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Fundamental groups of manifolds with S^1-category 2

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Abstract A closed topological n-manifold M^n is of S^1-category 2 if it can be covered by two open subsets W_1, W_2 such that the inclusions $W_i \to M^n$ factor homotopically through maps $W_i \to S^1 \to M^n$. We show that the fundamental group of such an n-manifold is a cyclic group or a free product of two cyclic groups with nontrivial amalgamation. In particular, if $n = 3$, the fundamental group is cyclic.

Keywords Lusternik-Schnirelmann category · Coverings of n-manifolds with open S^1-contractible subsets

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1 Introduction

The concept of the *A-category* of a manifold was introduced by Clapp and Puppe [1]. For a closed, connected 3-manifold M it is defined as follows: Let A be a closed connected k-manifold, $0 \leq k \leq 2$. A subset B in the 3-manifold M is *A-contractible* if there are maps $\varphi : B \longrightarrow A$ and $\alpha : A \longrightarrow M$ such that the inclusion map $i : B \longrightarrow M$ is homotopic to $\alpha \cdot \varphi$. The *A-category* $cat_A(M)$ of M is the smallest number of sets, open and A-contractible needed to cover M. Note that $2 \leq cat_A(M) \leq 4$. Endowing M with a (essentially unique) differential structure, an *A-function* on M is a smooth function $M \longrightarrow R$ whose critical set

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is a finite disjoint union of components each diffeomorphic to A. The invariant $crit_A(M)$ of M is the minimum number of components of the critical set over all A-functions on M.

If A is a point, then $crit_{point}(M) = crit(M)$ has been calculated by Takens [10]. He shows that $crit(M) = 2$ if and only if $M = S^3$ and $crit(M) = 3$ if and only if M is a connected sum of S^2-bundles over S^1. A related invariant of a more geometrical nature is $C(M)$, which is the smallest number of open 3-cells needed to cover M. Hempel–McMillan [6] (see also [4]) showed that in fact $C(M) = crit(M)$. Finally, $cat_{\text{point}}(M) = cat\,(M)$, is the Lusternik-Schnirelmann category of M, and in [2] it is shown that $cat\,(M) = 2$ if and only if $\pi_1(M) = 1$ and $cat\,(M) = 3$ if and only if $\pi_1(M)$ is a non-trivial free group (of finite rank). Hence, modulo the Poincaré conjecture, the three invariants $crit(M)$, $C(M)$, and $cat\,(M)$ coincide for closed 3-manifolds.

For the case $A = S^1$, Khimshiashvili and Siersma [8] show that for orientable 3-manifolds M, $crit_{S^1}(M) = 2$ if and only if M is a lens space. A related invariant of a more geometrical nature is $T(M)$, which is the smallest number of open solid 3-tori needed to cover M. In [3] it is shown that an orientable 3-manifold M has $T(M) = 2$ if and only if M is a lens pace, so that in this case $T(M)$ and $crit_{S^1}(M)$ agree.

In this paper we show that for a closed 3-manifold M we have $cat_{S^1}(M) = 2$ if and only if $\pi_1(M)$ is cyclic.

If M is orientable, then by results of Olum [9], M is homotopy equivalent to a lens space. Therefore, modulo the conjecture that homotopy lens spaces are lens spaces, it follows that for orientable 3-manifolds M, $crit_{S^1}(M) = 2$ if and only if $T(M) = 2$ if and only if $cat_{S^1}(M) = 2$.

The case that $cat_{S^1}(M) = 3$ seems to be difficult and one is lead to conjecture that the three invariants $crit_{S^1}(M)$, $T(M)$, and $cat_{S^1}(M)$ coincide for closed orientable 3-manifolds.

The paper is organized as follows: If a closed topological n-manifold M^n has $cat_{S^1}(M^n) = 2$, we show in Sect. 2 that then M can be constructed from two compact S^1-contractible submanifolds that intersect along their boundaries, and we prove some basic properties of S^1-contractible submanifolds and intersection numbers of their boundaries with closed curves. In Sect. 3 we show that all closed 2-manifolds with negative Euler characteristic have $cat_{S^1}(M^2) = 3$. Section 4 is devoted to the proof of the

Main Theorem *Suppose M^n is closed, $n \geq 3$ and $\text{cat}_{S^1} M^n = 2$. Then $\pi_1 M^n = A *_C B$ with A, B and C cyclic non-trivial or $\pi_1 M^n = 1$.*

Here is a sketch of the proof of this theorem.

Using the result from Sect. 2 (Corollary 1) we express M as a union of two compact S^1-contractible compact n-submanifolds W_0, W_1 such that $W_0 \cap W_1 = \partial W_0 = \partial W_1$. If W_0 and W_1 are connected we give a Seifert-van Kampen argument and use Poincaré duality in the orientable case (and consider the orientable 2-sheeted covering in the nonorientable case), to show that $\pi_1(M)$ is cyclic (Theorem 1). The case when W_0 or W_1 is not connected is considerably more complicated and our approach is best described by using the language of graphs of groups ([11], p. 155): $\pi_1(M)$ is the fundamental group of $\mathcal{G}$, a graph of cyclic groups. Here, for $F = W_0 \cap W_1$, the graph G of (M, F) is a tree (Lemma 4) whose vertices (resp. edges) are in one-to-one correspondence with the components W_i^j of W_i, $i = 0, 1$ (resp. with the components $F_{jk} = W_0^j \cap W_1^k$ of F). The group associated to a vertex v corresponding to W_i^j (resp. edge e corresponding to F_{jk}) is the cyclic group $\text{im}(\pi_1 W_i^j \longrightarrow \pi_1 M)$ (resp. $\text{im}(\pi_1 F_{jk} \longrightarrow \pi_1 M)$). We identify $\text{im}(\pi_1 W_i^j \longrightarrow \pi_1 M)$ (resp. $\text{im}(\pi_1 F_{jk} \longrightarrow \pi_1 M)$) with $\pi_1 \widehat{W}_i^j$ (resp. $\pi_1 \widehat{F}_{jk}$), where $\widehat{W}_i^j$ (resp. $\widehat{F}_{jk}$) is obtained from W_i^j (resp. F_{jk}) by attaching certain 2-cells (Lemmas 5 and 6). An important point here is that W_i can be deformed into a circle contained in $M - F$ $(i = 0, 1)$ (Proposition 1). From this we can show (Lemma 9) that there is a sub-graph G_Q of G homeomorphic to a point or a segment such that the fundamental group of the restriction of $\mathcal{G}$ to G_Q is all of $\pi_1(M)$. Furthermore at most two of the edge monomorphisms corresponding to edges of G_Q are not epimorhisms (Proof of Theorem 2, Claims 1 and 2). It follows that $\pi_1(M)$ is cyclic if G_Q is a point and $\pi_1(M) = A *_C B$ if G_Q is a segment. An additional argument is needed at the end of Sect. 4 to show that C is not trivial.

Finally, in Sect. 5 we apply the Main Theorem to infer that if $\text{cat}_{S^1} M^3 = 2$ then $\pi_1(M)$ is cyclic.

2 Preliminaries

A subspace W of the manifold M^n is S^1-contractible (in M^n) if there exist maps $f : W \to S^1$, $\alpha : S^1 \to M^n$ such that the inclusion $\iota : W \to M^n$ is homotopic to αf. If $H : W \times I \to M^n$ is a homotopy between ι and αf, and $* \in W$, we have a commutative diagram

$$\begin{array}{ccc} \pi_1(W, *) & \xrightarrow{\iota_*} & \pi_1(M^n, *) \\ f_* \downarrow & & \approx \downarrow \gamma_\# \\ \pi_1(S^1, f(*)) & \xrightarrow{\alpha_*} & \pi_1(M^n, \alpha f(*)), \end{array}$$

where $\gamma = H \mid_{\{*\}\times I}$ is the trace of the homotopy. Hence im ι_* is cyclic.

Notice that a subset of an S^1-contractible set is also S^1-contractible.

$\text{cat}_{S^1} M$ is the smallest m such that there exist m open S^1-contractible subsets of M whose union is M.

It is easy to show that cat_{S^1} is a homotopy type invariant.

We first note that for the case that $\text{cat}_{S^1} M^n = 2$ we can choose compact S^1-contractible submanifolds that intersect along their boundaries:

Lemma 1 *If U_0 and U_1 are open subsets of the closed manifold M^n whose union is M^n then there exist compact n-submanifolds W_0, W_1 such that $W_0 \cup W_1 = M^n$, $W_0 \cap W_1 = \partial W_0 = \partial W_1$ and $W_i \subset U_i$ $(i = 0, 1)$.*

Proof Let $g : M^n \to [0, 1]$ be a map such that $g(M^n - U_i) = \{i\}$, $(i = 0, 1)$. For ϵ with $0 < \epsilon < 1/2$ there is an $\epsilon-$approximation f of g such that $f^{-1}(1/2)$ is an $(n-1)-$submanifold of M (see [7], Theorem 1.1). Let $W_0 = f^{-1}([1/2, 1])$ and $W_1 = f^{-1}([0, 1/2])$. These submanifolds satisfy the conclusion of the lemma. □

Corollary 1 *Suppose* $\text{cat}_{S^1} M^n = 2$ *where M^n is a closed n-manifold. Then there exist S^1-contractible compact n-submanifolds W_0, W_1 such that $W_0 \cup W_1 = M^n$ and $W_0 \cap W_1 = \partial W_0 = \partial W_1$.*

Lemma 2 *If W^n is S^1-contractible in M^n and every loop in W^n is nullhomotopic in M^n, then W^n is contractible in M^n.*

Proof The inclusion $W \to M$ is homotopic to a composition $W \xrightarrow{\tilde{f}} A \xrightarrow{\tilde{\alpha}} M$ where $p : A \to S^1$ is the covering space of S^1 corresponding to $f_*(\pi_1 W, *) \subset \pi_1(S^1, f(*))$ and

$\tilde{f}$ is a lift of f, $\tilde{\alpha} = \alpha p$. If $A \approx R^1$, then W is contractible in M; if not, α must be null homotopic and, again, W is contractible in M. □

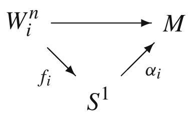
We think of S^1 as the space of complex numbers with modulus 1. If $\alpha : S^1 \to M$ and $m \in \mathbb{Z}$, we define α^m by $\alpha^m(z) = \alpha(z^m)$. Clearly, if $\beta \simeq \alpha$ then $\beta^m \simeq \alpha^m$ where $\simeq$ means "is homotopic in M^n to". If F is a compact $(n-1)$-submanifold of M^n with empty boundary and $\alpha : S^1 \to M$ is a loop, we define the intersection number $\alpha \cdot F = \min\{\#\beta^{-1}(F) : \beta \simeq \alpha\}$, where $\#\beta^{-1}(F)$ denotes the cardinality of $\beta^{-1}(F)$.

Lemma 3 *Let M^n be a closed n-manifold and let W_0^n and W_1^n be compact nonempty n-submanifolds of M^n such that $W_0^n \cup W_1^n = M^n$ and $W_0^n \cap W_1^n = \partial W_0^n = \partial W_1^n$ and let $\alpha : S^1 \to M$ be a loop. If $\alpha^m \cdot (W_0 \cap W_1) = 0$ and $m \neq 0$, then $\alpha \cdot (W_0 \cap W_1) = 0$.*

Proof We may assume $m > 0$. For $F = W_0^n \cap W_1^n$, the number $\alpha \cdot F$ is finite and we may assume that α is in general position with respect to F so that $\#\alpha^{-1}(F) = \alpha \cdot F = p$ say. Suppose $p > 0$. Since $\alpha^m \cdot F = 0$ there exists a loop β, homotopic to α^m, such that $\beta(S^1) \cap F = \emptyset$.

Consider a homotopy $\varphi : S^1 \times I \to M$ with $\varphi|_{S^1 \times \{0\}} = \alpha^m$ and $\varphi|_{S^1 \times \{1\}} = \beta$. Using transversality of maps between topological manifolds (for example Theorem 1.1 of [7]) we may assume that φ is in general position with respect to F. Then $S = \varphi^{-1}(F)$ consists of simple closed curves in $int(S^1 \times I)$ and arcs, with the endpoints of each arc in $S_1 \times \{0\}$. Each arc of S splits off a disk from $S^1 \times I$. Since $p > 0$ there is an innermost such disk D such that $\partial D = a \cup b$, where a is an arc of S and b is an arc on $S^1 \times 0$ and $D \cap S - a$ is empty or consists of simple closed curves only. Hence the restriction of α^m to b is homotopic rel boundary to a map into F and thus, for $b^m = \{z^m \mid z \in b\}$, the restriction of α to the arc b^m is homotopic rel boundary to a map from b^m into F, contradicting the fact that $\#\alpha^{-1}(F) = \alpha \cdot F$. Hence $0 = p = \alpha \cdot F$. □

Now consider again the case that $\mathrm{cat}_{S^1} M^n = 2$. Recall that we can write $M^n = W_0^n \cup W_1^n$ as a union of two compact submanifolds with $W_0^n \cap W_1^n = \partial W_0^n = \partial W_1^n$ such that for $i = 0, 1$ we have homotopy commutative diagrams



Proposition 1 *For $i = 0, 1$, we can take α_i so that $\alpha_i(S^1)$ does not intersect $W_0^n \cap W_1^n$.*

Proof If every loop in W_i^n is nullhomotopic in M^n then, by Lemma 2, W_i^n is contractible in M^n and therefore we can take as α_i a constant map with image in $\mathrm{int}(W_0^n)$ or $\mathrm{int}(W_1^n)$. If there is a loop γ in W_i^n that is not nullhomotopic in M^n, then $\gamma \simeq \alpha_i f_i \gamma \simeq \alpha_i^m$ for some $m \neq 0$. Hence $0 = \gamma \cdot (W_0^n \cap W_1^n) = \alpha_i^m \cdot (W_0^n \cap W_1^n)$ and, by Lemma 3, $\alpha_i \cdot (W_0^n \cap W_1^n) = 0$. Therefore, we can take as α_i a loop such that $\alpha_i(S^1) \cap W_0^n \cap W_1^n = \emptyset$. □

Lemma 4 *Suppose $n > 2$. Then every component of $W_0^n \cap W_1^n$ is separating.*

Proof Such a component C is S^1-contractible and so the inclusion induced homomorphism factors as

$$H_{n-1}(C; \mathbb{Z}_2) \to H_{n-1}(S^1; \mathbb{Z}_2) \to H_{n-1}(M^n; \mathbb{Z}_2).$$

Since $H_{n-1}(S^1; \mathbb{Z}_2) = 0$, C bounds in M^n and so C is separating. □

3 Two-manifolds

Note that disks, annuli, and Möbius bands embedded in a closed 2-manifold M are S^1-contractible in M.
Since

$S^2 =$ (disk) $\cup$ (disk)
$P^2 =$ (Möbius band) $\cup$ (disk)
$T^2 =$ (annulus) $\cup$ (annulus)
$K^2 =$ (Klein bottle) $=$ (annulus) $\cup$ (annulus)

we have $\text{cat}_{S^1}(S^2) = \text{cat}_{S^1}(P^2) = \text{cat}_{S^1}(T^2) = \text{cat}_{S^1}(K^2) = 2$.

We will see that all other closed 2-manifolds have cat_{S^1} equal to 3.

Proposition 2 *Let M^2 be a closed 2-manifold. Suppose there is a compact 1-submanifold of M^2, with empty boundary, such that, for every component X of its complement,* $\text{im}(\pi_1 X \to \pi_1 M^2)$ *is cyclic. Then $\chi(M^2) \geq 0$.*

Proof Let F be a compact 1-submanifold of M^2, with a minimal number of components, having the property of the statement. We claim that every component X of $M^2 - F$ has nonnegative Euler characteristic. For, if $\chi(X) < 0$ then $\partial \overline{X} \to \overline{X}$ is π_1-injective, $\pi_1 \overline{X}$ is not cyclic and $\text{im}(\pi_1 \overline{X} \to \pi_1 M^2)$ is cyclic. These three properties imply that $\partial \overline{X} \to M^2 - X$ is not π_1-injective and, therefore, some component C of $\partial \overline{X}$ bounds a 2-disk D in $M^2 - X$. But then $\text{im}(\pi_1(\overline{X} \cup D) \to \pi_1 M^2)$ is cyclic and $F - C$ is a compact 1-submanifold having the property of the statement, contradicting our minimality assumption. Hence $\chi(\overline{X}) = \chi(X) \geq 0$ for every component X of $M - F$. Therefore $\chi(M^2) = \sum \chi(\overline{X}) - \chi(F) = \sum \chi(\overline{X}) \geq 0$, where in the sum X runs over the components of $M^2 - F$. □

Corollary 2 *For a closed 2-manifold M^2,*

$$\text{cat}_{S^1} M^2 = \begin{cases} 2, & if\, M^2 = S^2, P^2, T^2, K^2 \\ 3, & otherwise. \end{cases}$$

Proof If $\text{cat}_{S^1} M^2 = 2$ then by Corollary 1, there are S^1-contractible submanifolds W_0, W_1 such that $W_0 \cup W_1 = M^2$, $W_0 \cap W_1 = \partial W_0 = \partial W_1$. Every component X of $M^2 - W_0 \cap W_1$ is S^1-contractible and so $\text{im}(\pi_1 X \to \pi_1 M^2)$ is cyclic. By Prop. 2, $\chi(M^2) \geq 0$. In any other case, since closed 2-manifolds can be covered with 3 open disks, it follows that $\text{cat}_{S^1} M^2 = 3$. □

4 *n*-manifolds

In this section we prove the Main Theorem:

Suppose M^n is closed, $n \geq 3$ and $\text{cat}_{S^1} M^n = 2$. *Then $\pi_1 M^n = A *_C B$ with A, B and C cyclic non-trivial or $\pi_1 M^n = 1$.*

Suppose $\text{cat}_{S^1} M^n = 2$. Recall that we can write $M^n = W_0^n \cup W_1^n$, where W_0^n and W_1^n are S^1-contractible compact n-submanifolds with $W_0^n \cap W_1^n = \partial W_0^n = \partial W_1^n$.

We first consider the case that W_i is connected:

Theorem 1 *If W_0 and W_1 are connected, then $\pi_1 M^n$ is cyclic.*

Proof By Lemma 4, $W_0^n \cap W_1^n$ is connected. Let $A = \operatorname{im}(\pi_1 W_0^n \to \pi_1 M^n)$, $B = \operatorname{im}(\pi_1 W_1^n \to \pi_1 M^n)$ and $C = \operatorname{im}(\pi_1(W_0^n \cap W_1^n) \to \pi_1 M^n)$. Since W_0^n, W_1^n and $W_0^n \cap W_1^n$ are S^1-contractible A, B and C are cyclic.

We have natural homomorphisms $\pi_1 W_0^n \to A \to A *_C B$ and similarly for $\pi_1 W_1^n$ and $\pi_1(W_0^n \cap W_1^n)$. We also have a natural homomorphism $\psi : A *_C B \to \pi_1(M)$. By Van Kampen's theorem and the universal property of $A *_C B$, we have the following commutative diagram with a homomorphism φ. Since $\psi\varphi$ and $\varphi\psi$ are the identity on $A \cup B$ we have $\psi\varphi = id$

$$\begin{array}{ccccc} & & \pi_1(M^n) & & \\ & \nearrow & \psi \uparrow \vdots \downarrow \varphi & \nwarrow & \\ \pi_1(W_0^n) & \longrightarrow & A *_C B & \longleftarrow & \pi_1(W_1^n) \\ & \nwarrow & \uparrow & \nearrow & \\ & & \pi_1(W_0^n \cap W_1^n) & & \end{array}$$

and $\varphi\psi = id$. Hence $\pi_1 M^n = A *_C B$ and $H_1 M^n = A \oplus_C B := (A \oplus B)/\{(c, -c) : c \in C\}$.

Observe that this implies that $A = \operatorname{im}(H_1(W_0^n) \to H_1(M^n))$, $B = \operatorname{im}(H_1(W_1^n) \to H_1(M^n))$ and $C = \operatorname{im}(H_1(W_0^n \cap W_1^n) \to H_1(M^n))$.

Case (i): M^n is orientable.

We have

$$\begin{array}{ccccccccc} H^n(W_i^n) & \longleftarrow & H^n(M^n) & \xleftarrow{\approx} & H^n(M^n, W_i^n) & \xleftarrow{0} & H^{n-1}(W_i^n) & \xleftarrow{0} & H^{n-1}(M^n) \\ \| & & \wr & & \wr \text{ excision} & & \nwarrow & \swarrow & \\ 0 & & \mathbb{Z} & & H^n\left(W_{1-i}^n, \partial W_{1-i}^n\right) & & H^{n-1}\left(S^1\right) = 0 & & \\ & & & & \wr \text{ Poincaré duality} & & & & \\ & & & & H_0\left(W_{1-i}^n\right) = \mathbb{Z} & & & & \end{array}$$

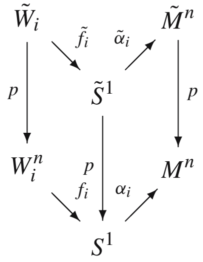
(Here $H^n(W_i^n) = 0$ since W_i^n is a compact orientable n-manifold with non-empty boundary). Hence $0 = H^{n-1}(W_i^n) = H_1(W_i^n, \partial W_i^n)$, so $H_1(\partial W_i^n) \to H_1(W_i^n)$ is onto. Therefore $C = \operatorname{im}(H_1(\partial W_0) \to H_1(M)) = \operatorname{im}(H_1(W_0) \to H_1(M)) = A$ and similarly $C = B$.

It follows that the three cyclic subgroups A, B, C coincide in $\pi_1 M^n$, which implies that $\pi_1 M^n$ is cyclic.

Case (ii): M^n is nonorientable.

By a similar proof as in case (i) taking $\mathbb{Z}_2$ coefficients, we obtain that C has odd index in A and in B. Hence $\operatorname{coker}(H_1 W_0^n \to H_1 M^n) = B/C$ is a finite cyclic group of odd order. Since the subgroup of $H_1 M^n$ consisting of all orientation-preserving loops has index two in $H_1 M^n$ it follows that $\operatorname{im}(H_1 W_0^n \to H_1 M^n)$ contains an orientation-reversing loop and hence W_0^n (and similarly W_1^n) is non orientable. Therefore for the orientable two-fold covering $p : \tilde{M}^n \to M^n$ the lift $\tilde{W}_i = p^{-1}(W_i^n)$ is connected. We may assume that α_i is an embedding. Since an orientation reversing loop is not null-homotopic in M it follows that

$\tilde{S}^1 = p^{-1}(S^1)$ is homeomorphic to S^1, α_i lifts to an embedding $\tilde{\alpha}_i$, f_i lifts to $\tilde{f}_i$ and we obtain the following diagram



Then $\tilde{\alpha}_i \tilde{f}_i$ is homotopic to the inclusion $\tilde{i} : \tilde{W}_i \to \tilde{M}^n$ and $\mathrm{cat}_{S^1} \tilde{M}^n = 2$ and by case (i) $\pi_1 \tilde{W}_i \to \pi_1 \tilde{M}^n$ is surjective.

Hence $\mathrm{im}(\pi_1 W_i^n \to \pi_1 M^n)$ contains $\mathrm{im}(\pi_1 \tilde{M}^n \to \pi_1 M^n)$, the index 2 subgroup of orientation preserving loops, and since W_i^n is nonorientable, $\mathrm{im}(\pi_1 W_i^n \to \pi_1 M^n) = \pi_1(M^n)$. Therefore $\pi_1 M^n$ is cyclic. □

We now consider the case that W_0 or W_1 is not connected.

By Proposition 1 we can assume $\alpha_i(S^1)$ does not intersect $W_0^n \cap W_1^n$. It turns out that the structure of the fundamental group of M depends on the images $\alpha_i(S^1)$, i.e. whether $\alpha_i(S^1)$ is in W_i or W_{1-i}.

To study $\pi_1 M$ we first embed $W_0, W_1, F = W_0 \cap W_1$ into spaces $\widehat{W}_0, \widehat{W}_1, \widehat{F} = \widehat{W}_0 \cap \widehat{W}_1$, respectively, such that $M = W_0 \cup W_1$ embeds in $\widehat{M} = \widehat{W}_0 \cup \widehat{W}_1$ and such that the components of $\widehat{F}$ are π_1-injective in the corresponding components of $\widehat{W}_i$ $(i = 0, 1)$, and inclusion induces an isomorphism $\pi_1(M) \cong \pi_1(\widehat{M})$.

Lemma 5 *Let $W_i^1, W_i^2, \ldots$ be the components of W_i and if $W_0^j \cap W_1^k \neq \emptyset$ let $F_{jk} = W_0^j \cap W_1^k$. By attaching 2-cells to W_0^j, W_1^k, F_{jk} we obtain embeddings of W_0^j, W_1^k, F_{jk} into spaces $\widehat{W}_0^j, \widehat{W}_1^k, \widehat{F}_{jk}$ such that*

(i) *$\pi_1 \widehat{F}_{jk}$ is cyclic for every jk.*
(ii) *$\pi_1 \widehat{W}_i^j$ is cyclic for every i and every j.*
(iii) *The inclusions $\widehat{F}_{jk} \longrightarrow \widehat{W}_0^j$, $\widehat{F}_{jk} \longrightarrow \widehat{W}_1^k$ are π_1-injective.*
(iv) *For $\widehat{W}_i = \cup_j \widehat{W}_i^j$, the inclusion $M = W_0 \cup W_1 \longrightarrow \widehat{M} = \widehat{W}_0 \cup \widehat{W}_1$ induces an isomorphism on fundamental groups.*

Proof Note that by Lemma 4 each F_{jk} is connected. Since subspaces of S^1-contractible spaces are S^1-contractible, the images of $\pi_1 W_i^j \longrightarrow \pi_1 M$ and $\pi_1 F_{jk} \longrightarrow \pi_1 M$ are cyclic. Let $K_{jk} = \ker\left(\pi_1 F_{jk} \longrightarrow \pi_1 M\right)$ and $K_i^j = \ker\left(\pi_1 W_i^j \longrightarrow \pi_1 M\right)$.
For every jk, attach 2-cells to F_{jk} along a collection of loops whose normal closure in $\pi_1 F_{jk}$ is K_{jk} and denote by E_{jk} the union of these 2-cells. For every i and every j attach 2-cells to W_i^j along a collection of loops whose normal closure in $\pi_1 W_i^j$ is K_i^j and denote by A_i^j the union of these 2-cells. Now let
$\widehat{F}_{jk} = F_{jk} \cup E_{jk}$,
$\widehat{W}_0^j = W_0^j \cup A_0^j \cup \left(\cup_k E_{jk}\right)$,
$\widehat{W}_1^k = W_1^k \cup A_1^k \cup \left(\cup_j E_{jk}\right)$.

Clearly the resulting spaces satisfy properties (i)–(iv). □

If Y is a union of subspaces of M which are components of W_0 or of W_1 we write $\widehat{Y} = \cup \left\{ \widehat{W}_i^j : W_i^j \text{ is a component of } W_0 \text{ or of } W_1 \text{ contained in } Y \right\}$. Observe that if Y is connected then $\pi_1 \widehat{Y} \to \pi_1 \widehat{M}$ is injective (use, for example, [5, Lemma 2.2]) and we have a commutative diagram with $\pi_1 Y \to \pi_1 \widehat{Y}$ surjective:

$$\begin{array}{ccc} \pi_1 Y & \longrightarrow & \pi_1 M \\ \downarrow & & \approx\downarrow \\ \pi_1 \widehat{Y} & \longrightarrow & \pi_1 \widehat{M} \end{array}$$

Hence we can identify the image of $\pi_1 Y$ in $\pi_1 M$ with $\pi_1 \hat{Y} \subset \pi_1 \widehat{M}$. A similar argument shows that the image of $\pi_1 F_{jk}$ in $\pi_1 M$ can be identified with $\pi_1 \widehat{F}_{jk}$.

Lemma 6 *Let β and γ be loops in $\widehat{F}_{jk}$ that are homotopic in $\widehat{W}_0^j$ or in $\widehat{W}_1^k$. Then they are homotopic in $\widehat{F}_{jk}$.*

Proof Since the fundamental groups of $\widehat{F}_{jk}$, $\widehat{W}_0^j$ and $\widehat{W}_1^k$ are abelian, the inclusions $\widehat{F}_{jk} \longrightarrow \widehat{W}_0^j$ and $\widehat{F}_{jk} \longrightarrow \widehat{W}_1^k$ are H_1-injective. Hence β and γ are homologous, and therefore homotopic, in $\widehat{F}_{jk}$ □

Recall that $F = W_0 \cap W_1$. In the following lemma we will use the graph G of (M, F) which is defined as follows. The vertices (resp. edges) of G are in one-to-one correspondence with the closures of the components of $M - F$ (resp. with the components of F). Vertices corresponding to components W_0^j and W_1^k of $\overline{M - F}$ are joined by an edge e of G if $W_0^j \cap W_1^k \neq \emptyset$. In this case e corresponds to the component $F_{jk} = W_0^j \cap W_1^k$ of F.

If $n > 2$, the graph G is a tree because of Lemma 4.

An example, in the form of a schematic diagram of $\widehat{M}$, is shown in Fig. 1. The graph G of (M, F) is obtained by collapsing each $\widehat{W}_i^j$ to a point.

Lemma 7 *Let β and γ be loops in different components of $M - F$ that are homotopic in M. Let $p : [0, 1] \longrightarrow M$ be a map, with $p(0) \in \operatorname{im} \beta$, $p(1) \in \operatorname{im} \gamma$, such that $p^{-1}(F)$ has minimal cardinality m. Let $p^{-1}(F) = \{t_1, \ldots, t_m\}$ where $t_1 < t_2 < \cdots < t_m$. Then there is a sequence of loops $\beta_0, \beta_1, \ldots, \beta_{m+1}$ such that*

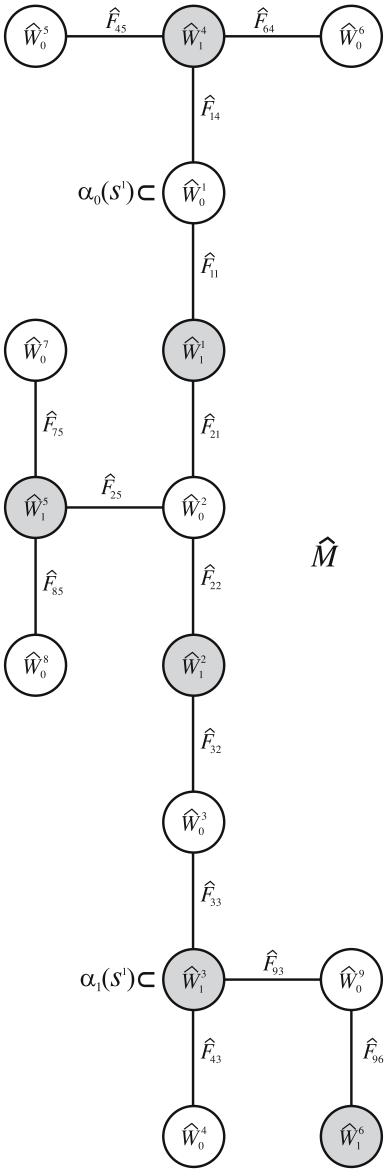
1. *$\beta_0 = \beta$ and $\beta_{m+1} = \gamma$*
2. *$\operatorname{im} \beta_j$ is contained in the component of F that contains $p(t_j)$ $(j = 1, \ldots, m)$, and*
3. *β_j is homotopic to β_{j+1} in $\widehat{W}_0$ or in $\widehat{W}_1$ $(j = 0, 1, \ldots, m)$*

Proof Let $\varphi : S^1 \times I \longrightarrow M$ be a homotopy between β and γ in M. By general position (transversality of maps between topological manifolds e.g. Theorem 1.1 of [7]) we may assume that $S = \varphi^{-1}(F)$ is a collection of simple closed curves in $int(S^1 \times I)$.

Let $D_1, \ldots, D_t$ be disjoint 2-disks embedded in $S^1 \times I$ such that $\partial D_1, \ldots, \partial D_t$ are components of S and all components of $S - \bigcup_{j=1}^t D_j$ are not null-homotopic in $S^1 \times I$. Since the inclusion of $\widehat{F}$ in $\widehat{M}$ is π_1-injective we can define a homotopy $\widehat{\varphi} : S^1 \times I \longrightarrow \widehat{M}$ such that $\widehat{\varphi}$ coincides with φ on $S^1 \times I - \bigcup_{j=1}^t \operatorname{int} D_j$ and $\widehat{\varphi}\left(\bigcup_{j=1}^t D_j\right) \subset \widehat{F}$. If the components of $S - \bigcup_{j=1}^t D_j$ are suitably indexed as $s_1, s_2, \ldots, s_{r-1}$, and $s_0 = S^1 \times \{0\}$, $s_r = S^1 \times \{1\}$, then $\varphi|_{s_i}$ $(i = 0, \ldots, r)$ defines a loop β_i' in M with β_i' homotopic to β_{i+1}' $(i = 0, \ldots, r)$ in $\widehat{W}_0$ or $\widehat{W}_1$.

The sequence of loops $\beta_0', \beta_1', \ldots, \beta_r'$ has the following properties

Fig. 1 A schematic diagram of $\widehat{M}$



(a) The first one is β and the last one is γ.
(b) Their images are contained in F, except the first one and the last one.
(c) Each loop in the sequence is homotopic to the next one in $\widehat{W}_0$ or in $\widehat{W}_1$.

Now let $\beta_0, \beta_1, \ldots, \beta_s$ be a sequence of loops satisfying (a), (b) and (c), such that s is minimal. We claim that $s = m + 1$ and that *2)* holds.

Let G be the graph of (M, F). Consider the path Δ in G associated to the sequence $(\beta_0, \beta_1, \ldots, \beta_s)$, that is, the sequence of edges $(e_1, \ldots, e_s)$ such that, for $0 < i < s$, im β_i is contained in the component of F associated to e_i. The loop β_0 (resp. β_s) is homotopic to

β_1 (resp. β_{s-1}) in the component of $\widehat{W}_0$ or of $\widehat{W}_1$ containing the component associated to u (resp. v) where u (resp. v) is a vertex of e_1 (resp. e_{s-1}). Δ is a path from u to v in G. Suppose Δ is not a simple path. Then $e_i = e_{i+1}$ for some i and, by Lemma 6, β_i is homotopic to β_{i+1} in the component of F associated to e_i; then, if we omit β_{i+1} in the sequence $(\beta_0, \beta_1, \ldots, \beta_s)$ we still have a sequence satisfying (a), (b) and (c) contradicting the minimality of s. Hence Δ is a simple path in G from u to v.

The map p also defines a path $\left(e'_1, \ldots, e'_s\right)$ of minimal length from u to v; the component associated to e'_j is the component of F containing $p\left(t_j\right)$. This path is also simple and, since G is a tree, we must have $e'_j = e_j$ for all j. Hence $s = s' = m + 1$ and the component of F containing im β_j is the one to which $p\left(t_j\right)$ belongs ($j = 1, \ldots, m$). □

In the following we wish to prove that in some cases the monomorphism $\pi_1 \widehat{F}' \longrightarrow \pi_1 \widehat{W}'$ is surjective, where F' is a component of F and W' is a component of W_0 or of W_1 containing F'. To do so it suffices to show that every loop in W' is homotopic in $\widehat{W}'$ to a loop in F'; this implies that every element of $\pi_1 \widehat{W}'$ is conjugate to an element of the image of $\pi_1 \widehat{F}' \longrightarrow \pi_1 \widehat{W}'$ but, since $\pi_1 \widehat{W}'$ is abelian, this image must be $\pi_1 \widehat{W}'$.

Lemma 8 *Assume that the images of* α_0 *and* α_1 *do not intersect* F. *Let* W_i^q *be a component of* W_i *which does not contain* $\alpha_i\left(S^1\right)$ *and let* F'_{jk} *be the component of* ∂W_i^q *separating* int W_i^q *from* $\alpha_i\left(S^1\right)$. *Then* $\pi_1 \widehat{F}'_{jk} \longrightarrow \pi_1 \widehat{W}_i^q$ *is an isomorphism.*

Proof Since $\pi_1 \widehat{F}'_{jk} \longrightarrow \pi_1 \widehat{W}_i^q$ is injective we only need to prove surjectivity. Let β be a loop in W_i^q. Then β is homotopic in M to a power of α_i. A map $p : [0, 1] \longrightarrow M$ with $p(0) \in \operatorname{im} \beta$, $p(1) \in \operatorname{im} \alpha_i$ and $\# p^{-1}(F)$ minimal is such that $p(t_1) \in F'_{jk}$ where $p^{-1}(F) = \{t_1, \ldots, t_m\}$ and $t_1 < t_2 < \cdots < t_m$. By Lemma 7, there is a sequence $(\beta, \beta_1, \ldots, \beta_{m+1})$ where β_{m+1} is a power of α_i, β is homotopic to β_1 in $\widehat{W}_i^q$ and im $\beta_1 \subset F'_{jk}$. Hence $\pi_1 \widehat{F}'_{jk} \longrightarrow \pi_1 \widehat{W}_i^q$ is surjective. □

In the next lemma we refer to the graph G of (M, F).

Lemma 9 *There is an n-submanifold* Q^n *of* M^n *with the following properties*:

(i) Q^n *is a union of components of* W_0 *and* W_1 *and the sub-graph* G_Q *of* G *corresponding to* $(Q^n, int\ Q^n \cap F)$ *is linear and connected*;
(ii) $\alpha_i\left(S^1\right)$, $(i = 0, 1)$ *lies in a component of* W_0 *or* W_1 *corresponding to a vertice of degree* 1 *in* G_Q;
(iii) *inclusion induces an isomorphism* $\pi_1 \widehat{Q}^n \cong \pi_1 \widehat{M}^n \cong \pi_1 M^n$.

For example, for the manifold pair (M, F) represented in Fig. 1, $\widehat{Q} = \widehat{W}_0^1 \cup \widehat{W}_1^1 \cup \widehat{W}_0^2 \cup \widehat{W}_1^2 \cup \widehat{W}_0^3 \cup \widehat{W}_1^3$.

Proof Recalling that G is a finite tree, let W^p be a component of W_0 or W_1 corresponding to a vertex of degree 1 in G and let $Q_1^n = \overline{M^n - W^p}$. If W^p does not contain $\alpha_i\left(S^1\right)$ for $i = 0, 1$ then by Lemma 8, $\pi_1 \widehat{F}'_{jk} \longrightarrow \pi_1 \widehat{W}_i^p$ is an isomorphism, where $F'_{jk} = W_i^p \cap Q_1^n$. By Van Kampen's Theorem inclusion induces an isomorphism $\pi_1 \widehat{Q_1}^n \cong \pi_1 \widehat{M}^n$. We now obtain Q^n by cutting off from M^n all those components of W_0 and W_1 corresponding to vertices of degree 1 which do not contain $\alpha_i\left(S^1\right)$ for $i = 0, 1$ and repeating this process inductively. □

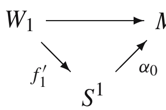
Corollary 3 *If* $\alpha_0\left(S^1\right)$ *and* $\alpha_1\left(S^1\right)$ *are contained in the same component of* $M - F$ *then* $\pi_1 M$ *is cyclic.*

Proof By Lemma 9, $\pi_1 M \approx \pi_1 \widehat{Q^n}$ where now Q^n is equal to the component W^p of W_0 or of W_1 containing $\alpha_0(S^1)$ and $\alpha_1(S^1)$. Hence $\pi_1 \widehat{Q}^n \cong \pi_1 \widehat{W}^p$ is cyclic and the result follows. □

Now we show how the structure of $\pi_1(M)$ depends on the images of α_i. Recall that by Proposition 1 we assume $\alpha_i(S^1)$ does not intersect $W_0^n \cap W_1^n$.

Theorem 2 (a) *If $\alpha_0(S^1) \subset W_1$ or $\alpha_1(S^1) \subset W_0$, then $\pi_1(M^n)$ is cyclic.*
(b) *If $\alpha_i(S^1) \subset W_i$ $(i = 0, 1)$ and F^{n-1} is any component of $W_0 \cap W_1$ separating $\alpha_0(S^1)$ from $\alpha_1(S^1)$, let X_i be the component of $M^n - F^{n-1}$ containing $\alpha_i(S^1)$. Then $C = \mathrm{im}(\pi_1 F^{n-1} \to \pi_1 M^n)$ is cyclic, $A_i = \mathrm{im}(\pi_1 X_i \to \pi_1 M^n)$ is cyclic $(i = 0, 1)$, and $\pi_1 M^n = A_0 *_C A_1$.*

Proof (a) Suppose $\alpha_1\left(S^1\right) \subset W_0$. We may assume $\alpha_1\left(S^1\right) \subset \mathrm{int}\, W_0$ and let $f_1' = f_0 \alpha_1 f_1$. Then



is also homotopy commutative and we can take $\alpha_1' = \alpha_0$ instead of α_1. By Corollary 3, $\pi_1 M$ is cyclic.
Similarly, if $\alpha_0\left(S^1\right) \subset W_1$ then $\pi_1 M$ is cyclic.
(b) Assume $\alpha_i\left(S^1\right) \subset W_i (i = 0, 1)$. Let Q^n be as in Lemma 9 and let W_0^p (resp. W_1^p) $(p = 1, \ldots, s)$ be the components of $W_0 \cap Q^n$ (resp. $W_1 \cap Q^n$) indexed such that $\mathrm{int}\, W_0^1 \supset \alpha_0\left(S^1\right)$, $\mathrm{int}\, W_1^s \supset \alpha_1\left(S^1\right)$ and $W_0^p \cap W_1^q \neq \emptyset$ if and only if $p = q$ or $p = q + 1$. Write $F_{q,q} = W_0^q \cap W_1^q$ and $F_{q+1,q} = W_0^{q+1} \cap W_1^q$.

Claim 1 $\pi_1 \widehat{F}_{q+1,q} \longrightarrow \pi_1 \widehat{W}_0^{q+1}$ and $\pi_1 \widehat{F}_{q+1,q} \longrightarrow \pi_1 \widehat{W}_1^q$ are isomorphisms.

This is an immediate consequence of Lemma 8.

Claim 2 If $1 < q \leq s$ then $\pi_1 \widehat{F}_{q,q} \longrightarrow \pi_1 \widehat{W}_0^q$ is an isomorphism and if $1 \leq q < s$ then $\pi_1 \widehat{F}_{q,q} \longrightarrow \pi_1 \widehat{W}_1^q$ is an isomorphism.

To see this, if $q > 1$, let β be any loop in W_0^q. Then, by Claim 1, β is homotopic in $\widehat{W}_0^q$ to a loop γ in $F_{q,q-1}$. Let δ be a loop in int W_1^{q-1} homotopic to γ in W_1^{q-1}. Then δ is homotopic in M to a loop in W_1^s and therefore, using Lemma 7, δ is homotopic in $\widehat{W}_1^{q-1}$ to a loop δ_1 in $F_{q,q-1}$ and δ_1 is homotopic in $\widehat{W}_0^q$ to a loop δ_2 in $F_{q,q}$. By Lemma 6, γ is homotopic to δ_1 in $\widehat{F}_{q,q-1}$. Hence, in $\widehat{W}_0^q$, $\beta \simeq \gamma \simeq \delta_1 \simeq \delta_2$. Therefore $\pi_1 \widehat{F}_{q,q} \longrightarrow \pi_1 \widehat{W}_0^q$ is an isomorphism.
Similarly, if $q < s$, we show that if β is any loop in W_1^q, then, in $\widehat{W}_1^q$ we have $\beta \simeq \gamma \simeq \delta_1 \simeq \delta_2$, where now γ and δ_1 are loops in $F_{q+1,q}$ and δ_2 is a loop in $F_{q,q}$. Therefore $\pi_1 \widehat{F}_{q,q} \longrightarrow \pi_1 \widehat{W}_1^q$ is an isomorphism.
Now let F' be any component of $W_0 \cap W_1$ separating $\alpha_0\left(S^1\right)$ from $\alpha_1\left(S^1\right)$, that is, $F' = F_{q,q}$ or $F' = F_{q+1,q}$ for some q. Let X_i be the closure of the component of $M - F'$ containing $\alpha_i\left(S^1\right)$. The argument in the proof of Lemma 9 shows that the inclusion of $\widehat{W}_0^1$ in $\widehat{X}_0$ and the inclusion of $\widehat{W}_0^s$ in $\widehat{X}_1$ induce isomorphisms of fundamental groups. Hence $\pi_1 \widehat{X}_0$, $\pi_1 \widehat{X}_1$ and $\pi_1 \widehat{F}'$ are cyclic and therefore A_0, A_1 and C are cyclic (see the remark before Lemma 6).
Since by Van Kampen's Theorem we have $\pi_1 \widehat{M} = \pi_1 \widehat{X}_0 *_{\pi_1 \widehat{F}'} \pi_1 \widehat{X}_1$ it follows that $\pi_1 M = A_0 *_C A_1$. □

To complete the proof of the Main Theorem it remains to show that if $\pi_1 M^n$ is not trivial, then the amalgamating subgroup C is non-trivial.

Lemma 10 *Let W^0 and W^1 be disjoint compact n-submanifolds of M^n where W^0 is S^1-contractible in M^n and W^1 is connected and contractible in M. Let $T = D^{n-1} \times [0,1]$ be a tube in M^n such that $W^i \cap T = D^{n-1} \times \{i\}$, $(i = 0, 1)$. Then $W^0 \cup T \cup W^1$ is S^1-contractible in M.*

Proof Let $a = \{0\} \times [0,1]$ be the core of T, $p = (0,0)$ and $q = (0,1)$ so $\partial a = \{p, q\}$. Then $W^0 \cup T \cup W^1$ deformation retracts to $W^0 \cup a \cup W^1$ in M so it suffices to show that $W^0 \cup a \cup W^1$ is S^1-contractible in M. Since it is easy to see that $W^0 \cup a$ is S^1-contractible in M, it suffices to show that the diagram below is homotopy commutative

$$\begin{array}{ccc} W^0 \cup a \cup W^1 & \xrightarrow{\text{inclusion}} & M \\ & {\scriptstyle r} \searrow \quad \nearrow {\scriptstyle \text{inclusion}} & \\ & W^0 \cup a & \end{array}$$

where r is the retraction with $r\left(W^1\right) = q$ and the other two maps are inclusions.

To construct the homotopy $H : (W^0 \cup a \cup W^1) \times I \longrightarrow M$ we note that since W^1 is contractible in M there is a map $H : W^1 \times \left[0, \frac{1}{2}\right] \longrightarrow M$ such that $H(x, 0) = x$ and $H\left(W^1 \times \left\{\frac{1}{2}\right\}\right)$ is a point. Extend H to $W^1 \times [0,1]$ by defining $H(x,t) = H(q, 1-t)$ for $\frac{1}{2} \leq t \leqq 1$. Since $H|_{q \times [0,1]}$ defines a nullhomotopic loop of the form $\gamma \cdot \gamma^{-1}$ we can extend H to $\left(a \cup W^1\right) \times [0,1]$ in such way that $H(p,t) = p$ for t$\in [0,1]$ and $H(x, 1) = x$ if $x \in a$. Finally, extend H to $\left(W^0 \cup a \cup W^1\right) \times [0,1]$ by defining $H(x,t) = x$ for $x \in W^0$, $t \in [0,1]$. □

We denote the number of components of a submanifold W of M^n by $|W|$.

Corollary 4 *Suppose that M^n admits a decomposition $M^n = W_0 \cup W_1$ where W_0 and W_1 are S^1-contractible submanifolds of M^n with $W_0 \cap W_1 = \partial W_0 = \partial W_1$ and such that $|W_0| + |W_1| = c$ is minimal. If $|W_0| > 1$ (resp. $|W_1| > 1$) then no component of W_0 (resp. W_1) is contractible in M^n.*

Proof Suppose, say, that $|W_0| > 1$ and W_0 has a contractible (in M^n) component W_0^1. Let $T = D^{n-1} \times [0,1]$ be a tube in M^n joining $W_0 - W_0^1$ to W_0^1 i.e. $T \cap (W_0 - W_0^1) = D^{n-1} \times \{0\}$ and $T \cap W_0^1 = D^{n-1} \times \{1\}$. Then by Lemma 10, $W_0 \cup T = (W_0 - W_0^1) \cup T \cup W_0^1$ is S^1-contractible and, as a submanifold of W_1, the manifold $\overline{W_1 - T}$ is S^1-contractible. This contradicts the minimality of c since $|W_0 \cup T| + \left|\overline{W_1 - T}\right| = c - 1$. □

We now finish the proof of the Main Theorem.

We express M^n as the union of two S^1-contractible submanifolds W_0, W_1 with $W_0 \cap W_1 = \partial W_0 = \partial W_1$ such that $|W_0| + |W_1| = c$ is minimal.

If $c = 2$ then $\pi_1 M$ is cyclic by Theorem 1. Hence we can assume $c > 2$. By Proposition 1 and Theorem 2 we can assume that $\alpha_i\left(S^1\right) \subset \operatorname{int} W_i^1$, $(i = 0, 1)$, where W_i^1 is a component of W_i. Furthermore for a component F' of ∂W_0^1 separating $\alpha_0\left(S^1\right)$ from $\alpha_1\left(S^1\right)$ and the closures X_i of the components of $M - F'$ containing $\alpha_i\left(S^1\right)$ $(i = 0, 1)$ we have $\pi_1 M = A_0 *_C A_1$ where $C = \operatorname{im}\left(\pi_1 F' \longrightarrow \pi_1 M\right)$ and $A_i = \operatorname{im}\left(\pi_1 X_i \longrightarrow \pi_1 M\right)$ are cyclic $(i = 0, 1)$.

We now show that C is not trivial.

Suppose, on the contrary, that C is trivial. If W_0^2 (resp. W_1^2) is a component of W_0 (resp W_1) contained in X_1 (resp. X_0) then every loop in W_0^2 (resp. W_1^2) is homotopic to a loop in W_0^1 (resp. W_1^1) and therefore, by Lemma 7, to a loop in F'. By assumption this loop is null homotopic in M^n and so, by Lemma 2, W_0^2 (resp. W_1^2) is contractible in M, which is impossible by Corollary 4 . Hence there are no components of W_0 (resp. W_1) contained in X_1 (resp. X_0) and so $X_1 = W_1^1$, $X_0 = W_0^1$ and $c = 2$, a contradiction.

5 Closed 3-manifolds

If the fundamental group of a closed 3-manifold M^3 is cyclic, then, by results of Olum [9], M^3 is homotopy equivalent to a lens space $L(p, q)$ including S^3 and $S^1 \times S^2$, or $S^1 \tilde{\times} S^2$. Since these spaces can be expressed as the union of two solid tori or two solid Klein bottles and since cat_{S^1} is a homotopy-type invariant it follows that $\mathrm{cat}_{S^1} M^3 = 2$.

This shows sufficiency for the following

Theorem 3 *Let M^3 be a closed 3-manifold. Then $\mathrm{cat}_{S^1} M^3 = 2$ if and only if $\pi_1 M^3$ is cyclic.*

Proof By the Main Theorem, if $\pi_1 M^3$ is not cyclic then $\pi_1 M^n = A *_C B$ is a non-trivial free product with amalgamation, with A, B and C cyclic. Hence $\pi_1 M^n$ is infinite with center $C \neq 1$ and so $\pi_1 M^n$ is not a non-trivial free product and it follows that every 2−sphere in M is homotopically trivial. Hence the prime decomposition of M shows that $\pi_1 M^n = \pi_1 M'$ where M' is a closed irreducible 3-manifold.

First assume that M is orientable or non-orientable but P^2-irreducible. Then Waldhausen's proof of Satz 1.2 [12], applies to show that M' contains a closed surface, different from S^2 or P^2, with fundamental group isomorphic to a subgroup of C, which is impossible. Hence $\pi_1 M^3$ is cyclic.

If M' is non-orientable and contains a 2−sided P^2 then $i_* \pi_1 P^2 \cong \mathbf{Z}_2$ is conjugate to a subgroup of A,B, or C and it follows that A,B and C are finite cyclic, hence $H_1(M')$ is finite, a contradiction, since the first Betti number of a closed and non-orientable 3-manifold M' is positive. □

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References

1. Clapp, M., Puppe, D.: Invariants of the Lusternik–Schnirelmann type and the topology of itical sets. Trans. Am. Math. Soc. **298**, 603–620 (1986)
2. Gómez-Larrañaga, J.C., González-Acuña, F.: Lusternik–Schnirelmann category of 3-manifolds. Topology **31**, 791–800 (1992)
3. Gómez-Larrañaga, J.C., González-Acuña, F., Heil, W.: 3-Manifolds that are covered by two open bundles, vol. 10. Bol. Soc. Mat. Mexicana, Número especial, pp. 171–180 (2004)
4. Gómez-Larrañaga, J.C., González-Acuña, F., Heil, W.: A note on Hempel–McMillan coverings of 3-manifolds. Topol. App. **154**, 1363–1367 (2007)
5. González-Acuña, F., Whitten, W.C.: Imbeddings of three-manifold groups. Mem. Am. Math. Soc. **99**(474), viii+55 pp (1992)
6. Hempel, J., McMillan, D.R.: Covering three-manifolds with open cells. Fund. Math. **64**, 99–104 (1969)

7. Kirby, R., Siebenmann, L.: Some basic theorems about topological manifolds. In: Foundational Essays on Topological Manifolds, Smoothings and Triangulations. Ann. Math. Stud. vol. 88. Princeton University Press, Princeton (1977)
8. Khimshiashvili, G., Siersma, D.: Remarks on minimal round functions. Geometry and topology of caustics—CAUSTICS '02, 159–172, Banach Center Publ., 62, Polish Acad. Sci., Warsaw (2004)
9. Olum, P.: Mappings of manifolds and the notion of degree. Ann. Math. (2) **58**, 458–480 (1953)
10. Takens, F.: The minimal number of itical points of a function on a compact manifold and the Lusternik–Schnirlemann category. Invent. Math. **6**, 197–244 (1968)
11. Scott, P., Wall, C.T.C.: Topological methods in group theory. In: Homological Group Theory, London Mathematical Society, Lecture Notes Series, vol. 36. Cambridge University Press, Cambridge (1979)
12. Waldhausen, F.: Gruppen mit Zentrum und 3−dimensionale Mannigfaltigkeiten. Topology **6**, 505–517 (1967)