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The elliptic genus on non-spin even 4-manifolds ☆

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Abstract

We prove the rigidity under circle actions of the elliptic genus on oriented non-spin closed smooth 4-manifolds with even intersection form.

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0. Introduction

The elliptic genus was introduced by S. Ochanine [11] as a topological genus and E. Witten conjectured its rigidity under circle actions on spin manifolds [13]. Witten's rigidity theorem was proved by Taubes [12], Bott and Taubes [5], etc. The aim of this note is to show that the elliptic genus is also rigid under circle actions on oriented *non-spin* 4-manifolds with even intersection form (see Theorem 1.1), and the proof is carried out along the lines of that of [5] for spin manifolds. Working in dimension 4, grants us several simplifying features on the fixed point sets and the opportunity to make certain calculations more explicit.

In Section 1 we recall some preliminaries concerning even 4-manifolds and the elliptic genus. In Section 2 we prove the Rigidity Theorem 1.1 for non-spin even 4-manifolds. In Section 3 we give an alternative proof of the vanishing of the signature and the $\widehat{A}$-genus on smooth even 4-manifolds with circle actions.

1. Preliminaries

1.1. Rigidity of the index of an elliptic operator

Let $D:\Gamma(E)\to\Gamma(F)$ be an elliptic operator acting on sections of the vector bundles E and F over a compact, connected, oriented, smooth manifold M. The index of D, $\mathrm{ind}(D)$, is the virtual dimension of the virtual vector space

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$$\mathrm{Ind}(D) = \ker(D) - \mathrm{coker}(D).$$

If M admits a circle action preserving D, i.e. such that S^1 acts on E and F, and commutes with D, the virtual vector space $\mathrm{Ind}(D)$ admits a (finite) Fourier decomposition into complex 1-dimensional irreducible representations of S^1

$$\ker(D) - \mathrm{coker}(D) = \sum a_m L^m,$$

where $a_m \in \mathbb{Z}$ and L^m is the representation of S^1 on $\mathbb{C}$ given by $\lambda \mapsto \lambda^m$. The elliptic operator D is called *rigid* for the given action if $a_m = 0$ for all $m \neq 0$, i.e. if $\mathrm{Ind}(D)$ consists of the trivial representation with multiplicity a_0. The elliptic operator D is called *universally rigid* if it is rigid under any S^1 action on M by isometries.

Note that this rigidity can be equivalently reflected by the trace of the action of $\lambda \in S^1$ on the corresponding spaces, i.e. by considering the equivariant index of D

$$\mathrm{ind}(D)_{S^1}(\lambda) = \mathrm{tr}_\lambda\big(\ker(D) - \mathrm{coker}(D)\big) = \sum_m a_m \lambda^m,$$

which is a (finite) Laurent polynomial on λ. The relevant feature of this expression is that it can be considered as the restriction to $S^1 \subset \mathbb{C}$ of a Laurent polynomial on $\lambda \in \mathbb{C}$ whose poles can only occur at 0 and ∞, a fact that will be used repeatedly later.

1.2. Elliptic genus

Let $\bigwedge_c^\pm$ be the even and odd complex differential forms on the oriented, compact, smooth 4-manifold M under the Hodge $*$-operator, respectively. The signature operator

$$d_s^M = d - {*d*} : \textstyle\bigwedge_c^+ \to \bigwedge_c^-$$

is elliptic and the virtual dimension of its index equals the signature of M, $\mathrm{sign}(M)$. If W is a complex vector bundle on M endowed with a connection, we can *twist* the signature operator to forms with values in W

$$d_s^M \otimes W : \textstyle\bigwedge_c^+(W) \to \bigwedge_c^-(W).$$

This operator is also elliptic and the virtual dimension of its index is denoted by $\mathrm{sign}(M, W)$.

Definition 1.1. Let $T = TM \otimes \mathbb{C}$ denote the complexified tangent bundle of M and let R_i be the sequence of bundles defined by the formal series

$$R(q, T) = \sum_{i=0}^{\infty} R_i q^i = \bigotimes_{i=1}^{\infty} \textstyle\bigwedge_{q^i} T \otimes \displaystyle\bigotimes_{j=1}^{\infty} S_{q^j} T,$$

where $S_t T = \sum_{k=0}^{\infty} S^k T t^k$, $\bigwedge_t T = \sum_{k=0}^{\infty} \bigwedge^k T t^k$, and $S^k T$, $\bigwedge^k T$ denote the kth symmetric and exterior tensor powers of T, respectively. The *elliptic genus* of M is defined as

$$\Phi(M) = \mathrm{ind}\big(d_s^M \otimes R(q, T)\big) = \sum_{i=0}^{\infty} \mathrm{sign}(M, R_i) \cdot q^i. \tag{1}$$

Note that the first few terms of the sequence $R(q, T)$ are $R_0 = 1$, $R_1 = 2T$, $R_2 = 2(T^{\otimes 2} + T)$. In particular, the constant term of $\Phi(M)$ is $\mathrm{sign}(M)$.

The equivariant elliptic genus with respect to the S^1 action is

$$\Phi(M)_{S^1}(\lambda) = \sum_{i=0}^{\infty} \mathrm{sign}(M, R_i)_{S^1}(\lambda) \cdot q^i. \tag{2}$$

The main theorem of this article is the following.

Theorem 1.1. *Let M be an oriented, compact, connected, non-spin, even, smooth 4-manifold admitting smooth S^1 actions. Then, each of the operators $d_s \otimes R_i$ is universally rigid.*

1.3. The complex projective plane

In order to become acquainted with the rigidity property, let us examine the elliptic genus on the complex projective plane $\mathbb{CP}^2$.

It is well known that the signature operator d_s is rigid on any oriented smooth manifold admitting (isometric) circle actions, including non-spin manifolds such as $\mathbb{CP}^2$. However, the twisted signature operator $d_s \otimes T$ (the so-called Rarita–Schwinger operator) fails to be universally rigid on $\mathbb{CP}^2$ (see [9,7]) as we shall see next. We denote by $T\mathbb{CP}^2_c = T\mathbb{CP}^2 \otimes \mathbb{C}$ the complexified tangent bundle of $\mathbb{CP}^2$. Since $\mathbb{CP}^2$ is a homogeneous space for the Lie group $SU(3)$ we can describe the relevant spaces as $SU(3)$ representations.

Let $F(\lambda_1, \lambda_2, \lambda_3)$ denote the complex irreducible representation of $SU(3)$ with dominant weight $(\lambda_1, \lambda_2, \lambda_3)$, where the coordinates are such that $F(1,0,0) = \mathbb{C}^3$ and $F(1,1,0) = \mathfrak{su}(3)$ are the standard and adjoint representations of $SU(3)$ respectively.

As shown in [7], the $SU(3)$-representation corresponding to $\operatorname{sign}(\mathbb{CP}^2)$ is the one-dimensional trivial representation

$$\operatorname{Ind}\big(d_s^{\mathbb{CP}^2}\big) = F(0,0,0),$$

so that

$$\operatorname{sign}\big(\mathbb{CP}^2\big) = 1.$$

The representation corresponding to $\operatorname{sign}(\mathbb{CP}^2, T\mathbb{CP}^2_c)$ is the 16-dimensional $SU(3)$-representation

$$\operatorname{Ind}\big(d_s^{\mathbb{CP}^2} \otimes T\mathbb{CP}^2_c\big) = 2F(0,0,0) \oplus F(1,0,1) \oplus F(1,0,0) \oplus F(0,0,-1),$$

where $\dim F(0,0,0) = 1$, $\dim F(1,0,1) = 8$, $\dim F(1,0,0) = 3$, $\dim F(0,0,-1) = 3$, so that

$$\operatorname{sign}\big(\mathbb{CP}^2, T\mathbb{CP}^2_c\big) = 16.$$

Let g be a projective involution of $\mathbb{CP}^2$ with fixed point set a projective line and a point. By the Weyl character formula, the trace of g on each one of them is

$$\begin{aligned}
\operatorname{sign}\big(\mathbb{CP}^2\big)^{S^1}(g) &= \operatorname{tr}_g\big(F(0,0,0)\big) = 1, \\
\operatorname{sign}\big(\mathbb{CP}^2, T\mathbb{CP}^2_c\big)^{S^1}(g) &= \operatorname{tr}_g\big(2F(0,0,0) \oplus F(1,0,1) \oplus F(1,0,0,) \oplus F(0,0,-1)\big) \\
&= 2 + 0 + (-1) + (-1) = 0,
\end{aligned}$$

thus showing the non-rigidity of the operator $d_s \otimes T\mathbb{CP}^2_c$.

1.4. Even intersection form

The intersection form of a closed oriented 4-manifold M is an unimodular symmetric bilinear form over the integers. Donaldson [6] proved that among all the *definite* bilinear forms, only the diagonalizable ones can be the intersections forms of smooth closed 4-manifolds. An indefinite bilinear form is called *odd* if there exists x such that $Q(x,x)$ is odd, and it is called *even* otherwise. The indefinite forms Q which are odd are diagonalizable, so that we are left to consider indefinite even unimodular bilinear forms. Let E_8 be the unique irreducible negative definite quadratic form of rank eight and let H be the hyperbolic quadratic form. It is known that any indefinite even bilinear form Q is of the form $aE_8 \oplus bH$, $a, b \in \mathbb{Z}$. A smooth 4-manifold is called *even* if its intersection form is even. It is a well-known fact that all spin manifolds are even, but the converse is not true [1].

Now, we give some interesting properties of S^1 actions on smooth even 4-manifolds. Assume M is endowed with a (non-trivial) smooth S^1-action. Let M^{S^1} denote the fixed point set of the circle action. At each point $p \in M^{S^1}$, the tangent space of M splits as a sum of S^1 representations

$$T_pM = L^{m_1(p)} \oplus L^{m_2(p)},$$

where L^m denotes the S^1 representation on which $\lambda \in S^1$ acts by multiplication by λ^m. The numbers $m_1(p), m_2(p)$ are called the *exponents* (or *weights*) of the S^1-action at the point p. The exponents of an action are not canonical and their sign can be changed in pairs. The space $T_pM^{S^1}$ is a trivial representation of S^1, i.e. in dimension 4 at most one exponent can equal 0.

Consider the sum of the exponents $\sigma(p) = m_1(p) + m_2(p)$. The number $\sigma(p)$ is constant along connected components of M^{S^1}, but may vary for different connected components. Note that in this dimension, the connected components P of M^{S^1} are oriented totally geodesic submanifolds of even codimension, i.e. oriented surfaces or isolated fixed points.

Definition 1.2. A circle action on an oriented 4-dimensional manifold M with non-empty fixed point set will be called either

- even if $\sigma(p) \equiv 0 \pmod 2$ for all $p \in M^{S^1}$, or
- odd if $\sigma(p) \equiv 1 \pmod 2$ for all $p \in M^{S^1}$.

Lemma 1.1. *Let M be an oriented, connected, compact even smooth 4-manifold. Assume M admits a effective smooth S^1 action. Then*

$$\sigma(p_1) \equiv \sigma(p_2) \pmod 2$$

for all $p_1, p_2 \in M^{S^1}$. In particular, the S^1-action is either even or odd.

Proof. First note that by the effectiveness assumption the fixed point set M_k of the finite subgroup $\mathbb{Z}_k$ of S^1 has codimension greater than or equal to 2. Thus, consider a path joining p_1 and p_2 whose interior points are different from p_1 and p_2, and which is also disjoint from submanifolds with finite isotropy. Let S be the sphere generated by letting S^1 act on the path. Then $TM|_S$ is an even dimensional, real, oriented, equivariant bundle on the sphere S, and, by [5, Lemma 9.2], $TM|_S$ can be considered as a complex equivariant vector bundle on S. Furthermore, by [5, Lemma 9.1],

$$\langle c_1(TM|_S), [S]\rangle = \sigma(p_1) - \sigma(p_2).$$

On the other hand,

$$TM|_S = TS \oplus \nu,$$

where ν is the normal bundle of S in M and $c_1(TM|_S) = c_1(S) + c_1(\nu)$. Thus

$$\langle c_1(TM|_S), [S]\rangle = \langle c_1(TS), [S]\rangle + \langle c_1(\nu), [S]\rangle = 2 + S \cdot S \equiv 0 \pmod 2$$

by the assumption on the intersection form, where $S \cdot S$ denotes the self-intersection number of S. □

From this we see the following.

Lemma 1.2. *Let M be an oriented, connected, compact even smooth 4-manifold admitting an effective smooth S^1 action.*

- *If M^{S^1} contains a surface, then the action is necessarily odd.*
- *If the action is even, the M^{S^1} consists of isolated fixed points only.*

2. Rigidity of the elliptic genus

Proof of Theorem 1.1. Let us assume that the S^1 action is effective. By the Atiyah–Segal G-signature theorem [4], the equivariant elliptic genus can be expressed as

$$\Phi(M)_{S^1}(\lambda) = \sum_{P \subset M^{S^1}} \mu_P(\lambda), \tag{3}$$

where the sum runs over the connected components P of the fixed point set M^{S^1} of the S^1 action.

Remark. The rigidity theorem is equivalent to showing that $\Phi(M)_{S^1}(\lambda)$ does not depend on λ.

Given (3), let us examine the contributions $\mu_P(\lambda)$. In this dimension, the connected components P of M^{S^1} are oriented totally geodesic submanifolds of even codimension, i.e. oriented surfaces or isolated fixed points. The tangent bundle of M restricted to P splits as

$$TM|_P = L^{m_1} \oplus L^{m_2},$$

where L^{m_i} denotes the complex line bundle L^{m_i} on which S^1 acts by λ^{m_i}, $m_i = m_i(P) \in \mathbb{Z}$. We have two possibilities

- $\dim P = 0$: $m_1, m_2 \neq 0$, and they must be coprime since the circle action is effective

$$\mu_P(\lambda) = \frac{(1+\lambda^{m_1})}{(1-\lambda^{m_1})}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda^{m_1})(1+q^k\lambda^{-m_1})}{(1-q^k\lambda^{m_1})(1-q^k\lambda^{-m_1})} \times \frac{(1+\lambda^{m_2})}{(1-\lambda^{m_2})}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda^{m_2})(1+q^k\lambda^{-m_2})}{(1-q^k\lambda^{m_2})(1-q^k\lambda^{-m_2})}, \tag{4}$$

- $\dim P = 2$: L^{m_1} is a trivial representation ($m_1 = 0$). Note that $L^{m_2} = \nu$ the normal bundle to P in M and that we can take $m_2 = 1$, since the circle action is effective. Thus,

$$\mu_P(\lambda) = \left\langle x_1 \frac{(1+e^{-x_1})}{(1-e^{-x_1})}\prod_{k=1}^{\infty}\frac{(1+q^k e^{-x_1})(1+q^k e^{x_1})}{(1-q^k e^{-x_1})(1-q^k e^{x_1})} \times \frac{(1+\lambda e^{-x_2})}{(1-\lambda e^{-x_2})}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda e^{-x_2})(1+q^k\lambda^{-1}e^{x_2})}{(1-q^k\lambda e^{-x_2})(1-q^k\lambda^{-1}e^{x_2})}, [P]\right\rangle, \tag{5}$$

  where $x_1 = c_1(P)$ and $x_2 = c_1(L^{m_2})$.

Claim 1. *The contributions $\mu_P(\lambda)$ are meromorphic functions of $\lambda \in \mathbb{T}_{q^2}$, where $\mathbb{T}_{q^2}$ denotes the 2-dimensional torus $\mathbb{C}^*/q^2$, the quotient of $\mathbb{C}^* = \mathbb{C} - \{0\}$ by the multiplicative subgroup generated by the element $q^2 \neq 0$.*

Let p denote an isolated S^1-fixed point and consider one of the factors of its contribution (4), say

$$\frac{(1+\lambda^{m_1})}{(1-\lambda^{m_1})}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda^{m_1})(1+q^k\lambda^{-m_1})}{(1-q^k\lambda^{m_1})(1-q^k\lambda^{-m_1})},$$

and substitute λ by $q\lambda$ so that

$$\frac{(1+q^{m_1}\lambda^{m_1})}{(1-q^{m_1}\lambda^{m_1})}\prod_{k=1}^{\infty}\frac{(1+q^k q^{m_1}\lambda^{m_1})(1+q^k q^{-m_1}\lambda^{-m_1})}{(1-q^k q^{m_1}\lambda^{m_1})(1-q^k q^{-m_1}\lambda^{-m_1})} = (-1)^{m_1}\frac{(1+\lambda^{m_1})}{(1-\lambda^{m_1})}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda^{m_1})(1+q^k\lambda^{-m_1})}{(1-q^k\lambda^{m_1})(1-q^k\lambda^{-m_1})},$$

so that

$$\mu_p(q\lambda) = (-1)^{m_1+m_2}\mu_p(\lambda),$$

which shows that $\mu_p(\lambda)$ is the pullback of a meromorphic function on the torus $\mathbb{T}_{q^2}$ since $\mu_p(q^2\lambda) = \mu_p(\lambda)$.

Secondly, let P denote an S^1-fixed surface and consider its contribution (5). Substitute λ by $q\lambda$

$$\mu_P(q\lambda) = \left\langle x_1 \frac{(1+e^{-x_1})}{(1-e^{-x_1})}\prod_{k=1}^{\infty}\frac{(1+q^k e^{-x_1})(1+q^k e^{x_1})}{(1-q^k e^{-x_1})(1-q^k e^{x_1})} \times \frac{(1+q\lambda e^{-x_2})}{(1-q\lambda e^{-x_2})}\prod_{k=1}^{\infty}\frac{(1+q^k q\lambda e^{-x_2})(1+q^k q^{-1}\lambda^{-1}e^{x_2})}{(1-q^k q\lambda e^{-x_2})(1-q^k q^{-1}\lambda^{-1}e^{x_2})}, [P]\right\rangle$$

$$= -\left\langle x_1 \frac{(1+e^{-x_1})}{(1-e^{-x_1})} \prod_{k=1}^{\infty} \frac{(1+q^k e^{-x_1})(1+q^k e^{x_1})}{(1-q^k e^{-x_1})(1-q^k e^{x_1})} \right.$$
$$\left. \times \frac{(1+\lambda e^{-x_2})}{(1-\lambda e^{-x_2})} \prod_{k=1}^{\infty} \frac{(1+q^k \lambda e^{-x_2})(1+q^k \lambda^{-1} e^{x_2})}{(1-q^k \lambda e^{-x_2})(1-q^k \lambda^{-1} e^{x_2})}, [P] \right\rangle$$
$$= -\mu_P(\lambda),$$

so that $\mu_P(q^2\lambda) = \mu_P(\lambda)$, which means that $\mu_P(\lambda)$ is the pullback of a meromorphic function on the torus $\mathbb{T}_{q^2}$.

Claim 2. *The meromorphic function $\Phi(M)_{S^1}(\lambda)$ has no poles at all on $\mathbb{T}_{q^2}$, i.e. it is holomorphic on a compact surface and, hence, constant on the variable λ.*

By looking at

$$\frac{(1+\lambda^{m_1})}{(1-\lambda^{m_1})} \prod_{k=1}^{\infty} \frac{(1+q^k\lambda^{m_1})(1+q^k\lambda^{-m_1})}{(1-q^k\lambda^{m_1})(1-q^k\lambda^{-m_1})},$$

we notice that poles may occur at the values of λ where the denominators vanish, i.e. we have the equations

$$1-\lambda^{m_1}=0, \qquad 1-q^k\lambda^{m_1}=0, \qquad 1-q^k\lambda^{-m_1}=0,$$

which means the poles may occur at

$$\lambda = 1^{1/m_1}, \qquad \lambda = q^{-k/m_1}, \qquad \lambda = q^{k/m_1}.$$

Secondly, by expanding $\mu_P(\lambda)$ up to dimension 2 we get the following expression

$$\mu_P(\lambda) \equiv 2\left\langle \prod_{k=1}^{\infty}\left[\left(\frac{1+q^k}{1-q^k} - \frac{2q^k x_1}{(1-q^k)^2}\right)\left(\frac{1+q^k}{1-q^k} + \frac{2q^k x_1}{(1-q^k)^2}\right)\right]\right.$$
$$\left. \times \left(\frac{1+\lambda}{1-\lambda} - \frac{2\lambda x_2}{(1-\lambda)^2}\right) \prod_{k=1}^{\infty}\left[\left(\frac{1+q^k\lambda}{1-q^k\lambda} - \frac{2q^k\lambda x_2}{(1-q^k\lambda)^2}\right)\left(\frac{\lambda+q^k}{\lambda-q^k} + \frac{2q^k\lambda x_2}{(\lambda-q^k)^2}\right)\right], [P]\right\rangle$$

which shows that its poles could occur only if

$$1-\lambda=0, \qquad 1-q^k\lambda=0, \qquad 1-q^k\lambda^{-1}=0,$$

i.e.

$$\lambda=1, \qquad \lambda=q^{-k}, \qquad \lambda=q^k.$$

Since we are now working on the torus $\mathbb{T}_{q^2}$ and $q^2 \equiv 1 \pmod{q^2}$, we need to check for poles at 1 and q.

- *No poles at roots of unity $\lambda^m = 1$.*

Observe that the coefficient of q^i in (2), for all i, is the equivariant index of a twisted signature operator $\mathrm{ind}(d_s^M \otimes W)_{S^1}(\lambda)$, where W is some finite rank (virtual) vector bundle and, therefore, it is a finite Laurent polynomial on λ with poles only at $\lambda = 0, \infty$. Moreover, it has no pole on the unit circle $|\lambda| = 1$. This property is shared by (3) so that $\Phi(M)_{S^1}(\lambda)$ has no pole on the circle $|\lambda| = 1$.

- *No pole at q.*

There could be a pole at q whenever an S^1-fixed point has one exponent $m_i = 1$.

Now, $\Phi(M)_{S^1}(\lambda)$ has a pole at $\lambda = q$ if and only if $\Psi(\lambda) = \Phi(M)_{S^1}(q\lambda)$ has a pole at 1. We have seen that the contributions behave as follows

$$\mu_p(q\lambda) = (-1)^{m_1+m_2}\mu_p(\lambda),$$

for an isolated S^1-fixed point, and

$$\mu_P(q\lambda) = -\mu_P(\lambda)$$

for an S^1-fixed surface.

By Lemma 1.2, if there is an S^1-fixed surface, then the action is odd, and $(-1)^{m_1(p)+m_2(p)} = -1$ for every fixed point p. This means that $\Psi(\lambda) = -\Phi(M)_{S^1}(\lambda)$ has no pole at 1.

On the other hand, if M^{S^1} consists of isolated fixed points only, then

$$m_1(p) + m_2(p) \equiv m_1(p') + m_2(p') \pmod 2$$

by Lemma 1.1, for any fixed points $p, p' \in M^{S^1}$, so that $\Psi(\lambda) = \pm\Phi(M)_{S^1}(\lambda)$ has no pole at 1.

- *No pole at $q^{k_0/m}$.*

In order to prove that $\Phi(M)_{S^1}(\lambda)$ is holomorphic on $\mathbb{T}_{q^2}$, we must show that it has no poles on the points

$$\lambda = q^{-k/m}, \qquad \lambda = q^{k/m},$$

for all k, and all m belonging to the collection of exponents of all the isolated fixed points. The pole $q^{k/m}$ will appear in $\mu_p(\lambda)$ if and only if $m = m_1(p)$ or $m = m_2(p)$ for some p.

Let k_0 denote a fixed integer. Notice that if a point p has an exponent $m_1(p) = m > 0$ then there is a submanifold S_p of $\mathbb{Z}_m$-fixed points containing p. The submanifold S_p is necessarily an S^1-invariant 2-sphere with two isolated S^1-fixed points: p and another point p'. Since the exponents of the tangent space to the sphere at the two points p and p' differ only by sign, let us assume that $m_1(p) = -m_1(p') = m$. Let $m_2 = m_2(p)$ and $m_2' = m_2(p')$. Thus, the contributions of p and p' add up to

$$\mu_p(\lambda) + \mu_{p'}(\lambda) = \left(\frac{(1+\lambda^m)}{(1-\lambda^m)}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda^m)(1+q^k\lambda^{-m})}{(1-q^k\lambda^m)(1-q^k\lambda^{-m})}\right)$$
$$\times\left[\frac{(1+\lambda^{m_2})}{(1-\lambda^{m_2})}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda^{m_2})(1+q^k\lambda^{-m_2})}{(1-q^k\lambda^{m_2})(1-q^k\lambda^{-m_2})} - \frac{(1+\lambda^{m_2'})}{(1-\lambda^{m_2'})}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda^{m_2'})(1+q^k\lambda^{-m_2'})}{(1-q^k\lambda^{m_2'})(1-q^k\lambda^{-m_2'})}\right].$$

Now, $\mu_p(\lambda) + \mu_{p'}(\lambda)$ will have a pole at $q^{k_0/m}$ if and only if

$$\mu_{k_0/m}(\lambda) = \mu_p\big(q^{k_0/m}\lambda\big) + \mu_{p'}\big(q^{k_0/m}\lambda\big) \tag{6}$$

has a pole at 1. Let us examine (6)

$$\mu_p\big(q^{k_0/m}\lambda\big) + \mu_{p'}\big(q^{k_0/m}\lambda\big)$$
$$= \left(\frac{(1+q^{mk_0/m}\lambda^m)}{(1-q^{mk_0/m}\lambda^m)}\prod_{k=1}^{\infty}\frac{(1+q^kq^{mk_0/m}\lambda^m)(1+q^kq^{-mk_0/m}\lambda^{-m})}{(1-q^kq^{mk_0/m}\lambda^m)(1-q^kq^{-mk_0/m}\lambda^{-m})}\right)$$
$$\times\left[\frac{(1+q^{m_2k_0/m}\lambda^{m_2})}{(1-q^{m_2k_0/m}\lambda^{m_2})}\prod_{k=1}^{\infty}\frac{(1+q^kq^{m_2k_0/m}\lambda^{m_2})(1+q^kq^{-m_2k_0/m}\lambda^{-m_2})}{(1-q^kq^{m_2k_0/m}\lambda^{m_2})(1-q^kq^{-m_2k_0/m}\lambda^{-m_2})}\right.$$
$$\left. - \frac{(1+q^{m_2'k_0/m}\lambda^{m_2'})}{(1-q^{m_2'k_0/m}\lambda^{m_2'})}\prod_{k=1}^{\infty}\frac{(1+q^kq^{m_2'k_0/m}\lambda^{m_2'})(1+q^kq^{-m_2'k_0/m}\lambda^{-m_2'})}{(1-q^kq^{m_2'k_0/m}\lambda^{m_2'})(1-q^kq^{-m_2'k_0/m}\lambda^{-m_2'})}\right]$$
$$= \left(\frac{(1+q^{k_0}\lambda^m)}{(1-q^{k_0}\lambda^m)}\prod_{k=1}^{\infty}\frac{(1+q^kq^{k_0}\lambda^m)(1+q^kq^{-k_0}\lambda^{-m})}{(1-q^kq^{k_0}\lambda^m)(1-q^kq^{-k_0}\lambda^{-m})}\right)$$
$$\times\left[\frac{(1+q^{[m_2k_0/m]}q^{w(p)/m}\lambda^{m_2})}{(1-q^{[m_2k_0/m]}q^{w(p)/m}\lambda^{m_2})}\prod_{k=1}^{\infty}\frac{(1+q^kq^{[m_2k_0/m]}q^{w(p)/m}\lambda^{m_2})(1+q^kq^{-[m_2k_0/m]}q^{-w(p)/m}\lambda^{-m_2})}{(1-q^kq^{[m_2k_0/m]}q^{w(p)/m}\lambda^{m_2})(1-q^kq^{-[m_2k_0/m]}q^{-w(p)/m}\lambda^{-m_2})}\right.$$

$$
\begin{aligned}
&- \frac{(1+q^{[m_2'k_0/m]}q^{w(p')/m}\lambda^{m_2'})}{(1-q^{[m_2'k_0/m]}q^{w(p')/m}\lambda^{m_2'})}\prod_{k=1}^{\infty}\frac{(1+q^kq^{[m_2'k_0/m]}q^{w'/m}\lambda^{m_2'})(1+q^kq^{-[m_2'k_0/m]}q^{-w(p')/m}\lambda^{-m_2'})}{(1-q^kq^{[m_2'k_0/m]}q^{w(p')/m}\lambda^{m_2'})(1-q^kq^{-[m_2'k_0/m]}q^{-w(p')/m}\lambda^{-m_2'})}\Bigg]\\
&= (-1)^{k_0}\left(\frac{(1+\lambda^m)}{(1-\lambda^m)}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda^m)(1+q^k\lambda^{-m})}{(1-q^k\lambda^m)(1-q^k\lambda^{-m})}\right)\\
&\times\Bigg[(-1)^{[m_2k_0/m]}\frac{(1+q^{w(p)/m}\lambda^{m_2})}{(1-q^{w(p)/m}\lambda^{m_2})}\prod_{k=1}^{\infty}\frac{(1+q^kq^{w(p)/m}\lambda^{m_2})(1+q^kq^{-w(p)/m}\lambda^{-m_2})}{(1-q^kq^{w(p)/m}\lambda^{m_2})(1-q^kq^{-w(p)/m}\lambda^{-m_2})}\\
&- (-1)^{[m_2'k_0/m]}\frac{(1+q^{w(p')/m}\lambda^{m_2'})}{(1-q^{w(p')/m}\lambda^{m_2'})}\prod_{k=1}^{\infty}\frac{(1+q^kq^{w(p')/m}\lambda^{m_2'})(1+q^kq^{-w(p')/m}\lambda^{-m_2'})}{(1-q^kq^{w(p')/m}\lambda^{m_2'})(1-q^kq^{-w(p')/m}\lambda^{-m_2'})}\Bigg],
\end{aligned}
\tag{7}
$$

where $[x]$ denotes the largest integer less than or equal x and $w(p) = k_0m_2(p) - [k_0m_2/m]m$ is the residue modulo m of k_0m_2.

In order to continue manipulating (7), we need the following lemma.

Lemma 2.1.

$$w(p) = w(p'),$$

and

$$[m_2k_0/m] \equiv [m_2'k_0/m] \pmod 2.$$

Proof. On the one hand, by Lemma 1.1

$$m + m_2 \equiv m + m_2' \pmod 2,$$

so that

$$m_2 \equiv m_2' \pmod 2,$$

On the other hand, since S_p is $\mathbb{Z}_m$-fixed, the infinitesimal action on ν must satisfy

$$m_2 \equiv m_2' \pmod m,$$
$$m_2 - m_2' = 2bm,$$

for some $b \in \mathbb{Z}$. This implies

$$k_0m_2 = l_pm + w(p) \equiv k_0m_2' = l_{p'}m + w(p') \pmod m,$$

where $l_p = [k_0m_2/m]$ and $l_{p'} = [k_0m_2'/m]$. Since

$$w(p) \equiv w(p') \pmod m \quad \text{and} \quad 0 \leqslant w(p), w(p') < m,$$

the two residues must be equal

$$w(p) = w(p'),$$

so that

$$l_pm - l_{p'}m \equiv 0 \pmod m,$$
$$l_pm - l_{p'}m = k_0(m_2 - m_2') = 2bmk_0,$$

and

$$l_p - l_{p'} = 2bk_0. \qquad \square$$

Thus, (7) becomes

$$
\begin{aligned}
&\mu_p\big(q^{k_0/m}\lambda\big)+\mu_{p'}\big(q^{k_0/m}\lambda\big)\\
&\quad=(-1)^{k_0+[m_2(p)k_0/m]}\left(\frac{(1+\lambda^m)}{(1-\lambda^m)}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda^m)(1+q^k\lambda^{-m})}{(1-q^k\lambda^m)(1-q^k\lambda^{-m})}\right)\\
&\qquad\times\Bigg[\frac{(1+q^{w(p)/m}\lambda^{m_2})}{(1-q^{w(p)/m}\lambda^{m_2})}\prod_{k=1}^{\infty}\frac{(1+q^kq^{w(p)/m}\lambda^{m_2})(1+q^kq^{-w(p)/m}\lambda^{-m_2})}{(1-q^kq^{w(p)/m}\lambda^{m_2})(1-q^kq^{-w(p)/m}\lambda^{-m_2})}\\
&\qquad-\frac{(1+q^{w(p)/m}\lambda^{m'_2})}{(1-q^{w(p)/m}\lambda^{m'_2})}\prod_{k=1}^{\infty}\frac{(1+q^kq^{w(p)/m}\lambda^{m'_2})(1+q^kq^{-w(p)/m}\lambda^{-m'_2})}{(1-q^kq^{w(p)/m}\lambda^{m'_2})(1-q^kq^{-w(p)/m}\lambda^{-m'_2})}\Bigg]\\
&\quad=\pm\left(\frac{(1+\lambda^m)}{(1-\lambda^m)}\prod_{k=1}^{\infty}\frac{(1+q^k\lambda^m)(1+q^k\lambda^{-m})}{(1-q^k\lambda^m)(1-q^k\lambda^{-m})}\right)\\
&\qquad\times\Bigg[\frac{(1+\alpha\lambda^{m_2})}{(1-\alpha\lambda^{m_2})}\prod_{k=1}^{\infty}\frac{(1+q^k\alpha\lambda^{m_2})(1+q^k\alpha^{-1}\lambda^{-m_2})}{(1-q^k\alpha\lambda^{m_2})(1-q^k\alpha^{-1}\lambda^{-m_2})}\\
&\qquad-\frac{(1+\alpha\lambda^{m'_2})}{(1-\alpha\lambda^{m'_2})}\prod_{k=1}^{\infty}\frac{(1+q^k\alpha\lambda^{m'_2})(1+q^k\alpha^{-1}\lambda^{-m'_2})}{(1-q^k\alpha\lambda^{m'_2})(1-q^k\alpha^{-1}\lambda^{-m'_2})}\Bigg] \qquad (8)
\end{aligned}
$$

where $\alpha=q^{w(p)/m}$. At this point, we recall Bott–Taubes' observation by which (8) is the equivariant version of the index

$$
\begin{aligned}
&\pm\mathrm{ind}\left(d_s^{S_p}\otimes R\big(q,(TS_p)_c\big)\otimes\left(\frac{\bigwedge_{\alpha}\nu}{\bigwedge_{-\alpha}\nu}\otimes\bigotimes_{k=1}^{\infty}\frac{\bigwedge_{\alpha q^k}\nu}{\bigwedge_{-\alpha q^k}\nu}\otimes\frac{\bigwedge_{\alpha^{-1}q^k}\nu^*}{\bigwedge_{-\alpha^{-1}q^k}\nu^*}\right)\right)\\
&\quad=\pm\left\langle x_1\frac{(1+e^{-x_1})}{(1-e^{-x_1})}\prod_{k=1}^{\infty}\frac{(1+q^ke^{-x_1})(1+q^ke^{x_1})}{(1-q^ke^{-x_1})(1-q^ke^{x_1})}\cdot\frac{(1+\alpha e^{-x_2})}{(1-\alpha e^{-x_2})}\prod_{k=1}^{\infty}\frac{(1+\alpha q^ke^{-x_2})(1+\alpha^{-1}q^ke^{x_2})}{(1-\alpha q^ke^{-x_2})(1-\alpha^{-1}q^ke^{x_2})},[S_p]\right\rangle,
\end{aligned}
$$

where $x_1=c_1(S_p)$, $\nu=\nu(S_p)$ denotes the normal bundle to S_p in M (which can be considered as a complex line bundle [5, Lemma 9.2]) and $x_2=c_1(\nu)$.

Observe that the coefficient of q^i, for all i, is an equivariant index $\mathrm{ind}(d_s^{S_p}\otimes W)(\lambda)$ for some finite rank (virtual) vector bundle W and therefore a finite Laurent polynomial on λ with poles only at 0 and ∞, and no pole at 1. Thus (6) has no pole at 1, i.e. $\Phi(M)_{S^1}(\lambda)$ has no pole at $q^{k_0/m}$. □

3. Vanishing of the signature

As a consequence, we obtain a new proof of the following vanishing result [10], which generalizes the Atiyah–Hirzebruch vanishing theorem of the $\widehat{A}$-genus on spin manifolds [3].

Corollary 3.1. *Let M be an even 4-manifold admitting smooth circle actions, and let $Q=aE_8\oplus bH$ denote its intersection form. Then the signature of M vanishes,* $\mathrm{sign}(M)=0$, *i.e., the intersection form is $Q=bH$.*

Proof. Since in dimension 4 we have that $\mathrm{sign}(M)=-8\widehat{A}(M)$, we shall prove Corollary 3.1 by proving the vanishing of $\widehat{A}(M)$. Since we are also considering the case when M may be non-spin, $\widehat{A}(M)$ may only be defined as a characteristic number and may not represent the index of an elliptic operator. Thus, $\widehat{A}(M)$ may, in principle, be a rational number.

According to Theorem 1.1, the value of $\Phi(M)_{S^1}(\lambda)$ does not depend on λ. Applying the Atiyah–Bott fixed point theorem [2], $\Phi(M)_{S^1}(\lambda)$ can be expressed in terms of the fixed point set of $\lambda\in S^1$ and the action of λ on its normal bundle of in M. In particular, let $\lambda=-1\in S^1$ be the orientation preserving involution in $\mathbb{Z}_2\subset S^1$, and let M_2 denote its fixed point set. We denote the transversal self-intersection of M_2 by $M_2\circ M_2$. In [9, p. 315], Hirzebruch and Slodowy showed that

$$\Phi(M)_{S^1}(-1)=\Phi(M_2\circ M_2).$$

On the other hand, applying Theorem 1.1

$$\Phi(M) = \Phi(M)_{S^1}(\lambda) = \Phi(M)_{S^1}(-1) = \Phi(M_2 \circ M_2). \tag{9}$$

The codimension of M_2 is positive and even, so that the elliptic genus $\Phi(M)$ can now be computed from the elliptic genera of submanifolds of M of codimension at least 4, i.e. isolated points.

Now, recall the expansion of $\Phi(M)$ at the other cusp [8]

$$\tilde{\Phi}(M) = \frac{1}{q^{\dim(M)/8}} \sum_{j=0}^{\infty} \widehat{A}(M, R'_j) \cdot q^j,$$

where R'_j is the sequence of virtual tensor bundles given by

$$R'(q, T) = \bigotimes_{k=2m+1} \bigwedge\nolimits_{-q^k} T \otimes \bigotimes_{k=2m+2} S_{q^k} T,$$

and the $\widehat{A}(M, R'_j) = \langle \widehat{A}(M) \cdot \mathrm{ch}(R'_j), [M] \rangle$ may only defined be as characteristic numbers. The first few terms of the sequence are $R'_0 = 1$, $R'_1 = -T$, $R'_2 = \bigwedge^2 T + T$, etc. This expansion is obtained by considering $q = e^{\pi i t}$ and changing the t coordinate in (1) by $t \to -1/t$, and then by $t \to 2t$ (cf. [8]). This expansion has, a priori, a pole of order $1/2$ in the variable q. On the other hand, by (9) we also have

$$\tilde{\Phi}(M) = \tilde{\Phi}(M_2 \circ M_2), \tag{10}$$

whose right-hand side has a pole of order at most 0 on the variable q, since the dimension of any connected component of $M_2 \circ M_2$ is at most 0. Therefore (10) implies that the first coefficient on the left-hand side vanishes,

$$\widehat{A}(M) = 0. \qquad \square$$

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