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Strong factorization of operators on spaces of vector measure integrable functions and unconditional convergence of series

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Abstract In this paper, we characterize the space of multiplication operators from an L^p-space into a space $L^1(\mathbf{m})$ of integrable functions with respect to a vector measure $\mathbf{m}$, as the subspace $L^1_{p,\mu}(\mathbf{m})$ of $L^1(\mathbf{m})$ defined by the functions that have finite p-semivariation. We prove several results concerning the Banach lattice structure of such spaces. We obtain positive results—for instance, they are always complete, and we provide counterexamples to prove that other properties are not satisfied—for example, simple functions are not in general dense. We study the operators that factorize through $L^1_{p,\mu}(\mathbf{m})$, and we prove an optimal domain theorem for such operators. We use our characterization to generalize the Bennet–Maurey–Nahoum Theorem on decomposition of functions that define an unconditionally convergent series in $L^1[0,1]$ to the case of 2-concave Banach function spaces.

Mathematics Subject Classification (2000) Strong factorizations · Unconditional summability · Vector valued integration

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1 Introduction

Let (Ω, Σ, μ) be a finite measure space and let E be a Banach. Consider a countably additive vector measure $\mathbf{m} : \Sigma \to E$. Then $\mathrm{L}^1(\mathbf{m})$ is the space of Bartle–Dunford–Schwartz integrable functions with respect to $\mathbf{m}$ (see [1,13]). Given $p \in [1, \infty]$ let p' its conjugate index, that is, the (extended) real number $p' \in [1, \infty]$ satisfying the equation $1/p+1/p' = 1$. The relationship between spaces of μ-continuous vector measures taking values in a Banach space E with finite p-semivariation and the space of operators $\mathrm{L}(\mathrm{L}^{p'}(\mu), E)$ is a classical result from vector measure theory (see for instance [11]). Using the same kind of relations, this paper is devoted to the study of a particular subspace of operators from $\mathrm{L}^{p'}(\mu)$ into $\mathrm{L}^1(\mathbf{m})$. This particular subspace is the space of the multiplication operators from $\mathrm{L}^{p'}(\mu)$ into $L^1(\mathbf{m})$. Actually, the first section of the paper shows that this space can be written as the subspace of $\mathrm{L}^1(\mathbf{m})$, denoted by $\mathrm{L}^1_{p,\mu}(\mathbf{m})$, consisting of all functions on Ω satisfying that the p-semivariation of the associated vector measure is finite.

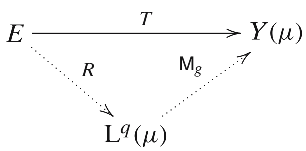
The interest of this characterization is given by the applications of these ideas in the context of the factorization theorems of the Maurey–Rosenthal framework, where multiplication operators appear in a natural way. The following scheme gives a (in a canonical sense) factorization result in this context. Let $1 \le q < \infty$ and let T be a q-convex operator from a Banach space E into a q-concave Banach function space $Y(\mu)$. Then there is a constant K and a measurable function g with

$$\sup_{h \in B_{L^q(\mu)}} \|g^{\frac{1}{q}} h\|_{Y(\mu)} \le K$$

such that for all $x \in E$,

$$\left(\int_\Omega \frac{|T(x)|^q}{g} d\mu \right)^{\frac{1}{q}} \le \|x\|_E.$$

This allows the factorization of the operator T as

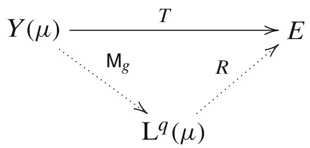


where M_g is the multiplication operator associated with the function g (see [6, Corollary 2]).

The dual factorization scheme can be written as follows. Let $1 \le q < \infty$ and let T be a q-concave operator from a q-convex Banach function space $Y(\mu)$ into a Banach space E. Then there is a positive linear functional $\varphi \in (Y(\mu)_{[q]})'$—where $Y(\mu)_{[q]}$ is the q-power of $Y(\mu)$, see [6] and the comments after Theorem 3—such that $\sup_{f \in B_{Y(\mu)}} \varphi(|f|^q)^{1/q} \le K$ and for all $f \in Y(\mu)$,

$$\|T(f)\|_E \le \varphi(|f|^q)^{\frac{1}{q}}.$$

If $Y(\mu)$ is order continuous, then φ can be identified with a measurable function, and then there is a measurable function g such that the operator factorizes as



where M_g is the corresponding multiplication operator (see [6, Corollary 5]).

Both factorization diagrams above involve multiplication operators that belong to $L(L^q(\mu), Y(\mu))$ and $L(Y(\mu), L^q(\mu))$, respectively. But no additional information about the structure of these operators is known. In this paper, we give a representation of these spaces as subspaces of the space $L^1(\mathbf{m})$ of integrable functions with respect to a vector measure $\mathbf{m}$ whenever the space $Y(\mu)$ is $L^1(\mathbf{m})$. Although this requirement may seem very strong, this is not the case since it is known that every order continuous Banach lattice with weak unit can be represented as an $L^1(\mathbf{m})$-space for a suitable vector measure $\mathbf{m}$ (see [3]).

In Sect. 3 we provide a description of the operators that factorize through the spaces $L^1_{p,\mu}(\mathbf{m})$ in a canonical way. In fact, we give an optimal domain theorem (see Theorem 6), showing that for these operators the maximal extension is the one that is defined from a space $L^1_{\mathsf{p},\mu}(\mathbf{m})$. This factorization can be easily related to the Maurey–Rosenthal cycle of ideas in order to obtain strong factorizations through the space $L^p(\mu)$. The last section of the paper is devoted to show an application of our representation of the spaces of multiplication operators to provide information about the decomposition of unconditionally convergent series in 2-concave Banach function spaces, obtaining in this way a generalization of the Bennet–Maurey–Nahoum Theorem (see [2,16]). A suitable abstract version of this result due to P. Ørno establishes that for an unconditional convergent series $\sum_{n=1}^{\infty} f_n$ in the space of Lebesgue integrable functions $L^p[0,1]$, $1 \le p \le 2$, it is possible to find a sequence $(\alpha_n)_{n=1}^{\infty} \in \ell^2$, a function $g \in L^2[0,2]$ and an orthonormal sequence $(h_n)_{n=1}^{\infty}$ in $L^2[0,2]$ such that for all $t \in [0,1]$, $f_n(t) = \alpha_n g(t) h_n(t)$ (see [16]). This allows the generalization to unconditionally summable sequences in $L^p[0,1]$ of several Menchoff–Rademacher type theorems concerning almost everywhere convergence of orthogonal series in $L^2[0,1]$. Recently, Defant and Junge have developed an abstract setting for a unified treatment of these almost everywhere convergence problems, that is applied in the setting of non-commutative L^p-spaces (see [7]).

Our results show that the elements of any unconditionally summable sequence $\sum_{n=1}^{\infty} f_n$ in $L^1(\mathbf{m})$—whenever this space is 2-concave Banach function space over the Lebesgue measure μ in $[0,1]$—can be written as a product of an scalar coming from a fixed two-summable sequence, a function h_n that belongs to a fixed orthonormal sequence in $L^2[0,2]$, and a function g belonging to the space $L^1_{2,\mu}(\mathbf{m})$.

2 Notation and preliminaries

Throughout this paper, (Ω, Σ, μ) will be a finite measure space and E be a Banach space. We denote by E' the topological dual of E and by $B_{E'}$ its open unit ball. $\mathcal{P}(A)$ will represent the set of partitions π of $A \in \Sigma$ where π has a finite number of disjoint measurable sets. The cardinality of the partition $\pi \in \mathcal{P}(A)$ will be denoted by $\#\pi$.

We denote the complementary of a set A by A^c or by $\Omega \backslash A$. As usual we indicate by $\mathrm{L}(E,F)$ the space of all bounded linear operators going from the Banach space E into the Banach space F.

The expression $s=\sum_{k=1}^{n}\alpha_k\chi_{A_k}$ represents a simple function such that $(A_k)_k \subseteq \mathcal{P}(\Omega)$ and $\alpha_k \in \mathbb{R}\setminus\{0\}$ for every $k=1,2,\ldots,n$ where χ_A is the characteristic function of the set A. The set of the simple functions is denoted by $\mathrm{sim}(\Sigma)$. The set consisting of all measurable functions on Ω is denoted by $\mathrm{L}^0(\mu)$.

If $1\le p\le\infty$ then $p'\in[1,\infty]$ is given by $1/p+1/p'=1$.

Let $\mathbf{m}:\Sigma\to E$ be a (countably additive) vector measure. The *semivariation of* $\mathbf{m}$ *over* $A\in\Sigma$ is defined by

$$\|\mathbf{m}\|(A)=\sup_{x'\in B_{E'}}|\langle\mathbf{m},x'\rangle|(A)=\sup_{x'\in B_{E'}}\sup_{\pi\in\mathcal{P}(A)}\sum_{B\in\pi}|\langle\mathbf{m}(B),x'\rangle|,$$

where we have employed the usual notation

$$\langle\mathbf{m},x'\rangle(A)=\langle\mathbf{m}(A),x'\rangle \quad \text{for each } A\in\Sigma.$$

A set $A\in\Sigma$ is called $\mathbf{m}$-null if $\|\mathbf{m}\|(A)=0$. A property which holds outside an $\mathbf{m}$-null is said to hold $\mathbf{m}$-almost everywhere (briefly $\mathbf{m}$-almost everywhere or simply $\mathbf{m}$-a.e.). In this paper, we assume that μ and $\mathbf{m}$ are mutually absolutely continuous, that is, $\mathbf{m}$ and μ have the same null sets (see [10, Theorem I.2.6]). In particular, μ can be a Rybakov measure; recall that a *Rybakov measure for a vector measure* $\mathbf{m}$ is a scalar measure ν defined as the variation of a measure $\langle\mathbf{m},x'\rangle$, where $x'\in E'$, whenever $\mathbf{m}$ is absolutely continuous with respect to ν. A Rybakov measure always exists for every vector measure $\mathbf{m}$ (see [10, IX.2.2]).

Definition 1 A function $f:\Omega\to\mathbb{R}$ is said to be integrable with respect to the measure $\mathbf{m}$ if

(a) for each $x'\in E'$ we have that $f\in\mathrm{L}^1(\langle\mathbf{m},x'\rangle)$,
(b) for each $A\in\Sigma$ there exists $x_A\in E$ such that

$$\langle x_A,x'\rangle=\int_A f\,\mathrm{d}\langle\mathbf{m},x'\rangle \quad \text{for every } x'\in E'.$$

The vector x_A is unique and will be denoted by $\int_A f\mathrm{d}\mathbf{m}$. Observe that

$$\left\langle\int_A f\mathrm{d}\mathbf{m},x'\right\rangle=\int_A f\,\mathrm{d}\langle\mathbf{m},x'\rangle \quad \text{for each } x'\in E'.$$

The space of the classes (equality $\mathbf{m}$-almost everywhere) of these functions is denoted by $\mathrm{L}^1(\mathbf{m})$. The expression

$$\|f\|_{\mathrm{L}^1(\mathbf{m})}=\sup_{x'\in B_{E'}}\int_\Omega |f|\mathrm{d}|\langle\mathbf{m},x'\rangle| \quad \text{for each } f\in\mathrm{L}^1(\mathbf{m}),$$

defines a lattice norm on $\mathrm{L}^1(\mathbf{m})$ for which $\mathrm{L}^1(\mathbf{m})$ is an order continuous Banach lattice with weak unit χ_Ω (see [3]). The indefinite integral $\mathbf{m}_f:\Sigma\to E$ of a function $f\in\mathrm{L}^1(\mathbf{m})$ is defined by

$$\mathbf{m}_f(A)=\int_A f\mathrm{d}\mathbf{m}, \quad A\in\Sigma.$$

The Orlicz-Pettis Theorem ensures that $\mathbf{m}_f$ is again a countably additive vector measure. An equivalent norm for $\mathrm{L}^1(\mathbf{m})$ is given by

$$|||f|||_{\mathrm{L}^1(\mathbf{m})} = \sup_{A\in\Sigma} \left\| \int_A f \mathrm{d}\mathbf{m} \right\|_E \qquad \text{for each } f \in \mathrm{L}^1(\mathbf{m}),$$

because $|||f|||_{\mathrm{L}^1(\mathbf{m})} \le \|f\|_{\mathrm{L}^1(\mathbf{m})} \le 2|||f|||_{\mathrm{L}^1(\mathbf{m})}$ (see [10, Chap. I.1.11]).

Let $1 \le p < \infty$. A measurable function $f : \Omega \to \mathbb{R}$ is said to be p-integrable if $|f|^p$ is integrable with respect to $\mathbf{m}$. The space consisting of the equivalence classes (with respect to $\mathbf{m}$-almost everywhere equality) of these functions is denoted by $\mathrm{L}^p(\mathbf{m})$. The expression

$$\|f\|_{\mathrm{L}^p(\mathbf{m})} = \sup_{x'\in B_{E'}} \left(\int_\Omega |f|^p \mathrm{d}|\langle \mathbf{m}, x'\rangle| \right)^{\frac{1}{p}} \qquad \text{for each } f \in \mathrm{L}^p(\mathbf{m}),$$

defines a lattice norm on $\mathrm{L}^p(\mathbf{m})$ (see [12]).

Definition 2 Let $(X(\mu), \|\cdot\|_{X(\mu)})$ be a Banach space consisting of (equivalence classes with respect to μ-a.e. equality) measurable functions $f : \Omega \to \mathbb{R}$. We say that $X(\mu)$ is a μ-Köthe function space when the following conditions hold:

(a) If f is a real measurable function defined on Ω and $|f| \le |g|$ for some $g \in X(\mu)$, then $f \in X(\mu)$ and $\|f\|_{X(\mu)} \le \|g\|_{X(\mu)}$.
(b) $\chi_A \in X(\mu)$ for each $A \in \Sigma$.
(c) $X(\mu) \subset \mathrm{L}^1(\mu)$ and the inclusion is continuous.

The corresponding Köthe dual of $X(\mu)$, that is, $X(\mu)^\times$ is the vector space of all measurable functions g on Ω such that $fg \in \mathrm{L}^1(\mu)$ for all $f \in X(\mu)$. The Köthe dual is often called the associated space of $X(\mu)$. Observe that $X(\mu)^\times$ is also a μ-Köthe function space. If $f \in X(\mu)$ and $g \in X(\mu)^\times$, then $fg \in \mathrm{L}^1(\mu)$ and $\left|\int_\Omega fg \,\mathrm{d}\mu\right| \le \|f\|_{X(\mu)}\|g\|_{X(\mu)^\times}$ (see X' in [15, p. 27]).

For every $1 \le p < \infty$, $\mathrm{L}^p(\mathbf{m})$ is a Köthe function space over any Rybakov measure for $\mathbf{m}$.

Remark 1 Assume that $(h_n)_n$ is a sequence in $X(\mu)$ converging to $h \in X(\mu)$. Then, property (c) in Definition 2 implies that we can find a subsequence $(h_{n_k})_k$ such that $h_{n_k} \to h$ μ-a.e.

Observe that $\mathrm{L}^1(\mathbf{m})$ is a ν-Köthe function space for any of its Rybakov measures ν. Hence, if $(h_n)_n$ is a sequence in $\mathrm{L}^1(\mathbf{m})$ that converges to $h \in \mathrm{L}^1(\mathbf{m})$, then there is a subsequence $(h_{n_k})_k$ such that $h_{n_k} \to h$ $\mathbf{m}$.a.e.

In what follows we recall several definitions regarding the semivariation of the vector measure $\mathbf{m}$.

Definition 3 Let $A \in \Sigma$. For $1 \leq p < \infty$ the p-semivariation, $\|\mathbf{m}\|_{p,\mu}(A)$ of a vector measure $\mathbf{m}$ of $A \in \Sigma$ with respect to μ is defined by

$$\|\mathbf{m}\|_{p,\mu}(A) = \sup_{\pi \in \mathcal{P}(A)} \sup_{x' \in B_{E'}} \left\| \sum_{B \in \pi} \frac{\langle \mathbf{m}(B), x' \rangle}{\mu(B)} \chi_B \right\|_{\mathrm{L}^p(\mu)}$$
$$= \sup_{\pi \in \mathcal{P}(A)} \sup_{x' \in B_{E'}} \left(\sum_{B \in \pi} \frac{|\langle \mathbf{m}(B), x' \rangle|^p}{\mu(B)^{p-1}} \right)^{\frac{1}{p}} \leq \infty.$$

For the case $p = \infty$ the ∞-semivariation is

$$\|\mathbf{m}\|_{\infty,\mu}(A) = \sup_{\pi \in \mathcal{P}(A)} \sup_{x' \in B_{E'}} \left\| \sum_{B \in \pi} \frac{\langle \mathbf{m}(B), x' \rangle}{\mu(B)} \chi_B \right\|_{\mathrm{L}^\infty(\mu)}$$
$$= \sup_{\Sigma \ni B \subseteq A} \sup_{x' \in B_{E'}} \frac{|\langle \mathbf{m}(B), x' \rangle|}{\mu(B)} \leq \infty.$$

Remark 2 (a) When $p = 1$, we have that $\|\mathbf{m}\|_{1,\mu} = \|\mathbf{m}\|$ in Σ.
(b) Using that $\mu(\Omega) < \infty$ we have that if $1 \leq q < p \leq \infty$ then

$$\|\mathbf{m}\|_{q,\mu}(\Omega) \leq \mu(\Omega)^{\frac{1}{q} - \frac{1}{p}} \|\mathbf{m}\|_{p,\mu}(\Omega). \tag{1}$$

Definition 4 Let $1 \leq p \leq \infty$. We say that a function $f : \Omega \to \mathbb{R}$ belongs to the vector space $\mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})$ if

(a) $f \in \mathrm{L}^1(\mathbf{m})$, and
(b) the vector measure associated to f, $\mathbf{m}_f$, has finite p-semivariation, i.e.,

$$\|f\|_{\mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})} = \|\mathbf{m}_f\|_{p,\mu}(\Omega) < \infty. \tag{2}$$

Note that $\|f\|_{\mathrm{L}^1_{\infty,\mu}(\mathbf{m})} = 0$ if and only if $f = 0$ ($\mathbf{m}$-almost everywhere). So, identifying functions which are equal $\mathbf{m}$-almost everywhere it is easy to see that the expression (2) defines a norm in $\mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})$, that is , given $f \in \mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})$, then

$$\|f\|_{\mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})} = \sup_{\pi \in P(\Omega)} \sup_{x' \in B_{E'}} \left(\sum_{A \in \pi} \frac{|\int_A f \mathrm{d}\langle \mathbf{m}, x' \rangle|^p}{\mu(A)^{p-1}} \right)^{\frac{1}{p}}, \quad \text{if } 1 \leq p < \infty,$$

and

$$\|f\|_{\mathrm{L}^1_{\infty,\mu}(\mathbf{m})} = \sup_{A \in \Sigma} \sup_{x' \in B_{E'}} \frac{|\int_A f \mathrm{d}\langle \mathbf{m}, x' \rangle|}{\mu(A)}, \quad \text{if } p = \infty.$$

Remark 3 Observe that for $1 \leq p \leq \infty$ we have that $\mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m}) \subseteq \mathrm{L}^1(\mathbf{m})$. Also, using the fact that $\|\mathbf{m}_f\|_{1,\mu} = \|f\|_{\mathrm{L}^1(\mathbf{m})}$ and *(a)* in Remark 2 we obtain that $\mathrm{L}^1_{1,\mu}(\mathbf{m}) = \mathrm{L}^1(\mathbf{m})$.

Remark 4 If we take $1 \leq q < p \leq \infty$ then we have, using (1) with the vector measure $\mathbf{m}_f$, that

$$\|f\|_{\mathrm{L}^1_{q,\mu}(\mathbf{m})} \leq \mu(\Omega)^{\frac{1}{q} - \frac{1}{p}} \|f\|_{\mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})} \quad \text{for each } f \in \mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m}).$$

In particular, taking $q = 1$, we have that

$$\|f\|_{L^1(\mathbf{m})} \le \mu(\Omega)^{\frac{1}{p'}} \|f\|_{L^1_{p,\mu}(\mathbf{m})} \quad \text{for each } f \in L^1_{p,\mu}(\mathbf{m}).$$

Remark 5 Observe that

$$\|\chi_A\|_{L^1_{p,\mu}(\mathbf{m})} = \|\mathbf{m}\|_{p,\mu}(A) \quad \text{for each } A \in \Sigma,$$

then if we suppose that the vector measure $\mathbf{m}$ has finite p-semivariation then the set of the simple functions, $\text{sim}(\Sigma)$, is contained in $L^1_{p,\mu}(\mathbf{m})$.

Let $1 \le r \le \infty$. Given a measurable function h on Ω let

$$h \cdot L^r(\mu) = \{hf : f \in L^r(\mu)\}.$$

When $h \cdot L^r(\mu) \subseteq L^1(\mathbf{m})$ we can define the associated *multiplication operator* $\mathsf{M}_h : L^r(\mu) \to L^1(\mathbf{m})$ by

$$\mathsf{M}_h(f) = hf, \quad \text{for every } f \in L^r(\mu).$$

Note that, if h and g are equal μ-a.e., then the multiplication operators M_h and M_g are equal. Therefore, we can define the vector space of (classes of) all multiplication operators between $L^r(\mu)$ into $L^1(\mathbf{m})$ as

$$\mathsf{M}(L^r(\mu), L^1(\mathbf{m})) = \{\mathsf{M}_h : h \cdot L^r(\mu) \subseteq L^1(\mathbf{m})\}.$$

If $h \cdot L^r(\mu) \subseteq L^1(\mathbf{m})$ the Closed Graph Theorem shows easily that M_h is bounded. This means that the vector space of classes of multiplication operators $\mathsf{M}(L^r(\mu), L^1(\mathbf{m}))$ equipped with the operator norm is a subspace of $L(L^r(\mu), L^1(\mathbf{m}))$. A norm in the vector space $\mathsf{M}(L^r(\mu), L^1(\mathbf{m}))$ is given by the expression

$$\|h\|_{\mathsf{M}} = \|\mathsf{M}_h\| = \sup\{\|hf\|_{L^1(\mathbf{m})} : f \in B_{L^r(\mu)}\}.$$

Let $1 \le q < \infty$, we refer the reader to [15] for information about the notions of q-convexity and q-concavity.

Definition 5 Let E and F Banach lattices. An operator $T \in L(E, F)$ is said to be q-convex if there exists $k \ge 0$ such that for every $N \in \mathbb{N}$ and $x_1, x_2, \ldots, x_N \in E$

$$\left\| \left(\sum_{i=1}^{N} |Tx_i|^q \right)^{\frac{1}{q}} \right\| \le k \left(\sum_{i=1}^{N} \|x_i\|^q \right)^{\frac{1}{q}},$$

and q-concave if there exists $k \ge 0$ such that for every $N \in \mathbb{N}$ and $x_1, x_2, \ldots, x_N \in E$

$$\left(\sum_{i=1}^{N} \|Tx_i\|^q \right)^{\frac{1}{q}} \le k \left\| \left(\sum_{i=1}^{N} |x_i|^q \right)^{\frac{1}{q}} \right\|.$$

As usual we write $M^{(q)}(T)$ and $M_{(q)}(T)$ for the best constants $k \ge 0$ in the inequalities above, respectively.

Definition 6 A Banach lattice E is said to be q-convex (resp. q-concave) if the identity operator on E is q-convex (resp. q-concave). In this case we write $M^{(q)}(E)$ and $M_{(q)}(E)$ for the corresponding constants.

3 The structure of the spaces $L^1_{p,\mu}(\mathbf{m})$

The aim of this section is to prove the general results concerning the structure of the space $L^1_{p,\mu}(\mathbf{m})$. We show that a lattice norm can be given for this space. This norm allows us to prove that the space is complete, and to provide a suitable representation of the space as a Banach function space. In this section we assume that $\mathbf{m}$ has finite p-semivariation. In this case recall that the set $\mathrm{sim}(\Sigma)$ is contained in $L^1_{p,\mu}(\mathbf{m})$ (see Remark (5)).

Lemma 1 *Let be ν a real measure on a measurable space (Ω, Σ) and take $x' \in E'$. If f is a ν-integrable function and $A \in \Sigma$, then there exists $B \in \Sigma$ such that $B \subseteq A$ and*

$$\int_A |f| \, d\nu \leq \left| \int_B f \, d\nu \right| + \left| \int_{A \setminus B} f \, d\nu \right|.$$

Proof We denote by ν^+ and ν^-, the positive and negative parts of the measure ν. According to the Hahn's Decomposition Theorem we can find $D_i \in \Sigma$, $i = 1, 2$, such that $D_1 \bigcap D_2 = \emptyset$, $D_1 \bigcup D_2 = \Omega$, $\nu = \nu^+$ in D_1 and $\nu = -\nu^-$ in D_2. Let us take $D_{i,+} = \{x \in D_i : f(x) \geq 0\}$ and $D_{i,-} = \{x \in D_i : f(x) < 0\}$, $i = 1, 2$. Then we have that,

$$\begin{aligned} \int_A |f| \, d\nu &= \int_{A \bigcap D_1} |f| \, d\nu - \int_{A \bigcap D_2} |f| \, d\nu \\ &= \int_{A \bigcap (D_{1,+} \bigcup D_{2,-})} f \, d\nu - \int_{A \bigcap (D_{1,-} \bigcup D_{2,+})} f \, d\nu \\ &\leq \left| \int_B f \, d\nu + \right| \int_{A \setminus B} f \, d\nu, \end{aligned}$$

where $B = A \bigcap (D_{1,+} \bigcup D_{2,-})$. □

Definition 7 Let $f \in L^1_{p,\mu}(\mathbf{m})$. For $1 \leq p < \infty$ we define

$$|||f|||_{L^1_{p,\mu}(\mathbf{m})} = \sup_{\pi \in \mathcal{P}(\Omega)} \sup_{x' \in B_{E'}} \left(\sum_{A \in \pi} \frac{(\int_A |f| d|\langle \mathbf{m}, x' \rangle|)^p}{\mu(A)^{p-1}} \right)^{\frac{1}{p}}.$$

For the case $p = \infty$ the analogous is

$$|||f|||_{L^1_{\infty,\mu}(\mathbf{m})} = \sup_{A \in \Sigma} \sup_{x' \in B_{E'}} \frac{\int_A |f| d|\langle \mathbf{m}, x' \rangle|}{\mu(A)}.$$

Obviously $||| \cdot |||_{L^1_{p,\mu}(\mathbf{m})}$ is also a norm in $L^1_{p,\mu}(\mathbf{m})$ for every $1 \leq p \leq \infty$.

Theorem 1 *The norms $\| \cdot \|_{L^1_{p,\mu}(\mathbf{m})}$ and $||| \cdot |||_{L^1_{p,\mu}(\mathbf{m})}$ are equivalent and satisfy*

$$\|f\|_{L^1_{p,\mu}(\mathbf{m})} \leq |||f|||_{L^1_{p,\mu}(\mathbf{m})} \leq 2^{\frac{1}{p'}} \|f\|_{L^1_{p,\mu}(\mathbf{m})} \quad \text{for each } f \in L^1_{p,\mu}(\mathbf{m}).$$

Proof Obviously, we have that $\|f\|_{L^1_{p,\mu}(\mathbf{m})} \leq |||f|||_{L^1_{p,\mu}(\mathbf{m})}$ for every $f \in L^1_{p,\mu}(\mathbf{m})$.

Suppose $1 \leq p < \infty$. Let $f \in L^1_{p,\mu}(\mathbf{m})$, $\pi \in \mathcal{P}(\Omega)$ and $x' \in E'$. Using Lemma 1 with the real measure $\nu = |\langle \mathbf{m}, x' \rangle|$, for each $A \in \pi$ there exists $B_A \in \Sigma$ such that $B_A \subseteq A$ and

$$\int_A |f| \, d\langle \mathbf{m}, x' \rangle| \leq |\int_{B_A} f \, d\langle \mathbf{m}, x' \rangle| + |\int_{A \setminus B_A} f \, d\langle \mathbf{m}, x' \rangle|.$$

Now, using the inequality, $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for each $a, b \geq 0$, we obtain that

$$\begin{aligned}
\sum_{A \in \pi} \frac{(\int_A |f| d\langle \mathbf{m}, x' \rangle|)^p}{\mu(A)^{p-1}} &\leq \sum_{A \in \pi} \frac{(|\int_{B_A} f \, d\langle \mathbf{m}, x' \rangle| + |\int_{A \setminus B_A} f \, d\langle \mathbf{m}, x' \rangle|)^p}{\mu(A)^{p-1}} \\
&\leq 2^{p-1} \sum_{A \in \pi} \frac{|\int_{B_A} f \, d\langle \mathbf{m}, x' \rangle|^p}{\mu(A)^{p-1}} + 2^{p-1} \sum_{A \in \pi} \frac{|\int_{A \setminus B_A} f \, d\langle \mathbf{m}, x' \rangle|^p}{\mu(A)^{p-1}} \\
&\leq 2^{p-1} \sum_{A \in \pi} \frac{|\int_{B_A} f \, d\langle \mathbf{m}, x' \rangle|^p}{\mu(B_A)^{p-1}} + 2^{p-1} \sum_{A \in \pi} \frac{|\int_{A \setminus B_A} f \, d\langle \mathbf{m}, x' \rangle|^p}{\mu(A \setminus B_A)^{p-1}} \\
&= 2^{p-1} \sum_{D \in \tilde{\pi}} \frac{|\int_D f \, d\langle \mathbf{m}, x' \rangle|^p}{\mu(D)^{p-1}}.
\end{aligned}$$

where $\tilde{\pi}$ is the partition given by $\tilde{\pi} = \{B_A : A \in \pi\} \cup \{A \setminus B_A : A \in \pi\}$.

For the case $p = \infty$ the proof is analogous. □

Theorem 2 *Given* $1 \leq p \leq \infty$, *then* $(L^1_{p,\mu}(\mathbf{m}), |||\cdot|||_{L^1_{p,\mu}(\mathbf{m})})$ *is a Banach space.*

Proof Let $(f_n)_n$ be a Cauchy sequence in $L^1_{p,\mu}(\mathbf{m})$. Taking into account that

$$\|f\|_{L^1(\mathbf{m})} \leq \mu(\Omega)^{\frac{1}{p'}} |||f|||_{L^1_{p,\mu}(\mathbf{m})},$$

we have that $(f_n)_n$ is a Cauchy sequence in $L^1(\mathbf{m})$. The completeness of $L^1(\mathbf{m})$ ensures that there exists a function $f \in L^1(\mathbf{m})$ such that the sequence $(f_n)_n$ converges to f in the norm $\|\cdot\|_{L^1(\mathbf{m})}$.

Let us see that $(f_n)_n$ converges to f in the norm $|||\cdot|||_{L^1_{p,\mu}(\mathbf{m})}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n, k \geq N$, then

$$|||f_n - f_k|||_{L^1_{p,\mu}(\mathbf{m})} \leq \frac{\varepsilon}{2}.$$

Since $(f_n)_n$ converges to f in $\|\cdot\|_{L^1(\mathbf{m})}$ given $n \geq N$ we have that

$$\int_A |f_n - f_k| d|\langle \mathbf{m}, x' \rangle| \xrightarrow{k} \int_A |f_n - f| d|\langle \mathbf{m}, x' \rangle|,$$

for each partition $\pi \in \mathcal{P}(\Omega)$ and every $A \in \pi$ and $x' \in E'$. It follows that for $n \geq N$ we have that $f_n - f \in L^1_{p,\mu}(\mathbf{m})$ and

$$|||f_n - f|||_{L^1_{p,\mu}(\mathbf{m})} \leq \varepsilon.$$

In order to see this, note that if for a fixed $n \geq N$ we have that $|||f_n - f|||_{\mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})} > \varepsilon$, then we can find a partition $\pi_n \in \mathcal{P}(\Omega)$ and $x'_n \in B_{E'}$ such that

$$\sum_{A \in \pi_n} \frac{(\int_A |f_n - f| \mathrm{d}|\langle \mathbf{m}, x'_n \rangle|)^p}{\mu(A)^{p-1}} > \varepsilon^p.$$

Taking $k \geq N$ we obtain

$$\begin{aligned}
\varepsilon^p &< \left| \sum_{A \in \pi_n} \frac{(\int_A |f_n - f| \mathrm{d}|\langle \mathbf{m}, x'_n \rangle|)^p}{\mu(A)^{p-1}} - \sum_{A \in \pi_n} \frac{(\int_A |f_n - f_k| \mathrm{d}|\langle \mathbf{m}, x'_n \rangle|)^p}{\mu(A)^{p-1}} \right| \\
&\quad + \sum_{A \in \pi_n} \frac{(\int_A |f_n - f_k| \mathrm{d}|\langle \mathbf{m}, x'_n \rangle|)^p}{\mu(A)^{p-1}} \\
&\leq \left| \sum_{A \in \pi_n} \frac{(\int_A |f_n - f| \mathrm{d}|\langle \mathbf{m}, x'_n \rangle|)^p}{\mu(A)^{p-1}} - \sum_{A \in \pi_n} \frac{(\int_A |f_n - f_k| \mathrm{d}|\langle \mathbf{m}, x'_n \rangle|)^p}{\mu(A)^{p-1}} \right| + \left(\frac{\varepsilon}{2}\right)^p,
\end{aligned}$$

that clearly gives a contradiction just taking k big enough. This implies that $f \in \mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})$. □

Corollary 1 *If the inclusion* $i : \mathrm{L}^1(\mathbf{m}) \to \mathrm{L}^1(\mu)$ *is continuous, then* $\mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})$ *is a* μ*-Köthe function space. In particular, this holds if* μ *is a Rybakov measure for* $\mathbf{m}$.

The main result of this section is the identification of the space of multiplication operators from $\mathrm{L}^{p'}(\mu)$ in $\mathrm{L}^1(\mathbf{m})$ and the space $\mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})$. In particular, we show that this identification is in fact an isometry. This characterization provides a different point of view for the understanding of the space of multiplication operators; for instance, it leave us to show some negative results. The main one is the counterexample given at the end of the section, that shows that the set of simple functions is not in the general case dense in $\mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})$. The same example proves that, if we do not assume that the measure $\mathbf{m}$ has finite p-semivariation, then the set of simple functions can not be embedded in $\mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})$. Through this section we assume that $\mathbf{m}$ has finite p-semivariation unless stated otherwise.

Proposition 1 *Let* $1 \leq p \leq \infty$ *and let* $f \in \mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})$ *and* $g \in \mathrm{L}^{p'}(\mu)$*. Then*

$$\|fg\|_{\mathrm{L}^1(\mathbf{m})} \leq \|g\|_{\mathrm{L}^{p'}(\mu)} |||f|||_{\mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})}.$$

Proof For $p = 1$ the result is obvious (see Remark 3). Suppose that $1 < p \leq \infty$ and consider a simple function $s = \sum_{k=1}^n \alpha_k \chi_{A_k}$. Then

$$\begin{aligned}
\|fs\|_{\mathrm{L}^1(\mathbf{m})} &= \sup_{x' \in B_{E'}} \int_\Omega |fs| \mathrm{d}|\langle \mathbf{m}, x' \rangle| = \sup_{x' \in B_{E'}} \sum_{k=1}^n |\alpha_k| \int_{A_k} |f| \mathrm{d}|\langle \mathbf{m}, x' \rangle| \\
&= \sup_{x' \in B_{E'}} \sum_{k=1}^n |\alpha_k| \mu(A_k)^{\frac{1}{p'}} \frac{\int_{A_k} |f| \mathrm{d}|\langle \mathbf{m}, x' \rangle|}{\mu(A_k)^{\frac{1}{p'}}} \\
&\leq \sup_{x' \in B_{E'}} \left(\sum_{k=1}^n |\alpha_k|^{p'} \mu(A_k) \right)^{\frac{1}{p'}} \left(\sum_{k=1}^n \frac{(\int_{A_k} |f| \mathrm{d}|\langle \mathbf{m}, x' \rangle|)^p}{\mu(A_k)^{\frac{p}{p'}}} \right)^{\frac{1}{p}}
\end{aligned}$$

$$= \|s\|_{L^{p'}(\mu)} \sup_{x' \in B_{E'}} \left(\sum_{k=1}^{n} \frac{(\int_{A_k} |f| d|\langle \mathbf{m}, x' \rangle|)^p}{\mu(A_k)^{p-1}} \right)^{\frac{1}{p}}$$
$$\leq \|s\|_{L^{p'}(\mu)} |||f|||_{L^1_{p,\mu}(\mathbf{m})}.$$

Suppose now that $g \in L^{p'}(\mu)$. There exists a sequence of simple functions $(s_n)_n$ that converges to g pointwise and also in the norm $\|\cdot\|_{L^{p'}(\mu)}$. In particular the sequence $(fs_n)_n$ converges pointwise to fg. On the other hand the inequality

$$\|fs\|_{L^1(\mathbf{m})} \leq \|s\|_{L^{p'}(\mu)} |||f|||_{L^1_{p,\mu}(\mathbf{m})} \quad \text{for each } s \in \text{sim}(\Sigma),$$

implies that $(fs_n)_n$ is a Cauchy sequence in the Banach space $L^1(\mathbf{m})$ so there exists a function $h \in L^1(\mathbf{m})$ such that $(fs_n)_n$ converges to h in the norm $\|\cdot\|_{L^1(\mathbf{m})}$. Take now a subsequence $(fs_{n_k})_k$ that converges to h almost everywhere. Since $(fs_n)_n$ converges to fg pointwise then $h = fg$ in almost every point of Ω. Thus $fg \in L^1(\mathbf{m})$ and

$$\|fg\|_{L^1(\mathbf{m})} = \lim_n \|fs_n\|_{L^1(\mathbf{m})} \leq \lim_n |||f|||_{L^1_{p,\mu}(\mathbf{m})} \|s_n\|_{L^{p'}(\mu)} = \|g\|_{L^{p'}(\mu)} |||f|||_{L^1_{p,\mu}(\mathbf{m})}.$$

□

Theorem 3 *Let* $1 \leq p < \infty$. *Then there exists an isomorphism between the space* $L^1_{p,\mu}(\mathbf{m})$ *and the space* $M(L^{p'}(\mu), L^1(\mathbf{m}))$. *Moreover*

$$|||f|||_{L^1_{p,\mu}(\mathbf{m})} = \|f\|_M \quad \textit{for every } f \in L^1_{p,\mu}(\mathbf{m}).$$

Proof Suppose first $1 < p < \infty$. Let $f \in L^1_{p,\mu}(\mathbf{m})$. By Proposition 1, the multiplication operator M_f, is continuous and satisfies $\|f\|_M = \|M_f\| \leq |||f|||_{L^1_{p,\mu}(\mathbf{m})}$. Let us show that $\|f\|_M = |||f|||_{L^1_{p,\mu}(\mathbf{m})}$.

Fixed $x' \in B_{E'}$ and $\pi \in \mathcal{P}(\Omega)$ consider the simple function

$$s = \sum_{A \in \pi} \left(\frac{\int_A |f| d|\langle \mathbf{m}, x' \rangle|}{\mu(A)} \right)^{p-1} \chi_A. \tag{3}$$

Observe that

$$\|s\|_{L^{p'}(\mu)} = \left(\sum_{A \in \pi} \left(\frac{\int_A |f| d|\langle \mathbf{m}, x' \rangle|}{\mu(A)} \right)^{p'(p-1)} \mu(A) \right)^{\frac{1}{p'}} = \left(\sum_{A \in \pi} \frac{(\int_A |f| d|\langle \mathbf{m}, x' \rangle|)^p}{\mu(A)^{p-1}} \right)^{\frac{1}{p'}}.$$

Consider now the norm one simple function $\widetilde{s} = s/\|s\|_{L^{p'}(\mu)}$. Note that

$$\int_\Omega |f \widetilde{s}| d|\langle \mathbf{m}, x' \rangle| = \frac{1}{\|s\|_{L^{p'}(\mu)}} \sum_{A \in \pi} \frac{(\int_A |f| d|\langle \mathbf{m}, x' \rangle|)^p}{\mu(A)^{p-1}} = \|s\|^{p'-1}_{L^{p'}(\mu)}. \tag{4}$$

Let now $\varepsilon > 0$. There exist a partition $\pi_0 \in \mathcal{P}(\Omega)$ and an element $x'_0 \in B_{E'}$ such that

$$|||f|||_{L^1_{p,\mu}(\mathbf{m})} < \left(\sum_{A \in \pi_0} \frac{(\int_A |f| d|\langle \mathbf{m}, x'_0 \rangle|)^p}{\mu(A)^{p-1}} \right)^{\frac{1}{p}} + \varepsilon.$$

Let us denote by s_0 the simple function defined as (3) associated to the partition π_0 and x'_0. Taking into account (4) we obtain

$$\|\!|f|\!\|_{\mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})} < \left(\sum_{A\in\pi_0} \frac{(\int_A |f|\mathrm{d}|\langle \mathbf{m}, x'_0\rangle|)^p}{\mu(A)^{p-1}}\right)^{\frac{1}{p}} + \varepsilon = \|s_0\|^{p'-1}_{\mathrm{L}^{p'}(\mu)} + \varepsilon$$
$$= \int_\Omega |f\widetilde{s}_0|\mathrm{d}|\langle \mathbf{m}, x'_0\rangle| + \varepsilon \le \|f\|_{\mathsf{M}} + \varepsilon.$$

Just take now limits when ε tends to 0^+.
Finally, we have to prove that $f \to \mathsf{M}_f$ is an isomorphism, or equivalently, that it is a surjection. In order to do this observe first that if $h \in \mathsf{M}(\mathrm{L}^{p'}(\mu), \mathrm{L}^1(\mathbf{m}))$ then just taking $g = \chi_\Omega \in \mathrm{L}^{p'}(\mu)$ we obtain that $\mathsf{M}_h(\chi_\Omega) = h \in \mathrm{L}^1(\mathbf{m})$. Thus the vector measure associated to the function h is well defined as

$$\begin{aligned} \mathbf{m}_h : \Sigma &\to E \\ E &\mapsto \textstyle\int_E h\mathrm{d}\mathbf{m}. \end{aligned}$$

Let us show now that $\mathbf{m}_h$ has finite p-semivariation. Let $\pi_0 \in P(\Omega)$ and let $x'_0 \in B_{E'}$. Then

$$\sum_{A\in\pi_0} \frac{|\langle \mathbf{m}_h(A), x'_0\rangle|^p}{\mu(A)^{p-1}} \le \sum_{A\in\pi_0} \frac{(\int_A |h|\mathrm{d}|\langle \mathbf{m}, x'_0\rangle|)^p}{\mu(A)^{p-1}}$$
$$= \left(\int_\Omega |h\widetilde{s}_0|\mathrm{d}|\langle \mathbf{m}, x'_0\rangle|\right)^p \le \|h\|^p_{\mathsf{M}}.$$

Therefore $\|h\|_{\mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})} \le \|\mathsf{M}_h\|_{\mathsf{M}}$ and then $h \in \mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})$.

For the case $p = 1$ is enough to take $s = \chi_\Omega$. □

Let us show as an example some particular representation of spaces of multiplication operators. If the Banach function space, $X(\mu)$, over the measure μ is a p'-concave Banach lattice then the results of [8, Proposition 3.5] allow us to write

$$\mathsf{M}(\mathrm{L}^{p'}(\mu), X(\mu)) = (((X(\mu)^\times)_{[p]})^\times)_{[\frac{1}{p}]}$$

where, if $Y(\mu)$ is a Banach function space, $Y(\mu)^\times$ is the corresponding Köthe dual (see E' in [15, p. 27]), and $Y(\mu)_{[p]}$ is defined as the function space

$$Y(\mu)_{[p]} = \{f \in \mathrm{L}^0(\mu) : |f|^{1/p} \in Y(\mu)\},$$

with the quasi norm $\|\cdot\|_{Y(\mu)_{[p]}}$ defined by

$$\|f\|_{Y(\mu)_{[p]}} = \|\, |f|^{1/p}\, \|^p_{Y(\mu)}, \quad f \in Y(\mu)_{[p]}.$$

In general this abstract result do not give useful information about these spaces—only in the cases where the Köthe duals can be directly computed. However, our representation provides an internal characterization of the functions belonging to the space, in the sense that gives the properties that a measurable function must satisfy to belong to the space. Thus if $\mathrm{L}^1(\mathbf{m})$ is p'-concave we obtain

$$\mathsf{M}(\mathrm{L}^{p'}(\mu), \mathrm{L}^1(\mathbf{m})) = \mathrm{L}^1_{p,\mu}(\mathbf{m}) = (((\mathrm{L}^1(\mathbf{m}))^\times)_{[p]})^\times)_{[\frac{1}{p}]}.$$

Proposition 2 *Let* $1 \le p < \infty$. *If* $\chi_\Omega \in \mathrm{L}^1_{\infty,\mu}(\mathbf{m})$, *then* $\mathrm{L}^p(\mathbf{m}) \subseteq \mathrm{L}^1_{p,\mu}(\mathbf{m})$ *and*

$$|||f|||_{\mathrm{L}^1_{p,\mu}(\mathbf{m})} \le |||\chi_\Omega|||^{\frac{1}{p'}}_{\mathrm{L}^1_{\infty,\mu}(\mathbf{m})} \|f\|_{\mathrm{L}^p(\mathbf{m})}.$$

Proof For $p = 1$ the result is obvious. Suppose that $1 < p < \infty$. Take $f \in \mathrm{L}^p(\mathbf{m})$ and consider $\pi \in \mathcal{P}(\Omega)$, $x' \in B_{E'}$. Then

$$\begin{aligned}\sum_{A\in\pi} \frac{\left(\int_A |f|\, \mathrm{d}\,|\langle \mathbf{m}, x'\rangle|\right)^p}{\mu(A)^{p-1}} &\le \sum_{A\in\pi} \left(\int_A |f|^p \,\mathrm{d}\,|\langle \mathbf{m}, x'\rangle|\right)\left(\frac{|\langle \mathbf{m}, x'\rangle|(A)}{\mu(A)}\right)^{p-1} \\ &\le |||\chi_\Omega|||^{p-1}_{\mathrm{L}^1_{\infty,\mu}(\mathbf{m})} \sum_{A\in\pi} \int_A |f|^p \,\mathrm{d}\,|\langle \mathbf{m}, x'\rangle| \\ &\le |||\chi_\Omega|||^{p-1}_{\mathrm{L}^1_{\infty,\mu}(\mathbf{m})} \|f\|^p_{\mathrm{L}^p(\mathbf{m})}.\end{aligned}$$

□

Corollary 2 *Let* $1 < p < \infty$ *and assume that* $\chi_\Omega \in \mathrm{L}^1_{\infty,\mu}(\mathbf{m})$. *If* $f \in \mathrm{L}^p(\mathbf{m})$, *then there is a sequence* $(s_n)_n$ *of simple functions in* Ω *such that* $(s_n)_n$ *converges to* f *in* $\mathrm{L}^1_{p,\mu}(\mathbf{m})$.

Lemma 2 *Let* $1 \le q \le \infty$. *If* $\mathrm{L}^q(\mu) \subseteq \mathrm{L}^1(\mathbf{m})$, *then the inclusion* $i : \mathrm{L}^q(\mu) \to \mathrm{L}^1(\mathbf{m})$ *is continuous.*

Proof Just observe that a linear positive operator between two Banach lattices is always continuous (see [15], p. 2). □

Theorem 4 *Let* $1 \le p < \infty$. *The following properties are equivalent*:

(a) $\mathbf{m}$ *has finite p-semivariation.*
(b) $\chi_\Omega \in \mathrm{L}^1_{p,\mu}(\mathbf{m})$.
(c) $\mathrm{L}^{p'}(\mu) \subseteq \mathrm{L}^1(\mathbf{m})$.

Proof The equivalence of (a) with (b) is clear. If we suppose (b) is true then (c) follows immediately from Proposition 1. Suppose finally that the inclusion in (c) is verified and let us see that (b) is true. Using Lemma 2 the inclusion $i : \mathrm{L}^{p'}(\mu) \to \mathrm{L}^1(\mathbf{m})$ is continuous. Thus there is some $C > 0$ such that $\|s\|_{\mathrm{L}^1(\mathbf{m})} \le C\|s\|_{\mathrm{L}^{p'}(\mu)}$, for each $s \in \mathrm{L}^{p'}(\mu)$. Let us fix $x' \in B_{E'}, \pi \in \mathcal{P}(\Omega)$ and consider s as in (3) (in Theorem 3) with $f = \chi_\Omega$. Finally, observe that

$$\sum_{A\in\pi} \frac{(\int_A \mathrm{d}\,|\langle \mathbf{m}, x'\rangle|)^p}{\mu(A)^{p-1}} = \int_\Omega |s|\,\mathrm{d}\,|\langle \mathbf{m}, x'\rangle| \le C\,\|s\|_{\mathrm{L}^{p'}(\mu)} = C.$$

This implies $|||\chi_\Omega|||_{\mathrm{L}^1_{p,\mu}(\mathbf{m})} \le C.$ □

Theorem 5 *Let* $1 \le p < \infty$. *Then* $\mathrm{L}^1_{p,\mu}(\mathbf{m}) \subseteq \mathrm{L}^p(\mathbf{m})$ *and* $|||f|||_{\mathrm{L}^p(\mathbf{m})} \le |||f|||_{\mathrm{L}^1_{p,\mu}(\mathbf{m})}$.

Proof Let $f \in \mathrm{L}^1(\mathbf{m})$. We will prove that if $M \in \mathbb{R}$ and $\|f\|_{\mathrm{L}^p(\mathbf{m})} > M$, then $|||f|||_{\mathrm{L}^1_{p,\mu}(\mathbf{m})} > M$.

So, let $M \in \mathbb{R}$ and assume that $\|f\|_{\mathrm{L}^p(\mathbf{m})} > M$. Hence, there is $x' \in B_{E'}$ such that

$$\int_{\Omega} |f|^p \, \mathrm{d}\,|\langle \mathbf{m}, x' \rangle| > M.$$

Since $\mathrm{L}^p(|\langle \mathbf{m}, x' \rangle|)' = \mathrm{L}^{p'}(|\langle \mathbf{m}, x' \rangle|)$ isometrically, by the density of simple functions in $\mathrm{L}^{p'}(|\langle \mathbf{m}, x' \rangle|)$, there is $\pi \in \Omega$, $s = \sum_{A \in \pi} \alpha_A \chi_A$, $\alpha_A \in \mathbb{R}$, such that $s \in B_{\mathrm{L}^{p'}(|\langle \mathbf{m}, x' \rangle|)}$ and $|\int_{\Omega} |f|\, s \, \mathrm{d}\,|\langle \mathbf{m}, x' \rangle||> M$. That is

$$\left| \sum_{A \in \pi} \alpha_A \int_A |f| \, \mathrm{d}\,|\langle \mathbf{m}, x' \rangle| \right| > M.$$

We obtain

$$|||f|||_{\mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})} \geq \left(\sum_{A \in \pi} \alpha_A^{p'} \mu(A) \right)^{\frac{1}{p'}} \left(\sum_{A \in \pi} \frac{(\int_A |f| \, \mathrm{d}\,|\langle \mathbf{m}, x' \rangle|)^p}{\mu(A)^{\frac{p}{p'}}} \right)^{\frac{1}{p}} > M.$$

where we have used Hölder's inequality. □

Note that if $1 \leq p < \infty$ and $\chi_\Omega \in \mathrm{L}^1_{\infty,\mu}(\mathbf{m})$ then $\mathrm{L}^p(\mathbf{m}) = \mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})$. For this particular case, this identification gives more information about the decomposition given in Theorem 8, since the properties of the space $\mathrm{L}^p(\mathbf{m})$ are nowadays well-known (see [12]). A direct computation shows that the condition $\chi_\Omega \in \mathrm{L}^1_{\infty,\mu}(\mathbf{m})$ is equivalent to the existence of a constant $k > 0$ such that

$$\|\mathbf{m}(A)\| \leq k\, \mu(A) \quad \text{for every } A \in \Sigma.$$

It is well known that *the set of simple functions is dense in the space* $\mathrm{L}^1(\mathbf{m})$. However, this is not the case for the spaces $\mathrm{L}^1_{p,\mu}(\mathbf{m})$, as the following construction shows. The rest of this section is devoted to present an example in which simple functions are not dense in $\mathrm{L}^1_{p,\mu}(\mathbf{m})$. In this final part, we can not suppose that the vector measure $\mathbf{m}$ has finite p-semivariation. Recall that for a vector measure $\mathbf{m}$ the space $\mathrm{L}^1_w(\mathbf{m})$ of weakly $\mathbf{m}$-integrable (classes of) functions is defined by all the measurable functions that satisfy (1) in Definition 1. This is a Banach function space when the almost everywhere order and the norm $\|\cdot\|_{\mathrm{L}^1(\mathbf{m})}$ are considered. $\mathrm{L}^1(\mathbf{m})$ can be considered as a sublattice of $\mathrm{L}^1_w(\mathbf{m})$ but these spaces are not in general equal; $\mathrm{L}^1_w(\mathbf{m})$ can be no σ-order continuous. The reader can find more information about this space in [17] and for the corresponding $\mathrm{L}^p_w(\mathbf{m})$ spaces in [12].

The following example is due to S. Okada. Let us define the function $\varphi : \mathbb{N} \to \mathbb{R}$ given by $\varphi(n) = n$. Let $\Sigma = 2^{\mathbb{N}}$ and define a vector measure

$$\begin{aligned} \mathbf{m} : \Sigma &\to c_0 \\ A &\mapsto \mathbf{m}(A) = \chi_A / \varphi, \end{aligned}$$

where the notation is $\chi_A/\varphi = (\chi_A/n)_{n=1}^{\infty} \in c_0$.

Then we can obtain by means of a direct computation that

(a) $\mathrm{L}^1(\mathbf{m}) = \varphi \cdot c_0 = \{\varphi f : f \in c_0\}$, and
(b) $\mathrm{L}^1_w(\mathbf{m}) = \varphi \cdot \ell^\infty = \{\varphi f : f \in \ell^\infty\}$.

Here, the norm on the spaces $\varphi \cdot c_0$ and $\varphi \cdot \ell^\infty$ are defined by

$$\|f\| = \|f/\varphi\|_{\ell^\infty}, \quad f \in \varphi \cdot \ell^\infty.$$

We will use this notation through the following construction. Define a finite measure $\mu : \Sigma \to [0, \infty)$ by

$$\mu(A) = \sum_{n \in A} \frac{1}{\varphi(n)^3} = \sum_{n \in A} \frac{1}{n^3}, \quad A \in \Sigma.$$

Let us denote by ℓ^1_+ and $(c_0)_+$ the positive cone of ℓ^1 and c_0, respectively.

Proposition 3 *Let* $0 < r < \infty$. *Then*

(a) $\ell^1_+ = \{|f|^r/\varphi^3 : f \in \mathrm{L}^r(\mu)\}$.
(b) $\mathsf{M}(\mathrm{L}^r(\mu), \mathrm{L}^1(\mathbf{m})) = \varphi^{(r-3)/r} \cdot \ell^\infty$.

Proof First we *claim* that for every $g \in \mathbb{C}^\mathbb{N}$ then $g \cdot \ell^1 \subseteq \ell^\infty$ if and only if $g \in \ell^\infty$. Hence, $g \cdot \ell^1 \subseteq c_0$ if and only if $g \in \ell^\infty$. To see this, first note that if $g \in \ell^\infty$, then clearly $g \cdot \ell^1 \subseteq \ell^1 \subseteq c_0 \subseteq \ell^\infty$. For the converse, assume that $g \notin \ell^\infty$. Choose $k(1) < k(2) < k(3) < \cdots$ such that $|g(k(n))| \geq n^3$. Let

$$f = \sum_{n=1}^\infty \frac{1}{n^2} e_{k(n)} \in \ell^1,$$

where $e_{k(n)} = \chi_{\{k(n)\}}$. Then

$$|g(k(n)) f(k(n))| \geq n^3 \cdot \frac{1}{n^2} = n \to \infty.$$

Thus $gf \notin \ell^\infty$. So we must have $g \in \ell^\infty$.

Now let us prove the Proposition. For the proof of (a), note that a function $f \in \mathbb{R}^\mathbb{N}$ belongs to $\mathrm{L}^r(\mu)$ if and only if $|f|^r/\varphi^3 \in \ell^1_+$. Therefore, $\ell^1_+ \supseteq \{|f|^r/\varphi^3 : f \in \mathrm{L}^r(\mu)\}$. Conversely, let $h \in \ell^1_+$. Then let $f = h^{1/r} \cdot \varphi^{3/r}$. Thus, $|f|^r/\varphi^3 = h \cdot \varphi^3/\varphi^3 = h \in \ell^1$, and then $f \in \mathrm{L}^r(\mu)$.

For the proof of (b), let $g \in \mathbb{C}^\mathbb{N}$. Then $g \in \mathsf{M}(\mathrm{L}^r(\mu), \mathrm{L}^1(\mu))$. This means that $gf \in \varphi \cdot c_0$ for every $f \in \mathrm{L}^r(\mu)$; equivalently, for all $f \in \mathrm{L}^r(\mu)$ we have that $|g|^r \varphi^{3-r} |f|^r/\varphi^3 = |g|^r |f|^r/\varphi^r \in c_0$. Therefore, by (*a*) we obtain that $|g|^r \varphi^{3-r} \cdot \ell^1_+ \subseteq (c_0)_+$, and by the claim at the beginning of the proof we obtain that $|g|^r \varphi^{3-r} \in \ell^\infty$. A direct calculation shows that $g \in \varphi^{(r-3)/r} \cdot \ell^\infty$. □

Lemma 3 *Given* $\alpha \in \mathbb{R}$,

(a) $\varphi^\alpha \cdot \ell^\infty \not\supseteq \mathrm{sim}(\Sigma)$ *if* $\alpha < 0$.
(b) $\varphi^\alpha \cdot \ell^\infty \supsetneq \ell^\infty \supsetneq \mathrm{sim}(\Sigma)$ *if* $\alpha > 0$.
(c) *If* $\alpha > 0$, *then* $\mathrm{sim}(\Sigma)$ *is not dense in the space* $\varphi^\alpha \cdot \ell^\infty$.

Proof (a) The constant function $\chi_\mathbb{N}$ do not belong to $\varphi^\alpha \cdot \ell^\infty$ because $\varphi^{-\alpha}$ is not in ℓ^∞ for all $-\alpha > 0$.

For (b), note that for every $f \in \ell^\infty$, $f = \varphi^\alpha \cdot (f/\varphi^\alpha) \in \varphi^\alpha \cdot \ell^\infty$.

Let us prove now (c). We show that given $g \in \ell^\infty$,

$$\|\varphi^\alpha - g\|_{\varphi^\alpha \cdot \ell^\infty} \geq 1/2,$$

recalling that $\varphi^\alpha = \varphi^\alpha \cdot \chi_{\mathbb{N}} \in \varphi^\alpha \cdot \ell^\infty$. Find $n_0 \in \mathbb{N}$ such that $|g(n_0)/\varphi^\alpha(n_0)| < 1/2$. So

$$\|\varphi^\alpha - g\|_{\varphi^\alpha \cdot \ell^\infty} = \|\chi_{\mathbb{N}} - \frac{g}{\varphi^\alpha}\|_{\ell^\infty} \geq \left|1 - \frac{g(n_0)}{\varphi^\alpha(n_0)}\right| \geq \left|1 - |\frac{g(n_0)}{\varphi^\alpha(n_0)}|\right| > \frac{1}{2}.$$

□

Summing up the results above, we have the following proposition for the vector measure **m** that we are working with.

Proposition 4 *Given* $r \in \mathbb{R}$,

(a) *If* $0 < r < 3$, *then* $\mathrm{sim}(\Sigma) \nsubseteq \mathsf{M}(\mathrm{L}^r(\mu), \mathrm{L}^1(\mathbf{m}))$.
(b) *If* $r = 3$, *then* $\mathrm{sim}(\Sigma)$ *is dense in* $\mathsf{M}(\mathrm{L}^r(\mu), \mathrm{L}^1(\mathbf{m}))$.
(c) *If* $r > 3$, *then* $\mathrm{sim}(\Sigma)$ *is not dense in* $\mathsf{M}(\mathrm{L}^r(\mu), \mathrm{L}^1(\mathbf{m}))$.

4 Operators factorizing through $\mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})$

In this section, we define and characterize a class of operators that satisfy a certain vector norm inequality that is related to the one for which the space $\mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})$ is the optimal domain. The theory of optimal domains for continuous operators from Banach function spaces has been developed recently by Curbera and Ricker (see [4,5]). The optimal domain for an operator can be defined as *the bigger function space to which an operator satisfying a particular property can be extended preserving the same property*. In our case, this property is given in the following definition.

Definition 8 Let $X(\mu)$ be a μ-Köthe function space and E a Banach space and let $1 \leq p \leq \infty$. We say that an operator $T : X(\mu) \to E$ is L^p-product extensible if there is a constant $K > 0$ such that the inequality

$$\sup\{\|T(hf)\| : h \in \mathrm{sim}(\Sigma) \cap B_{\mathrm{L}^{p'}(\mu)}\} \leq K\|f\|_{X(\mu)}$$

holds for every function $f \in X(\mu)$.

An easy example of L^p-product extensible operator is given by the following construction. Let $1 < p < \infty$, μ a finite measure and take an operator $S : \mathrm{L}^1(\mu) \to \mathrm{L}^1(\mu)$. Then the composition $S \circ i$ -where $i : L^p(\mu) \to L^1(\mu)$ is the inclusion map- clearly provides an L^p-product extensible operator.

Recall that if $T : X(\mu) \to E$ is an operator from the σ-order continuous μ-Köthe function space $X(\mu)$ to the Banach space E, the expression $\mathbf{m}_T(A) = T(\chi_A)$, $A \in \Sigma$, defines a countably additive vector measure. It is said that T is μ-determined if the measure $\mathbf{m}_T$ controls μ, i.e. if $\mu(A) = 0$ whenever the semivariation of $\mathbf{m}_T$ on this set equals 0, for every $A \in \Sigma$ (see [4,5]). The following theorem gives the characterization of the L_p-product extensible operators and the optimality of $\mathrm{L}^1_{\mathrm{p},\mu}(\mathbf{m})$ with respect to this class of operators.

Theorem 6 *Let* $X(\mu)$ *be a* σ*-order continuous* μ*-Köthe function space with weak unit* χ_Ω, *E a Banach space and* $T : X(\mu) \to E$ *a* μ*-determined operator. The following statements are equivalent.*

(a) *T is* L^p*-product extensible operator.*

(b) *There is a constant* $K > 0$ *such that*

$$\|(\mathbf{m}_T)_g\|_{p,\mu}(\Omega) \le K\|g\|_{X(\mu)} \quad \textit{for every function } g \in X(\mu).$$

(c) T *satisfies the following factorization diagram*

$$\begin{array}{ccc} X(\mu) & \xrightarrow{\quad T \quad} & E \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle \mathrm{I}_{\mathbf{m}_T}} & \\ & \mathrm{L}^1_{p,\mu}(\mathbf{m}_T) & \end{array}$$

where i *is the inclusion map and* $\mathrm{I}_{\mathbf{m}_T}$ *is the corresponding integration map.*

Moreover, if $Y(\mu)$ *is a* σ*-order continuous* μ*-Köthe function space with weak unit* χ_Ω *such that* $X(\mu) \subseteq Y(\mu)$*, and* T *can be extended to an* L^p*-product extensible operator* $S : Y(\mu) \to E$*, then* $Y(\mu) \subseteq \mathrm{L}^1_{p,\mu}(\mathbf{m}_T)$.

Proof The following argument shows that (a) implies (b); note first that the vector measure $\mathbf{m}_T$ is well-defined since $X(\mu)$ is σ-order continuous. Thus $\mathrm{L}^1(\mathbf{m}_T)$ is the optimal domain of the operator T (see [4]) and $X(\mu) \subseteq \mathrm{L}^1(\mathbf{m}_T)$. Take a simple function $s = \sum_{k=1}^n \alpha_k \chi_{A_k} \in B_{\mathrm{L}^{p'}(\mu)}$ and a function $g \in X(\mu)$. Then $gs \in X(\mu) \subseteq \mathrm{L}^1(\mathbf{m}_T)$ and for every $A \in \Sigma$,

$$T(gs\chi_A) = \int_A gs \, \mathrm{d}\mathbf{m}_T.$$

Note also that

$$\sup_{s \in B_{\mathrm{L}^{p'}(\mu)}} \sup_{A \in \Sigma} \left\| \int_A gs \mathrm{d}\mathbf{m}_T \right\| \le \sup_{s \in B_{\mathrm{L}^{p'}(\mu)}} \|gs\|_{\mathrm{L}^1(\mathbf{m}_T)} \le 2 \sup_{s \in B_{\mathrm{L}^{p'}(\mu)}} \sup_{A \in \Sigma} \left\| \int_A gs \mathrm{d}\mathbf{m}_T \right\|,$$

where only simple functions are considered as functions s, as a consequence of the equivalent norm for the space $\mathrm{L}^1(\mathbf{m})$ given after Definition 1. Therefore, Theorem 3 gives the equivalence between (a) and (b). (Observe that the proof of Theorem 3 actually gives the equivalence between the norm of $g \in \mathrm{L}^1_{\mathsf{p},\mu}(\mathbf{m})$ and the norm of the multiplication operator M_g computed just when acting on the subspace of simple functions of $\mathrm{L}^{p'}(\mu)$.)

To see the equivalence between (b) and (c) it is enough to use Remark 3 and Remark 4; if $g \in \mathrm{L}^1_{p,\mu}(\mathbf{m}_T)$,

$$\left\| \int_\Omega g \mathrm{d}\mathbf{m}_T \right\| \le \|g\|_{\mathrm{L}^1(\mathbf{m}_T)} \le \|g\|_{\mathrm{L}^1_{p,\mu}(\mathbf{m}_T)}.$$

Therefore, the inequality in (b) holds if and only if the factorization holds.

To prove the optimal domain property given by the last statement of the Theorem, suppose that $Y(\mu)$ is an σ-order continuous Köthe function space with weak unit χ_Ω such that $X(\mu) \subseteq Y(\mu)$. Assume also that the operator T can be extended to an L^p-product extensible operator $S : Y(\mu) \to E$. Since the vector measure $\mathbf{m}_T$ is the same than the one defined by S, $\mathbf{m}_S$, the same argument that has been used for proving the equivalence between (a) and (b) gives the continuous inclusion $Y(\mu) \subseteq \mathrm{L}^1_{p,\mu}(\mathbf{m}_T)$.

□

In what follows we use the factorization Theorem given above for obtaining factorizations for L^p-product extensible operators through $\mathrm{L}^p(\mu)$ by means of the Maurey–Rosenthal cycle of ideas that has been explained in the first section.

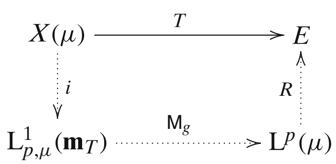
Lemma 4 *The Banach space* $(\mathrm{L}^1_{p,\mu}(\mathbf{m}), ||| \cdot |||_{\mathrm{L}^1_{p,\mu}(\mathbf{m})})$ *is p-convex with constant* $M^{(p)}(\mathrm{L}^1_{p,\mu}(\mathbf{m})) = 1$.

Proof The following inequalities gives the proof of this technical result; we use essentially Theorem 3. For $p = 1$ the result is trivial, so suppose that $1 < p < \infty$. Note that for every $x' \in E'$, if we denote by $\mathbf{m}_{x'} = \langle \mathbf{m}, x' \rangle$, then the measure $|\mathbf{m}_{x'}|$ is absolutely continuous with respect to μ. Let us write $\mathrm{d}|\mathbf{m}_{x'}|/\mathrm{d}\mu$ for the corresponding Radon–Nikodým derivative. Take a finite set of functions $f_1, \ldots, f_n \in \mathrm{L}^1_{p,\mu}(\mathbf{m})$. Then

$$
\begin{aligned}
\left|\left|\left| \left(\sum_{i=1}^n |f_i|^p \right)^{\frac{1}{p}} \right|\right|\right|_{\mathrm{L}^1_{p,\mu}(\mathbf{m})} &= \sup_{h \in B_{L^{p'}(\mu)}} \sup_{x' \in E'} \left(\int_\Omega \left(\sum_{i=1}^n |f_i|^p \right)^{\frac{1}{p}} |h| \mathrm{d}|\mathbf{m}_{x'}| \right) \\
&= \sup_{x' \in E'} \sup_{h \in B_{L^{p'}(\mu)}} \left(\int_\Omega \left| \sum_{i=1}^n |f_i|^p \right|^{\frac{1}{p}} |h| \frac{\mathrm{d}|\mathbf{m}_{x'}|}{\mathrm{d}\mu} \mathrm{d}\mu \right) \\
&\leq \sup_{x' \in E'} \left(\int_\Omega \left(\left| \sum_{i=1}^n |f_i|^p \right|^{\frac{1}{p}} \right)^p \left(\frac{\mathrm{d}|\mathbf{m}_{x'}|}{\mathrm{d}\mu} \right)^p \mathrm{d}\mu \right)^{\frac{1}{p}} \\
&= \sup_{x' \in E'} \left(\sum_{i=1}^n \int_\Omega |f_i|^p \left(\frac{\mathrm{d}|\mathbf{m}_{x'}|}{\mathrm{d}\mu} \right)^p \mathrm{d}\mu \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^n \sup_{x' \in E'} \int_\Omega |f_i|^p \left(\frac{\mathrm{d}|\mathbf{m}_{x'}|}{\mathrm{d}\mu} \right)^p \mathrm{d}\mu \right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^n \sup_{x' \in E'} \sup_{h \in B_{L^{p'}(\mu)}} \left(\int_\Omega |f_i| \frac{\mathrm{d}|\mathbf{m}_{x'}|}{d\mu} |h| \mathrm{d}\mu \right)^p \right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^n \|f_i\|^p_{\mathrm{L}^1_{p,\mu}(\mathbf{m})} \right)^{\frac{1}{p}}.
\end{aligned}
$$

This proves the result. □

Corollary 3 *Let* $T : X(\mu) \to E$ *be an* L^p*-product extensible* μ*-determined operator such that* $\mathrm{L}^1_{p,\mu}(\mathbf{m}_T)$ *is* σ*-order continuous. Suppose that the integration map* $\mathrm{I}_{\mathbf{m}_T}$ *is p-concave. Then T factorizes through* $\mathrm{L}^p(\mu)$ *as*



where $g \in \mathsf{M}(\mathrm{L}^1_{p,\mu}(\mathbf{m}_T), \mathrm{L}^p(\mu))$, *R is a continuous operator and i is the inclusion map.*

Proof An application of one of the Maurey–Rosenthal type theorems explained in Sect. 1 gives the result. We apply first Theorem 6 to factorize T through $\mathrm{L}^1_{p,\mu}(\mathbf{m}_T)$. Since this space is always p-convex, we obtain the complete factorization scheme. □

Corollary 4 *Let* $T : X(\mu) \to E$ *be an* L^p*-product extensible positive operator such that* $\mathrm{L}^1_{p,\mu}(\mathbf{m}_T)$ *is* σ*-order continuous. Suppose that T is* μ*-determined and E is p-concave. Then T factorizes through* $\mathrm{L}^p(\mu)$ *as in Corollary 3.*

Proof Since the integration map is positive and E is p-concave, we obtain that $\mathrm{I}_{\mathbf{m}_T}$ is p-concave just by applying [15, Proposition 1.d.9.]. Corollary 3 gives the result. □

5 A generalization of the Bennet–Maurey–Nahoum Theorem

In this section, we show that a classical result on decomposition of functions that define an unconditionally summable sequence on $\mathrm{L}^1([0,1],\mu)$, where μ is Lebesgue measure, can be extended to any unconditionally sequence of functions in 2-concave $\mathrm{L}^1(\mathbf{m})$-spaces. The key of our Theorem is the characterization of the multiplication operators from $\mathrm{L}^p(\mu)$ in $\mathrm{L}^1(\mathbf{m})$ that we have obtained in the previous section. The framework where these results are used is the so called Maurey–Rosenthal Factorization Theory for linear operators between Banach lattices. For the proof we use the Nagy's Dilation Theorem whose proof can be found in [9, p. 253].

Theorem 7 (Nagy's Dilation Theorem) *Let* $u \in \mathrm{L}(H_1, H_2)$ *be a Hilbert space operator with* $u \in B_{\mathrm{L}(H_1,H_2)}$. *There is a Hilbert space G such that u admits a factorization*

$$\begin{array}{ccc} H_1 & \xrightarrow{\quad u \quad} & H_2 \\ & {}_{v}\searrow \quad \nearrow_{\pi} & \\ & G \oplus_2 H_2 & \end{array}$$

where v is an isometric embedding and π *is a orthogonal projection of* $G \oplus_2 H_2$ *onto* H_2.

Theorem 8 *Suppose that* μ *is a Rybakov measure for a vector measure* $\mathbf{m}$ *and that* $\mathrm{L}^1(\mathbf{m})$ *is a* 2*-concave space. Let* $(f_n)_n$ *be an unconditionally summable sequence in* $\mathrm{L}^1(\mathbf{m})$. *Then we can find* $(a_n)_n \in \ell_2$, $g \in \mathrm{L}^1_{2,\mu}(\mathbf{m})$, *a Hilbert space G and an orthonormal sequence* $(x_n)_n \in G \oplus_2 \mathrm{L}^2(\mu)$ *such that for every* $w \in \Omega$ *and all* $n \in \mathbb{N}$

$$f_n(w) = g(w) a_n g_n(w)$$

where g_n *is the projection of* x_n *onto* $\mathrm{L}^2(\mu)$.

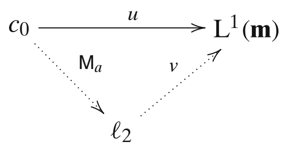
Proof We assume that all the functions in the sequence are not zero.
Since $(f_n)_n$ is an unconditionally summable sequence, then we can define a compact operator by

$$\begin{aligned} u : c_0 &\longrightarrow \mathrm{L}^1(\mathbf{m}) \\ (t_n)_n &\mapsto \sum_{n=1}^{\infty} t_n f_n. \end{aligned}$$

Recall that c_0 is σ-order continuous and 2-convex and that $\mathrm{L}^1(\mathbf{m})$ is 2-concave. From [9, Theorem 1.f.14] we obtain that for every $h_1, h_2, \ldots, h_n \in c_0$

$$\frac{1}{M_{(2)}(\mathrm{L}^1(\mathbf{m}))}\left(\sum_{i=1}^n \|u(h_i)\|^2\right)^{\frac{1}{2}} \leq \left\|\left(\sum_{i=1}^n |u(h_i)|^2\right)^{\frac{1}{2}}\right\|_{\mathrm{L}^1(\mathbf{m})} \leq k_G\|u\| \left\|\left(\sum_{i=1}^n |h_i|^2\right)^{\frac{1}{2}}\right\|_{c_0}.$$

This means that u is a 2-concave operator. Taking into account [6, Corollary 5] we can find a sequence $a = (a_n)_n \in B_{\ell_2}$ such that u allows the factorization

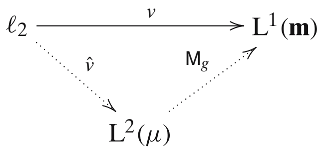


where v is a continuous linear operator for which this diagram commutes. (Observe that $u = v \circ \mathsf{M}_a$ gives $v = u \circ \mathsf{M}_{\frac{1}{a}}$).

In fact the Banach lattice ℓ_2 is σ-order continuous and 2-convex so using again [9, Theorem 1.f.14] but now with the operator $v : \ell_2 \to \mathrm{L}^1(\mathbf{m})$ we have that for every $h_1, h_2, \ldots, h_n \in \ell_2$

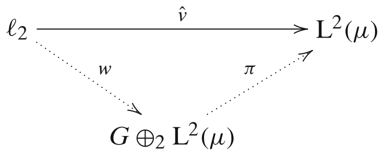
$$\left\|\left(\sum_{i=1}^n |v(h_i)|^2\right)^{\frac{1}{2}}\right\|_{\mathrm{L}^1(\mathbf{m})} \leq k_G\|v\| \left\|\left(\sum_{i=1}^n |h_i|^2\right)^{\frac{1}{2}}\right\|_{\ell_2} \leq k_G\|v\| \left(\sum_{i=1}^n \|h_i\|^2\right)^{\frac{1}{2}},$$

since $M^{(2)}(\ell_2) = 1$. Then we can find a function $g \in \mathsf{M}(\mathrm{L}^2(\mu), \mathrm{L}^1(\mathbf{m})) = \mathrm{L}^1_{2,\mu}(\mathbf{m})$ such that v can be factorized as



where now the commutativity of the diagram above provides that $\hat{v} = \mathsf{M}_{g^{-1}} \circ v$. Observe that we can suppose that $\hat{v}$ has norm less or equal to one (changing the function g if it is necessary).

Take now the operator $\hat{v} : \ell_2 \to \mathrm{L}^2(\mu)$. Since $\|\hat{v}\| \leq 1$ then applying the *Dilation Theorem* there is a Hilbert space G such that $\hat{v}$ admits the factorization



where w is an isometric embedding and π is a orthogonal projection of $G \oplus_2 \mathrm{L}^2(\mu)$ onto $\mathrm{L}^2(\mu)$.

Since w is an isometry then the sequence $(x_n)_n$ given by $x_n = w(e_n)$, $n \geq 1$, is an orthonormal sequence in $G \oplus_2 \mathrm{L}^2(\mathbf{m})$. For each $n \in \mathbb{N}$

$$\begin{aligned} f_n = T(e_n) &= \mathsf{M}_g(\pi(w(\mathsf{M}_a(e_n)))) = \mathsf{M}_g(\pi(w(a_n e_n))) \\ &= \mathsf{M}_g(a_n \pi(x_n)) = \mathsf{M}_g(a_n g_n) = g a_n g_n, \quad \text{in } \mathrm{L}^1(\mathbf{m}). \end{aligned}$$

Then there exists a μ-null set $A_n \in \Sigma$ such that $f_n = g a_n g_n$ in $\Omega \setminus A_n$. Let us consider the function

$$\hat{g}_n = g\chi_{A_n^c} + \frac{f_n}{a_n g}\chi_{A_n}.$$

Finally just observe that for every $n \in \mathbb{N}$ we have that $\hat{g}_n = g_n$ in $L^1(\mathbf{m})$ and clearly $f_n(t) = g(t)a_n g_n(t)$ for each $t \in [0, 1]$. □

Remark 6 Note that the characterization of unconditionally summable sequence given in [9, Theorem 1.9] provides a compact operator from c_0 onto $L^1(\mathbf{m})$ although we only use the continuity of u in the proof of Theorem 8 so the reader can think that the same argument should provide a more general result regarding sequences of elements of $L^1(\mathbf{m})$ satisfying that the corresponding operator is continuous. However a simple argument shows that every operator from c_0 onto $L^1(\mathbf{m})$ is compact whenever $L^1(\mathbf{m})$ is 2-concave. The reason is that we can always give for such operator a factorization through an operator $R : c_0 \to \ell_2$. These operators are always compact as a consequence of Pitt's Theorem (see [14, Proposition 2.c.3.]).

Remark 7 Since every $L^1(\mathbf{m})$ can be included in the space $L^1(\mu)$, where μ is a Rybakov measure for $\mathbf{m}$, every unconditionally convergent series in $L^1(\mathbf{m})$ can be considered as an unconditionally convergent series in $L^1(\mu)$ (see [16]). Thus, a function $g \in L^2(\mu)$ can be obtained for the decomposition given in Theorem 8. Our result provides a more specialized information about the function g.

In the particular case that the vector measure is defined on the measurable space $([0, 1], \mathcal{B}([0, 1]))$ and $L^1(\mathbf{m})$ is a Köthe function space for Lebesgue measure μ then the Hilbert space G that appears in the factorization is $L^2(\mu)$. In this case the space $G \oplus_2 L^2(\mu)$ that gives can be chosen to be $L^2([0, 2], \mu)$ (see [9, Theorem 12.31]). This enables us to present the following

Corollary 5 *Let* $\mathbf{m}$ *be a vector measure on* $([0, 1], \mathcal{B}([0, 1]))$ *such that* $L^1(\mathbf{m})$ *is a Köthe function space for Lebesgue measure* μ *and* $L^1(\mathbf{m})$ *is 2-concave. Let* $(f_n)_n$ *be an unconditionally summable sequence in* $L^1(\mathbf{m})$. *Then we can find* $(a_n)_n \in \ell_2$, $g \in L^1_{2,\mu}(\mathbf{m})$ *and an orthonormal sequence* $(g_n)_n \in L^2[0, 2]$ *such that for every* $t \in [0, 1]$ *and all* $n \in \mathbb{N}$

$$f_n(t) = g(t)a_n g_n(t).$$

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