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Dissection of solutions in cooperative game theory using representation techniques

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Abstract We compute a decomposition for the space of cooperative TU-games under the action of the symmetric group S_n. In particular we identify all irreducible subspaces that are relevant to the study of symmetric linear solutions – namely those that are isomorphic to the irreducible summands of $\mathbb{R}^n$. We then use such decomposition to derive, in a very economical way, some old and some new results for linear symmetric solutions.

1 Introduction

In this article we study linear, symmetric solutions for the space of cooperative TU games with n players using basic representation theory of the group of permutations S_n.

Representation theory is a general tool for organizing linear algebra data in the presence of a group of symmetries. It makes sense to use it, first, as a "bookkeeping" tool, converting arguments that would typically require some ingenuity on the part of the researcher into routine exercises. More importantly, it presents the information in a more clear and concise way, thus shedding new light into the relations (sometimes hidden until then) between the elements

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that participate in a given problem. We believe we have shown how this is true in the context of linear symmetric solutions in cooperative game theory.

Briefly, what we do is the following. We derive direct sum decompositions of the space of games and the space of payoffs into "elementary" pieces. Moreover, any linear, symmetric solution when restricted to any such elementary subspace is either zero or multiplication by a single scalar, regardless of the dimension of the elementary subspace – this follows from the so called Schur's lemma; therefore, all linear, symmetric solutions may be written (simultaneously) as a sum of trivial maps.

Once we have such a global description of all linear and symmetric solutions, it is easy to understand the restrictions imposed by other conditions or axioms, for example: efficiency, dummy player axiom, self and anti-self-duality, etc.

The reader will find here new proofs, following the above scheme, of well known results as well as new theorems and characterizations for certain classes of linear symmetric solutions. Besides presenting these results, one of the main objectives of the present work is to advertise representation theory as a natural tool for research in cooperative game theory. We believe it is natural and powerful and nevertheless its use has been neglected (with the notable exception of Kleinberg and Weiss (1985), we have not been able to find a single reference where the theory is used).

2 Framework

Consider $\mathbb{R}^n$, the space of payoff vectors for n players. Every permutation σ of $N = \{1, 2, \ldots, n\}$ may be thought of as a linear map $L_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ by permuting the coordinates of any vector $x \in \mathbb{R}^n$ according to σ. The assignment $\sigma \mapsto L_\sigma$ is called a representation, since we represent each permutation by a certain linear map. One can also say that there is a linear "action" on $\mathbb{R}^n$ by the group of permutations.

From linear algebra we know that the matrix of the linear map L_σ may have a simple block decomposition with respect to some basis of $\mathbb{R}^n$ (e.g., L_σ might be diagonalizable). But we are not really interested in only finding a basis that will set a single map L_σ in a simpler form. What we want is to write them all in a simpler way at the same time.

For example, consider the vector $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$, then $\mathbf{1}$ is at the same time an eigenvector to every linear map L_σ, for all σ, since permuting its coordinates does nothing to it. Let $\Delta_n = \{(t, t, \ldots, t) \mid t \in \mathbb{R}\}$ be the diagonal generated by $\mathbf{1}$; we say that Δ_n is "trivial", in the sense that every map L_σ is the identity map when restricted to Δ_n .

Look also at the orthogonal complement to $\mathbf{1}$, which we denote by $\Delta_n^\perp$. Clearly $\Delta_n^\perp = \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \mid \sum_{i=1}^n y_i = 0\}$. Now, if we pick $y \in \Delta_n^\perp$ and apply to it any linear map L_σ we obtain another vector also inside of $\Delta_n^\perp$; that is

$$y \in \Delta_n^\perp \Rightarrow L_\sigma(y) \in \Delta_n^\perp, \qquad \text{for every } \sigma,$$

since permuting the coordinates of a vector does not change the fact that their sum is zero. We say that $\Delta_n^\perp$ is an invariant subspace.

Thus, we can "decompose" $\mathbb{R}^n$ as a direct sum, $\mathbb{R}^n = \Delta_n \oplus \Delta_n^\perp$, in such a way that each summand is "invariant". So if we were to choose a basis of $\mathbb{R}^n$ consisting of $\mathbf{1}$ and a basis of $\Delta_n^\perp$, then all L_σ would have a matrix, with respect to that basis, of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A_\sigma \end{pmatrix}.$$

We show in the Appendix – although it is perhaps intuitively clear – that there is no further decomposition; that is, the subspace $\Delta_n^\perp$ does not contain any invariant subspace smaller than itself (and different to $\{0\}$). We say that $\Delta_n^\perp$ is "irreducible".

The space of cooperative games $G = \{v : 2^N \to \mathbb{R} \mid v(\emptyset) = 0\}$ is also a vector space and the group S_n has a natural action on G as follows: Given a permutation σ let $T_\sigma : G \to G$ be the linear map defined by

$$[T_\sigma(v)](S) = v(\sigma^{-1}S)$$

for every game $v \in G$ and every coalition $S \subset N$, where $\sigma^{-1}S$ is the coalition that contains a player i if and only if σ_i is a player in S.

The main result of Sect. 3 is a statement about a decomposition of G into invariant subspaces for this action.

We first observe the following obvious decomposition

$$G = \bigoplus_{j=1}^{n} G_j,$$

where G_j consists of those games that vanish on every coalition not containing exactly j players. Clearly, every G_j is invariant under all permutations of the players.

Let us identify certain types of games within each G_j, $j < n$ (G_n is a 1-dimensional trivial subspace generated by the game that assigns 1 to the grand coalition, and zero to every other one). For each $j < n$, and each $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ define the game $x^j \in G_j$ as follows

$$x^j(S) = \begin{cases} \sum_{i \in S} x_i & \text{if } |S| = j; \\ 0 & \text{if } |S| \neq j. \end{cases}$$

Let $h_j : \mathbb{R}^n \to G_j$ denote the map $h_j(x) = x^j$; h_j is a linear 1–1 map that commutes with the actions of S_n, that is, $h_j \circ L_\sigma = T_\sigma \circ h_j$.

Set $C_j = h_j(\Delta_n)$ and $U_j = h_j(\Delta_n^\perp)$. We note that h_j is an isomorphism between U_j and $\Delta_n^\perp$ (similarly, between C_j and Δ_n) since it is a linear map which is 1–1 and onto and it commutes with the respective actions of S_n. Isomorphic spaces are indistinguishable from the point of view of linear algebra together with an action of the group of permutations. Thus we may consider C_j as the same as Δ_n and U_j same as $\Delta_n^{\perp}$. In particular, both spaces are irreducible.

Typically (as soon as $n > 3$ and $1 < j < n-1$), within G_j there are games not necessarily lying in $C_j \oplus U_j$. In G we have defined a natural inner product:

$$\langle v_1, v_2 \rangle = \sum_{S \subset N} v_1(S) v_2(S);$$

moreover, all T_σ are orthogonal transformations with respect to this inner product: $\langle T_\sigma(v_1), T_\sigma(v_2) \rangle = \langle v_1, v_2 \rangle$, for all $v_1, v_2 \in G$ (when this happens it is said that the inner product is invariant under the action).

Call W_j the orthogonal complement to $C_j \oplus U_j$ within G_j. We have

$$G_j = C_j \oplus U_j \oplus W_j,$$

and we have arrived to the desired decomposition

$$G = C_1 \oplus \cdots \oplus C_n \oplus U_1 \oplus \cdots \oplus U_{n-1} \oplus W,$$

where $W = \bigoplus_{j=1}^{n-1} W_j$, and every C_j is isomorphic to Δ_n and every U_j is isomorphic to $\Delta_n^{\perp}$. Note that $C = C_1 \oplus \cdots \oplus C_n$ is precisely the space of symmetric games (that is, games whose values depend only on the cardinality of the given coalition).

Each W_j could be further decomposed so that in the end we have expressed W as a sum of irreducible subspaces. Now, the main theorem (Proposition 1) asserts that none of the irreducible summands of W is isomorphic to either Δ_n or to $\Delta_n^{\perp}$.

We use the above decomposition to study solutions, more precisely we look at solutions $\phi : G \to \mathbb{R}^n$ which are linear and symmetric (i.e., $\phi \circ T_\sigma = L_\sigma \circ \phi$, $\forall \sigma$). Rather than keep on carrying the L_σ's and T_σ's, one abuses notation and denotes the linear transformations by the same letter σ; thus, symmetry is simply expressed as

$$\phi(\sigma \cdot v) = \sigma \cdot \phi(v).$$

Schur's lemma (see the Appendix for a precise statement) implies that every linear, symmetric ϕ is zero on W, is a multiple of $h_j^{-1} : C_j \to \Delta_n$ when restricted to C_j and is a multiple of $h_k^{-1} : U_k \to \Delta_n^{\perp}$ when restricted to U_k. This is the punch line of the decomposition theorem.

As a result, the block decomposition (relative to the decomposition of G) of every symmetric, linear solution

$$\phi : G = C_1 \oplus \cdots \oplus C_n \oplus U_1 \oplus \cdots \oplus U_{n-1} \oplus W \to R^n = \Delta_n \oplus \Delta_n^{\perp}$$

is as simple as possible: with most entries zeroes, and where the non-zero blocks are all diagonal and multiples of the identity matrix:

$$\phi = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \mu_1 I_{n-1} & \cdots & \mu_{n-1} I_{n-1} & 0 & \cdots & 0 \end{pmatrix}.$$

In particular, the space of all such solutions is $(2n-1)$-dimensional and W is the common kernel of all of them.

The rest of the article consists of using this information to study solutions satisfying further hypotheses.

In Sect. 4 we give some applications of this method. First we add to linearity and symmetry, the efficiency axiom and characterize very easily all such solutions:

A linear symmetric solution is efficient if and only if its restriction to the symmetric games C coincides with the egalitarian solution.

Next, we study and characterize the linear symmetric solutions that further satisfy the null (or dummy player) axiom.

In Sect. 4.3, we turn to Shapley's value and its relationship to additive games. It turns out that there is an invariant inner product on G (should this be the "natural" inner product on G?) that makes Shapley's value, Sh, the adjoint to the additive games map, i.e., the map $\mathbb{R}^n \to G$ such that

$$x \mapsto [S \mapsto x(S) = \sum_{j \in S} x_j].$$

Equivalently, there is an invariant inner product on G such that Shapley's value is the same as orthogonal projection – with respect to this inner product – onto the subspace of additive games.

Another application is to the study of the notion of self-duality. Recall that the duality operator on games $* : G \to G$, is defined by $(*v)(S) = v(N) - v(N \setminus S)$. A self-dual solution is one for which $\phi(*v) = \phi(v)$, for every game v. We study and characterize self-duality, and its relationship to the other most common axioms (efficiency and nullity).

The last section of the article is devoted to a few results regarding the kernel of linear symmetric solutions: we compute the common kernel of all linear, symmetric and efficient solutions; we give an expression for the kernel of any given linear symmetric solution, in particular we compute it for the Shapley value.

3 Group representation preliminaries

Precise definitions and some proofs for this section may be found in the Appendix at the end of the article. Nevertheless, for the sake of easier reading we repeat here a few definitions and give an idea of proofs, sometimes in a less rigorous but more accessible manner.

Let $N = \{1, \ldots, n\}$ be the set of players. Let $G = G^{(n)} = \{v : 2^N \to \mathbb{R} \mid v(\emptyset) = 0\}$ be the real vector space of games in n players.

The group of permutations of N, S_n, acts naturally on G via linear transformations (i.e., G is a representation of S_n). That is, each permutation $\theta \in S_n$ corresponds to a linear, invertible transformation, which we still call θ, of the vector space G; namely, let

$$(\theta \cdot v)(S) := (\rho(\theta)(v))(S) = v(\theta^{-1} \cdot S)$$

for every $\theta \in S_n$, $v \in G$ and $S \subset N$, where $\theta \cdot S$ is the set obtained from S by permuting its elements according to θ (i.e., $\theta \cdot S = \{\theta_i := \theta(i) \mid i \in S\}$).

Moreover, this assignment preserves multiplication (i.e., is a group homomorphism) in the sense that the linear map corresponding to the product of the two permutations $\theta\sigma$ is the product (or composition) of the maps corresponding to θ and σ, in that order. We will sometimes say that S_n acts (linearly) on G.

Similarly, $\mathbb{R}^n$ is also a representation space for S_n:

$$\theta \cdot (x_1, x_2, \ldots, x_n) = (x_{\theta_1}, x_{\theta_2}, \ldots, x_{\theta_n}).$$

Definition 1 *A linear symmetric solution ϕ is a linear map $\phi : G \to \mathbb{R}^n$ that is symmetric in the following sense: for every $\theta \in S_n$ and $v \in G$ we have that*

$$\phi(\theta \cdot v) = \theta \cdot \phi(v).$$

In other words, a symmetric ϕ "commutes" with the actions on the domain and range of ϕ.

We denote by $\mathcal{LS}(G)$ the vector space of all linear symmetric solutions on G.

In the language of representation theory, what we are calling a linear, symmetric map is usually referred to as an S_n-equivariant map.

3.1 Decomposition of G under S_n

Definition 2

- *A subspace V of G or $\mathbb{R}^n$ is invariant (for the action of S_n) if for every vector $v \in V$ and every permutation $\theta \in S_n$ we have that*

$$\theta \cdot v \in V.$$

- *A subspace V of G or $\mathbb{R}^n$ is irreducible if V itself has no invariant subspaces other than $\{0\}$ and V itself.*

We begin with the decomposition of $\mathbb{R}^n$ into irreducible representations, which is easier, and then proceed to do the same thing for G; that is, we wish to write $\mathbb{R}^n$ as a direct sum of subspaces, each invariant for all permutations in S_n

and in such a way that the summands cannot be further decomposed (i.e., they are irreducible).

For this, set $\mathbf{1} = (\overbrace{1, 1, \ldots, 1}^{n}) \in \mathbb{R}^n$, $\Delta_n = \{(t, t, \ldots, t) \in \mathbb{R}^n\} = \mathbb{R}\mathbf{1}$ and $\Delta_n^{\perp} = \{x \in \mathbb{R}^n \mid x \cdot \mathbf{1} = 0\}$ the orthogonal complement to the diagonal Δ_n. The space $\Delta_n^{\perp}$ is usually called the "*standard representation*" of S_n. Notice that Δ_n is a "trivial" subspace in the sense that every permutation acts as the identity transformation.

Every permutation fixes every element of the diagonal line Δ_n, so, in particular, this line is an invariant subspace of $\mathbb{R}^n$. Being 1-dimensional, it is automatically irreducible. Its orthogonal complement, $\Delta_n^{\perp}$, consists of all vectors such that the sum of their coordinates is zero. Clearly, if we permute the coordinates of any such vector, its sum will still be zero. Hence $\Delta_n^{\perp}$ is also an invariant subspace. The next result tells us that this subspace is also irreducible.

Lemma 1 *The decomposition of* $\mathbb{R}^n$*, under* S_n*, into irreducible subspaces is*

$$\mathbb{R}^n = \Delta_n \oplus \Delta_n^{\perp}.$$

Thus, Lemma 1 tells us that $\mathbb{R}^n$ as a vector space with group of symmetry S_n as defined above, can be written as an orthogonal sum of two subspaces (Δ_n and $\Delta_n^{\perp}$) which are invariant under permutations and which can no longer be further decomposed.

The proof of this lemma is an induction argument that can be found in the Appendix.

For each $j : 1, \ldots, n$, let $G_j = \{v \in G \mid v(S) = 0 \text{ if } |S| \neq j\}$. G_j is a vector subspace of G and, moreover, $G = \bigoplus_{j=1}^{n} G_j$, each G_j is invariant under S_n and the direct sum is orthogonal with respect to the invariant inner product on G given by $\langle v, w\rangle = \sum_{S \subset N} v(S)w(S)$. [1]

Here, invariance of the inner product means that every permutation $\theta \in S_n$ is not only a linear map of G, but an orthogonal map with respect to this inner product. Formally,

$$\langle \theta \cdot v, \theta \cdot w\rangle = \langle v, w\rangle$$

for every $v, w \in G$.

The following games play an important role in describing the decomposition of the space of games G.

For each $j : 1, \ldots, n$ define $c_j \in G_j$ as follows

$$c_j(S) = \begin{cases} 1 & \text{if } |S| = j, \\ 0 & \text{if } |S| \neq j. \end{cases}$$

[1] This seems like the natural inner product to consider. Nevertheless, later on we will see that there is another choice for which the Shapley value can be characterized as the adjoint of the map $\mathbb{R}^n \to G$ that takes $x \in \mathbb{R}^n$ to the game $\hat{x}(S) = \sum_{i \in S} x_i$.

Notice that $G_n = \mathbb{R}c_n$.

Also, for each $j : 1, \ldots, n$, and for each $x \in \mathbb{R}^n$, let $x^j \in G_j$ be given by

$$x^j(S) = \begin{cases} x(S) & \text{if } |S| = j, \\ 0 & \text{if } |S| \neq j. \end{cases}$$

where $x(S) = \sum_{i \in S} x_i$.

Definition 3 *Suppose V_1 and V_2 are two representations for the group S_n i.e., we have two vector spaces V_1 and V_2 where the group S_n is acting by linear maps. We say that V_1 and V_2 are isomorphic if there is a linear map between them, which is* 1–1 *and onto and that commutes with the respective S_n-actions. Formally, there is an invertible linear map $h : V_1 \to V_2$, such that*

$$h(\theta \cdot v_1) = \theta \cdot (h(v_1)) \qquad \textit{for all } \theta \in S_n.$$

We then write $V_1 \simeq V_2$.

For our purposes, V_1 will be an irreducible subspace of G and V_2 an irreducible subspace of $\mathbb{R}^n$

Isomorphic representations are essentially "equal"; not only are they spaces of the same dimension, but the actions are equivalent under some linear invertible map between them. A concrete example may be found in the Appendix.

Proposition 1 *For $j < n$,*

$$G_j = C_j \oplus U_j \oplus W_j,$$

where $C_j = \mathbb{R}c_j \simeq \mathbb{R}$, $U_j = \{x^j \mid x \in \Delta_n^\perp\} \simeq \Delta_n^\perp$ and W_j does not contain any summands isomorphic to either $\mathbb{R}$ nor $\Delta_n^\perp$. The decomposition is orthogonal.

Let us give an idea of how the proof goes; the complete proof of this proposition can be found at the end of the Appendix as the proof of Proposition 9.

We define the map $T_j : \mathbb{R}^n \to G_j$ by $T_j(x) = x^j$. This map is linear (and S_n-equivariant) and 1–1. From Lemma 1 we have the splitting $\mathbb{R}^n = \Delta_n \oplus \Delta_n^\perp$. Thus, inside of G_j, we have the images of these two subspaces: $C_j = T_j(\Delta_n)$ and $U_j = T_j(\Delta_n^\perp)$.

Denote by W_j the orthogonal complement to $C_j \oplus U_j$ within G_j. The last, and hardest, part is to show that W_j does not contain any summands isomorphic to either C_j or U_j.

Proposition 1 does not quite give us a decomposition of G_j into irreducible summands. The subspaces C_j and U_j are irreducible (each isomorphic to Δ_n and $\Delta_n^\perp$, respectively) and together give us a copy of $\mathbb{R}^n$ inside of G. Whereas W_j may or may not be irreducible (depending on j and N), but as we shall see the exact nature of this subspace plays no role in the study of linear, symmetric solutions since (as it will be promptly proved) it lies in the kernel of any such solution.

Proposition 1 gives us a decomposition of the space of games that is a key ingredient in our subsequent analysis.

Set $C = \bigoplus_{j=1}^{n} C_j$. This is the space of **symmetric** games, i.e., games whose value on a given set depends only on its cardinality. According to Proposition 1, C is the largest subspace of G where S_n acts trivially. Let $U = \bigoplus_{j=1}^{n-1} U_j$ and $W = \bigoplus_{j=1}^{n-1} W_j$. Then

$$G = C \oplus U \oplus W.$$

The following result gives a good example of how Proposition 1 is to be used (in conjunction with Schur's Lemma) to gain information about linear symmetric solutions.

Corollary 1

- *Every linear symmetric solution vanishes in W.*
- $\dim \mathcal{LS}(G) = 2n - 1$.

Proof Let $\phi : C \oplus U \oplus W \to \Delta_n \oplus \Delta_n^{\perp}$ be a linear symmetric solution. Assume $X \subset W$ is an irreducible summand in the decomposition of W (even while we do not know the decomposition of W as a sum of irreducible subspaces, it is known that such a decomposition exists). Let $\pi_\alpha, \alpha \in \{1, 2\}$, denote orthogonal projection of $\mathbb{R}^n$ onto Δ_n and $\Delta_n^{\perp}$, respectively. Now, $\phi : G \to \mathbb{R}^n = \Delta_n \oplus \Delta_n^{\perp}$, may be written as $\phi = (\pi_1 \circ \phi, \pi_2 \circ \phi)$. Denote by $\iota : X \to G$ the inclusion, then, the restriction of ϕ to X may be expressed as

$$\phi_{|X} = \phi \circ \iota = (\pi_1 \circ \phi \circ \iota, \pi_2 \circ \phi \circ \iota).$$

Now, $\pi_\alpha \circ \phi \circ \iota$ is a linear symmetric map from X to either Δ_n or $\Delta_n^{\perp}$; according to Proposition 1, X is not isomorphic to either of these two spaces, thus Schur's Lemma (see Appendix for the statement) says that $\pi_\alpha \circ \phi \circ \iota$ must be zero. Since this is true for every irreducible summand X of W, ϕ is zero on all of W.

Schur's Lemma also implies that ϕ maps each C_j into Δ_n and each U_j into $\Delta_n^{\perp}$, and that its restriction to each C_j or U_j is unique up to multiplication by a scalar (i.e., any two linear symmetric solutions when restricted to C_j – or U_j – differ only by multiplication by a constant).

So, define the following linear symmetric solutions. For $j : 1, \ldots, n$ and $k : 1, \ldots, n-1$, we set

$$\phi_j(c_l) = \delta_{jl}, \ \phi_{j_{|U \oplus W}} \equiv 0,$$

$$\psi_k(x^l) = \delta_{kl} x, \ \psi_{k_{|C \oplus W}} \equiv 0$$

and where δ_{jl} is Kronecker's delta.

Clearly $\{\phi_j\} \cup \{\psi_k\}$ is a linearly independent set. Moreover, as discussed above, Schur's Lemma tells us that for any $\phi \in \mathcal{LS}(G)$, every $j : 1, \ldots, n$ and

$k : 1, \ldots, n-1$, the restriction of ϕ to C_j is a multiple of ϕ_j and ϕ restricted to U_k is a multiple of ψ_k. Thus ϕ is a linear combination of the ϕ_j's and ψ_k's. Thus $\{\phi_j\} \cup \{\psi_k\}$ is a basis for $\mathcal{LS}(G)$ □

Remark 1 Proposition 1 and Corollary 2 imply that in order to study symmetric solutions, one needs to look only at those games inside $C \oplus U$; in the next section we refine this further to a subspace of dimension $2n-1$ inside of G.

3.1.1 The space W

Although we have remarked that to the study of linear symmetric solutions the space W plays only the role of the common kernel of every such solution, it may be interesting, nevertheless, to characterize the games that lie in this subspace. That is the content of Proposition 2 of this subsection.

Lemma 2

$$C \oplus U = \left\{ \sum_{j=1}^{n} x_j^j \mid x_j \in \mathbb{R}^n,\ \textit{for } j = 1, 2, \ldots, n \right\}.$$

Proof Let $V = \left\{ \sum_{j=1}^{n} x_j^j \mid x_j \in \mathbb{R}^n \right\}$. We first show $V \subset C \oplus U$.

Recall that to every vector $x \in \mathbb{R}^n$, and to every $j : 1, \ldots, n$, we have associated a game x^j via

$$x^j(S) = \begin{cases} x(S) & \text{if } |S| = j, \\ 0 & \text{if } |S| \neq j. \end{cases}$$

Thus, for the vector $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n$ we have that

$$\mathbf{1}^j = \begin{cases} j & \text{if } |S| = j, \\ 0 & \text{if } |S| \neq j; \end{cases}$$

which in particular shows that $\mathbf{1}^j = jc_j$.

Now, choose n arbitrary vectors $x_1, x_2, \ldots, x_n \in \mathbb{R}^n$. Each of them may be decomposed as a sum with respect to the direct sum decomposition $\mathbb{R}^n = \Delta_n \oplus \Delta_n^{\perp}$. Thus we write $x_j = a_j \mathbf{1} + z_j$ where $z_j \in \Delta_n^{\perp}$.

Then,

$$\sum_{j=1}^{n} x_j^j = \sum_{j=1}^{n} a_j \mathbf{1}^j + \sum_{j=1}^{n} z_j^j = \sum_{j=1}^{n} j a_j c_j + \sum_{j=1}^{n-1} z_j^j \in C \oplus U,$$

since $z^n = 0$ for all $z \in \Delta_n^{\perp}$. This shows that $V \subset C \oplus U$.

Now, $c_j = \frac{1}{j} \mathbf{1}^j \in V$, thus $C_j \subset V$. Also, every element in U_j is, by definition, of the form x^j with $x \in \Delta_n^{\perp}$, thus $U_j \subset V$. Hence $C \oplus U \subset V$ □

Proposition 2

$$W = \left\{ w \in G \mid \forall i,j : 1,\ldots,n, \sum_{\substack{|S| = j \\ S \ni i}} w(S) = 0 \right\}.$$

Proof First, $w \in W \Leftrightarrow w \in (C \oplus U)^{\perp} \Leftrightarrow \langle w, \sum_1^n x_j^j \rangle = 0,$ for all $x_j \in \mathbb{R}^n$. Thus, if $\{e_i\}$ stands for the standard basis in $\mathbb{R}^n$, then $w \in W \Leftrightarrow \langle w, e_i^j \rangle = 0$, for every $i, j : 1, \ldots, n$. Now,

$$\langle w, e_i^j \rangle = \sum_{|S|=j} w(S) e_i(S),$$

but

$$e_i(S) = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

□

In Kleinberg and Weiss (1985) the number of irreducible summands appearing in W is computed, whereas in Amer et al. (2003) a basis for W is given.

Example 1 Let us compute a basis for W for the case of four players, i.e., $n = 4$.
If $w \in W$, then for the coalition $\{i\}$ we have

$$w(\{i\}) = \sum_{\substack{|S|=1 \\ S \ni i}} w(S) = 0.$$

For subsets of cardinality two we get a system of four equations (one for each i) in the six unknowns $w(\{1,2\}), w(\{1,3\}), w(\{1,4\}), w(\{2,3\}), w(\{2,4\}), w(\{3,4\})$. The solution space for this system is 2-dimensional with basis:

$$(u(\{1,2\}), u(\{1,3\}), u(\{1,4\}), u(\{2,3\}), u(\{2,4\}), u(\{3,4\})) = \begin{cases} (1, 0, -1, -1, 0, 1) \\ (0, 1, -1, -1, 1, 0) \end{cases}.$$

For subsets of cardinality three we again get four linearly independent equations in four unknowns. Thus the value of w is zero on all subsets with three elements. Finally, on the total set it also vanishes.

Thus, W consists of games that vanish on subsets of cardinality different to 2, and, on sets of cardinality two satisfy the relations:
$w(\{1,2\}) = w(\{3,4\}), w(\{1,3\}) = w(\{2,4\})$,
$w(\{1,4\}) = w(\{2,3\}) = -w(\{1,2\}) - w(\{1,3\})$.

Remark 2 It is not difficult to show that always $W_1 = W_{n-1} = 0$.

3.2 Symmetric solutions and the action of H_n

Every solution $\phi \in \mathcal{LS}(G)$ is determined by its nth-coordinate, ϕ_n. Moreover, ϕ_n is an H_n-invariant functional, where $H_n = \{\theta \in S_n \mid \theta_n = n\}$ is the subgroup fixing n. More precisely, let

$$\mathrm{Hom}_{H_n}(G,\mathbb{R}) = \{f : G \to \mathbb{R} \mid \ f \text{ is linear and } f(\theta \cdot v) = f(v), \forall v \in G, \\ \text{and } \forall \theta \in H_n\}$$

denote the space of H_n-invariant linear functions on G. Then

Lemma 3 *The linear map* $\mathcal{LS}(G) \to \mathrm{Hom}_{H_n}(G,\mathbb{R})$ *given by* $\phi \mapsto \phi_n$ *is an isomorphism of vector spaces.*

Proof First, let us show that if $\phi \in \mathcal{LS}(G)$ then ϕ_n is an H_n-invariant functional. Let $v \in G$ be any game, and $\sigma \in H_n$ be a permutation fixing player n. Thus, symmetry of ϕ implies

$$\begin{aligned}(\phi_1(\sigma \cdot v), \ldots, \phi_{n-1}(\sigma \cdot v), \phi_n(\sigma \cdot v)) &= \phi(\sigma \cdot v) = \sigma \cdot (\phi_1(v), \ldots, \phi_{n-1}(v), \phi_n(v)) \\ &= (\phi_{\sigma_1}(v), \ldots, \phi_{\sigma_{n-1}}(v), \phi_n(v))\end{aligned}$$

since $\sigma_n = n$. Thus $\phi_n(\sigma \cdot v) = \phi_n(v)$ for every $\sigma \in H_n$.

In order to check that $\phi \mapsto \phi_n$, is an isomorphism it suffices to give the inverse: If $f \in \mathrm{Hom}_{H_n}(G,\mathbb{R})$, then let

$$\Psi(f) = (\phi_1, \phi_2, \ldots, \phi_{n-1}, f),$$

where $\phi_j(v) = f((jn) \cdot v)$, $v \in G$, and (jn) is the transposition interchanging j with n and fixing every other player. The map Ψ is clearly linear, and the inverse to $\phi \mapsto \phi_n$. The remaining question is whether the map $\Psi(f)$ is indeed symmetric. We finish the proof by showing this.

We use the fact that any permutation $\theta \in S_n$ is a composition of transpositions. Hence, to prove symmetry it is enough to check it for an arbitrary transposition (ik). There are two cases, when both i and k are different to n, and when (ik) is of the form (in).

Let us consider the first case, and for simplicity of exposition assume $(ik) = (12)$ and $n \geq 3$. For $j \geq 3$, we have

$$\phi_j((12) \cdot v) = f((jn)(12) \cdot v) = f((12)(jn) \cdot v) = f((jn) \cdot v);$$

where the second equality holds because (jn) and (12) commute and the last equality is the H_n-invariance of f.

Also,

$$\phi_1((12)\cdot v) = f((1n)(12)\cdot v) = f((12)(2n)\cdot v) = f((2n)\cdot v)$$

since $(1n)(12) = (12)(2n)$ and invariance of f. Similarly, $\phi_2((12)\cdot v) = f((1n)\cdot v)$. Hence,

$$\begin{aligned}\phi((12)\cdot v) &= (f((2n)\cdot v), f((1n)\cdot v), f((3n)\cdot v), \ldots, f(((n-1)n)\cdot v), f(v)) \\ &= (12)\cdot\phi(v).\end{aligned}$$

Finally, we take the case of a transposition of the form (in). In this case, for $j \neq i$,

$$\phi_j((in)\cdot v) = f((jn)(in)\cdot v) = f((ij)(jn)\cdot v) = f((jn)\cdot v),$$

whereas

$$\phi_i((in)\cdot v) = f((in)(in)\cdot v) = f(v) = \phi_n(v).$$

Thus the ith and the nth coordinates of ϕ have been permuted. $\square$

Just as we decomposed $\mathbb{R}^n$ and G under the action of S_n, we want to find a decomposition under the action of the smaller group $H_n \subset S_n$. Clearly, every space that is invariant under the larger group is still invariant under H_n, though it may happen that a space that was irreducible for S_n decomposes into more than one piece under H_n. Let us see first what happens to $\mathbb{R}^n$.

Recall that $\mathbb{R}^n = \Delta_n \oplus \Delta_n^\perp$. Now, Δ_n is 1-dimensional so it is also irreducible for H_n. What about $\Delta_n^\perp$? Consider the vector $\omega \in \Delta_n^\perp$ given by

$$\omega = (1, 1, \ldots, 1, 1-n).$$

Clearly ω and its multiples are fixed by every element of H_n. Likewise, the orthogonal complement (within $\Delta_n^\perp$) to the line through ω is also invariant. Call that space A. The simplest way to see that A is invariant is by noticing that

$$A = \{(x_1, x_2, \ldots, x_{n-1}, 0) \mid x_1 + \cdots + x_{n-1} = 0\},$$

thus permuting the first $n-1$ entries of a vector leaves the vector inside of A.

The subgroup H_n can be identified with S_{n-1}, the group of permutations of the first $n-1$ players, and A is then seen to be isomorphic to the standard representation of S_{n-1}, i.e., $\Delta_{n-1}^\perp$. Therefore, it is irreducible.

Summarizing, the irreducible decomposition of $\mathbb{R}^n$ under the action of the group H_n is

$$\mathbb{R}^n = \mathbb{R}\mathbf{1} \oplus \mathbb{R}\omega \oplus A.$$

We turn now to the decomposition of G under H_n.

Theorem 1 *Let $n \geq 3$.*

1. *The space of games G decomposes under H_n as*

$$G = C \oplus T \oplus V;$$

where
(a) $C = \bigoplus_{j=1}^{n} C_j$ *as before, (in particular it is an n-dimensional trivial representation);*
(b) $T = \bigoplus_{j=1}^{n-1} \mathbb{R}\omega^j$ *is a trivial (for H_n) representation of dimension $n-1$;*
(c) *V does not contain any trivial summands;*
(d) *The decomposition is orthogonal.*

2. *Any $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ vanishes in V, hence any symmetric solution $\phi \in \mathcal{LS}(G)$ is determined by the values of ϕ_n in the $(2n-1)$-dimensional trivial subspace $C \oplus T$.*

Proof Each C_j is 1-dimensional, so remains an irreducible piece under the action of H_n.

Each U_j is isomorphic to $\Delta_n^{\perp}$, which we have just seen splits as a sum of two subspaces, a trivial one and a space isomorphic to A. Set $A^j = \{z^j \mid z \in A\}$, then

$$U_j = \mathbb{R}\omega^j \oplus A^j$$

is the decomposition of U_j into irreducibles under the H_n-action.

Notice that, even when $n = 3$, the action of H_n on A (and thus the one on A^j) is not the trivial action (e.g., when $n = 3$, A is 1-dimensional, but the action can be seen to be multiplication by the sign of the permutation).

So we get a decomposition of G of the form

$$G = C \oplus T \oplus V,$$

C and T as in the statement of the theorem and $V = \bigoplus_{j=1}^{n-1} A^j \oplus W$. The A^j's are irreducible and non-trivial, so to prove (c) we need to check only that there are no trivial summands (for the H_n-action) within W.

Suppose W contains a 1-dimensional trivial subspace Y, say. Take any non-zero element $y \in Y$ and define the linear map $f : G \to \mathbb{R}$ by setting $f(y) = 1$ and $f \equiv 0$ on $Y^{\perp}$. Such an f is H_n-invariant (since H_n does nothing on Y and f is set to be zero on its orthogonal complement), and thus, as we have seen, it determines a linear symmetric solution ϕ. But we have proved that every linear symmetric solution vanishes on W. This is a contradiction. Therefore, W cannot contain trivial H_n-representations.

Orthogonality of the decomposition follows from that of the S_n -decomposition, plus the fact that $\mathbb{R}\omega^j \perp A^j$.

To prove 2, notice that, by Schur's Lemma, any H_n-invariant $f : G \to \mathbb{R}$ vanishes when restricted to any irreducible subspace other than a trivial one. Thus $f_{|V} \equiv 0$. On the other hand, any linear $f : C \oplus T \to \mathbb{R}$, when extended as zero to V, is an H_n-invariant functional and gives rise to a linear symmetric solution. □

Remark 3 In other words, any symmetric solution can be uniquely determined as follows: given arbitrary real numbers $a_1, \ldots, a_n, b_1, \ldots, b_{n-1}$ set $\phi_n(c_j) = a_j$, $\phi_n(\omega^j) = b_j, \phi_{n|_V} \equiv 0$. Then ϕ_n is H_n-invariant and determines via Ψ a symmetric solution ϕ. Notice that this shows again that $\mathcal{LS}(G)$ is $(2n-1)$-dimensional.

Theorem 1 identifies the minimal possible subspace of games $(C \oplus T)$ that determines every possible linear symmetric solution. Thus, in principle, to understand any given linear symmetric solution one has only to know its values on $c_1, \ldots, c_n, \omega^1, \ldots, \omega^{n-1}$.

4 Applications

4.1 Efficient symmetric solutions

Definition 4 *A solution is efficient if*

$$\sum_{j=1}^{n} \phi_j(v) = \phi(v) \cdot \mathbf{1} = v(N), \qquad \textit{for every } v \in G.$$

In the previous section we saw that any linear symmetric solution ϕ is completely determined by its nth coordinate ϕ_n. Also, at the end of that section, we saw that ϕ_n is itself determined by its values on $C \oplus T$. Thus one expects that efficiency can be translated into some simple condition on these values. Thus,

Proposition 3 *Let $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$. The linear symmetric solution determined by f is efficient if and only if*

1. $f(c_j) = 0$, *for all $j < n$; and*
2. $f(c_n) = \frac{1}{n}$.

In particular, the set of efficient symmetric solutions is an affine space of dimension $n-1$.

Proof Let ϕ be the symmetric solution corresponding to f. Thus, $\phi_k(v) = f((kn) \cdot v)$.

First of all, $C_n^{\perp}$ is exactly the subspace of games v for which $v(N) = 0$. Of these, those in $U \oplus W$ trivially satisfy $\phi(v) \cdot 1_n = 0$, since (by Schur) their image lies in $\Delta_n^{\perp}$.

Therefore, efficiency need only be checked on C.

Since c_j is fixed by all permutations in S_n,

$$\sum_{k=1}^{n} \phi_k(c_j) = nf(c_j)$$

and so, ϕ is efficient if and only if $nf(c_j) = c_j(N) = \delta_{jn}$.

Finally, for the last assertion, let $f_0 \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ be given by $f_0(c_j) = 0$, $j < n$, $f_0(c_n) = \frac{1}{n}$, and $f_{0|_T} \equiv 0$ (i.e., f_0 is the invariant functional that gives the "egalitarian solution" $\phi_0(v) = \frac{v(N)}{n}$). Then the set of efficient symmetric solutions corresponds to the following affine set of H_n-invariant functionals

$$\{f + f_0 \mid f \in \mathrm{Hom}_{H_n}(G, \mathbb{R}) \text{ and } f_{|C} \equiv 0\}. \qquad \square$$

Those games v whose values, $v(S)$, depends only on the cardinality of the coalition, S, are called symmetric games. In our notation these are exactly the games $v \in C$. The next Corollary characterizes the solutions to these games in terms of linearity, symmetry and efficiency.

Corollary 2 *Among all linear, symmetric solutions, the egalitarian solution is characterized as the unique efficient solution on symmetric games.*

Proof If we restrict a solution ϕ to the symmetric games space C, then efficiency is equivalent to $\phi_n(c_j) = \frac{1}{n}\delta_{jn}$, i.e., $\phi(v) = \frac{v(N)}{n}$. $\square$

In other words, all linear, symmetric, efficient solutions (e.g., Shapley's value) coincide with the egalitarian solution when restricted to the space of symmetric games.

4.1.1 Formula for all efficient symmetric solutions

We have seen that every efficient symmetric solution is uniquely determined via a functional $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ such that

1. $f(c_j) = 0, j < n$;
2. $f(c_n) = \frac{1}{n}$.

Now, we want to translate this information into more standard game theoretic terminology. We compute a formula for $f(v)$, for any game v, and, finally, for $\phi(v)$.

Observe that, since $\omega = (1, 1, \ldots, 1, 1-n)$,

$$\omega^j(S) = \begin{cases} \sum_{i \in S} \omega_i & |S| = j \\ 0 & |S| \neq j \end{cases} = \begin{cases} 0 & |S| \neq j & & \\ j & |S| = j & \text{and} & n \notin S \\ j - n & |S| = j & \text{and} & n \in S \end{cases}.$$

Given a game $v \in G$ we first compute its orthogonal projection onto $C_n \oplus T$ (since it is the only part that contributes in the computation of any efficient symmetric solution). The projection of v onto $C_n = \mathbb{R}c_n$ is just $v(N)c_n$. Next we compute its projection, v^T, onto T: this is $v^T = \sum_j \frac{\langle v, \omega^j \rangle}{\langle \omega^j, \omega^j \rangle} \omega^j$, where

$$\langle v, \omega^j \rangle = \sum_{S \subset N} v(S)\omega^j(S) = \sum_{\substack{|S|=j \\ n \notin S}} jv(S) + \sum_{\substack{|S|=j \\ n \in S}} (j-n)v(S),$$

and

$$\langle \omega^j, \omega^j \rangle = \sum_{\substack{|S|=j \\ n \notin S}} j^2 + \sum_{\substack{|S|=j \\ n \in S}} (j-n)^2 = \binom{n-1}{j} j^2 + \binom{n-1}{j-1} (n-j)^2 = \frac{n!}{(j-1)!(n-j-1)!}.$$

Note that $\langle \omega^j, \omega^j \rangle = \langle \omega^{n-j}, \omega^{n-j} \rangle$.

Remark 4 At this point, we do not really care about the value of $\langle \omega^j, \omega^j \rangle$, since it is absorbed by the parameters $b_j = f(\omega^j)$. Nevertheless it will be used later on.

Since

$$f(v) = f(v(N)c_n + v^T) = \frac{v(N)}{n} + f(v^T) = \frac{v(N)}{n} + \sum_{j=1}^{n-1} \frac{\langle v, \omega^j \rangle}{\langle \omega^j, \omega^j \rangle} f(\omega^j)$$

for all $v \in G$, we get:

Proposition 4 *All efficient linear symmetric solutions are given by $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ of the following form:*

$$f(v) = \frac{v(N)}{n} + \sum_{j=1}^{n-1} t_j \left[\sum_{\substack{|S|=j \\ n \notin S}} jv(S) + \sum_{\substack{|S|=j \\ n \in S}} (j-n)v(S) \right],$$

where $t_1, \ldots, t_{n-1}$ are arbitrary real numbers.

In the previous formula, the coefficients t_j are related to $f(\omega^j)$ via

$$t_j = \frac{f(\omega^j)}{\langle \omega^j, \omega^j \rangle}.$$

Theorem 2 *The efficient linear symmetric solutions are precisely those $\phi \in \mathcal{LS}(G)$ of the form*

$$\phi_i(v) = \frac{v(N)}{n} + \sum_{\substack{S \ni i \\ S \neq N}} (n-s) \left[\beta_s v(S) - \beta_{n-s} v(N \setminus S) \right], \tag{1}$$

where $s = |S|$ and $\beta_s \in \mathbb{R}$ are arbitrary.

Proof From Proposition 4 it follows that

$$\phi_i(v) = \frac{v(N)}{n} + \sum_{j=1}^{n-1} t_j \left[\sum_{\substack{|S|=j \\ i \notin S}} jv(S) + \sum_{\substack{|S|=j \\ i \in S}} (j-n)v(S) \right].$$

Now,

$$\begin{aligned}\sum_{j=1}^{n-1}\sum_{\substack{|S|=j\\ i\notin S}} t_j j v(S) &= \sum_{j=1}^{n-1}\sum_{\substack{|T|=n-j\\ i\in T}} t_j j v(N\setminus T)\\ &= \sum_{l=1}^{n-1}\sum_{\substack{|T|=l\\ i\in T}} t_{n-l}(n-l)v(N\setminus T)\end{aligned}$$

from which we get the expression of the theorem by setting $\beta_s = -t_s$. □

We should mention that an equivalent formula to (1) has been obtained by Driessen and Radzik (2002).

4.2 Nullity

Definition 5

- *For a game $v \in G$, we say that player i is null for v if*

$$v(S) = v(S \cup \{i\})$$

  for all $S \subset N$.
- *A solution $\phi \in \mathcal{LS}(G)$ is said to be null (or to satisfy the nullity axiom) if $\phi_i(v) = 0$ for every game v for which i is a null player.*

Here we study the nullity axiom in the same spirit of the previous section. Namely, first we understand the nullity axiom as a condition on the the last coordinate, ϕ_n, of a linear symmetric solution. Thus, in terms of H_n-invariant functionals, the condition of being null may be restated as follows:

$f \in \mathrm{Hom}_{H_n}(G,\mathbb{R})$ *gives rise to a null linear symmetric solution if and only if* $f(v) = 0$ *for every game v for which n is a null player.*

Let $M_n = \{v \in G \mid \ n \text{ is null for } v\} \subset G$. Note that M_n is an invariant subspace for the action of H_n. It is in fact isomorphic to the representation space $G^{(n-1)}$ of all games in $n-1$ players:

If we set $L : G^{(n-1)} = \{g : 2^{\{1,2,\ldots,n-1\}} \to \mathbb{R} \mid g(\emptyset) = 0\} \to M_n$ by

$$L(g)(S) = \begin{cases} g(S) & \text{if } n \notin S \\ \\ g(S\setminus\{n\}) & \text{if } n \in S \end{cases}$$

then L is an H_n-equivariant isomorphism.

Now, any $f \in \mathrm{Hom}_{H_n}(G,\mathbb{R})$ is determined by its values on the subspace $C \oplus T$, thus we want to identify the null games for n inside $C \oplus T$. By definition, this is the space $\mathcal{N} = (C \oplus T) \cap M_n$.

Since $C \oplus T$ is the largest trivial subspace – for H_n – in G, we conclude that $\mathcal{N}$ is the largest trivial subspace inside M_n. Therefore, it corresponds under the isomorphism L to the subspace of symmetric games in $n-1$ players, that is

$$\mathcal{N} = L(C^{(n-1)}).$$

So immediately we get that $\dim \mathcal{N} = n-1$ and, since $\dim(C \oplus T) = 2n-1$, that the space of null symmetric linear solutions is n-dimensional.

Next we compute an explicit basis for $\mathcal{N}$.

Lemma 4 *Let $\mathcal{N} = (C \oplus T) \cap M_n$. Then $\mathcal{N}$ is the $(n-1)$ -dimensional trivial subspace with basis $\{v_j\}$, $j : 1, \ldots, n-1$, where*

$$v_j = \frac{n-j}{n} c_j + \frac{j+1}{n} c_{j+1} + \frac{1}{n}\left(\omega_n^j - \omega_n^{j+1}\right).$$

Proof Let $\{z_j\}_1^{n-1}$ be the basis of $C^{(n-1)}$ obtained by restricting each c_j, $j : 1, \ldots, n-1$, to subsets of $\{1, \ldots, n-1\}$. Then, by the above observation, $v_j = L(z_j)$ is a basis of $\mathcal{N}$; more explicitly,

$$v_j(S) = \begin{cases} c_j(S) & \text{if } n \notin S \\ c_j(S \setminus \{n\}) & \text{if } n \in S \end{cases} = \begin{cases} 1 & \text{if } |S| = j \quad \text{and} \quad n \notin S \\ 1 & \text{if } |S| = j+1 \quad \text{and} \quad n \in S \\ 0 & \text{otherwise} \end{cases}.$$

Now, write $v_j = \sum a_i c_i + \sum b_k \omega^k$. To compute the coefficients a_i, b_k we proceed as follows:

$$\left.\begin{aligned} 1 &= v_j(\{1, \ldots, j\}) = a_j + j b_j \\ 0 &= v_j(\{1, \ldots, j-1, n\}) = a_j + (j-n) b_j \end{aligned}\right\} \Rightarrow\ b_j = \tfrac{1}{j} \text{ and } a_j = 1 - \tfrac{j}{n}$$

similarly, evaluating v_j on $\{1, \ldots, j+1\}$ and $\{1, \ldots, j, n\}$ one gets $a_{j+1} = \frac{j+1}{n}$ and $b_{j+1} = \frac{-1}{n}$ (if $j < n-1$, for $j = n-1$ one has $a_{j+1} = 1$, $b_{j+1} = 0$).

Finally, in the same way one computes that $a_k = b_k = 0$ for the remaining coefficients. □

Example 2 Let us assume that $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ is null and efficient. Efficiency implies $f(c_j) = 0$, for $j \in N \backslash \{n\}$, and $f(c_n) = \frac{1}{n}$. Therefore, if $r_j = f(\omega_n^j)$,

$$0 = f(v_{n-1}) = \frac{1}{n}\left(1 + r_{n-1}\right) \Rightarrow r_{n-1} = -1,$$

and so, if $j < n-1$

$$0 = f(v_j) = \frac{1}{n}(r_j - r_{j+1}) \Rightarrow r_j = r_{j+1};$$

thus $r_j = -1$, for all $j \in N \backslash \{n\}$.

Hence, we conclude that there is exactly one linear symmetric solution which is efficient and null (the Shapley value, Shapley (1953)). It is given by $f(c_n) = \frac{1}{n}$, $f(c_j) = 0$ and $f(\omega_n^j) = -1$ for $j \in N\backslash\{n\}$. □

We finish this subsection by stating the general formula for null linear symmetric solutions, i.e., solutions that satisfy all the axioms that traditionally characterize the Shapley value except for the efficiency axiom. Dubey et al. (1981) have given a similar formula. See also Weber (1988) for further discussion about solutions satisfying the nullity axiom.

Proposition 5 *The space of null linear symmetric solutions is n-dimensional. The general formula for such a solution is given by:*

$$\phi_i(v) = \sum_{S \not\ni i} r_s\,[v(S \cup \{i\}) - v(S)].$$

for arbitrary $r_0, \ldots, r_{n-1} \in \mathbb{R}$, *and where* $s = |S|$.

Proof Define, for $j : 1, \ldots, n-1$,

$$\mu_j(S) = \begin{cases} 1 & \text{if } |S| = j \text{ and } n \notin S \\ -1 & \text{if } |S| = j+1 \text{ and } n \in S \\ 0 & \text{otherwise} \end{cases},$$

and

$$\chi_n(S) = \begin{cases} 1 & \text{if } S = \{n\} \\ 0 & \text{if } S \neq \{n\} \end{cases}.$$

The μ_j's and χ_n are H_n-invariant, and so they belong to $C \oplus T$. Let us show that they form a basis for the orthogonal complement to $\mathcal{N}$ inside $C \oplus T$.

Since for every game v,

$$\langle v, v_k \rangle = \sum_{S \subset N} v(S) v_k(S) = \sum_{\substack{|S|=k \\ n \notin S}} v(S) + \sum_{\substack{|S|=k+1 \\ n \in S}} v(S) = \sum_{\substack{|S|=k \\ n \notin S}} [v(S) + v(S \cup \{n\})]$$

it follows that $\langle \mu_j, v_k \rangle = 0$ and $\langle \chi_n, v_k \rangle = 0$. It is also not hard to check that $\{\mu_j\} \cup \{\chi_n\}$ is an orthogonal set.

Let $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ be an arbitrary null solution, and $v \in G$. Then

$$f(v) = f\left(\frac{\langle v, \chi_n \rangle}{\langle \chi_n, \chi_n \rangle} \chi_n + \sum_{j=1}^{n-1} \frac{\langle v, \mu_j \rangle}{\langle \mu_j, \mu_j \rangle} \mu_j \right).$$

Since

- $\langle v, \chi_n \rangle = v(\{n\})$,
- $\langle \chi_n, \chi_n \rangle = 1$,
- $\langle v, \mu_j \rangle = \sum_{\substack{|S|=j \\ n \notin S}} [v(S) - v(S \cup \{n\})]$ and
- $\langle \mu_j, \mu_j \rangle = 2 \binom{n-1}{j}$,

we obtain

$$f(v) = f(\chi_n) v(\{n\}) + \sum_{j=1}^{n-1} \left(\sum_{\substack{|S|=j \\ n \notin S}} [v(S) - v(S \cup \{n\})] \frac{f(\mu_j)}{2 \binom{n-1}{j}} \right).$$

Set $r_0 = f(\chi_n)$, $r_j = \frac{-f(\mu_j)}{2\binom{n-1}{j}}$ for $j : 1, \ldots, n-1$. Then,

$$f(v) = \sum_{S \not\ni n} r_s \left[v(S \cup \{n\}) - v(S) \right],$$

where the sum includes $S = \emptyset$ and where s stands for the cardinality of S.
Now recall that $\phi_i(v)$ is obtained from $f = \phi_n$ by interchanging i with n. □

For future computations, we state the formulas of the μ_k, $k : 1, \ldots, n-1$, and χ_n, in terms of the orthogonal basis $\{c_i, \omega^j\}$, $i : 1, \ldots, n$, $j : 1, \ldots, n-1$:

$$\mu_k = \frac{n-k}{n} c_k - \frac{k+1}{n} c_{k+1} + \frac{1}{n} \left(\omega^k + \omega^{k+1} \right),$$

$$\chi_n = \frac{1}{n} \left(c_1 - \omega^1 \right).$$

4.3 Shapley's solution

Shapley's solution can be characterized (as is well known) as the unique linear symmetric solution which is both efficient and null. As seen in the previous section, it is characterized by saying that its nth coordinate functional Sh_n is the unique element of $\mathrm{Hom}_{H_n}(G, \mathbb{R})$ such that

- $Sh_n(c_n) = \frac{1}{n}$,
- $Sh_n(c_j) = 0, j < n$, and
- $Sh_n(\omega^j) = -1, j < n$.

From formula (1) for all efficient, symmetric and linear solutions, we obtain the following formula for Shapley's value [see Myerson (1991) for a similar expression]:

$$Sh_i(v) = \sum_{S \ni i} \frac{(s-1)!(n-s)!}{n!}[v(S) - v(N \setminus S)],$$

where $s = |S|$.

4.3.1 Shapley's value as the adjoint of the additive games map

Throughout the article we have been using an inner product on G which seems very "natural" (we call it the standard inner product on G); namely,

$$\langle v_1, v_2 \rangle = \sum_{S \subset N} v_1(S) v_2(S).$$

In fact, any other S_n-invariant inner product on G would have done as well. By Schur's Lemma, on each irreducible component these inner products are unique up to a constant. We see next, that a suitable choice of conformal factors can be chosen which make the new inner product have a unique relation to Shapley's value.

Look at the following S_n-equivariant map: $\hat{}: \mathbb{R}^n \to G$, which maps a vector x to the game $S \mapsto x(S) = \sum_{i \in S} x_i$, which we denote by $\hat{x}$. The image $\widehat{\mathbb{R}^n} \subset G$, of this map, is precisely the subspace of additive games (i.e., those games v such that $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$).

Set a new inner product $\langle\langle , \rangle\rangle$ on G as follows:

- On C define it by declaring $\{c_1, c_2, \ldots, c_{n-1}, \widehat{\mathbf{1}}\}$ orthogonal, and of length $\sqrt{n}$.
- On each U_j set $\langle\langle , \rangle\rangle$ with

$$\langle\langle x^j, y^j \rangle\rangle = \frac{1}{n-1} x \cdot y$$

  for every $x, y \in \Delta_n^{\perp}$.
- Set $U_i \perp U_j$, for all $i \neq j$.
- Set $\langle\langle , \rangle\rangle = \langle , \rangle$ on W.

Then $\langle\langle , \rangle\rangle$ is S_n-invariant; moreover

Theorem 3 *Shapley's value is characterized as being the adjoint, with respect to $\langle\langle , \rangle\rangle$, of the map $x \mapsto \hat{x}$. In other words, for every game $v \in G$ and every vector $x \in \mathbb{R}^n$ we have that*

$$Sh(v) \cdot x = \langle\langle v, \hat{x} \rangle\rangle.$$

Proof Let $\varphi : G \to \mathbb{R}^n$ be the adjoint to $\hat{}$ with respect to $\langle\langle , \rangle\rangle$, i.e., define the linear solution φ by

$$\varphi(v) \cdot x = \langle\langle v, \hat{x} \rangle\rangle \qquad \text{for every } v \in G, x \in \mathbb{R}^n.$$

First of all, notice that φ is symmetric. Thus $\varphi(\widehat{\mathbf{1}}) = \lambda \mathbf{1}$, for some $\lambda \in \mathbb{R}$, and since

$$\lambda n = \lambda \mathbf{1} \cdot \mathbf{1} = \varphi(\widehat{\mathbf{1}}) \cdot \mathbf{1} = \langle\langle \widehat{\mathbf{1}}, \widehat{\mathbf{1}} \rangle\rangle = n$$

then $\varphi(\widehat{\mathbf{1}}) = \mathbf{1}$.

Also, for every $j < n$, $\varphi(c_j) \cdot \mathbf{1} = \langle\langle c_j, \widehat{\mathbf{1}} \rangle\rangle = 0$, therefore $\varphi(c_j) = 0$.

To finish, we need only check that $\varphi_n(\omega^j) = -1$ for every $j < n$. Thus, it is enough to check that $\varphi(\omega_n^j) = \frac{1}{n-1}\omega$; but, for every $y \in \Delta_n^{\perp}$,

$$\varphi(\omega^j) \cdot y = \langle\langle \omega^j, \hat{y} \rangle\rangle = \langle\langle \omega^j, y^j \rangle\rangle = \frac{1}{n-1}\omega \cdot y. \quad \square$$

Remark 5 The proof shows that given any linear symmetric, efficient solution ψ, for which $\psi_n(\omega^j) < 0$, for every j, we can find an invariant positive definite inner product on G so that ψ is the adjoint of the map $\hat{}$.

The condition $\psi_n(\omega^j) < 0$, for every j, for efficient ψ, is a natural one, in that it reflects on the fact that the only negative values of the game ω^j happen for coalitions that contain player n.

Corollary 3 *For any $v \in G$,*

$$Sh_i(v) = \langle\langle v, \hat{e}_i \rangle\rangle$$

where $e_i \in \mathbb{R}^n$ is the usual ith vector of the standard basis of $\mathbb{R}^n$.

Remark 6 Equivalently, with respect to $\langle\langle , \rangle\rangle$, Shapley's value is characterized by saying that

- $Sh(\hat{x}) = x$, for every additive game $\hat{x}$, and
- $Sh(v) = 0$, for every game v perpendicular to the additive games.

So Shapley's value, in this sense, is nothing more than the orthogonal projection (with respect to $\langle\langle , \rangle\rangle$) to the additive games space. This should be compared with results by Kultti and Salonen (2005) relating certain efficient linear solutions with different types of inner products on G.

4.4 Duality

The interpretation of $v(S)$ changes accordingly to what people want to model. For example, $v(S)$ could be the joint benefit that the coalition S could generate if they decide to play together; in this case we would say that v is a benefit game. In a second interpretation, we could assume that the players in N want to hire a service, then $v(S)$ could be thought of as the joint cost (for the players in S) if they act together. In the latter case we say that v is a cost game. In both cases,

$v(S)$ is the "worth" assigned to the coalition S when it is formed, i.e., when the players in S decide to play together.

The duality operator, as defined next, allows us to move from one of these interpretations to the other. Thus it is a natural concept to study.

The duality operator $* : G \to G$ is defined by

$$(*v)(S) = v(N) - v(N \setminus S).$$

Observe that $*^2 = I$ and that $*$ is S_n-equivariant (i.e., symmetric).

Definition 6 *A solution $\phi \in \mathcal{LS}(G)$ is*

- *self-dual, if $\phi(*v) = \phi(v)$ for all $v \in G$;*
- *anti-self-dual, if $\phi(*v) = -\phi(v)$ for all $v \in G$.*

In order to understand better the (anti-)self-dual solutions we need to understand the action of $*$ on the H_n-invariant subspace $C \oplus T$. Since $*$ is S_n-equivariant, C is an invariant subspace: i.e., $*c \in C$ for every $c \in C$; similarly, T is also an invariant space for $*$.

Lemma 5 *The action of $*$ on $C \oplus T$ is as follows:*

1. *For $j < n$, $*c_j = -c_{n-j}$.*
2. *$*c_n = c_1 + \cdots + c_n$.*
3. *For $j < n$, $*\omega^j = \omega^{n-j}$.*

Proof 1. Let S be such that $|S| = k$, then

$$*c_j(S) = c_j(N) - c_j(N \setminus S) = -\delta_{j,n-k}.$$

2.

$$*c_n(S) = c_n(N) - c_n(N \setminus S) = 1 - \delta_{n,n-k},$$

thus, $*c_n(S) = 1$ for every $S \neq \emptyset$.

3.

$$*\omega^j(S) = -\omega^j(N \setminus S) = \begin{cases} 0 & \text{if } |N \setminus S| \neq j \\ -\omega(N \setminus S) & \text{if } |N \setminus S| = j \end{cases} = \begin{cases} 0 & \text{if } |S| \neq n - j \\ \omega(S) & \text{if } |S| = n - j \end{cases}$$
$$= \omega^{n-j}(S).$$

$\square$

Remark 7 Note that $*\hat{\mathbf{1}} = \hat{\mathbf{1}}$ (in fact for every $x \in \mathbb{R}^n$ $*\hat{x} = \hat{x}$). Thus, one can check that $*$ is orthogonal with respect to the inner product $\langle\langle , \rangle\rangle$ (defined in the previous section) that sets Shapley's solution as the adjoint of $x \mapsto \hat{x}$. From this one can give a quick proof of the self-duality of Shapley's value: For every $v \in G, x \in \mathbb{R}^n$

$$Sh(*v) \cdot x = \langle\langle *v, \hat{x} \rangle\rangle = \langle\langle v, *\hat{x} \rangle\rangle = \langle\langle v, \hat{x} \rangle\rangle = Sh(v) \cdot x.$$

Duality of Shapley's value was already noticed in Kalai and Samet (1987).

Clearly, every symmetric solution decomposes uniquely as a sum of a self and an anti-self dual solution ($\phi(v) = \frac{\phi(v)+\phi(*v)}{2} + \frac{\phi(v)-\phi(*v)}{2}$), thus the spaces of these solutions have complementary dimensions. In fact,

Corollary 4 *Let $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$, with $f(c_j) = a_j$ and $f(\omega^j) = b_j$.*

1. *The symmetric solution corresponding to f is self-dual if and only if $a_j = -a_{n-j}$, $b_j = b_{n-j}$, $j < n$. In particular, the space of symmetric self-dual solutions has dimension n.*
2. *The symmetric solution corresponding to f is anti-self-dual if and only if $a_j = a_{n-j}$, $b_j = -b_{n-j}$, $j < n$, and $a_1 + \cdots + a_{n-1} + 2a_n = 0$. In particular, the space of symmetric anti-self-dual solutions has dimension $n - 1$.*

Proof Let us show the first statement, the other one is proven similarly.

Let ϕ be the linear symmetric solution such that $\phi_n = f$. Then, for every game v

$$\phi(*v) = \phi(v) \Leftrightarrow f(*v) = f(v) \Leftrightarrow \ \forall j \begin{cases} f(*c_j) = f(c_j) \\ f(*\omega^j) = f(\omega^j) \end{cases} \Leftrightarrow \begin{cases} -a_{n-j} = a_j, \ j < n \\ a_1 + \cdots + a_n = a_n \\ b_{n-j} = b_j \end{cases};$$

now, notice that the middle equation is a consequence of the first one.

Finally, for the dimension count, assume first that n is even, then $a_{n/2} = 0$, and we have the following "free" variables: $a_1, \ldots, a_{n/2-1}, a_n, b_1, \ldots, b_{n/2}$, i.e., n variables in total. Whereas if n is odd, then we can specify values for: $a_1, \ldots, a_{[n/2]}, a_n, b_1, \ldots, b_{[n/2]}$, so we have $2[n/2] + 1 = n$ variables. □

4.4.1 *Efficient and self-dual solutions*

An efficient and self-dual solution assigns equal importance to the amount that a particular coalition claims as well as to that amount which the players outside of the coalition fail to claim.

Corollary 5 *The space of linear symmetric solutions which are both efficient and self-dual has dimension $\left[\frac{n}{2}\right]$. This space is the space of solutions of the form*

$$\phi_i(v) = \frac{v(N)}{n} + \sum_{\substack{S \ni i \\ S \neq N}} (n-s)\beta_s \left[v(S) - v(N \setminus S)\right]$$

with $\beta_s = \beta_{n-s}$.

Proof With same notation as above, efficiency implies $a_j = 0, j < n$, and $a_n = \frac{1}{n}$; while self-duality imposes the extra $b_j = b_{n-j}$. Clearly, then, the dimension is $\left[\frac{n}{2}\right]$. The general formula follows from formula (1) for efficient solutions, since $\beta_{n-s} = \beta_s$. □

Since $a_1 + \cdots + a_{n-1} + 2a_n = 0$ for any anti-self-dual solution, there can be none which is also efficient.

4.4.2 Nullity and duality

Recall the bases $\{\nu_j\}$ and $\{\mu_j\} \cup \{\chi_n\}$ of the null space $\mathcal{N}$ and its orthogonal complement $\mathcal{N}^\perp$ (respectively) inside $C \oplus T$. The next lemma gives us the action of $*$ with respect to these bases.

Lemma 6

1. *For* $j = 1, \ldots, n-2$ *we have*

$$*\nu_j = -\nu_{n-j-1} \text{ and } *\mu_j = \mu_{n-j-1}.$$

2. $*\nu_{n-1} = *c_n - \chi_n$ *and* $*\mu_{n-1} = - * c_n - \chi_n$.
3. $*\chi_n = \frac{-1}{n}(c_{n-1} + \omega^{n-1})$.

Proof The proof follows from Lemma 4, the formulas at the end of 2.2 and Lemma 5. □

Recall (Proposition 5) that $f \in \mathrm{Hom}_{H_n}(G, \mathbb{R})$ gives rise to a null solution if and only if it is of the form:

$$f(v) = \sum_{S \not\ni n} r_s \left[v(S \cup \{n\}) - v(S) \right]$$

where $r_0 = f(\chi_n)$, and

$$r_j = \frac{-f(\mu_j)}{2\binom{n-1}{j}}$$

for $j : 1, \ldots, n-1$.

Proposition 6 *The null solution* $f(v) = \sum_{S \not\ni n} r_s \left[v(S \cup \{n\}) - v(S) \right]$ *is self-dual (respectively, anti-self-dual) if and only if* $r_j = r_{n-j-1}$ *(respectively,* $r_j = -r_{n-j-1}$*), for all* $j < n$.

Proof Let us give a proof for the self-duality case.

The null f gives a self-dual solution if and only if it satisfies self-duality on any basis for $C \oplus T$. Thus, f gives rise to a self-dual solution if and only if

$$f(*\nu_j) = f(\nu_j), f(*\mu_j) = f(\mu_j) \quad \text{and} \quad f(*\chi_n) = f(\chi_n).$$

A null f automatically satisfies $f(*\nu_j) = 0 = f(\nu_j)$ for $j \le n-1$, thus the first $n-2$ of these equations impose no restriction.

The equation $f(*\nu_{n-1}) = f(\nu_{n-1}) = 0$ is equivalent to

$$0 = f(\nu_{n-1}) = f(*c_n) - r_0.$$

The set of equations $f(*\mu_j) = f(\mu_j)$ give

$$\begin{aligned} r_j &= r_{n-j-1}, \qquad \text{for } j < n-1, \\ -2r_{n-1} &= -f(*c_n) - r_0 \end{aligned}$$

which, when combined with the previous one, gives

$$\begin{aligned} r_j &= r_{n-j-1}, \qquad \text{for } j < n-1, \\ r_{n-1} &= r_0; \end{aligned}$$

i.e., $r_j = r_{n-j-1}$ for all $j \leq n-1$.

Finally, the last equation, $f(*\chi_n) = f(\chi_n)$, is already implied by the others. Notice that, since $2c_n = \nu_{n-1} - \mu_{n-1}$ then $f(*c_n) = f(c_n)$ follows from the first two sets of equations. Then,

$$f(\chi_n) = f(- * c_n - *\mu_{n-1}) = f(-c_n - \mu_{n-1}) = f(*\chi_n).$$

Anti-self-duality is treated in the same way. □

5 Kernels

Corollary 1 is the statement that the common kernel of all symmetric linear solutions is W, i.e.,

$$\bigcap_{\phi \in \mathcal{LS}(G)} \ker \phi = W.$$

From the basic results for efficient solutions discussed in Sect. 3.1 (namely Proposition 3), the next result follows easily.

Proposition 7 *The common kernel of all linear, symmetric and efficient solutions is*

$$C_1 \oplus \cdots \oplus C_{n-1} \oplus W.$$

Remark 8 In Amer et al. (2003) the common kernel of all linear, symmetric and null solutions is computed. It is shown that it coincides with W!

In what follows we concentrate on the kernel of a single solution.

Let $\phi : G \to \mathbb{R}^n$ be any linear symmetric solution. As we saw before it is uniquely determined by the numbers $a_i = \phi_n(c_i)$, $b_j = \phi_n(\omega^j)$.

Proposition 8 *The kernel of ϕ consists of all games $c + u + w \in C \oplus U \oplus W$, such that*

- $\langle c, c_\phi \rangle = 0$, *where* $c_\phi = \sum_{i=1}^{n} \frac{a_i}{\|c_i\|^2} c_i$.
- $u = \sum_{j=1}^{n-1} z_j^j$, *with* $z_j \in \Delta_n^\perp$ *such that* $\sum_j b_j z_j = 0$.
- $w \in W$ *is arbitrary.*

Compare the following corollary to a similar result in Kleinberg and Weiss (1985).

Corollary 6 *The kernel of Shapley's value consists of all games $c + u + w \in C \oplus U \oplus W$, such that*

- $c \in C_1 \oplus \cdots \oplus C_{n-1}$.
- $u = \sum_{j=1}^{n-1} z_j^j$, *with* $z_j \in \Delta_n^{\perp}$ *such that* $\sum_j z_j = 0$.
- $w \in W$ *is arbitrary.*

Proof of Corollary: For Shapley's value $a_1 = a_2 = \cdots = a_{n-1} = 0$, $a_n = \frac{1}{n}$ and $b_j = -1$. □

Remark 9 Recall that at the end of Sect. 2.3 we have shown that the kernel of Shapley's value is the orthogonal complement -with respect to the inner product $\langle\langle , \rangle\rangle$ defined there- to the subspace of additive games. This corollary is an equivalent formulation without mention to $\langle\langle , \rangle\rangle$.

Proof of Proposition: Since $\ker \phi$ is an invariant subspace of G then

$$\ker \phi = (\ker \phi \cap C) \oplus (\ker \phi \cap U) \oplus (\ker \phi \cap W)\,;$$

to prove this, let A be an irreducible summand of $\ker \phi$, then A is an irreducible summand of G, and, thus, is contained in either C, U or W.

Now, for $c \in C$

$$\langle c, c_\phi \rangle = \sum_i \frac{a_i}{\|c_i\|^2} \langle c, c_i \rangle = \sum_i \frac{\langle c, c_i \rangle}{\|c_i\|^2} \phi_n(c_i) = \phi_n(c),$$

and thus, $c \in \ker \phi$ if and only if $\langle c, c_\phi \rangle = 0$.

Every $u \in U$ can be written as $u = \sum_{j=1}^{n-1} z_j^j$, with $z_j \in \Delta_n^{\perp}$. Then,

$$\phi(u) = \sum_j \phi(z_j^j) = \frac{1}{1-n} \sum_j b_j z_j.$$

Finally, $W \subset \ker \phi$. □

Appendix

A reference for basic representation theory is Fulton and Harris (1991). Nevertheless, we recall all the basic facts that we need.

Definition 7 *Let H be an arbitrary group. A representation for H is a homomorphism $\rho : H \to GL(V)$, where V is a vector space and $GL(V)$ denotes the group of invertible linear maps of V.*

In other words, a representation of H is a map assigning to each element $h \in H$ a linear map $\rho(h) : V \to V$ that respects multiplication:

$$\rho(h_1h_2) = \rho(h_1)\rho(h_2).$$

One usually abuses notation and talks about the representation V without explicitly mentioning the homomorphism ρ. Thus, when applying the linear transformation corresponding to $h \in H$ on a vector $v \in V$, we write $h \cdot v$ rather than $(\rho(h))(v)$.

Definition 8 *Let V and W be two representations for the group H.*

- *A linear map $T : V \to W$ is H-equivariant if $T(h \cdot v) = h \cdot (T(v))$, for every $v \in V$ and every $h \in H$.*
- *V and W are isomorphic H-representations, $V \simeq W$, if there exists an H-equivariant isomorphism between them.*

Thus, two representations that are isomorphic are, as far as all problems dealing with linear algebra with a group of symmetries, the same. They are vector spaces of the same dimension where the actions are seen to correspond under a linear isomorphism.

Example 3 Let $A \subset G$ denote the subspace of additive games, i.e., $v \in A$ if and only if $v(S \cup T) = v(S) + v(T)$ for every pair of coalitions S, T with $S \cap T = \emptyset$. S_n acts on A; in other words, if $\theta \in S_n$ and $v \in A$ then $\theta \cdot v \in A$ as is readily seen.

We claim that A as a representation is isomorphic to $\mathbb{R}^n$ with the usual action $\theta \cdot (x_1, \ldots, x_n) = (x_{\theta_1}, x_{\theta_2}, \ldots, x_{\theta_n})$.

The S_n-equivariant isomorphism $\phi : \mathbb{R}^n \to A$ is given by $\phi(x) = \hat{x}$ where

$$\hat{x}(S) = \sum_{j \in S} x_j.$$

Clearly ϕ is a linear map and $\hat{x}$ is an additive game for every x. Let us show equivariance. We need to prove that

$$\phi(\theta \cdot x) = \theta \cdot \hat{x}.$$

Now,

$$(\theta \cdot \hat{x})(S) = \hat{x}(\theta^{-1}S) = \sum_{\{j \mid \theta_j \in S\}} x_j;$$

on the other hand, if we set $y = \theta \cdot x$, then

$$(\phi(\theta \cdot x))(S) = \widehat{\theta \cdot x}(S) = \hat{y}(S) = \sum_{i \in S} y_i,$$

but $y_i = x_{\theta^{-1}i}$ and, hence, both sums are identical (just set $i = \theta_j$).

Clearly ϕ is 1–1, and hence it is onto by counting dimensions of both spaces.

So, from the point of view of linear algebra with S_n-action, the space of additive games and $\mathbb{R}^n$ are the same. □

Definition 9 *A representation V is irreducible if it does not contain a nontrivial invariant subspace. That is, if $U \subset V$ is also a representation for H (meaning that $h \cdot u \in U$ for every $h \in H$ and every $u \in U$), then U is either $\{0\}$ or all of V.*

The following theorem is one of the reasons why it is worth carrying around the group action when there is one. Its simplicity hides the fact that it is a very powerful tool.

Theorem 4 *(Schur's Lemma) Let V, W be irreducible representations of a group H. If $\phi : V \to W$ is H-equivariant, then $\phi \equiv 0$ or ϕ is an isomorphism.*

Moreover, if V and W are complex vector spaces, then ϕ is unique up to multiplication by a scalar $\lambda \in \mathbb{C}$.

Proof Kerϕ and Imϕ are invariant subspaces of V and W, respectively, thus it is zero or the total space. From this follows the first part.

If the vector spaces are complex, $\phi, \psi : V \to W$ H-equivariant, then $T = \phi^{-1} \circ \psi : V \to V$ must have an eigenvalue $\lambda \in \mathbb{C}$. Since the eigenspace corresponding to λ is invariant, it must be all of V, i.e., $T = \lambda I$ or $\psi = \lambda\phi$. □

Corollary 7 *Let V be a real irreducible representation, such that its complexification $V^{\mathbb{C}} = V \otimes \mathbb{C} = V \oplus iV$ is also irreducible (as a complex representation). Let W be a real irreducible representation. If $\phi : V \to W$ is equivariant, then ϕ is unique up to multiplication by a real scalar.*

Proof Schur's Lemma implies that ϕ is zero or an isomorphism. Suppose ϕ is an isomorphism, then $W^{\mathbb{C}}$ is also isomorphic to $V^{\mathbb{C}}$ and $\phi^{\mathbb{C}} : V^{\mathbb{C}} \to W^{\mathbb{C}}$ (the complex-linear extension of ϕ) is an isomorphism. If $\psi : V \to W$ is any equivariant map, then by the previous theorem $\psi^{\mathbb{C}} = \lambda\phi^{\mathbb{C}}$, for some $\lambda \in \mathbb{C}$. Since $\phi^{\mathbb{C}}$ and $\psi^{\mathbb{C}}$ preserve real parts (i.e., send V to W) λ must be real. □

Lemma 7 *The decomposition of $\mathbb{R}^n$, under S_n, into irreducible subspaces is*

$$\mathbb{R}^n = \mathbb{R}\mathbf{1} \oplus \Delta_n^{\perp}.$$

Proof We need only to check that $\Delta_n^{\perp}$ is irreducible. This is done by induction on n. If $n = 2$, then $\Delta_n^{\perp}$ is 1-dimensional and thus, irreducible.

$\Delta_n^{\perp}$ is generated by $e_1 - e_n, e_2 - e_n, \ldots, e_{n-1} - e_n$, where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$. The isotropy subgroup of e_n, $H_n = \{\theta \in S_n \mid \theta_n = n\}$, is isomorphic to S_{n-1}, and $\Delta_n^{\perp}$ is the standard representation for H_n.

Therefore, induction hypothesis says that

$$\Delta_n^{\perp} \simeq \mathbb{R} \oplus 1_{n-1}^{\perp}$$

is the decomposition (up to isomorphism) into irreducible H_n-representations. Actually, the precise summands of this decomposition are $\mathbb{R}\omega$ and its orthogonal complement inside $\Delta_n^{\perp}$, where $\omega = (e_1 - e_n) + (e_2 - e_n) + \cdots + (e_{n-1} - e_n) = (1, 1, \ldots, 1, 1 - n)$.

Now, if $\Delta_n^\perp$ were a reducible S_n-representation, then it would split into subspaces also invariant under H_n. Thus, the only possible splitting would be into $\mathbb{R}\omega$ and its complement. But these subspaces are not S_n-invariant. □

Remark 10 Observe that the above proof also shows that the real irreducible representations $\mathbb{R}$ and $\Delta_n^\perp$ have irreducible complexifications.

Proposition 9 *For $j < n$,*

$$G_j = C_j \oplus U_j \oplus W_j,$$

where $C_j = \mathbb{R}c_j \simeq \mathbb{R}$, $U_j = \{x^j \mid x \in \Delta_n^\perp\} \simeq \Delta_n^\perp$ and W_j does not contain any summands isomorphic to either $\mathbb{R}$ nor $\Delta_n^\perp$. The decomposition is orthogonal.

Proof Let X_j be the subset of 2^N consisting of all subsets of cardinality j. So, we may identify G_j with $\mathbb{R}[X_j]$ the space of real functions on X_j. Let $K = S_j \times S_{n-j} \prec S_n$ be the subgroup fixing $\{1, 2, \ldots, j\}$. Thus, $X_j = S_n/K$ and

$$G_j \simeq \mathbb{R}[S_n/K].$$

Hence, if τ is the trivial 1-dimensional representation for K, then G_j is the induced representation for S_n. Now, Frobenius reciprocity implies that

$$\langle V, G_j\rangle_{S_n} = \langle V_{|K}, \tau\rangle_K$$

for every S_n-representation V. Here, $\langle V, W\rangle_G$ denotes the dimension of the space of G-equivariant maps from V to W; therefore, if V is irreducible, $\langle V, W\rangle_G$ is the number of copies of V in W.

For example, for $V = \mathbb{R}$, the trivial representation, we get

$$\langle \mathbb{R}, G_j\rangle_{S_n} = \langle \tau, \tau\rangle_K = 1.$$

And if $V = \Delta_n^\perp$, then again

$$\langle \Delta_n^\perp, G_j\rangle_{S_n} = \langle \Delta_n^\perp|_K, \tau\rangle_K = 1$$

since the standard representation for S_n, restricted to $S_j \times S_{n-j}$ contains exactly two copies of τ.

The invariant inner product $\langle , \rangle$ gives an equivariant isomorphism $G_j \to G_j^*$ via $v \mapsto [u \mapsto \langle v, u\rangle]$; in particular must preserve the decomposition. This implies orthogonality of the decomposition. □

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