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A note on Hempel–McMillan coverings of 3-manifolds

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Abstract

Motivated by the concept of $\mathcal{A}$-category of a manifold introduced by Clapp and Puppe, we give a different proof of a (slightly generalized) Theorem of Hempel and McMillan: If M is a closed 3-manifold that is a union of three open punctured balls then M is a connected sum of S^3 and S^2-bundles over S^1.

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1. Introduction

The concept of an $\mathcal{A}$-category of a manifold was introduced in [1]. A special case of this concept for a closed, connected 3-manifold M is as follows: Let A be a point, a 1-sphere S^1, a 2-sphere S^2, a projective plane P^2, a 2-dimensional torus T^2, or a 2-dimensional Klein bottle K^2. An open set C of M is *A-categorical* if there exist maps $\phi : C \to A$ and $\rho : A \to M$ such that the inclusion map $\iota : C \to M$ is homotopic to $\rho \cdot \phi$.

The A-category of M, A-cat(M) is the minimal number of A-categorical open sets that cover M. When A is a point, the A-category of M is the classical Lusternik–Schnirelmann category cat(M) of M. This invariant was studied in [3]. In a forthcoming paper [5] we will study the case $A = S^1$.

In order to better understand the A-category invariant we start by studying what we will call the "Hempel–McMillan" coverings of 3-manifolds. These are coverings of M by the interiors of given I^k-bundles over a fixed A, where $k + \dim(A) = 3$. When A is a point then this is a covering of M by open balls. It is well known that if M is covered by two balls then $M = S^3$ (see e.g. [4]) and the existence of a Heegaard-splitting shows that every M can be covered by four open balls. Hempel and McMillan [8] proved that if M is covered by three open balls, then M is a connected sum of finitely many S^2-bundles over S^1. Up to the Poincarè Conjecture the same is true for cat(M) [3].

When A is as above the manifolds covered by the interiors of two I^k-bundles were classified in [4]. In order to study the classification of 3-manifolds covered by three sets of this type, we start with the case that A is a point or S^2

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and give in this paper a new proof of (a generalized) Hempel–McMillan Theorem, which possibly can be adapted to classify manifolds M covered by three open I^k-bundles over S^1, P^2, T^2, or K^2.

2. Preliminaries

We first establish a corollary that allows us to work in the pl-category.

The following lemma is well known (see e.g. [2, Chapter VII, Theorem 6.1]) and easy to prove:

Lemma 1. *If $\{U_1, \dots, U_m\}$ is an open cover of the normal space X, then there is a closed cover $\{C_1, \dots, C_m\}$ of X with $C_i \subset U_i$ $(i = 1, \dots, m)$.*

Lemma 2. *Let $M^n, W_1, \dots, W_m$ be smooth compact n-manifolds with M^n closed. Let $\{U_1, \dots, U_m\}$ be an open cover of M^n with U_i diffeomorphic to* $\operatorname{int} W_i$ *$(i = 1, \dots, m)$. Then there exist smooth embeddings $f_i : W_i \to M^n$ such that*

(1) $\bigcup_{i=1}^m \operatorname{int} f_i(W_i) = M^n$ *and*
(2) $f_i(\partial W_i)$ *is transversal to* $\bigcap_{j<i} f_j(\partial W_j)$ *for* $i = 2, \dots, m$.

Proof. By Lemma 1 there exist $C_1, \dots, C_m$ compact with $C_i \subset U_i$ $(i = 1, \dots, m)$ and $\bigcup_{i=1}^m C_i = M^n$. For each i there are submanifolds of U_i diffeomorphic to W_i and with interior containing C_i. Let $f_1 : W_1 \to U_1$ be a smooth embedding with $C_1 \subset \operatorname{int} f_1(W_1)$.

Suppose that inductively we have defined for $i = 1, \dots, k$ smooth embeddings $f_i : W_i \to U_i$ with $C_i \subset \operatorname{int} f_i(W_i)$ and such that (2) holds. Then, if $k < m$, by the Transversality Theorem and the Stability Theorem for embeddings [7, p. 68, p. 35(e), respectively] there exists an embedding $f_{k+1} : W_{k+1} \to U_{k+1}$ with $C_{k+1} \subset \operatorname{int} f_{k+1}(W_{k+1})$ such that (2) holds, completing the inductive construction of the f_i. Note that (1) holds also since $C_i \subset \operatorname{int} f_i(W_i)$, $i = 1, \dots, m$. □

Remark. The second condition is equivalent to the following:

If $x \in f_{i_1}\partial(W_{i_1}) \cap f_{i_2}\partial(W_{i_2}) \cap \dots \cap f_{i_r}\partial(W_{i_r})$ with $i_1 < i_2 < \dots < i_r$, $r \geqslant 2$, and if $n_{i_j}(x)$ is a nonzero vector of $T_x(M^n)$ perpendicular to the tangent space of $f_{i_j}\partial(W_{i_j})$ at x $(j = 1, \dots, r)$, then $n_{i_1}(x), n_{i_2}(x), \dots, n_{i_r}(x)$ are linearly independent.

In particular, for $m = 3$, we obtain the following

Corollary 3. *Suppose M is a closed 3-manifold covered by three open sets H_1, H_2, H_3, such that H_i is homeomorphic to the interior of a compact connected 3-manifold V_i $(i = 1, 2, 3)$. Then M admits a covering $M = V_1 \cup V_2 \cup V_3$ such that ∂V_1 is transversal to ∂V_2, and $\partial V_3 \subset \operatorname{int}(V_1 \cup V_2)$, and V_1, V_2, V_3 are pl embedded.*

We will use the following notations throughout this paper:

$\mathbb{B}$	denotes a connected sum of S^3 and S^2-bundles over S^1 (with finitely many factors);
H or H_i	denotes a punctured ball with finitely many punctures (possibly no punctures);
W or W_i	denotes a handlebody (orientable or non-orientable).

By an n-times *punctured M* we mean a manifold obtained from M by removing interiors of n disjoint balls in $\operatorname{int}(M)$. We allow $n = 0$. Note that a connected *punctured* $M = M \# H$, for some punctured ball H.

By an open punctured ball we mean a manifold homeomorphic to an open ball with a finite number of points removed.

Lemma 4. *Suppose N is a connected 3-manifold that is a union of punctured balls $B_1, \dots, B_n$ such that $\partial B_i \cap \partial B_j = \emptyset$ for $i \neq j$, then $N = \mathbb{B} \# H$.*

Proof. For a fixed index i $(1 \leqslant i \leqslant n)$ the collection of 2-spheres $(\partial B_1 \cup \dots \cup \partial B_n) \cap \operatorname{int} B_i$ cuts B_i into punctured balls $B_{i_1}, \dots, B_{i_{n_i}}$. Now N is obtained from a collection of punctured balls by identifying (some) boundary spheres in pairs. The result follows. □

A 3-manifold N is obtained from a collection of 3-manifolds $N_1, \dots, N_n$ by *successive* 1-*handle attachments* if we start by attaching a 1-handle to $N_1 \cup \dots \cup N_n$ (either to one component N_i or two components N_i, N_j) and then successively repeat attaching 1-handles to the resulting collections of 3-manifolds (a finite number of times).

The following lemma is easily proved by induction on the number of 1-handle attachments (see e.g. [6, Lemma 2(a)]).

Lemma 5. *If N is a connected 3-manifold obtained from a collection of punctured balls $B_1, \dots, B_m$ by successive 1-handle attachments then $N = \mathbb{B} \# W_1 \# \cdots \# W_n \# H$, for some $n \geqslant 0$.*

3. Union of two balls

Suppose B_1, B_2 are two punctured balls embedded in the interior of some 3-manifold with ∂B_1 transversal to ∂B_2. Let $N = B_1 \cup B_2$. If F is an innermost planar surface of $\partial B_1 \cap B_2$, not a disk, we attach 2-handles to B_2 (near F) to obtain a new punctured ball B_2^* so that N is homeomorphic to $B_1 \cup B_2^*$ and the component F of $\partial B_1 \cap B_2$ is replaced by a disk component $\widehat{F}$ of $\partial B_1 \cap B_2^*$. We call this process a 2-*handle move on B_2 near F* (see Fig. 1).

Theorem 6. *Suppose B_1, B_2 are two punctured balls embedded in the interior of some 3-manifold with ∂B_1 transversal to ∂B_2 and let $N = B_1 \cup B_2$. Then $N = \mathbb{B} \# W_1 \# \cdots \# W_n \# H$ for some $n \geqslant 0$.*

Proof. If $\partial B_1 \cap \partial B_2 = \emptyset$ then Lemma 4 applies. Otherwise the components of $\partial B_1 \cap B_2$ are planar surfaces.

Step 1: Suppose there is a disk component $\widehat{F}$ of $\partial B_1 \cap B_2$.
Do surgery on $\widehat{F}$ to cut B_2 into two punctured balls with copies $\widehat{F}'$ and $\widehat{F}''$ of $\widehat{F}$ in their boundaries.

Step 2: Suppose F is an innermost planar surface of $\partial B_1 \cap B_2$, not a disk.
Perform a 2-handle move on B_2 near F and then do step (1) on the resulting disk component $\widehat{F}$.

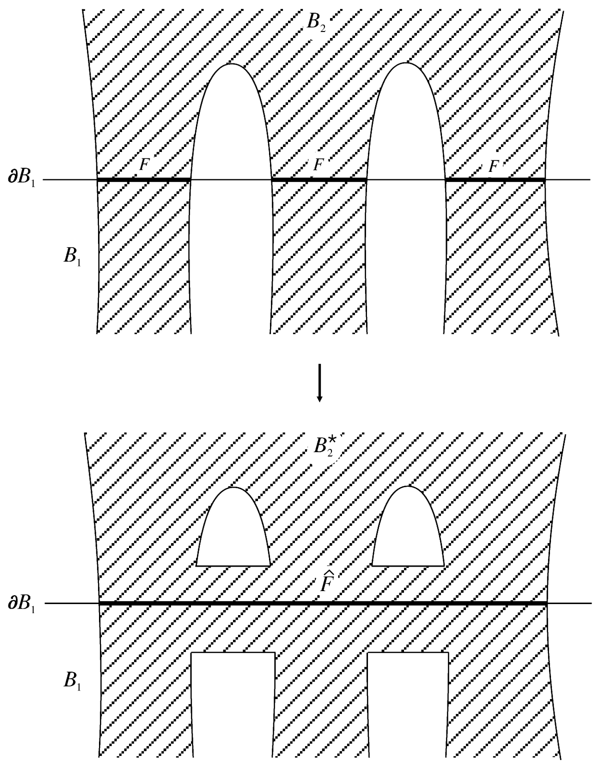

Fig. 1. A 2-handle move.

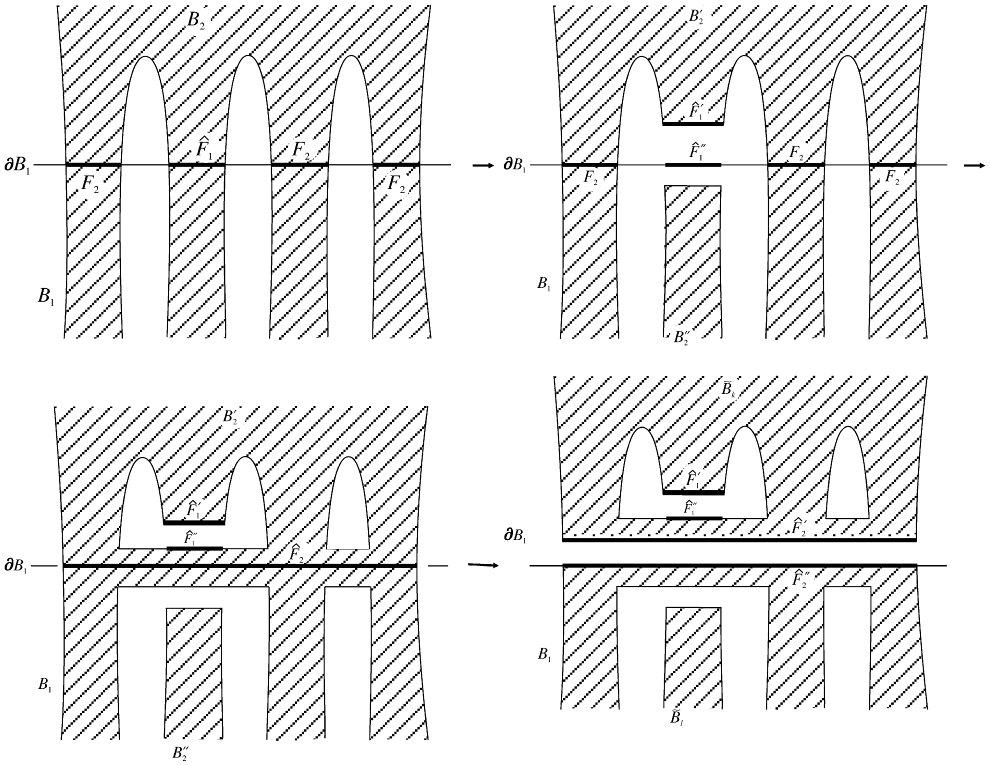

Fig. 2.

Doing steps 1 and 2 repeatedly starting with disk components of $\partial B_1 \cap B_2$ and then with innermost planar components, we convert B_2 into a collection of punctured balls $\widetilde{B}_k$. This is illustrated in Fig. 2, doing step 1 on $\widehat{F}_1$ and then step 2 on F_2. We may ignore those $\widetilde{B}_k$'s that lie in B_1. Then N is obtained from B_1 and a collection of punctured balls $\widetilde{B}_k$ by successive 1-handle attachments (in the picture first identity the two copies $\widehat{F}_2'$, $\widehat{F}_2''$ of $\widehat{F}_2$, then two copies $\widehat{F}_1'$, $\widehat{F}_1''$ of $\widehat{F}_1$) and the Theorem follows from Lemma 5. □

4. Unions of three balls

We now prove the main theorem.

Theorem 7. *If M is a closed 3-manifold that is a union of three open punctured balls then $M = \mathbb{B}$.*

Proof. By Corollary 3 we may assume that ∂B_1 is transversal to ∂B_2 and $\partial B_3 \subset \operatorname{int}(B_1 \cup B_2)$. Then the manifold $N = B_1 \cup B_2$ is as in Theorem 6 and $M = N \cup B_3$, with $\partial B_3 \cap N = \emptyset$.

We represent N as

$$N = H \cup K_1 \cup \cdots \cup K_m \cup W_1 \cup \cdots \cup W_n$$

where H is a punctured ball, K_j is a once-punctured S^2-bundle over S^1 $(j = 1, \ldots, m)$ and W_i is a once-punctured handlebody; furthermore $K_j \cap K_i = W_j \cap W_i = \emptyset$ for $i \neq j$, $H \cap K_j = \partial H \cap \partial K_j = C_j'$ is a 2-sphere $(j = 1, \ldots, m)$ and $H \cap W_i = \partial H \cap \partial W_i = C_i$ is a 2-sphere $(i = 1, \ldots, n)$.

Let S_j be a non-separating 2-sphere in $\operatorname{int} K_j$. We may assume that C_i, C_j', S_j are transversal to ∂B^3.

If $B_3 \cap S_j$ consists of planar surfaces perform 2-handle moves on B_3 and cut along disks in a regular neighborhood of S_j as in the proof of Theorem 6. Do the same for planar surfaces of $B_3 \cap C_j'$ and $B_3 \cap C_i$ $(j = 1, \ldots, m$, $i = 1, \ldots, n)$.

Since S_j, C_j', C_i are in $\operatorname{int}(N)$ this process converts B_3 into a disjoint collection $\widetilde{B}_k$ of punctured balls so that $M = N \cup \bigcup_k \widetilde{B}_k$ where $\partial \widetilde{B}_k \cap C_j' = \partial \widetilde{B}_k \cap C_i = \partial \widetilde{B}_k \cap S_j = \emptyset$ for all k and $i = 1, \ldots, n$, $j = 1, \ldots, m$.

We now cut N along the non-separating 2-spheres S_j into $N' = \widetilde{H} \cup W_1 \cup \cdots \cup W_n$ where $W_i \cap \widetilde{H} = \partial W_i \cap \partial \widetilde{H} = C_i$ $(i = 1, \ldots, n)$ and let

$$M' = N' \cup \bigcup_k \widetilde{B}_k = \widetilde{H} \cup W_1 \cup \cdots \cup W_n \cup \bigcup_k \widetilde{B}_k \qquad (*)$$

Note that M is obtained from M' by identifying some 2-spheres in $\partial M'$ in pairs (corresponding to the S'_j).

Let $\partial W_i = T_i \cup C_i$. Since M is closed we have $\partial \widetilde{B}_k \cap T_i = \emptyset$ hence $\partial \widetilde{B}_k \subset \operatorname{int} \widetilde{H} \cup \operatorname{int} W_i$ $(i = 1, \ldots, m)$.

If a component S of $\partial \widetilde{B}_k \cap \operatorname{int} W_i$ bounds a ball B in W_i we look at an innermost such B. Then either $\widetilde{B}_k = B$, in which case we delete $\widetilde{B}_k$ from the collection in $(*)$, or $\widetilde{B}_k \cap B = S$, in which case we replace $\widetilde{B}_k$ in $(*)$ by $\widetilde{B}_k \cup B$. Thus we may assume (since handlebodies are irreducible) that each component S of $\partial \widetilde{B}_k \cap W_i$ is parallel in W_i to C_i, and we can push all components of $\bigcup_k \partial \widetilde{B}_k \cap W_i$ across C_i into $\operatorname{int} \widetilde{H}$ by an isotopy.

Hence we now assume that in $(*)$ $\partial \widetilde{B}_k \subset \operatorname{int} \widetilde{H}$ for all k. Since M is closed, $T_i \subset \operatorname{int} \widetilde{B}_k$ for some k.

Let P be a point of $W_i \setminus T_i$. We join P by an arc α in W_i to a point Q in T_i such that $\operatorname{int} \alpha \subset \operatorname{int} W_i$. Suppose P does not lie in $\widetilde{B}_k$. Then since $Q \subset \widetilde{B}_k$, the arc α must intersect $\partial \widetilde{B}_k$. This is impossible since $\alpha \subset W_i$ and $\partial \widetilde{B}_k \cap W_i = \emptyset$.

Hence $W_i \subset \widetilde{B}_k$ and we may delete W_i in $(*)$ to obtain $M' = \widetilde{H} \cup \cup_k \widetilde{B}_k$ as in Lemma 4 (since $\partial \widetilde{H} \cap \partial \widetilde{B}_k = \emptyset$). Hence $M' = \mathbb{B} \# H$ and $M = \mathbb{B}$. $\square$

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