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A control approach to robust utility maximization with logarithmic utility and time-consistent penalties

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Abstract

We propose a stochastic control approach to the dynamic maximization of robust utility functionals that are defined in terms of logarithmic utility and a dynamically consistent convex risk measure. The underlying market is modeled by a diffusion process whose coefficients are driven by an external stochastic factor process. In particular, the market model is incomplete. Our main results give conditions on the minimal penalty function of the robust utility functional under which the value function of our problem can be identified with the unique classical solution of a quasilinear PDE within a class of functions satisfying certain growth conditions. The fact that we obtain classical solutions rather than viscosity solutions facilitates the use of numerical algorithms, whose applicability is demonstrated in examples.

Keywords: Robust utility maximization; Stochastic factor model; Stochastic control; Convex risk measure; Dynamic consistency; Hamilton–Jacobi–Bellman equation

1. Introduction

One of the fundamental problems in mathematical finance and mathematical economics is the construction of investment strategies that maximize the utility functional of a risk-averse investor. In the majority of the corresponding literature, the optimality criterion is based on a classical expected utility functional of von Neumann–Morgenstern form, which requires the choice of a single probabilistic model $\mathbb{P}$. In reality, however, the choice of $\mathbb{P}$ is often subject to model uncertainty. Schmeidler [28] and Gilboa and Schmeidler [12] therefore proposed the use

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of *robust utility functionals* of the form

$$X \longmapsto \inf_{Q \in \mathcal{Q}} E_Q[U(X)], \tag{1}$$

where $\mathcal{Q}$ is a set of prior probability measures. In analogy to the move from coherent to convex risk measures, Maccheroni et al. [17] recently suggested modeling investor preferences by robust utility functionals of the form

$$X \longmapsto \inf_{Q} \left(E_Q[U(X)] + \gamma(Q) \right), \tag{2}$$

where γ is a penalty function defined on the set of all possible probabilistic models.

Optimal investment problems for robust utility functionals (1) were considered by, among others, Talay and Zheng [29], Quenez [23], Schied [24,25], Burgert and Rüschendorf [5], Schied and Wu [27], Föllmer and Gundel [10], and the authors [15]. For the generalized utility functionals of type (2), the most popular choice for the penalty function has so far been the entropic penalty function $\gamma(Q) = kH(Q|\mathbb{P})$ for a constant $k > 0$ and a reference probability measure $\mathbb{P}$; see, e.g., Hansen and Sargent [14] and Bordigoni et al. [4] for studies of the optimal consumption problem. The duality theory for the optimal investment problem with a general penalty function γ was developed by Schied [26]. Robust utility maximization is also closely related to other optimization problems involving convex and coherent risk measures, and these problems have also received a lot of attention recently; see, for instance, Barrieu and El Karoui [3], Klöppel and Schweizer [16], Gundel and Weber [13], and the references therein.

In this paper, we propose a stochastic control approach to the dynamic maximization of robust utility functionals of the form (2). The penalty function γ will be defined in a Brownian setting and, apart from certain basic requirements such as time consistency, has a rather general form. In particular, we will go beyond the very particular situation of entropic penalties and include the 'coherent' setting (1) as a special case. Our setting will involve logarithmic utility $U(x) = \log x$ and an incomplete financial market model, whose volatility, interest rate process, and trend are driven by an external stochastic factor process.

Our goal consists in characterizing the value function and the optimal investment strategy via the solution of a quasilinear Hamilton–Jacobi–Bellman PDE. As a by-product, we also obtain a formula for the least favorable martingale measure in the sense of Föllmer and Gundel [10]. In contrast to earlier approaches such as [29], ours avoids the use of viscosity solutions and we concentrate our effort on obtaining strong regularity results, which allow us to identify the value function as a unique classical solution of the PDE in question. Regularity of solutions is important because it facilitates the use of standard numerical methods for solving the PDE, and we will use such methods in illustrating some interesting qualitative properties of the optimal strategy.

Our method consists in combining the duality results from [26] with a PDE approach to the dual problem of determining optimal martingale measures. This technique has already been applied by Castañeda-Leyva and Hernández-Hernández [6,7] to the maximization of von Neumann–Morgenstern expected utility and by the authors [15] in the maximization of 'coherent' robust utility functionals of the form (1). It turns out, however, that the introduction of the penalty function γ yields new kinds of problems, in particular when measures $Q \ll \mathbb{P}$ with $\gamma(Q) < \infty$ have to be described by unbounded control processes (this is the case, e.g., for entropic penalties). To deal with this case, we have to introduce new arguments both on the probabilistic and on the analytic side of the problem.

This paper is organized as follows. In Section 2 we describe the set-up of the problem and state the theorems containing our main findings. These theorems will be proved in the subsequent sections. Section 3 analyzes how certain classes of probability measures $Q \ll \mathbb{P}$ can be described by a suitable set of control processes. The dual problem for our robust utility maximization problem is formulated in Section 4. In Section 5 we derive the Hamilton–Jacobi–Bellman PDE for the value function via the dual problem and we prove a verification result. This verification result will suffice to prove our results in the special case where the effective domain of γ is a compact set of probability measures that are all equivalent to the reference measure $\mathbb{P}$. In Section 6 we consider the case in which $\gamma(Q)$ can also be finite for measures $Q \ll \mathbb{P}$ that are not equivalent to $\mathbb{P}$. Since the market model may admit arbitrage opportunities under such a measure Q, it is clear that the corresponding problem must become more involved, and it turns out that complications also appear on the analytical side of the problem.

2. Statement of main results

We consider a financial market model with a locally riskless money market account

$$\mathrm{d}S_t^0 = S_t^0 r(Y_t)\,\mathrm{d}t$$

and a risky asset defined under a reference measure $\mathbb{P}$ through the SDE

$$\mathrm{d}S_t = S_t b(Y_t)\,\mathrm{d}t + S_t \sigma(Y_t)\,\mathrm{d}W_t^1.$$

Here W^1 is a standard $\mathbb{P}$-Brownian motion and Y denotes an external economic factor process modeled by the SDE

$$\mathrm{d}Y_t = g(Y_t)\,\mathrm{d}t + \rho\,\mathrm{d}W_t^1 + \overline{\rho}\,\mathrm{d}W_t^2, \tag{3}$$

where $\rho \in [-1, 1]$ is some correlation factor, $\overline{\rho} := \sqrt{1-\rho^2}$, and W^2 is a standard $\mathbb{P}$-Brownian motion, which is independent of W^1 under $\mathbb{P}$. We suppose that the economic factor cannot be traded directly so that the market model will typically be incomplete. Market models of this type are popular in mathematical finance and economics, in particular if Y follows an Ornstein–Uhlenbeck dynamic with mean reversion term $g(y) = \kappa(\theta - y)$ for constants $\kappa > 0$, $\theta \geq 0$; see, e.g., [6–8] and the references therein.

We assume that g is in $C^2(\mathbb{R})$ with derivative $g' \in C_b^1(\mathbb{R})$, and $r,\ b$, and σ belong to $C_b^2(\mathbb{R})$, where $C_b^k(\mathbb{R})$ denotes the class of bounded functions with bounded derivatives up to order k. The 'market price of risk' with respect to $\mathbb{P}$ is defined via the function

$$\theta(y) := \frac{b(y) - r(y)}{\sigma(y)},$$

and we will assume that $\sigma \geq \sigma_0$ for some constant $\sigma_0 > 0$. The assumption of time-independent coefficients is for notational convenience only and can easily be relaxed.

In most economic situations, investors typically face *model uncertainty* in the sense that the dynamics of the relevant quantities are not precisely known. One common approach to coping with model uncertainty is to allow in principle all probability models corresponding to probability measures $Q \ll \mathbb{P}$ and to penalize each such model with a penalty $\gamma(Q)$; see [11,17]. To define $\gamma(Q)$, we assume henceforth that everything is modeled on the canonical path space $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ of $W = (W^1, W^2)$. Then every probability measure $Q \ll \mathbb{P}$ admits a progressively measurable process $\eta = (\eta_1, \eta_2)$ such that

$$\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(\int \eta_{1t}\,\mathrm{d}W_t^1 + \int \eta_{2t}\,\mathrm{d}W_t^2\right)_T \quad Q\text{-a.s.},$$

where $\mathcal{E}(M)_t = \exp(M_t - \langle M \rangle_t / 2)$ denotes the Doleans–Dade exponential of a continuous semimartingale M; see Lemma 3.1 below. Such a measure Q will receive a penalty

$$\gamma(Q) := E_Q\left[\int_0^T h(\eta_t)\,\mathrm{d}t\right], \tag{4}$$

where $h : \mathbb{R}^2 \to [0, \infty]$ is convex and lower semicontinuous. For simplicity, we will suppose $h(0) = 0$ so that $\gamma(\mathbb{P}) = 0$. We will also assume that h is continuously differentiable on its effective domain $\operatorname{dom} h := \{\eta \in \mathbb{R}^2 \mid h(\eta) < \infty\}$ and satisfies the coercivity condition

$$h(x) \geq \kappa_1 |x|^2 - \kappa_2 \quad \text{for some constants } \kappa_1,\ \kappa_2 > 0. \tag{5}$$

The choice $h(x) = |x|^2/2$ corresponds to the entropic penalty function considered in Hansen and Sargent [14] and Bordigoni et al. [4]; see Remark 2.6 below. Again, our assumption that h does not depend on time is for notational convenience only.

Let $\mathcal{A}$ denote the set of all progressively measurable process π such that $\int_0^T \pi_s^2\,\mathrm{d}s < \infty \mathbb{P}$-a.s. For $\pi \in \mathcal{A}$ we define

$$\begin{aligned} X_t^{x,\pi} &:= x \cdot \exp\left(\int_0^t \pi_s \sigma(Y_s)\,\mathrm{d}W_s^1\right. \\ &\quad \left. + \int_0^t \left[r(Y_s) + \pi_s\,(b(Y_s) - r(Y_s)) - \frac{1}{2}\sigma^2(Y_s)\pi_s^2\right] \mathrm{d}s\right). \end{aligned} \tag{6}$$

Then $X^{x,\pi}$ satisfies

$$X_t^{x,\pi} = x + \int_0^t \frac{X_s^{x,\pi}(1-\pi_s)}{S_s^0}\mathrm{d}S_s^0 + \int_0^t \frac{X_s^{x,\pi}\pi_s}{S_s}\mathrm{d}S_s$$

and thus describes the evolution of the wealth process $X^{x,\pi}$ of an investor with initial endowment $X_0^{x,\pi} = x > 0$ investing the fraction π_s of the current wealth into the risky asset at time $s \in [0, T]$.

The objective of the investor consists in

$$\text{maximizing} \quad \inf_{Q \ll \mathbb{P}} \left(E_Q[U(X_T^{x,\pi})] + \gamma(Q)\right) \quad \text{over } \pi \in \mathcal{A}, \tag{7}$$

where the utility function $U : (0, \infty) \to \mathbb{R}$ will be specified in the sequel as a HARA utility function with risk aversion parameter $\alpha = 0$, i.e.,

$$U(x) = \log x. \tag{8}$$

This choice has the advantage that the initial capital x can be separated from the problem, thus resulting in a dimension reduction. Our goal is to characterize the value function

$$u(x) := \sup_{\pi \in \mathcal{A}} \inf_{Q \ll \mathbb{P}} \left(E_Q[\log X_T^{x,\pi}] + \gamma(Q)\right)$$

of the robust utility maximization problem (7) in terms of the solution v of the quasilinear parabolic initial value problem

$$\begin{cases} v_t = \dfrac{1}{2} v_{yy} + \phi(v_y) + g v_y + r \\ v(0, \cdot) = 0, \end{cases} \tag{9}$$

where the nonlinearity $\phi(v_y) = \phi(y, v_y(t, y))$ is given by

$$\phi(y, z) := \psi(y, (\rho, \overline{\rho})z) \quad y, z \in \mathbb{R}$$

for the function

$$\psi(y, x) := \inf_{\eta \in \mathbb{R}^2} \left\{ \eta \cdot x + \frac{1}{2}(\eta_1 + \theta(y))^2 + h(\eta) \right\}, \quad y \in \mathbb{R},\ x \in \mathbb{R}^2.$$

Here, $\eta \cdot x$ denotes the inner product of η and x. The easy case is the one in which the effective domain of h is compact:

Theorem 2.1. *Suppose that* dom h *is compact. Then there exists a unique classical solution* v *to* (9) *within the class of functions in* $C^{1,2}((0, T) \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ *satisfying a polynomial growth condition. The value function* u *of the robust utility maximization problem is given by*

$$u(x) = \log\ x + v(T, Y_0).$$

Suppose furthermore that $\eta^* : [0, T] \times \mathbb{R} \to \mathbb{R}$ *is a measurable function such that* $\eta^*(t, y)$ *belongs to the supergradient of the concave function* $x \mapsto \psi(y, x)$ *at* $x = (\rho, \overline{\rho})v_y(t, y)$. *Then an optimal strategy* $\widehat{\pi}$ *for the robust problem can be obtained by letting*

$$\widehat{\pi}_t = \frac{\eta_1^*(T - t, Y_t) + \theta(Y_t)}{\sigma(Y_t)}, \quad 0 \le t \le T.$$

Moreover, by defining a measure $\widehat{Q} \sim \mathbb{P}$ *via*

$$\frac{\mathrm{d}\widehat{Q}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(\int_0 \eta^*(T - t, Y_t)\, \mathrm{d}W_t \right)_T, \tag{10}$$

we obtain a saddle point $(\widehat{\pi}, \widehat{Q})$ *for the maximin problem* (7).

The regularity of the value function obtained in the preceding theorem is important, because it facilitates the use of standard numerical methods for solving the PDE (9). In Example 2.7, we will use such methods in illustrating some qualitative properties of the optimal strategy.

Remark 2.2. The proof of Theorem 2.1 will show that the probability measure P^* with density

$$\frac{\mathrm{d}P^*}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(-\int \theta(Y_s)\, \mathrm{d}W_s^1 + \int \eta_2^*(T - s, Y_s)\, \mathrm{d}W_s^2 \right)_T$$

is a least favorable martingale measure in the sense of Föllmer and Gundel [10]. This will also be true in the setting of Theorems 2.3 and 2.5.

The problem becomes more difficult when dom h is noncompact, because then we can no longer apply standard theorems on the existence of classical solutions to (9). Other difficulties appear when dom h is not only noncompact but also unbounded. For instance, we then may have $\gamma(Q) < \infty$ even if Q is not equivalent but merely absolutely continuous with respect to $\mathbb{P}$, and this leads to difficulties when one tries to work directly on the primal problem; see Remark 4.2. Moreover, since the optimal η^* takes values in the unbounded set dom h, one needs an additional argument to ensure that the stochastic exponential in (10) is a true martingale and so defines a probability measure $\widehat{Q} \ll \mathbb{P}$. Our strategy for getting the necessary integrability of the process $\eta_1^*(T - t, Y_t)$ is using qualitative properties of solutions v to (9) so as to control the growth of the gradient v_y. In doing so, we have to eliminate the possible competition between the linear term gv_y and the nonlinear term $\phi(v_y)$ by imposing a growth condition on ϕ.

Theorem 2.3. *Suppose that g is bounded and that there exists some $\varepsilon > 0$ such that*

$$\liminf_{|p|\to\infty} \left| \frac{\phi(y, p)}{p} \right| \geq \varepsilon + |g(y)|. \tag{11}$$

Then there exists a unique classical solution v to (9) within the class of functions in $C^{1,2}((0, T) \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ with bounded gradient v_y. The value function u of the robust utility maximization problem satisfies $u(x) = \log\, x + v(T, Y_0)$, and also the conclusions on the optimal strategy $\widehat{\pi}$ and the measure $\widehat{Q}$ in Theorem 2.1 remain true.

The most interesting case is the one in which both dom h and the function g are unbounded. Here we need an additional condition on the shape of the function ψ. Note that g is unbounded if, e.g., Y is an Ornstein–Uhlenbeck process.

Definition 2.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an upper semicontinuous concave function. We will say that f satisfies a radial growth condition in direction $x \in \mathbb{R}^2 \setminus \{0\}$ if there exist positive constants p_0 and C such that

$$\max\{|z| \mid z \in \partial f(px)\} \leq C \left(1 + |\partial_p^+ f(px)| \vee |\partial_p^- f(px)|\right) \quad \text{for } p \in \mathbb{R},\ |p| \geq p_0,$$

where $\partial f(px)$ denotes the supergradient of f at px and $\partial_p^+ f(px)$ and $\partial_p^- f(px)$ are the right-hand and left-hand derivatives of the concave function $p \mapsto f(px)$.

Note that if f is of the form $f(x) = f_0(|x|)$ for some convex increasing function f_0, then the radial growth condition is satisfied in any direction $x \neq 0$ with constant $C = 1/|x|$.

Theorem 2.5. *Suppose that $|(\phi(y, p)/p)| \to \infty$ as $|p| \to \infty$ and assume that $\psi(y, \cdot)$ satisfies a radial growth condition in direction $(\rho, \overline{\rho})$, uniformly in y. Then there exists a unique classical solution v to (9) within the class of polynomially growing functions in $C^{1,2}((0, T) \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ whose gradient satisfies a growth condition of the form*

$$|\partial_p^- \phi(y; v_y(t, y))| \vee |\partial_p^+ \phi(y; v_y(t, y))| \leq C_1(1 + |y|)$$

for some constant C_1. The value function u of the robust utility maximization problem satisfies $u(x) = \log\, x + v(T, Y_0)$, and also the conclusions on the optimal strategy $\widehat{\pi}$ and the measure $\widehat{Q}$ in Theorem 2.1 remain true.

Remark 2.6. For $q > 0$, the choice $h(x) = \frac{1}{2q}|x|^2$ corresponds to the penalty function $\gamma(Q) = \frac{1}{q} H(Q|\mathbb{P})$, where

$$H(Q|\mathbb{P}) = \int \frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}} \log \frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}}\, \mathrm{d}\mathbb{P} = \sup_{Y \in L^\infty} \left(E_Q[Y] - \log \mathbb{E}[e^Y]\right)$$

is the relative entropy of Q with respect to $\mathbb{P}$. Due to the classical duality formula

$$\log \mathbb{E}[e^X] = \sup_{Q \in \mathcal{Q}} \left(E_Q[X] - H(Q|\mathbb{P})\right),$$

the above choices for h and γ correspond to the utility functional

$$\inf_{Q \ll \mathbb{P}} \left(E_Q[\log X] + \gamma(Q)\right) = -\frac{1}{q} \log \mathbb{E}\left[\mathrm{e}^{-q \log X}\right] = -\frac{1}{q} \log \mathbb{E}[X^{-q}].$$

In this case, the robust utility maximization problem (7) is equivalent to the maximization of the standard expected utility $\mathbb{E}[U(X_T^{x,\pi})]$ for the HARA utility function $U(x) = -x^{-q}$. This standard utility maximization problem is covered as a special case of Theorem 2.5. Indeed, the function ψ has the quadratic form

$$\psi(y, x) = -\frac{1}{2}\left(\frac{q}{1+q}(x_1 + \theta(y))^2 + qx_2^2 - \theta(y)^2\right),$$

and it is easily checked that it satisfies the radial growth condition in any direction.

The numerical computations in the following example were carried out using a multigrid Howard algorithm as explained in Akian [1] and Kushner and Dupuis [20]. For convergence results of such numerical schemes see [20], Krylov [19], Barles and Jakobsen [2], and the references therein.

Example 2.7. Here we will give some numerical results for the case in which $\mathbb{P}$ is such that Y is an Ornstein–Uhlenbeck process with $g(y) = 100 - y$ and S follows the SDE

$$\mathrm{d}S_t = S_t\left(\frac{1}{2}\,\mathrm{d}W_t^1 + b(Y_t)\,\mathrm{d}t\right),$$

where b smooth, bounded, increasing, and satisfies

$$b(y) = \begin{cases} -100.1 & \text{for } y \le -1001, \\ y/10 & \text{for } |y| \le 1000, \\ 100.1 & \text{for } y \ge 1001. \end{cases}$$

We suppose that $r = 0$. Then θ is given by $\theta(y) = y/5$ as long as $|y| \le 1000$. Let us first consider the 'coherent' case

$$h_1(\eta) = \begin{cases} 0 & \text{if } |\eta_1| \le 20 \text{ and } \eta_2 = 0, \\ \infty & \text{otherwise.} \end{cases}$$

The corresponding penalty function $\gamma_1(Q)$ takes only the values 0 and ∞. If $\rho = 0$ then the optimal η^* is given by $\eta_2^*(t, y) = 0$ and

$$\eta_1^*(t, y) = \begin{cases} -\theta(y) & \text{if } |y| \le 100, \\ -20\,\mathrm{sign}(y) & \text{otherwise.} \end{cases}$$

In particular, our formula for $\widehat{\pi}$ shows that there will be no investment into the risky asset as long as the factor process Y stays in the interval $[-100, 100]$. This corresponds to the fact that S has a local martingale dynamic under the 'worst-case measure' $\widehat{Q}$ as long as $-100 \le Y_t \le 100$.

A nonzero correlation factor ρ, however, can change the picture. This is illustrated in Fig. 1, which shows the function v for the 'coherent' penalty function h_1 but with nonzero correlation $\rho = 1/2$. This figure clearly exhibits a nonvanishing gradient of v, even within the interval $[-100, 100]$. But according to our formula for the optimal strategy $\widehat{\pi}$, a nonvanishing gradient v_y results in a nontrivial investment in the risky asset—despite the fact that for $-100 \le Y_t \le 100$ we can still turn S locally into a martingale by choosing an appropriate probability measure Q with $\gamma_1(Q) = 0$. Such a measure, however, will no longer be the 'worst-case measure'. This effect occurs as a tradeoff between the tendencies of minimizing asset returns and driving Y further away from 'favorable regions' under the 'worst-case measure' $\widehat{Q}$.

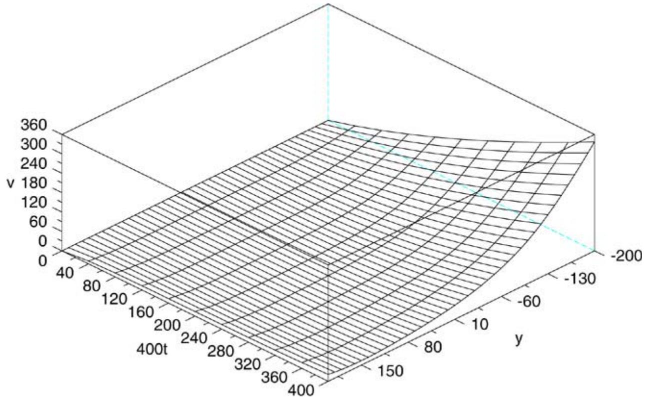

Fig. 1. The function v for the choices h_1 and $\rho = 1/2$.

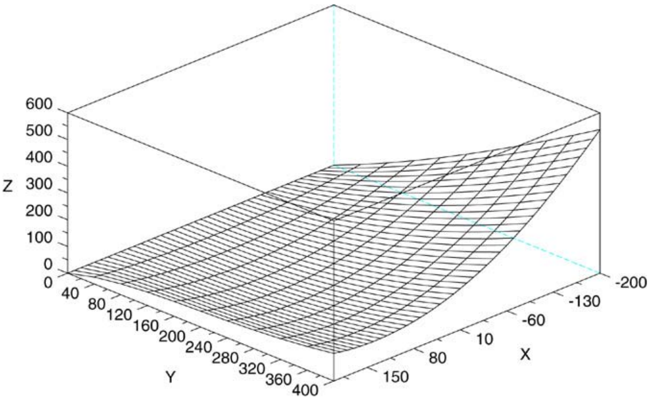

Fig. 2. The function v for h_2 and $\rho = 1/2$.

Fig. 2 shows the function v for the case in which we add to the relative entropy $H(Q|\mathbb{P})$ to the penalty function γ_1. That is, we use the function

$$h_2(\eta) = \begin{cases} \frac{1}{2}\eta_1^2 & \text{if } |\eta_1| \le 20 \text{ and } \eta_2 = 0, \\ \infty & \text{otherwise.} \end{cases}$$

It can be compared to the value function for the standard utility maximization problem with subjective measure $\mathbb{P}$, which is plotted in Fig. 3.

3. Control processes associated with absolutely continuous measure changes

The following lemma is well known, but we include it here since its statement and the arguments employed in the proof will be important in the sequel.

Lemma 3.1. *For any $Q \ll \mathbb{P}$ there exists a progressive process $\eta = (\eta_1, \eta_2)$ such that*

$$\int_0^t |\eta_s|^2 \, \mathrm{d}s < \infty \quad Q\text{-a.s. for all } t \tag{12}$$

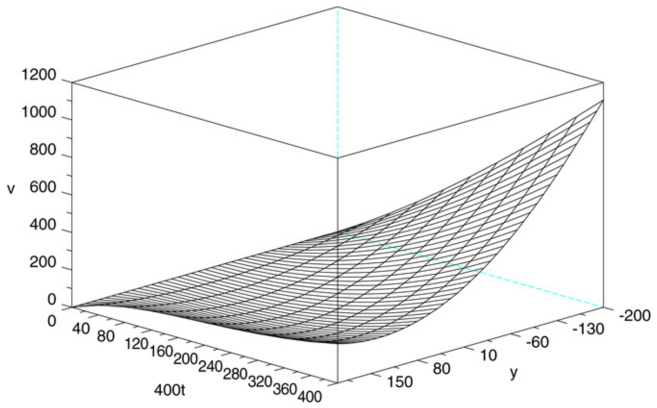

Fig. 3. The function v for the choices $h(\eta) = \infty \mathbb{I}_{\{\eta \neq 0\}}$ and $\rho = 1/2$.

and

$$\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0 \eta_{1s}\,\mathrm{d}W_s^1 + \int_0 \eta_{2s}\,\mathrm{d}W_s^2\right)_t \qquad Q\text{-}a.s. \tag{13}$$

Proof. If $Q \ll \mathbb{P}$ is given, we let $D_t := \mathrm{d}Q/\mathrm{d}\mathbb{P}|_{\mathcal{F}_t}$ and define $\tau_n := \inf\{t \geq 0 \mid D_t \leq 1/n\}$. By representing the local $\mathbb{P}$-martingale $\int_0^{t\wedge\tau_n} D_s^{-1}\,\mathrm{d}D_s$ as a stochastic integral with respect to $W = (W^1, W^2)$, we obtain the existence of a progressive process $\eta_s^{(n)}$, $s \leq \tau_n$, such that $\int_0^{t\wedge\tau_n} |\eta_s^{(n)}|^2\,\mathrm{d}s < \infty$ and

$$D_{t\wedge\tau_n} = \mathcal{E}\left(\int_0 \eta_s^{(n)}\,\mathrm{d}W_s\right)_{t\wedge\tau_n}$$

$\mathbb{P}$-a.s. for all t. Consistency requires that $\eta_t^{(n)} = \eta_t^{(n+1)}\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-a.e. on $\{t \leq \tau_n\}$. Using that $\tau_n \nearrow \infty$ Q-a.s., we obtain a Q-a.s. defined process η, which is as desired. □

In the following, we apply in our setting the concept of an *extended martingale measure* as introduced by Föllmer and Gundel [10]. To this end, recall that we assume that everything is modeled on the canonical path space $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ of $W = (W^1, W^2)$. By $\mathcal{M}$ we will denote the set of all progressive processes ν such that $\int_0^t \nu_s^2(\omega)\,\mathrm{d}s < \infty$ for all t and ω. Every $\nu \in \mathcal{M}$ gives rise to the strictly positive local $\mathbb{P}$-martingale

$$Z_t^\nu := \mathcal{E}\left(-\int \theta(Y_s)\,\mathrm{d}W_s^1 - \int \nu_s\,\mathrm{d}W_s^2\right)_t. \tag{14}$$

Remark 3.2. If $\mathbb{E}[Z_T^\nu] = 1$, then Z_T^ν is the $\mathbb{P}$-density of an equivalent martingale measure on $(\Omega, \mathcal{F}_T)$. If however $\mathbb{E}[Z_T^\nu] < 1$ then this interpretation is no longer possible. To deal with this situation, we will follow Föllmer and Gundel [10] (see also the references therein) and introduce the enlarged sample space $\bar{\Omega} := \Omega \times (0, \infty]$ endowed with the filtration

$$\bar{\mathcal{F}}_t := \sigma\left(A \times (s, \infty] \mid A \in \mathcal{F}_s, s \leq t\right), \qquad t \geq 0.$$

Any finite $(\mathcal{F}_t)$-stopping time τ is lifted up to an $(\bar{\mathcal{F}}_t)$-stopping time $\bar{\tau}$ by setting $\bar{\tau}(\omega, s) := \tau(\omega)\mathbb{I}_{(\tau(\omega),\infty]}(s)$. Now let $\nu \in \mathcal{M}$ be given. Although we may have $\mathbb{E}[Z_t^\nu] < 1$ it is possible to associate Z^ν with a *probability* measure $\bar{P}_\nu$ on $(\bar{\Omega}, \bar{\mathcal{F}}_\infty)$, where $\bar{\mathcal{F}}_\infty := \sigma\left(\bigcup_t \bar{\mathcal{F}}_t\right)$ as usual. This measure is called the *Föllmer measure* associated with the positive supermartingale Z^ν, and it is characterized by

$$\bar{P}_\nu[A \times (t, \infty]] = \mathbb{E}[Z_t^\nu \mathbb{I}_A], \qquad 0 \le t < \infty, A \in \mathcal{F}_t;$$

see [10] and the references therein.

Lemma 3.3. *Let $\nu \in \mathcal{M}$ and $Q \ll \mathbb{P}$ be given and denote by D the density process of Q with respect to $\mathbb{P}$. Then, for any bounded $(\mathcal{F}_t)$-stopping time τ, the probability measure $\bar{Q} := Q \otimes \delta_\infty$ is absolutely continuous with respect to $\bar{P}_\nu$ on the sigma field $\bar{\mathcal{F}}_{\bar{\tau}}$, and the relative entropy of $\bar{Q}$ with respect to $\bar{P}_\nu$ on $\bar{\mathcal{F}}_{\bar{\tau}}$ is given by*

$$H_{\bar{\mathcal{F}}_{\bar{\tau}}}(\bar{Q}|\bar{P}_\nu) = E_Q\left[\log \frac{D_\tau}{Z_\tau^\nu}\right]. \tag{15}$$

Proof. Since Z^ν is strictly positive, we obtain that for $A \in \mathcal{F}_t$

$$\begin{aligned}\bar{Q}[A \times (t, \infty]] &= \mathbb{E}[D_t \mathbb{I}_A] = \mathbb{E}\left[Z_t^\nu \frac{D_t}{Z_t^\nu} \mathbb{I}_A\right] \\ &= \int \frac{D_t(\omega)}{Z_t^\nu(\omega)} \mathbb{I}_A(\omega) \mathbb{I}_{(t,\infty]}(s)\, \bar{P}_\nu(\mathrm{d}\omega, \mathrm{d}s).\end{aligned}$$

Hence, $\overline{Q} \ll \bar{P}_\nu$ on $\bar{\mathcal{F}}_t$, and the corresponding density is given by

$$\left.\frac{\mathrm{d}\bar{Q}}{\mathrm{d}\bar{P}_\nu}\right|_{\bar{\mathcal{F}}_t}(\omega, s) = \frac{D_t(\omega)}{Z_t^\nu(\omega)} \mathbb{I}_{(t,\infty]}(s).$$

Replacing t by $\tau(\omega)$ everywhere on the right, we thus obtain the density on $\bar{\mathcal{F}}_{\bar{\tau}}$, due to the optional stopping theorem. Hence,

$$H_{\bar{\mathcal{F}}_{\bar{\tau}}}(\bar{Q}|\bar{P}_\nu) = \int \log \frac{D_\tau(\omega)}{Z_\tau^\nu(\omega)} \mathbb{I}_{(\tau(\omega),\infty]}(s)\, \bar{Q}(\mathrm{d}\omega, \mathrm{d}s) = E_Q\left[\log \frac{D_\tau}{Z_\tau^\nu}\right]$$

as desired. □

The preceding lemma can now be applied to obtain a formula that will be crucial in reformulating the dual problem as a stochastic control problem.

Lemma 3.4. *Define Z^ν as in (14), and suppose that η is a progressive process corresponding to some $Q \ll \mathbb{P}$. Then*

$$E_Q\left[\log \frac{D_t}{Z_t^\nu}\right] = \frac{1}{2} E_Q\left[\int_0^t (\eta_{1s} + \theta(Y_s^y))^2 + (\eta_{2s} + \nu_s)^2\, \mathrm{d}s\right]. \tag{16}$$

Proof. Take

$$\sigma_n := \inf\left\{t \ge 0 \mid \int_0^t (\nu_s^2 + |\eta_s|^2)\, \mathrm{d}s \ge n\right\}.$$

A straightforward computation shows that then

$$E_Q\left[\log \frac{D_{\sigma_n \wedge t}}{Z_{\sigma_n \wedge t}^\nu}\right] = \frac{1}{2} E_Q\left[\int_0^{t \wedge \sigma_n} (\eta_{1s} + \theta(Y_s^y))^2 + (\eta_{2s} + \nu_s)^2\, \mathrm{d}s\right].$$

Since $\sigma_n \nearrow \infty$ Q-a.s., the right-hand side of this equation increases to the right-hand side in (16). On the other hand, applying (15) we get

$$\sup_n E_Q\left[\log \frac{D_{\sigma_n\wedge t}}{Z^\nu_{\sigma_n\wedge t}}\right] = \sup_n H_{\bar{\mathcal{F}}_{\overline{\sigma_n\wedge t}}}(\bar{Q}|\bar{P}_\nu) = H_{\bar{\mathcal{F}}_t}(\bar{Q}|\bar{P}_\nu),$$

where the last identity follows from the fact that $\bar{\sigma}_n \nearrow \infty$ $\bar{Q}$-a.s. and by standard continuity properties of the relative entropy. Applying (15) with $\tau := t$ now gives the result. $\square$

4. Formulation of the dual problem

In this section, we will first apply results from Schied [26] in preparation for the application of stochastic control techniques. To check for the applicability of the results in [26], note first that our utility function (8) belongs to C^1, is increasing and strictly concave, and satisfies the Inada conditions $U'(0+) = \infty$ and $U'(\infty-) = 0$. It also has asymptotic elasticity $AE(U) = \limsup_{x\uparrow\infty} xU'(x)/U(x) = 0 < 1$. The following lemma states that the penalty function γ satisfies [26, Assumption 2.1], which is needed for the applicability of the duality results in [26].

Lemma 4.1. *The penalty function $\gamma(Q)$ defined in (4) is the minimal penalty function of the convex risk measure*

$$\rho(X) := \sup_{Q\ll\mathbb{P}} \left(E_Q[-X] - \gamma(Q)\right),$$

that is, γ satisfies the biduality relation

$$\gamma(Q) = \inf_{X\in L^\infty} \left(E_Q[-X] - \rho(X)\right), \qquad Q \ll \mathbb{P}.$$

Moreover, ρ is continuous from below on $L^\infty(\mathbb{P})$.

Proof. By the biduality theorem and the general representation theory for convex risk measures on $L^\infty(\mathbb{P})$ as described in [11], γ will be identified as the minimal penalty function of ρ once we have shown that it is convex and lower semicontinuous for the strong (and hence the weak) topology on $L^1(\mathbb{P})$.

We first show convexity. Take $Q, \widetilde{Q} \ll \mathbb{P}$ such that both $\gamma(Q)$ and $\gamma(\widetilde{Q})$ are finite and let $Q^\lambda := \lambda Q + (1-\lambda)\widetilde{Q}$ for $\lambda \in [0, 1]$. To this end, suppose that η and $\widetilde{\eta}$ are two progressive processes associated via (12) and (13) with Q and $\widetilde{Q}$, respectively. Let D_t and $\widetilde{D}_t$ denote the corresponding density processes. Since

$$\infty > \gamma(Q) \geq \kappa_1 \mathbb{E}\left[\int_0^T D_t|\eta_t|^2\,\mathrm{d}t\right] - T\kappa_2$$

due to (5), we have $D_t|\eta_t| < \infty$ $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-a.e., and so we can define the process

$$\xi_t := \frac{\lambda D_t\eta_t + (1-\lambda)\widetilde{D}_t\widetilde{\eta}_t}{\lambda D_t + (1-\lambda)\widetilde{D}_t} \cdot \mathbb{I}_{\{\lambda D_t + (1-\lambda)\widetilde{D}_t > 0\}}.$$

We use next that $(x, y) \mapsto xh(y/x)$ is a convex function on $(0, \infty) \times [0, \infty)$; see, e.g., [27, Eq. (21)]. Hence,

$$E_{Q^\lambda}\left[\int_0^T h(\xi_t)\,\mathrm{d}t\right] \leq \lambda\gamma(Q) + (1-\lambda)\gamma(\widetilde{Q}) < \infty,$$

where we have used that $D^\lambda := \lambda D + (1-\lambda)\widetilde{D}$ is the density process of Q^λ with respect to $\mathbb{P}$. In particular, we get $\int_0^T |\xi_t|^2 \,\mathrm{d}t < \infty$ Q^λ-a.s. Moreover, one easily checks that D^λ satisfies $\mathrm{d}D_t^\lambda = D_t^\lambda \xi_t \,\mathrm{d}W_t$, and we obtain the identity $\gamma(Q^\lambda) = E_{Q^\lambda}\left[\int_0^T h(\xi_t)\,\mathrm{d}t\right]$. This proves the convexity of γ.

Next, we will show the lower semicontinuity of γ for L^1-convergence. To this end, let $D_T^n := \mathrm{d}Q_n/\mathrm{d}\mathbb{P}$ be a sequence of probability densities converging to $D_T := \mathrm{d}Q/\mathrm{d}\mathbb{P}$ in $L^1(\mathbb{P})$. Let (η^n) be associated with (Q_n) and η associated with Q via (12) and (13). Then $\sup_{t\le T}|D_t^n - D_t| \to 0$ in $\mathbb{P}$-probability. A localization argument combined with the Burkholder–Davis–Gundy inequalities then shows that $\langle D^n - D\rangle_T \to 0$ in $\mathbb{P}$-probability. It follows that the processes $D_t^n \eta_t^n$ converge in $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-measure to the process $D_t \eta_t$. Hence, (η^n) converges in $\mathrm{d}t \otimes \mathrm{d}Q$-measure to η. Fatou's lemma now yields $\liminf_n \gamma(Q_n) \le \gamma(Q)$.

Let us now show that ρ is continuous from below. Due to our coercivity assumption (5), we have $\gamma(Q) + \kappa_2 \ge 2\kappa_1 H(Q|\mathbb{P}) = 2\kappa_1 \mathbb{E}\left[\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}} \log \frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}}\right]$ for $Q \ll \mathbb{P}$. Hence, the level sets $\left\{\frac{\mathrm{d}Q}{\mathrm{d}\mathbb{P}} \mid \gamma(Q) \le c\right\}$ are uniformly integrable. Therefore continuity from below follows from [18, Lemma 2] together with [11, Corollary 4.35] and the Dunford–Pettis theorem. □

Remark 4.2. Given the preceding lemma, it follows from [26, Theorem 2.4] that the value function u of the primal problem satisfies

$$u(x) = \sup_{\pi\in\mathcal{A}} \inf_{Q\ll\mathbb{P}} \left(E_Q[\log X_T^{x,\pi}] + \gamma(Q)\right) = \inf_{Q\ll\mathbb{P}} \sup_{\pi\in\mathcal{A}} \left(E_Q[\log X_T^{x,\pi}] + \gamma(Q)\right).$$

Due to (6), one might thus guess that

$$\sup_{\pi\in\mathcal{A}} E_Q[\log X_T^{x,\pi}] = \log x + \frac{1}{2}\int_0^T E_Q[(\eta_{1t} + \theta(Y_t))^2 + r(Y_t)]\,\mathrm{d}t, \tag{17}$$

if η is associated with $Q \ll \mathbb{P}$ via (12) and (13). Moreover, this argument suggests that the optimal strategy for Q is given by

$$\pi_t^Q = \frac{\eta_{1t} + \theta(Y_t)}{\sigma(Y_t)}. \tag{18}$$

Minimizing over $Q \ll \mathbb{P}$ would then formally yield the HJB equation (9) for our value function. There are, however, some subtleties associated with this approach. First of all, one needs a proper localization argument to justify (17). While this localization argument can be carried out via arguments similar to those in Lemma 3.4, another difficulty arises from the fact that the strategy in (18) is defined Q-a.s. only. Therefore one would have to check whether it can be extended to a $\mathbb{P}$-a.s. defined strategy in $\mathcal{A}$. In fact, if a strategy is admissible under some $Q \ll \mathbb{P}$ but not under $\mathbb{P}$ itself it may be an arbitrage opportunity in the model Q; see [27, Example 2.5]. For these reasons, we do not pursue further the control approach on the primal problem and work on the dual problem instead.

It follows from [26, Theorems 2.4 and 2.6] that the dual value function of the robust utility maximization problem is given as

$$\widetilde{u}(\lambda) := \inf_{\nu\in\mathcal{M}} \inf_{Q\ll\mathbb{P}} \left(\mathbb{E}\left[D_T^Q \widetilde{U}\left(\frac{\lambda Z_T^\nu}{D_T^Q S_T^0}\right)\right] + \gamma(Q)\right), \tag{19}$$

where $\widetilde{U}(z) = \sup_{x \geq 0}(U(x) - zx)$. Due to [26, Theorem 2.4], the primal value function

$$u(x) = \sup_{\pi \in \mathcal{A}} \inf_{Q \ll \mathbb{P}} \left(E_Q[\log X_T^{x,\pi}] + \gamma(Q) \right)$$

can then be obtained as

$$u(x) = \min_{\lambda > 0}(\widetilde{u}(\lambda) + \lambda x). \tag{20}$$

In our specific setting (8), we have $\widetilde{U}(z) = -\log z - 1$. Thus, we can simplify the duality formula (20) as follows. First, the expectation in (19) can be computed as

$$\mathbb{E}\left[D_T \widetilde{U}\left(\frac{\lambda Z_T^\nu}{D_T S_T^0} \right) \right] = \mathbb{E}\left[D_T \log \frac{D_T S_T^0}{Z_T^\nu} \right] - \log \lambda - 1 =: \Lambda_{Q,\nu} - \log \lambda - 1.$$

Hence,

$$u(x) = \log x + \inf_{Q \ll \mathbb{P}} \inf_{\nu \in \mathcal{M}} (\Lambda_{Q,\nu} + \gamma(Q)).$$

Lemma 4.3. *For $Q \sim \mathbb{P}$ such that $\gamma(Q) < \infty$, we have $\Lambda_{Q,0} < \infty$. In particular, condition* (2.10) *in* [26] *is satisfied.*

Proof. Our conditions on h yield that $\kappa_1 H(Q|\mathbb{P}) \leq \gamma(Q) + \kappa_2 < \infty$. Let P^* be the equivalent local martingale measure defined by $\mathrm{d}P^*/\mathrm{d}\mathbb{P} = Z_T^0$. Then

$$\begin{aligned}
\mathbb{E}\left[D_T \log \frac{D_T}{Z_T^0} \right] &= H(Q|P^*) = H(Q|\mathbb{P}) + E_Q\left[\log \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}P^*} \right] \\
&= H(Q|\mathbb{P}) + E_Q\left[\int_0^T \theta(Y_t)\,\mathrm{d}W_t^1 + \frac{1}{2}\int_0^T \theta(Y_t)^2\,\mathrm{d}t \right] \\
&= H(Q|\mathbb{P}) + E_Q\left[\int_0^T \left(\theta(Y_t)\eta_{1t} + \frac{1}{2}\theta(Y_t)^2 \right) \mathrm{d}t \right].
\end{aligned}$$

Using again $\gamma(Q) < \infty$ one sees that the last term is finite, and this implies the assertion. $\square$

Due to the preceding lemma, we may now apply [26, Theorem 2.6]. It yields that, if the pair $(\widehat{Q}, \widehat{\nu})$ minimizes (19), then there exists an optimal strategy $\widehat{\pi} \in \mathcal{A}$, whose terminal wealth is given by

$$X_T^{x,\widehat{\pi}} = I\left(\frac{\widehat{\lambda} Z_T^{\widehat{\nu}}}{D_T^{\widehat{Q}} S_T^0} \right), \tag{21}$$

where $I(y) := -\widetilde{U}'(y) = \log y + 1$ and $\widehat{\lambda} > 0$ minimizes (20).

5. HJB approach to the dual problem and proof of Theorem 2.1

In this section, we will tackle the dual problem by stochastic control techniques. Our aim is to minimize $\Lambda_{Q,\nu}$ over $Q \in \mathcal{Q}$ and $\nu \in \mathcal{M}$. Let us first heuristically derive the HJB equation for the dual problem; a rigorous argument will be provided at a later stage. To this end, we will use

Lemma 3.1 to write the density process of $Q \ll \mathbb{P}$ as

$$D_t^\eta = \mathcal{E}\left(\int \eta_s \, \mathrm{d}W_s\right)_t \qquad Q\text{-a.s.}$$

and denote by $\mathcal{N}$ the set of all processes η arising in this way. Note that the stochastic exponential need not be defined under $\mathbb{P}$ if Q is not equivalent to $\mathbb{P}$. We will use both $\eta \in \mathcal{N}$ and $\nu \in \mathcal{M}$ as control processes. Let us write $(Y_t^y)_{t\geq 0}$ to indicate the starting point $y = Y_0^y$ of the solution to the SDE (3). We then introduce the function

$$J(t, y, \eta, \nu) := \mathbb{E}\left[D_t^\eta \log \frac{D_t^\eta S_t^0}{Z_t^\nu}\right] + \mathbb{E}\left[D_t^\eta \int_0^t h(\eta_s)\, \mathrm{d}s\right],$$

where Z^ν and S^0 depend on y via Y^y. In particular, $J(T, Y_0, \eta, \nu) = \Lambda_{Q,\nu} + \gamma(Q)$. Our aim is to study the value function

$$V(t, y) := \inf_{\eta\in\mathcal{N}} \inf_{\nu\in\mathcal{M}} J(t, y, \eta, \nu).$$

Remark 5.1. The process $\mathrm{d}W^{(\eta)} := \mathrm{d}W_t - \eta_t\, \mathrm{d}t$ is a two-dimensional Q-Brownian motion. Hence, if the processes η and ν are sufficiently bounded, then their stochastic integrals with respect to $W^{(\eta)}$ are Q-martingales, and we get

$$\begin{aligned} J(t, y, \eta, \nu) = E_Q\Bigg[\int_0^t \bigg(&\frac{1}{2}|\eta_s|^2 + r(Y_s^y) + \theta(Y_s^y)\eta_{1s} + \nu_s\eta_{2s} \\ &+ \frac{1}{2}\left(\theta^2(Y_s^y) + \nu_s^2\right) + h(\eta_s)\bigg)\, \mathrm{d}s\Bigg]. \end{aligned}$$

Under Q, the process Y^y follows an SDE of the form

$$\mathrm{d}Y_t^y = g(Y_t^y)\, \mathrm{d}t + \rho\eta_{1t}\, \mathrm{d}t + \overline{\rho}\eta_{2t}\, \mathrm{d}t + \mathrm{d}\widetilde{W}_t^{(\eta)}, \tag{22}$$

where $\widetilde{W}^{(\eta)}$ is a one-dimensional Q-Brownian motion. Standard control theory now suggests that the function V is (formally) a solution to the Hamilton–Jacobi–Bellman (HJB) equation

$$v_t = \frac{1}{2}v_{yy} + gv_y + r + \inf_{\nu\in\mathbb{R}} \inf_{\eta\in\mathbb{R}^2} \left([\rho\eta_1 + \overline{\rho}\eta_2]\, v_y + \tfrac{1}{2}(\eta_2 + \nu)^2 + \tfrac{1}{2}(\eta_1 + \theta)^2 + h(\eta)\right)$$

with initial condition

$$v(0, y) = 0. \tag{23}$$

Eliminating the control parameter ν by taking $\nu = -\eta_2$ yields the reduced equation

$$\begin{aligned} v_t &= \frac{1}{2}v_{yy} + gv_y + r + \inf_{\eta\in\mathbb{R}^2}\left([\rho\eta_1 + \overline{\rho}\eta_2]\, v_y + \frac{1}{2}(\eta_1 + \theta)^2 + h(\eta)\right) \\ &= \frac{1}{2}v_{yy} + gv_y + r + \phi(v_y). \end{aligned} \tag{24}$$

The preceding heuristic argument is made precise by the following verification result.

Proposition 5.2 (*Verification Result*). *Suppose the PDE* (24) *and* (23) *admits a classical solution* $v \in C^{1,2}((0, T) \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ *satisfying a polynomial growth condition in* y *and suppose* *that one of the following three conditions is satisfied:*

(a) dom h *is bounded;*
(b) v_y *is bounded;*
(c) $\psi(y,\cdot)$ *satisfies a radial growth condition in direction* $(\rho,\overline{\rho})$, *uniformly in* y, *and* v_y *satisfies*

$$|\partial_p^- \phi(y; v_y(t,y))| \vee |\partial_p^+ \phi(y; v_y(t,y))| \le C_1(1+|y|)$$

for some constant C_1.

Then $v = V$. *Suppose furthermore that* $\eta^* : [0,T]\times\mathbb{R}\to\mathbb{R}$ *is a measurable function realizing the infimum in* (24). *Then* $\widehat{\eta}_t := \eta^*(T-t, Y_t)$ *belongs to the set* $\mathcal{N}$, $\widehat{\nu}_t := -\widehat{\eta}_{2t}$ *belongs to* $\mathcal{M}$, *and we have* $V(T,y) = J(T,y,\widehat{\eta},\widehat{\nu})$.

Proof. Let $\nu\in\mathcal{M}$ and $\eta\in\mathcal{N}$ be control processes such that $J(t,y,\eta,\nu)<\infty$ and consider the localized martingale measure $\widehat{P}$ associated with ν and let $\eta\in\mathcal{N}$ be associated with $Q\ll\mathbb{P}$. Then

$$J(t,y,\eta,\nu) = E_Q\left[\log\frac{D_t^\eta}{Z_t^\nu}\right] + E_Q[\log S_t^0] + E_Q\left[\int_0^t h(\eta_s)\,\mathrm{d}s\right].$$

The control process ν occurs only in the first term on the right, which according to Lemma 3.4 is given by

$$E_Q\left[\log\frac{D_t^\eta}{Z_t^\nu}\right] = \frac{1}{2}E_Q\left[\int_0^t (\eta_{1s}+\theta(Y_s^y))^2 + (\eta_{2s}+\nu_s)^2\,\mathrm{d}s\right].$$

This term is minimized by taking $\nu_s(\omega) := -\eta_{2s}(\omega)$ for $s\le t$ and $\omega\in\{\int_0^t \eta_{2s}^2\,\mathrm{d}s<\infty\}$ and $\nu_s(\omega):=0$ otherwise, since $E_Q[\int_0^t|\eta_s|^2\,\mathrm{d}s]<\infty$ by (5). Thus, we arrive at

$$\widetilde{J}(t,y,\eta) := \inf_{\nu\in\mathcal{M}} J(t,y,\eta,\nu) = E_Q\left[\int_0^t \frac{1}{2}(\eta_{1s}+\theta(Y_s^y))^2 + r(Y_s^y) + h(\eta_s)\,\mathrm{d}s\right]. \tag{25}$$

Due to (22), we have under Q that

$$\begin{aligned}
\mathrm{d}v(u-t,Y_t^y) &= v_y(u-t,Y_t^y)\,\mathrm{d}\widetilde{W}_t^{(\eta)} \\
&\quad + \left\{v_y(u-t,Y_t^y)\left(g(Y_t^y)+\rho\eta_{1t}+\overline{\rho}\eta_{2t}\right) - v_t(u-t,Y_t^y) + \frac{1}{2}v_{yy}(u-t,Y_t^y)\right\}\mathrm{d}t \\
&\ge v_y(u-t,Y_t^y)\,\mathrm{d}\widetilde{W}_t^{(\eta)} - \left\{\frac{1}{2}\left(\eta_{1t}+\theta(Y_t^y)\right)^2 + r(Y_t^y) + h(\eta_t)\right\}\mathrm{d}t,
\end{aligned} \tag{26}$$

where we have used (24) in the latter inequality. Letting $\sigma_n := \inf\{t\ge 0 \mid |Y_t^y|\ge n\}$, by the continuity of v_y and the boundedness of the process Y_t^y for $0\le t\le\sigma_n$, we get

$$v(u,y) \le E_Q\left[\int_0^{u\wedge\sigma_n} \frac{1}{2}\left(\eta_{1t}+\theta(Y_t^y)\right)^2 + r(Y_t^y) + h(\eta_t)\,\mathrm{d}t \; + v(u-u\wedge\sigma_n, Y_{u\wedge\sigma_n}^y)\right]. \tag{27}$$

Since $\sigma_n\nearrow\infty$ Q-a.s., we obtain $v(u,y)\le\widetilde{J}(u,y,\eta)$ and in turn $v\le V$. Here we have also used the initial condition $v(0,\cdot)=0$, the fact that r is bounded, and the assumption that v satisfies a polynomial growth condition in y together with dominated convergence and

Theorem 4.7 in [22], which states that

$$\sup_{0\le t\le T} \mathbb{E}\left[\exp\{\delta|Y_t|^2\}\right] < \infty \quad \text{for some } \delta > 0. \tag{28}$$

Now we shall prove the reverse inequality. The coercivity condition (5) and the lower semicontinuity of h imply that for each t and y there exists

$$\eta^*(t, y) \in \arg\min_{\eta\in\mathbb{R}^2} \left[(\rho\eta_1 + \overline{\rho}\eta_2)v_y(t, y) + \frac{1}{2}(\eta_1 + \theta(y))^2 + h(\eta)\right].$$

By a standard measurable selection argument, $\eta^*(t, y)$ can be chosen as a measurable function of t and y. To prove that $\hat{\eta}_s := \eta^*(u - s, Y_s)$ is an admissible Markov control, i.e., $\hat{\eta} \in \mathcal{N}$, we need to verify that

$$D_t^{\hat{\eta}} := \mathcal{E}\left(\int \hat{\eta}_{1s}\,\mathrm{d}W_s^1 + \int \hat{\eta}_{2s}\,\mathrm{d}W_s^2\right)_t, \quad 0 \le t \le u,$$

is a $\mathbb{P}$-martingale. Once this has been proved, we get an equality in (26) and hence in (27).

According to Liptser and Shiryayev [22], p. 220, $D^{\hat{\eta}}$ is a martingale if we can show that for some $\varepsilon > 0$

$$\sup_{0\le t\le u} \mathbb{E}\left[\exp\{\varepsilon|\hat{\eta}_t|^2\}\right] < \infty. \tag{29}$$

This is clear when dom h is bounded or when v_y is bounded, i.e., under conditions (a) or (b). Assuming condition (c), note that $\eta^*(t, y)$ belongs in fact to the supergradient of $x \mapsto \psi(y, x)$ at $x = (\rho, \overline{\rho})v_y(t, y)$. Hence, the radial growth condition together with the estimate on $\partial_p^{\pm}\phi(y; v_y(t, y))$ implies that $|\eta^*(t, y)| \le c(1 + |y|)$ for some constant c. Therefore (29) now follows from (28). □

Proof of Theorem 2.1. If dom h is compact, we can restrict the infimum in (24) to controls η in the compact set dom h, and Theorem IV.4.3 and Remark IV.3.3 in [9] yield the existence of a classical solution v to the PDE (24) and (23) satisfying a polynomial growth condition. Thus, condition (a) of Proposition 5.2 is satisfied, and we get the identification $v = V$. The form of the optimal strategy $\widehat{\pi}$ and the fact that $(\widehat{Q}, \widehat{\pi})$ is a saddle point follow immediately from (6) and (21), and the results in [26]. □

6. Existence of a classical solution for a noncompact control domain and proofs of Theorems 2.3 and 2.5

In this section, we will derive existence results for the PDE (24) and (23) in the case of a noncompact effective domain dom h. We will need the following estimate.

Lemma 6.1. *For $\delta > 0$, the value function V satisfies*

$$K_- \le \frac{V(t + \delta, y) - V(t, y)}{\delta} \le K_+,$$

where

$$K_- = -\|r^-\|_\infty \quad \textit{and} \quad K_+ = \frac{1}{2}\|\theta\|_\infty^2 + \|r^+\|_\infty.$$

In particular, we have $tK_- \le V(t, y) \le tK_+$.

Proof. To obtain the lower bound, note that by (25)

$$\begin{aligned} V(t+\delta, y) - V(t, y) &\geq \inf_\eta \big(\widetilde{J}(t+\delta, y, \eta) - \widetilde{J}(t, y, \eta)\big) \\ &= \inf_\eta \mathbb{E}\left[D^\eta_{t+\delta} \int_t^{t+\delta} \left(\frac{1}{2}\left(\eta_{1s} + \theta(Y^y_s)\right)^2 + r(Y^y_s) + h(\eta_s)\right) \mathrm{d}s \right] \\ &\geq -\|r^-\|_\infty \delta. \end{aligned}$$

To prove the upper bound, take $\varepsilon > 0$ and a process $\widehat{\eta}$ such that $V(t, y) + \varepsilon\delta \geq \widetilde{J}(t, y, \widehat{\eta})$ and $\widehat{\eta}_s = 0$ for $s \in [t, t+\delta]$. It follows that

$$\begin{aligned} V(t+\delta, y) - V(t, y) - \varepsilon\delta &\leq \widetilde{J}(t+\delta, y, \widehat{\eta}) - \widetilde{J}(t, y, \widehat{\eta}) \\ &\leq \mathbb{E}\left[D^{\widehat{\eta}}_{t+\delta} \int_t^{t+\delta} \left(\frac{1}{2}\theta(Y^y_s)^2 + r(Y^y_s)\right) \mathrm{d}s \right], \end{aligned}$$

which gives the upper bound. □

For $n \in \mathbb{N}$, let us introduce the auxiliary functions

$$h_n(\eta) := \begin{cases} h(\eta) & \text{if } h(\eta) \leq n, \\ +\infty & \text{otherwise.} \end{cases}$$

Then h_n also satisfies the assumptions made on h, and its effective domain dom h_n is compact. Thus, according to Theorem 2.1, its proof, and Lemma 6.1, the value function V^n obtained by replacing h with h_n coincides with the unique bounded classical solution v^n of the corresponding HJB equation. On the basis of the preceding lemma we now deduce an estimate on the growth of the gradients v^n_y.

Lemma 6.2. *Suppose first that $p \mapsto \phi(y; p)$ has superlinear growth. Then for every $R > 0$ there exist $C_R > 0$ and $n_0 \in \mathbb{N}$, both depending only on R, T, and the model parameters, such that $|v^n_y(t, y)| \leq C_R$ whenever $n \geq n_0$, $|y| \leq R$, and $0 < t < T$.*

If, alternatively, g is bounded and (11) holds, then n_0 can be chosen independently of R, and $v^n_y(t, y)$ can be bounded uniformly for $n \geq n_0$, $t \in (0, T)$, and $y \in \mathbb{R}$.

Proof. Let $R > 0$ be given. Recall from Lemma 6.1 that $-K \leq v^n(t, y) \leq K$ for some constant K depending only on the model parameters and T. Therefore, due to the mean value theorem, there exist $y^n_+ \in (R, R+1)$ and $y^n_- \in (-R-1, -R)$ such that

$$|v^n_y(t, y^n_\pm)| \leq 2K.$$

If $|v^n_y(t, \cdot)|$ exceeds $2K$ in (y^n_-, y^n_+), and hence in $[-R, R]$, this implies the existence of a local maximum of the continuous function $|v^n_y(t, \cdot)|$. Hence, it is enough to estimate $|v^n_y(t, y)|$ at critical points y of $v^n_y(t, \cdot)$, which are located in $[-R-1, R+1]$. At such points y, v^n satisfies the equation

$$v^n_t = \phi^n(v^n_y) + g v^n_y + r, \tag{30}$$

where ϕ^n corresponds to h_n. Due to Lemma 6.1, the left-hand side is bounded in absolute value by $K_+ - K_-$. Next, let c_R be an upper bound for $|g(y)|$ when $|y| \leq R+1$. Due to the superlinear growth assumption on $p \mapsto \phi(y; p)$, there exists some n_0 such that

$$\liminf_{|p|\to\infty} \left| \frac{\phi^n(y; p)}{p} \right| \geq c_R + 1 \quad \text{for } n \geq n_0 \text{ and all } y. \tag{31}$$

But in view of (30) and the uniform bound on v_t^n, this clearly implies a uniform bound of the form $|v_y^n(t, y)| \le c_0$ whenever $n \ge n_0$, $0 \le t \le T$, and y is a critical point of v_y^n with $|y| \le R + 1$. Taking $C_R := c_0 \vee (2K)$ yields the first part of the result.

If g is bounded and (11) holds, then c_R can be chosen independently of R, and (31) holds with $c_R + \varepsilon/2$ instead of $c_R + 1$, where ε is taken from (11). □

Note that the functions $v^n = V^n$ decrease pointwise to a function v, which also satisfies the bounds

$$tK_- \le v(t, y) \le tK_+.$$

Lemma 6.3. *Suppose that $p \mapsto \phi(y; p)$ has superlinear growth or g is bounded and (11) holds. Then $v(t, y) = \lim_n v^n(t, y)$ is a bounded classical solution in $C^{1,2}((0, T) \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ to the Cauchy problem*

$$\begin{cases} v_t = \frac{1}{2} v_{yy} + \phi(v_y) + g v_y + r \\ v(0, \cdot) = 0. \end{cases} \tag{32}$$

Moreover, $|v_t(t, y)| \le K_+ - K_-$.

Proof. Take $R > 0$ and let C_R and n_0 be as in Lemma 6.2. Note then that for $|p| \le C_R$ there exists some n_1 such that $\phi^n(y; p) = \phi(y; p)$ for $n \ge n_1$ and all y. Hence, Lemma 6.2 yields that for $n \ge n_0 \vee n_1$ and $|y| \le R$

$$v_t^n = \frac{1}{2} v_{yy}^n + \phi(v_y^n) + g v_y^n + r. \tag{33}$$

Since the terms $v_t^n(t, y)$, $\phi(v_y^n(t, y))$, $g(y) v_y^n(t, y)$, and $r(y)$ are uniformly bounded for $0 \le t \le T$, $|y| \le R$, and $n \ge n_0 \vee n_1$, the same must be true of $v_{yy}^n(t, y)$. Hence, for each t, the Arzela–Ascoli theorem yields the existence of a subsequence (n_k) such that $v_y^{n_k}(t, \cdot)$ converges locally uniformly to some function $w(t, \cdot)$, which is continuous in y. The pointwise convergence $v^{n_k}(t, y) \to v(t, y)$ implies that $w(t, \cdot)$ is equal to the y-derivative $v_y(t, \cdot)$ of $v(t, \cdot)$. Since this uniquely determines the limit $w(t, \cdot)$, we actually have $v_y^n(t, \cdot) \to v_y(t, \cdot)$ locally uniformly. In particular, we have $\phi^n(v_y^n(t, \cdot)) \to \phi(v_y(t, \cdot))$ locally uniformly as $n \uparrow \infty$. Moreover, the locally uniform bound on v_{yy}^n implies that v_y satisfies a local Lipschitz condition in y, uniformly in $t \le T$.

Further regularity properties of v will be obtained by applying regularity results for linear parabolic partial differential equations. To this end, we will use notation and results from Ladyzenskaya et al. [21]. Let us fix $R > 0$ and define $Q_T := (-R, R) \times (0, T)$. Taking φ in the space $C_c^\infty(Q_T)$ of C^∞-functions on Q_T with compact support and writing (33) in integral form yields

$$\iint \left(v^n \varphi_t - \frac{1}{2} v_y^n \varphi_y + \left[g v_y^n + r + \phi(v_y^n) \right] \varphi \right) \mathrm{d}y\, \mathrm{d}t = 0$$

whenever n is large enough. Taking the limit when $n \uparrow \infty$ it follows that

$$\iint \left(v \varphi_t - \frac{1}{2} v_y \varphi_y + \left[g v_y + f \right] \varphi \right) \mathrm{d}y\, \mathrm{d}t = 0,$$

where $f := r + \phi(v_y)$. Due to the boundedness of v, v_y, g, and f in Q_T, we may replace the smooth function φ by any function in the Sobolev space $\overset{\circ}{W}{}_2^{1,1}(Q_T)$, in which $C_c^\infty(Q_T)$ is dense.

That is, v is a generalized solution of the linear parabolic equation

$$v_t = \frac{1}{2} v_{yy} + g v_y + f \quad \text{in } Q_T \tag{34}$$

in the sense of [21, Chapter III]. Moreover, it follows from the already established regularity properties of v that v belongs to the space $V_2^{0,1}(Q_T)$ of all functions in $L^2(Q_T)$ that are continuous when considered as a map from $(0, T)$ into $L^2((-R, R))$ and possess a generalized y-derivative in $L^2(Q_T)$ (note that our order of notation is (t, y) rather than (y, t) in [21]).

Next, g, g', and f are bounded in Q_T. Hence it follows from [21, Theorem 12.1] that, for any $\alpha \in (0, 1)$, we have $v_y \in H^{\alpha/2,\alpha}(Q_T)$, i.e., v_y satisfies both a Hölder condition of order α in $y \in (-R, R)$, uniformly in $t \in (0, T)$, and a Hölder condition of order $\alpha/2$ in $t \in (0, T)$, uniformly in $y \in (-R, R)$.

Since $\phi(y, p)$ is concave in p, it is locally Lipschitz continuous in p, and it is easy to see that the Lipschitz constant can be taken uniformly in y. Hence, it follows that $f = \phi(v_y) + r$ belongs to $H^{\alpha/2,\alpha}(Q_T)$ for some $\alpha \in (0, 1)$. Thus, an application of the second part of [21, Theorem 12.2] yields that v belongs to $C^{1,2}(Q_T)$ and actually has derivatives in $H^{\alpha/2,\alpha}(Q_T)$.

Since R was arbitrary, we may conclude that v belongs to $C^{1,2}((0, T) \times \mathbb{R})$ and hence is a classical solution to (32). □

Proof of Theorem 2.3. It follows from Lemmas 6.2 and 6.3, and its proof that there exists a bounded classical solution v with a bounded gradient v_y. Hence, Proposition 5.2(b) applies, and the first part of Theorem 2.3 follows. The part on $\widehat{Q}$ and $\widehat{\pi}$ follows as in Theorem 2.1. □

The application of our verification result in Proposition 5.2 requires a growth condition on the gradient of v.

Lemma 6.4. *Suppose that $p \mapsto \phi(y; p)$ has superlinear growth. Then there exists a constant C_1, depending only on T and the model parameters, such that*

$$|\partial_p^- \phi(y; v_y(t, y))| \vee |\partial_p^+ \phi(y; v_y(t, y))| \le C_1(1 + |y|).$$

Proof. The C^2-function $y \mapsto v(t, y)$ is bounded from above and below by the two constants TK_+ and TK_-, which are independent of $t \le T$. Therefore, the function $y \mapsto |v_y(t, y)|$ cannot increase to its supremum, and we conclude that it is enough to estimate $|v_y(t, y)|$ in such points y that are critical points of $v_y(t, \cdot)$. In these points y, $v_{yy}(t, y)$ vanishes, and we obtain $v_t = \phi(v_y) + g v_y + r$. Dividing by $|v_y|$ and using Lemma 6.1, we hence get that for $|v_y| \ge 1$

$$\left| \frac{\phi\left(y; v_y(t, y)\right)}{v_y(t, y)} \right| \le K_+ - K_- + |g(0)| + \|g'\|_\infty |y| + \|r\|_\infty.$$

The right-hand side can be bounded by $c_1(1 + |y|)$ for an appropriate constant c_1.

The coercivity condition (5) implies that the concave function $p \mapsto \phi(y; p)$ grows at most quadratically as $|p| \to \infty$. Hence, there are constants $p_0, c_2 \ge 1$ such that

$$|\partial_p^+ \phi(p)| \vee |\partial_p^- \phi(p)| \le c_2 |\partial_p^+ \phi(p/2)| \vee |\partial_p^- \phi(p/2)| \quad \text{for } |p| \ge p_0.$$

Next, choose p_1 such that $\phi(y; p) \le 0$ and $\partial_p^- \phi(y; p) \le 0$ for $p \ge p_1/2$. Such a p_1 exists due to concavity. Then we obtain that for $p \ge p_0 \vee p_1$

$$\left| \frac{1}{p} \phi(p) \right| \ge \frac{-1}{p} (\phi(p) - \phi(p/2)) \ge \frac{1}{2} |\partial_p^+ \phi(p/2)| \ge \frac{1}{2c_2} |\partial_p^+ \phi(p)|.$$

An analogous inequality holds for p less than some $p_2 \leq 0$ and $\partial_p^- \phi$. Putting everything together yields the assertion. □

Proof of Theorem 2.5. From Lemma 6.3 we know that there exists a classical solution v to the Eq. (13). Lemma 6.4 gives the conditions for applying part (c) of Proposition 5.2. This proposition then implies the uniqueness of v, while the rest of the theorem follows as before. □

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