y \psi)) \cdot q_0 = \nabla_Y (J_0(X)) \cdot \psi \cdot q_0\end{aligned}$$

gives

$$(\nabla_Y X) \cdot \psi = \nabla_Y (J_0(X)) \cdot \psi \cdot q_0.$$

Substitute $\nabla_Y(X)$ for X in (6)

$$(\nabla_Y X) \cdot \psi = J_0(\nabla_Y X) \cdot \psi \cdot q_0.$$

Subtracting the last two equations gives

$$(\nabla_Y(J_0(X)) - J_0(\nabla_Y X)) \cdot \psi \cdot q_0 = 0,$$

where ψ has no zeroes. As before, this implies

$$(\nabla J_0)(X, Y) = \nabla_Y(J_0(X)) - J_0(\nabla_Y X) = 0.$$

Since X and Y are arbitrary, $\nabla J_0 = 0$, which means M is Kähler.

If $\mathcal{E}^\perp = TM$, we proceed as follows.

Step 3. By Step 2, the following intersections are trivial

$$TM \cdot \psi \cap TM \cdot \psi \cdot i = TM \cdot \psi \cap TM \cdot \psi \cdot j = TM \cdot \psi \cap TM \cdot \psi \cdot k = \{0\},$$

which imply

$$TM \cdot \psi \cdot i \cap TM \cdot \psi \cdot j = TM \cdot \psi \cdot j \cap TM \cdot \psi \cdot k = TM \cdot \psi \cdot i \cap TM \cdot \psi \cdot k = \{0\}.$$

Thus, the bundle $TM \cdot \psi \cdot i \oplus TM \cdot \psi \cdot j \oplus TM \cdot \psi \cdot k$ is a direct sum. Indeed, if

$$0 = X \cdot \psi \cdot i + Y \cdot \psi \cdot j + Z \cdot \psi \cdot k$$

for vectors X, Y and Z, then

$$0 = X \cdot X \cdot \psi \cdot i + X \cdot Y \cdot \psi \cdot j + X \cdot Z \cdot \psi \cdot k,$$
$$0 = Y \cdot X \cdot \psi \cdot i + Y \cdot Y \cdot \psi \cdot j + Y \cdot Z \cdot \psi \cdot k,$$
$$0 = Z \cdot X \cdot \psi \cdot i + Z \cdot Y \cdot \psi \cdot j + Z \cdot Z \cdot \psi \cdot k,$$

which imply

$$\psi \cdot q_1 = 2X \cdot Z \cdot \psi \cdot k,$$
$$\psi \cdot q_2 = 2Z \cdot Y \cdot \psi \cdot j,$$
$$\psi \cdot q_3 = 2Y \cdot X \cdot \psi \cdot i,$$

where

$$q_1 = (2\langle Y, Z\rangle + (|X|^2 - |Y|^2 + |Z|^2)i + 2\langle X, Y\rangle j - 2\langle X, Z\rangle k),$$
$$q_2 = (2\langle X, Y\rangle + 2\langle X, Z\rangle i - 2\langle Z, Y\rangle j + (-|X|^2 + |Y|^2 + |Z|^2)k),$$
$$q_3 = (2\langle X, Z\rangle - 2\langle X, Y\rangle i + (|X|^2 + |Y|^2 - |Z|^2)i + 2\langle Y, Z\rangle k).$$

If $q_1 \neq 0$, then

$$X \cdot \psi = (2|X|^2 |kq_1^{-1}| Z) \cdot \psi \cdot \frac{(-kq_1^{-1})}{|kq_1^{-1}|},$$

which by assumption does not happen. Similarly for q_2 and q_3. Thus $q_1 = q_2 = q_3 = 0$ and

$$|Y|^2 = |X|^2 + |Z|^2,$$
$$|X|^2 = |Y|^2 + |Z|^2,$$
$$|Z|^2 = |X|^2 + |Y|^2,$$

so that $|X| = |Y| = |Z| = 0$.

Now, consider the distribution

$$\mathcal{F} = \Phi^{-1}((TM \cdot \psi) \cap (TM \cdot \psi \cdot i \oplus TM \cdot \psi \cdot j \oplus TM \cdot \psi \cdot k)),$$

whose fiber at $x \in M$

$$\mathcal{F}_x = \{X \in T_xM \mid \exists Y_1, Y_2, Y_3 \in T_xM, X \cdot \psi = Y_1 \cdot \psi \cdot i + Y_2 \cdot \psi \cdot j + Y_3 \cdot \psi \cdot k\},$$

i.e. for any vector field $X \in \Gamma(\mathcal{F})$, $X \cdot \psi$ can be uniquely written as

$$X \cdot \psi = Y_1 \cdot \psi \cdot i + Y_2 \cdot \psi \cdot j + Y_3 \cdot \psi \cdot k.$$

The distribution $\mathcal{F}$ and its orthogonal complement $\mathcal{F}^{\perp}$ are parallel. Let $X \in \Gamma(\mathcal{F})$ and Z be a vector field, then

$$\begin{aligned}\nabla_Z X \cdot \psi &= \nabla_Z X \cdot \psi + X \cdot \nabla^q_Z \psi = \nabla^q_Z (X \cdot \psi)\\ &= (\nabla_Z Y_1 \cdot \psi + Y_1 \cdot \nabla^q_Z \psi) \cdot i + (\nabla_Z Y_2 \cdot \psi + Y_2 \cdot \nabla^q_Z \psi) \cdot j + (\nabla_Z Y_3 \cdot \psi + Y_3 \cdot \nabla^q_Z \psi) \cdot k\\ &= (\nabla_Z Y_1 \cdot \psi) \cdot i + (\nabla_Z Y_2 \cdot \psi) \cdot j + (\nabla_Z Y_3 \cdot \psi) \cdot k,\end{aligned}$$

so $\nabla_Z X \in \Gamma(\mathcal{F})$. On the other hand, if $W \in \Gamma(\mathcal{F}^{\perp})$, then $\langle W, X\rangle = 0$ for all $X \in \Gamma(\mathcal{F})$, and $\langle \nabla_Z W, X\rangle = Z\langle W, X\rangle - \langle W, \nabla_Z X\rangle = 0$, for any vector field Z, so that $\nabla_Z W \in \Gamma(\mathcal{F}^{\perp})$.

Since M is irreducible, either $\mathcal{F} = TM$ or $\mathcal{F}^{\perp} = TM$. If $\mathcal{F} = TM$, set $I(X) = Y_1$, $J(X) = Y_2$, $K(X) = Y_3$,

$$X \cdot \psi = I(X) \cdot \psi \cdot i + J(X) \cdot \psi \cdot j + K(X) \cdot \psi \cdot k, \tag{9}$$

which multiplied by i, j and k gives the following equations

$$\begin{aligned}I(X) \cdot \psi &= (-X) \cdot \psi \cdot i + (-K(X)) \cdot \psi \cdot j + J(X) \cdot \psi \cdot k,\\ J(X) \cdot \psi &= K(X) \cdot \psi \cdot i + (-X) \cdot \psi \cdot j + (-I(X)) \cdot \psi \cdot k,\\ K(X) \cdot \psi &= (-J(X)) \cdot \psi \cdot i + I(X) \cdot \psi \cdot j + (-X) \cdot \psi \cdot k.\end{aligned}$$

Therefore I, J, K are three almost complex structures satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = -1, \qquad IJ = -JI = K, \ldots.$$

Furtheremore, they are mutually orthogonal and of the same length. Indeed, after some manipulation, (9) leads to equations such as

$$\begin{aligned}&\psi \cdot ((-|X|^2 + |I(X)|^2 - |J(X)|^2 + |K(X)|^2) - 2(\langle J(X), K(X)\rangle - \langle X, I(X)\rangle)i - 2(\langle X, J(X)\rangle\\ &\quad + \langle K(X), I(X)\rangle)j + 2(\langle X, K(X)\rangle + \langle I(X), J(X)\rangle)k) = 2(X \cdot J(X) + K(X) \cdot I(X)) \cdot \psi \cdot j,\end{aligned}$$

which eventually lead, by the previous case, to

$$|X| = |I(X)| = |J(X)| = |K(X)|$$

and

$$\langle X, I(X)\rangle = \langle X, J(X)\rangle = \langle X, K(X)\rangle = \langle I(X), J(X)\rangle = \langle I(X), K(X)\rangle = \cdots = 0,$$

for all X.

By taking the covariant derivative of (9) yields

$$\nabla_Z X \cdot \psi = \nabla_Z(I(X)) \cdot \psi \cdot i + \nabla_Z(J(X)) \cdot \psi \cdot j + \nabla_Z(K(X)) \cdot \psi \cdot k, \tag{10}$$

where Z is a vector field. Now, by substituting $\nabla_Z X$ for X in (9)

$$\nabla_Z X \cdot \psi = I(\nabla_Z X) \cdot \psi \cdot i + J(\nabla_Z X) \cdot \psi \cdot j + K(\nabla_Z X) \cdot \psi \cdot k, \tag{11}$$

and subtracting (11) from (10), we get

$$0 = (\nabla_Z(I(X)) - I(\nabla_Z X)) \cdot \psi \cdot i + (\nabla_Z(J(X)) - J(\nabla_Z X)) \cdot \psi \cdot j + (\nabla_Z(K(X)) - K(\nabla_Z X)) \cdot \psi \cdot k.$$

Since such a linear combination is unique

$$\begin{aligned}\nabla_Z(I(X)) - I(\nabla_Z X) &= 0,\\ \nabla_Z(J(X)) - J(\nabla_Z X) &= 0,\\ \nabla_Z(K(X)) - K(\nabla_Z X) &= 0,\end{aligned}$$

and therefore the three almost complex structures are parallel $\nabla I = \nabla J = \nabla K = 0$. Hence, the manifold M is hyperkähler, Spin and $\mathrm{Ric}_M \equiv 0$.

If $\mathcal{F}^\perp = TM$, at each $x \in M$

$$(T_xM \cdot \psi) \cap (T_xM \cdot \psi \cdot i \oplus T_xM \cdot \psi \cdot j \oplus T_xM \cdot \psi \cdot k) = \{0\}.$$

Thus

$$\mathcal{V} = T_xM \cdot \psi \oplus T_xM \cdot \psi \cdot i \oplus T_xM \cdot \psi \cdot j \oplus T_xM \cdot \psi \cdot k$$

is a direct sum in $\Delta_x^q(M)$. Therefore we can find the underlying real vector space $V \subset \mathcal{V}$ as the fixed point set of the conjugation involution of Δ_x^q, so that $\mathcal{V}$ is the "*right quaternionification*" of V, i.e.

$$\mathcal{V} = V \otimes_{\mathbb{R}} \mathbb{H}.$$

In this fashion,

$$T_xM \cdot \psi + iT_xM \cdot \psi + jT_xM \cdot \psi + kT_xM \cdot \psi$$

is the "*left quaternionification*" of V, and must be a direct sum, i.e.

$$(TM \cdot \psi) \cap (iTM \cdot \psi \oplus jTM \cdot \psi \oplus kTM \cdot \psi) = \{0\},$$

which contradicts our working assumption that $\mathcal{D} = TM$. □

Let us examine the Spin^c and Spin^q structures of the Kähler piece of Theorem 3.1 in terms of representation theory. For a Kähler manifold, let $\Lambda^{0,1}$ denote the (0, 1) forms and $\kappa \cong \Lambda^{m,0}$ the canonical line bundle. The standard Spin representation branches under $SU(m) \times U(1)$ (see [13])

$$\Delta_{2m} \cong \left(\bigoplus_{l=0}^{m} \Lambda^{l,0}\right) \otimes \kappa^{-1/2} = \left(\bigoplus_{l=0}^{m-1} \Lambda^{l,0} \oplus \kappa\right) \otimes \kappa^{-1/2}.$$

The square root of the canonical bundle is globally defined only if the Kähler manifold is Spin. Note that in this case, the restricted spin representation does not contain any trivial summands, unless the Kähler manifold is special Kähler. In general, we can still tensor it with $\kappa^{-1/2}$ to get the globally defined Spin^c bundle

$$\Delta_{2m} \otimes \kappa^{-1/2} = \left(\bigoplus_{l=0}^{m-1} \Lambda^{l,0}\right) \otimes \kappa^{-1} \oplus \mathbb{C}.$$

This product contains a trivial subbundle of rank 1 corresponding to non-zero parallel spinors ψ of the canonical Spin^c structure of the Kähler manifold [8]. On the other hand, the canonical Spin^q structure of a Spin^c structure is given by tensoring Δ_{2m} with the (locally defined) rank 2 bundle $\kappa^{1/2} \oplus \kappa^{-1/2}$

$$\Delta_{2m} \otimes (\kappa^{1/2} \oplus \kappa^{-1/2}) = \left(\mathbb{C} \oplus \bigoplus_{l=1}^{m-1} \Lambda^{l,0} \oplus \kappa\right) \otimes (\mathbb{C} \oplus \kappa^{-1}).$$

This decomposition contains two trivial summands corresponding to the existence of non-zero parallel spinors for the canonical and anti-canonical Spin^c structures.

The previous remark and [8, Prop. 3.1] prove the following.

Proposition 3.1. *Let M be an n-dimensional simply-connected Riemannian* Spin^q *manifold admitting a non-zero parallel spinor. Then, the* Spin^q *structure of the (non-Ricci-flat) piece N_1 of M is the one associated to the canonical (or anticanonical)* Spin^c *structures of the Kähler metric.* □

Example. Consider again the complex Grassmannian $\mathcal{G} = \mathbb{G}r_2(\mathbb{C}^5)$. This manifold is not Spin, but admits both a canonical Spin^q structure as a Kähler manifold and a different Spin^q structure as a quaternion-Kähler manifold. While it admits non-zero parallel spinors for the first Spin^q structure, it does not for the latter, since that would imply the existence of a Kähler structure compatible with the quaternion-Kähler structure, which is not possible [1].

In order to clarify the last example, let us examine the standard Spin representation. The manifold $\mathcal{G}$ is (irreducible) quaternion-Kähler, so that the complexified tangent bundle factors as

$$T\mathcal{G}_c = E \otimes H,$$

where E and H correspond to the standard complex representations of the holonomy factors $Sp(3)$ and Sp(1) respectively. Recall that $\mathcal{G}$ is a homogeneous space

$$\mathcal{G} = \frac{U(5)}{U(3) \times U(2)} = \frac{SU(5)}{S(U(3) \times U(2))},$$

and that its Kähler structure implies

$$T\mathcal{G}_c \cong F \oplus F^*,$$

where $F \cong \mathbb{C}^6$ corresponds to the standard representation of $U(6)$. Furthermore, the isotropy group $U(3) \times U(2)$ determines the following factorization

$$F \cong V \otimes U^*,$$

where $V \cong \mathbb{C}^3$ and $U \cong \mathbb{C}^2$ denote the standard representations of $U(3)$ and $U(2)$ respectively. In particular, since

$$U(2) = SU(2) \times_{\mathbb{Z}_2} U(1) = Sp(1) \times_{\mathbb{Z}_2} U(1),$$
$$U = H \otimes L,$$

where

$$\det(U) = \bigwedge^2 U = \bigwedge^2 H \otimes L^2 = L^2,$$

i.e.

$$L = (\det(U))^{1/2}.$$

Hence,

$$\begin{aligned} T\mathcal{G}_c &= V \otimes U^* \oplus V^* \otimes U \\ &= [V \otimes L^* \otimes H] \oplus [V^* \otimes L \otimes H] \\ &= ([V \otimes L^*] \oplus [V^* \otimes L]) \otimes H \end{aligned}$$

so that

$$E = [V \otimes L^*] \oplus [V^* \otimes L].$$

Let us examine the term $\bigwedge_0^3 E$ in the decomposition (2). Indeed,

$$\bigwedge^3 E = \bigwedge^3 V \otimes L^{-3} \oplus \bigwedge^2 V \otimes V^* \otimes L^{-1} \oplus V \otimes \bigwedge^2 V^* \otimes L \oplus \bigwedge^3 V^* \otimes L^3$$

and

$$\bigwedge_0^3 E = \bigwedge^3 V \otimes L^{-3} \oplus S^2 V^* \otimes L^{-1} \oplus S^2 V \otimes L \oplus \bigwedge^3 V^* \otimes L^3.$$

Since $\kappa^{-1} = (\bigwedge^3 V)^2 \otimes L^{-6}$, when we multiply $\bigwedge_0^3 E$ by $\kappa^{1/2} = (\bigwedge^3 V)^{-1} \otimes L^{-3}$ we get a trivial summand, as expected for the Kähler Spin^c structure. On the other hand, there is no trivial summand after tensoring Δ_{12} with H and using the Clebsch–Gordan formula

$$\Delta_{12} \otimes H = \bigwedge_0^3 E \otimes H \oplus \bigwedge_0^2 E \otimes (S^2 H + 1) \oplus E \otimes (S^3 H + H) \oplus (S^4 H \oplus S^2 H).$$

4. Real Killing spinors

In this section, we classify the simply-connected Spin^q manifolds admitting a real Killing spinor ψ, i.e. a spinor satisfying

$$\nabla_X^q \psi = \lambda X \cdot \psi,$$

for every vector field X and for some $\lambda \in \mathbb{R}$. One can assume that $\lambda = \pm 1/2$ (by rescaling the metric).

First, recall that a Sasakian manifold is a Riemannian manifold (M, g) that has

(a) a Killing vector field ζ of unit length;
(b) two tensors $\phi := -\nabla\zeta$ and $\eta := g(\zeta, \cdot)$ satisfying $\phi^2 = -Id + \eta \otimes \zeta$;
(c) for any vector fields X, Y, $(\nabla_X \phi)Y = g(X, Y)\zeta - \eta(Y)X$.

Theorem 4.1. *A simply connected complete* Spin^q *manifold M admits a real Killing spinor if and only if either*
1. *M is a simply-connected spin manifold (with flat auxiliary bundle) admitting real Killing spinors, or*
2. *M is a Sasakian manifold.*

Let g denote the metric on M. As in [2,8], we consider the cone $\overline{M} = M \times_{r^2} \mathbb{R}^2$ over M with metric $\overline{g} = r^2 g + dr^2$. Let ∂_r denote the vertical unit vector field and $\overline{\nabla}$ the corresponding Levi-Civita connection. They satisfy the following equations [10],

$$\overline{\nabla}_{\partial_r} \partial_r = 0,$$
$$\overline{\nabla}_{\partial_r} X = \overline{\nabla}_X \partial_r = \frac{1}{r} X,$$
$$\overline{\nabla}_X Y = \nabla_X Y - rg(X, Y)\partial_r,$$

where X and Y denote both vector fields on M and their canonical extensions to $\overline{M}$. Thus, the curvature tensor $\overline{R}$ satisfies

$$\overline{R}(X, \partial_r)\partial_r = \overline{R}(X, Y)\partial_r = \overline{R}(X, \partial_r)Y = 0,$$
$$\overline{R}(X, Y)Z = R(X, Y)Z + g(X, Z)Y - g(Y, Z)X,$$

so that if $\overline{M}$ is flat, M is a space form. Also recall that if M is complete, then $\overline{M}$ is irreducible or flat [4, Prop. 3.1].

The main purpose in using the cone is to lift the Killing spinor ψ on M to a non-zero parallel spinor $\pi^*\psi$ on $\overline{M}$, where $\pi: \overline{M} \longrightarrow M$ is the canonical projection.

Proposition 4.1. *Every* Spin^q *structure on M with a chosen connection on* $P_{\mathrm{SO}(3)}(M)$ *induces a canonical* Spin^q *structure on* $\overline{M}$ *with an induced connection on* $P_{\mathrm{SO}(3)}(\overline{M})$. *If* ψ *is a spinor on M then* $\pi^*\psi$ *is a spinor on* $\overline{M}$ *satisfying*

$$\overline{\nabla}^q_X(\pi^*\psi) = \pi^*\left(\nabla^q_{\pi_* X}\psi - \frac{1}{2}(\pi_* X)\cdot\psi\right)$$

for any vector field X on $\overline{M}$.

Proof. Since $\mathrm{SO}(n) \subset \mathrm{SO}(n+1)$, we can enlarge the structure groups of the double cover

$$\xi: P_{\mathrm{Spin}^q(n)}(M) \longrightarrow P_{\mathrm{SO}(n)}(M) \times P_{\mathrm{SO}(3)}(M)$$

to

$$\xi: P_{\mathrm{Spin}^q(n+1)}(M) \longrightarrow P_{\mathrm{SO}(n+1)}(M) \times P_{\mathrm{SO}(3)}(M).$$

It is not hard to check that pulling back by means of π gives a Spin^q structure on $\overline{M}$. That is, $\pi^* P_{\mathrm{SO}(n+1)} M \cong P_{\mathrm{SO}(n+1)}\overline{M}$, and the induced map $\pi^*\xi : P_{\mathrm{Spin}^q(n+1)}\overline{M} \longrightarrow P_{\mathrm{SO}(n+1)}\overline{M} \times P_{\mathrm{SO}(3)}$ is $\mathrm{Spin}^q(n+1)$-equivariant.

The Levi-Civita connection of $\overline{M}$ and the pull-back of the connection on $P_{\mathrm{SO}(3)}M$ to $P_{\mathrm{SO}(3)}\overline{M}$ give a connection on $P_{\mathrm{Spin}^q(n+1)}\overline{M}$.

Let's consider $\mathrm{Spin}(n)$ inside the Clifford algebra Cl_n so that the Lie algebra $\mathfrak{spin}(n) = \mathrm{span}\{e_i \cdot e_j, 1 \leq i < j \leq n, \} \subset \mathrm{Cl}_n$, where $e_1, \ldots, e_n$ is a standard basis of $\mathbb{R}^n$. In this way,

$$\mathfrak{spin}(n+1) = \mathfrak{spin}(n) \oplus \mathbb{R}^n \subset \mathrm{Cl}_n.$$

Since the irreducible complex representation $\rho_n: \mathrm{Spin}(n) \longrightarrow U(\Delta_n)$ is the restriction of a representation of the complexified Clifford algebra $\mathrm{Cl}_n \otimes \mathbb{C}$, one can also restrict to $\mathrm{Spin}(n+1)$. There is an isomorphism $\mathrm{Cl}_n \stackrel{\cong}{\longrightarrow} \mathrm{Cl}^0_{n+1}$, by extending $\mathbb{R}^n \longrightarrow \mathrm{Cl}^0_{n+1}$, $v \mapsto v \cdot e_{n+1}$, which gives the inclusion

$$i: \mathrm{Spin}(n+1) \longrightarrow \mathrm{Cl}_n.$$

If n is even, $\rho_n|_{\mathrm{Spin}(n+1)}$ is the spinor representation of $\mathrm{Spin}(n+1)$; if n is odd, it is one of the half-spin representations. Therefore we have the following identifications,

$$\text{if } n = 2k, \quad \pi^*\Delta^q_{2k}(M) \cong \Delta^q_{2k+1}(\overline{M}),$$
$$\text{if } n = 2k+1, \quad \pi^*\Delta^q_{2k+1}(M) \cong \Delta^{q+}_{2k+2}(\overline{M}),$$

so that for any vector fields X and Y, and a spinor ψ

$$\pi^*(X \cdot \psi) = \frac{1}{r}X \cdot \partial_r \cdot \pi^*\psi, \qquad \pi^*(X \cdot Y \cdot \psi) = \frac{1}{r}X \cdot \frac{1}{r}Y \cdot \partial_r \cdot \pi^*\psi.$$

We shall now compute $\overline{\nabla}^q_X(\pi^*\psi)$ in the vertical and horizontal directions separately. Since the computation is local, let $U \subset M$ be an open neighborhood. A spinor may be represented by $\psi = [\sigma, \beta]$, where $\sigma \in \Gamma(U, P_{\mathrm{Spin}^q(n)}M)$, $\beta\colon U \longrightarrow \Delta^q_n$ is a Spin^q equivariant function, $\xi_1(\sigma) = (X_1, \ldots, X_n) \in \Gamma(U, P_{\mathrm{SO}(n)}M)$ is a local orthonormal frame and $\xi_2(\sigma) = s \in \Gamma(U, P_{Sp(1)}M)$. Then $\pi^*\psi = [\pi^*\sigma, \pi^*\beta]$, $\overline{\xi}_1(\pi^*\sigma) = \left(\frac{1}{r}X_1, \ldots, \frac{1}{r}X_n, \partial_r\right) \in \Gamma(\overline{U}, P_{\mathrm{SO}(n+1)}\overline{M})$ and $\overline{\xi}_2(\sigma) = \pi^*s \in \Gamma(\overline{U}, P_{Sp(1)}\overline{M})$.

Notice that $\overline{\nabla}_{\partial_r}\frac{1}{r}X = -\frac{1}{r^2}X + \frac{1}{r}\overline{\nabla}_{\partial_r}X = 0$, for every vector field X on M extended canonically to $\overline{M}$.

According to (3) applied to $\overline{M}$

$$\begin{aligned}
\overline{\nabla}^q_{\partial_r}\pi^*\psi &= [\pi^*\sigma, \partial_r(\pi^*\beta)] + \frac{1}{2}\sum_{j<k}\overline{g}\left(\overline{\nabla}_{\partial_r}\left(\frac{1}{r}X_j\right), \frac{1}{r}X_k\right)\frac{1}{r}X_j \cdot \frac{1}{r}X_k \cdot \pi^*\psi \\
&\quad + \frac{1}{2}\sum_{j}\overline{g}\left(\overline{\nabla}_{\partial_r}\left(\frac{1}{r}X_j\right), \partial_r\right)\frac{1}{r}X_j \cdot \partial_r \cdot \pi^*\psi + \frac{1}{2}\pi^*\theta((\pi^*s)_*(\partial_r))\pi^*\psi \\
&= \frac{1}{2}\theta((s\circ\pi)_*(\partial_r))\pi^*\psi = 0,
\end{aligned}$$

and for X a vector field of M extended to $\overline{M}$

$$\begin{aligned}
\overline{\nabla}^q_{X}\pi^*\psi &= [\pi^*\sigma, X(\pi^*\beta)] + \frac{1}{2}\sum_{j<k}\overline{g}\left(\overline{\nabla}_{X}\left(\frac{1}{r}X_j\right), \frac{1}{r}X_k\right)\frac{1}{r}X_j \cdot \frac{1}{r}X_k \cdot \pi^*\psi \\
&\quad + \frac{1}{2}\sum_{j}\overline{g}\left(\overline{\nabla}_{X}\left(\frac{1}{r}X_j\right), \partial_r\right)\frac{1}{r}X_j \cdot \partial_r \cdot \pi^*\psi + \frac{1}{2}\pi^*\theta((\pi^*s)_*(X))\pi^*\psi \\
&= \pi^*\left(\nabla^q_X\psi - \frac{1}{2}X \cdot \psi\right),
\end{aligned}$$

which proves the proposition. □

Proof of Theorem 4.1. From here we see that if M admits a real Killing Spin^q spinor, the cone $\overline{M}$ admits a parallel Spin^q spinor. By Theorem 3.1, the simply-connected manifold $\overline{M}$ is either Spin and Ricci-flat or a Kähler manifold with its canonical Spin^q structure. This implies that either

- M is Spin with flat connection on $P_{\mathrm{SO}(3)}$ admitting a real Killing spinor [2], or
- M is Sasakian, since this is equivalent to the cone $\overline{M}$ being Kähler [2]. □

Proposition 4.2. *Let M^{2k+1} denote a Sasakian manifold, then*

- *M admits a canonical* Spin^q *structure.*
- *If M is Einstein, the connection of the canonical bundle $P_{\mathrm{SO}(3)}(M)$ is flat. If M is also simply-connected, then M is Spin.*

Proof. First note that the Sasakian manifolds are necessarily odd dimensional. Next, M is Sasakian if and only if its cone $\overline{M}$ is Kähler. Thus, $\overline{M}$ has a canonical Spin^q structure (induced by its canonical Spin^c structure $\bigwedge^{k+1,0}\overline{M}$ and its canonical connection), which restricts to M.

If M is Einstein, $\overline{M}$ is Ricci-flat, so that its canonical Spin^c structure $\bigwedge^{k+1,0}\overline{M}$ is canonically flat, and so are the canonical Spin^q structure and its restriction to M. □

Finally, observe that the non-Einstein Sasakian manifolds admit (up to a constant) only one real Killing $Spin^q$ spinor.

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