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A generalization of uniform smoothness

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Abstract

We give a generalization of uniform smoothness, study its properties and give some examples.

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1. The EIS-property

We recall that an equivalent definition for uniform smoothness in a Banach space is given as follows:

Definition 1. A Banach space X is uniformly smooth if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $f, g \in S_{X^*}$ with $\|f - g\| \geq \varepsilon$, then $S(f, g, \delta) = \emptyset$.

It is well known that a uniformly smooth Banach space is superreflexive and has normal structure. Here we will give a generalization of this concept, which we will call the empty slice property, and study some of its consequences.

For this we need to give some prior definitions.

Definition 2. Let X be a Banach space.

We define

$$s_k(X) = \sup\{r : \exists x_1, x_2, \ldots, x_{k+1} \in S_X \text{ with } \|x_i - x_j\| \geq r \text{ for } i \neq j\}.$$

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Observe that if $k = 1$, then $s_k(X) = 2$ and if $\dim X = \infty$, then $s_k(X) \geq 1$ for every $k \in \mathbb{N}$.

$$s(X) = \sup\{r : \exists \{x_n\}_n \subset S_X \text{ with } \|x_i - x_j\| \geq r \text{ for } i \neq j\}.$$

Definition 3. Let X be a Banach space. Given $t \in \mathbb{N}$ and $0 < \varepsilon < s_t(X^*)$, we say that X has the "$\varepsilon, t-$ empty intersection of slices" property ($\varepsilon, t - EIS$) if there exists $0 < \delta$ so that if $g_1, \ldots, g_{t+1} \in S_{X^*}$ are such that $\|g_i - g_j\| \geq \varepsilon$ for every $i \neq j, i, j = 1, \ldots, t+1$, then $S(g_1, \ldots, g_{t+1}, \delta) = \phi$. We will say that X has *EIS*, if X has $\varepsilon, t - EIS$ for some ε and t.

Thus X is uniformly smooth if and only if it is $\varepsilon, 1 - EIS$ for every $\varepsilon \leq 2$.

Definition 4. A Banach space X with $\dim X = \infty$ has the "dual $\varepsilon, t-$ empty intersection of slices" property ($\varepsilon, t -^* EIS$) if there exist $t \in \mathbb{N}, 0 < \varepsilon < s_t(X)$, and $0 < \delta$ so that if $x_1, \ldots, x_{t+1} \in S_X$ are such that $\|x_i - x_j\| \geq \varepsilon$ for every $i \neq j, i, j = 1, \ldots, t+1$, then

$$\{f \in B_{X^*} : f(x_i) > 1 - \delta, i = 1, \ldots, t+1\} = \emptyset.$$

We will say that X has $^*EIS$, if X has $\varepsilon, t -^* EIS$ for some ε and t.

In [1] the authors proved that an (r, k, l) somewhat uniformly noncreasy Banach space is superreflexive. It is easy to see that if X has property *EIS* and $s_t(X^*) > \varepsilon > 0$ where ε, δ and t are as in the definition, then X is $(s(X^*), 1, t) - SUNC$. Thus we have the following corollary.

Corollary 1. *If X has property EIS then it is superreflexive.*

Lemma 1. *If X has $^*EIS$, then it is reflexive.*

Proof. Let $s_t(X) > \varepsilon > 0$ and $t \in \mathbb{N}, \delta > 0$ such that if $x_1, \ldots, x_{t+1} \in S_X$ and $\|x_i - x_j\| \geq \varepsilon$ for every $i \neq j, i, j = 1, \ldots, t+1$, then

$$\{f \in B_{X^*} : f(x_i) > 1 - \delta, i = 1, \ldots, t+1\} = \emptyset.$$

Further suppose that $\delta < 1 - \varepsilon$.

If X is not reflexive, by James' theorem there exists $\{x_n\}_n \subset S_X$ and $\{f_n\}_n \subset S_{X^*}$ such that $f_n(x_k) = 1 - \delta$ if $n \leq k$ and $f_n(x_k) = 0$ if $n > k$. Hence, since

$$f_1(x_1) = \cdots = f_1(x_{t+1}) = 1 - \delta,$$

there are $i, j \in \{1, \ldots, t+1\}$ with $i \neq j$ and $\|x_i - x_j\| < \varepsilon$. Suppose that $i < j$. Then $f_j(x_j) = 1 - \delta$ and $f_j(x_i) = 0$. Consequently

$$1 - \delta = f_j(x_j) = f_j(x_j - x_i) \leq \|x_i - x_j\| < \varepsilon$$

which is a contradiction. $\square$

Corollary 2. *If X has $\varepsilon, t - EIS$, then X^* has $\varepsilon, t -^* EIS$. Also, if X has $\varepsilon, t -^* EIS$, then X^* has $\varepsilon, t - EIS$.*

Corollary 3. *If X has $^*EIS$ then it is superreflexive.*

Similarly to García Falset et al. [3] the following can be shown:

Lemma 2. *If X has either EIS or $^*EIS$ and Y is finitely representable in X, then Y has EIS, respectively $^*EIS$.*

In their other work [2], García Falset et al. defined the following function

$$\widetilde{\delta}_X^k : [0, s_k(X)) \to [0,1]$$

by

$$\widetilde{\delta}_X^k(\varepsilon) = \inf\left\{1 - \left\|\frac{x_1 + \cdots + x_{k+1}}{k+1}\right\| : x_i \in B_X, i = 1, \ldots, k+1, \min_{i\neq j} \|x_i - x_j\| \geq \varepsilon\right\}.$$

They proved the following theorems:

Theorem 1. *If X is a Banach space such that there exist $k \in \mathbb{N}$ and $0 < \varepsilon < \min(s_k(X), 1)$ such that $\widetilde{\delta}_X^k(\varepsilon) > 0$, then X has normal structure.*

Theorem 2. *If X is a Banach space with strongly bimonotone basis and there exist $k \in \mathbb{N}$ and $0 < \varepsilon < \min(s_k(X), 2)$ such that $\widetilde{\delta}_X^k(\varepsilon) > 0$, then X has the weak fixed point property.*

We will see how the above is related to the *EIS* property, but first we introduce a lemma:

Lemma 3. *Let X be a Banach space and $g, h \in B_X$. Then*

$$\left\|\frac{g}{\|g\|} - \frac{h}{\|h\|}\right\| \geq \|g - h\| - |\,\|g\| - \|h\|\,|.$$

Proof. Since $\|g\| \leq 1$, we have that

$$\left\|\frac{g}{\|g\|} - \frac{h}{\|h\|}\right\| \geq \|g\| \left\|\frac{g}{\|g\|} - \frac{h}{\|h\|}\right\| = \left\|g - \|g\|\frac{h}{\|h\|}\right\|$$
$$\geq \|g - h\| - \|h\| \left|1 - \frac{\|g\|}{\|h\|}\right| = \|g - h\| - |\|h\| - \|g\||. \quad \square$$

Proposition 1. *Let X be a Banach space. Then*

(1) *If X has $(\varepsilon - \gamma), k - EIS$ for some $0 < \gamma < \varepsilon$, then $\widetilde{\delta}_{X^*}^k(\varepsilon) > 0$.*
(2) *If $\widetilde{\delta}_{X^*}^k(\varepsilon) > 0$, then X has $\varepsilon, k - EIS$.*

Proof. We will only prove 1. The proof of 2 is similar but easier. Suppose $\widetilde{\delta}_{X^*}^k(\varepsilon) = 0$ and $0 < \gamma < \varepsilon$. Then for every $\delta > 0$ there exist $f_i \in B_{X^*}, i = 1, \ldots, k+1, \min_{i\neq j} \|f_i - f_j\| \geq \varepsilon$ such that $1 - \left\|\frac{f_1+\cdots+f_{k+1}}{k+1}\right\| < \min\left(\frac{\gamma}{k+1}, \frac{\delta}{k+1}\right)$. Since X is reflexive, there exists $x \in S_X$ such that

$$\frac{f_1 + \cdots + f_{k+1}}{k+1}(x) = \left\|\frac{f_1 + \cdots + f_{k+1}}{k+1}\right\| > \max\left(1 - \frac{\gamma}{k+1}, 1 - \frac{\delta}{k+1}\right).$$

Hence $f_i(x) > \max(1 - \delta, 1 - \gamma)$ and $1 - \gamma < \|f_i\| \leq 1$ for $i = 1, \ldots, k+1$. By Lemma 3, if $i \neq j$

$$\left\|\frac{f_i}{\|f_i\|} - \frac{f_j}{\|f_j\|}\right\| \geq \varepsilon - \gamma.$$

But $\frac{f_i}{\|f_i\|}(x) > f_i(x) > 1 - \delta$ and thus $x \in S\left(\frac{f_1}{\|f_1\|}, \ldots, \frac{f_{k+1}}{\|f_{k+1}\|}, \delta\right)$ and X does not have $\varepsilon - \gamma, kEIS$ for any $\gamma > 0$. $\square$

Corollary 4. *Let X be a Banach space. Then*

(1) *If X has $\varepsilon, k-EIS$ for $\varepsilon < \min(s_k(X), 1)$, then X has normal structure.*
(2) *If X has a strongly bimonotone basis, and X has $\varepsilon, k-EIS$ for $\varepsilon < \min(s_k(X), 2)$, then X has the weak fpp.*

2. Permanence results

First we will see in what case we can assure that a space isomorphic to a space with the *EIS* property inherits this property.

Theorem 3. *Let X be an $\varepsilon, t-EIS$ space with $\varepsilon < 2$. Let $\delta > 0$ be as in the definition of $\varepsilon, t-EIS$. If $|\|\cdot\||$ is a norm in X such that for $x \in X$*

$$\|x\| \le |\|x\|| \le (1+\rho)\|x\|$$

where $\rho < \frac{\delta}{1+\delta}$, $s_t(X^*) > \varepsilon(1+\rho)$, and $n \in \mathbb{N}$ is such that

$$s_t(X^*) - \varepsilon(1+\rho) > \rho\frac{\rho+1}{n},$$

then $Y = (X, |\|\cdot\||)$ is $(1+\rho)(\varepsilon + \frac{\rho}{n}), n(t+1) - EIS$.

Proof. Let $\delta' = (1+\rho)\delta - \rho$, $\varepsilon' = (1+\rho)(\varepsilon + \frac{\rho}{n})$ and let $f_1, \dots, f_{n(t+1)+1} \in S_{Y^*}$ be such that $|\|f_i - f_j\|| \ge \varepsilon'$ for $i \ne j$, $i, j \in \{1, \dots, t+1\}$. Then, since for $f \in Y^*$

$$|\|f\|| \le \|f\| \le (1+\rho)|\|f\||,$$

and since $|\|f_i\|| = 1$, we have that there exists $l \in \{0, 1, \dots, n-1\}$ and A with $\#A \ge t+1$ such that for $i \in A$,

$$1 \le 1 + l\frac{\rho}{n} \le \|f_i\| \le 1 + (l+1)\frac{\rho}{n} \le 1+\rho.$$

Then, if for $i \in A$ we write $g_i = \frac{f_i}{1+(l+1)\frac{\rho}{n}}$, we get $\|g_i\| \le 1$ and

$$\begin{aligned}\left\|\frac{f_i}{\|f_i\|} - \frac{f_j}{\|f_j\|}\right\| &= \left\|\frac{g_i}{\|g_i\|} - \frac{g_j}{\|g_j\|}\right\| \\ &\ge \|g_i - g_j\| - |\|g_i\| - \|g_j\|| \\ &\ge |\|g_i - g_j\|| - \frac{\frac{\rho}{n}}{1+(l+1)\frac{\rho}{n}} \\ &\ge \frac{\varepsilon'}{1+\rho} - \frac{\rho}{n} = \varepsilon.\end{aligned}$$

Suppose that $x \in S(f_i, \delta')$ in Y. Then $\|x\| \le |\|x\|| \le 1$ and

$$\frac{f_i}{\|f_i\|}(x) \ge \frac{1-\delta'}{1+\rho} = 1-\delta.$$

Hence, if $x \in \bigcap_{i=1}^{n(t+1)+1} S(f_i, \delta')$ in Y, then $x \in \bigcap_{i\in A} S\left(\frac{f_i}{\|f_i\|}, \delta\right)$ in X and this proves our assertion. $\square$

We will see what the last theorem means for Hilbert spaces. First we need the following result.

Lemma 4. *A Hilbert space H is $\varepsilon, k - EIS$ for any ε in $(0, \sqrt{2})$ and $k \in \mathbb{N}$, with*

$$\delta < 1 - \sqrt{1 - \frac{k\varepsilon^2}{2(k+1)}}.$$

Proof. The proof of this fact follows from the next equality: Let $f_1, \dots, f_{k+1} \in H^* = H$, then

$$\|f_1 + \cdots + f_{k+1}\|^2 + \frac{1}{2}\sum_{j=1}^{k+1}\sum_{i=1}^{k+1}\|f_i - f_j\|^2 = (k+1)\sum_{i=1}^{k+1}\|f_i\|^2. \tag{2.1}$$

So suppose that $\|f_i\| = 1$ for $i = 1, \dots, k+1$, that $0 < \varepsilon < \sqrt{2}$ and $\|f_i - f_j\| \geq \varepsilon$ and that $x \in S(f_1, \dots, f_{k+1}, \delta)$ for some $\delta > 0$. Then $f_i(x) \geq 1 - \delta$ for $i = 1, \dots, k+1$ and thus

$$\|f_1 + \cdots + f_{k+1}\| \geq (f_1 + \cdots + f_{k+1})(x) \geq (k+1)(1-\delta).$$

By (2.1) this implies that

$$(k+1)^2 \geq (k+1)^2(1-\delta)^2 + \frac{\varepsilon^2}{2}k(k+1)$$

and hence

$$(k+1)\delta^2 - 2\delta(k+1) + \frac{\varepsilon^2}{2}k \leq 0.$$

Thus, if $0 < \delta < 1 - \sqrt{1 - \frac{k\varepsilon^2}{2(k+1)}}$, we have that $S(f_1, \dots, f_{k+1}, \delta) = \emptyset$. □

A consequence of Theorem 3 is the following.

Corollary 5. *Let H be a Hilbert space and $X = (H, \|\cdot\|)$ with $\|\cdot\|_2 \leq \|\cdot\| \leq \phi\|\cdot\|_2$ where $\phi < \sqrt{2}$ and such that $s(X^*) \geq \sqrt{2}$. Then X has EIS for some $\varepsilon < \sqrt{2}$.*

Proof. Let $\varepsilon = 1$ and $0 < \delta < 1 - \sqrt{1 - \frac{k\varepsilon^2}{2(k+1)}} = 1 - \sqrt{1 - \frac{k}{2(k+1)}}$. Then

$$\frac{\delta}{1-\delta} < \frac{1 - \sqrt{1 - \frac{k}{2(k+1)}}}{\sqrt{1 - \frac{k}{2(k+1)}}}.$$

Let $k \in \mathbb{N}$ be big enough so that $\phi < \frac{1}{\sqrt{1-\frac{k}{2(k+1)}}}$. Then $\phi\varepsilon < \frac{1}{\sqrt{1-\frac{k}{2(k+1)}}} < \sqrt{2}$. Hence, if $n \in \mathbb{N}$ satisfies $\varepsilon' = \phi(\varepsilon + \frac{\rho}{n}) < \sqrt{2}$, by Theorem 3, X is $\varepsilon', k - EIS$.

On the other hand, we will see later (Lemma 8) that $X = H \bigoplus_1 H$ does not have $\varepsilon, k - EIS$ for any $\varepsilon < \sqrt{2}$ and $k \in \mathbb{N}$. □

Corollary 6. *Let $X_\beta = (l_2, |\|\cdot\||)$ where $|\|x\|| = \max(\|x\|_2, \beta\|x\|_\infty)$. If $1 < \beta < \sqrt{2}$, then X_β has EIS and by Corollary 4 since it has a strongly bimonotone basis, it has the wfpp.*

Proof. Let $f = \{b_n\}_n$ in X_β^* and let $x = \{a_n\}_n \in X_\beta$ with $|\|x\|| = 1$. Then

$$\left|\sum_{i=1}^{\infty} a_i b_i\right| \leq \left(\sum_{i=1}^{\infty} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} b_i^2\right)^{\frac{1}{2}} \leq |\|x\|| \left(\sum_{i=1}^{\infty} b_i^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{\infty} b_i^2\right)^{\frac{1}{2}}.$$

On the other hand, if $\frac{|b_n|}{(\sum_{i=1}^{\infty} b_i^2)^{\frac{1}{2}}} \le \frac{1}{\beta}$ for $n \in \mathbb{N}$ and $x = \{c_n\}_n$ is given by

$$c_n = \frac{b_n}{\left(\sum\limits_{i=1}^{\infty} b_i^2\right)^{\frac{1}{2}}},$$

clearly $|||x||| = 1$ and $f(x) = \left(\sum_{i=1}^{\infty} b_i^2\right)^{\frac{1}{2}}$. Thus $|||f||| = \left(\sum_{i=1}^{\infty} b_i^2\right)^{\frac{1}{2}}$.

Now let $k \in \mathbb{N}$ be such that $\frac{1}{\sqrt{k}} < \beta$ and $f_{k+m} = \frac{1}{\sqrt{k+m}} \sum_{i=\frac{m(2k+m-1)}{2}}^{\frac{(m+1)(2k+m)}{2}-1} e_i^*$ where $\{e_i^*\}_i$ is the canonical basis of X_β^*. Then clearly $|||f_i||| = 1$ and, if $m \ne n$,

$$|||f_{k+m} - f_{k+n}||| = \sqrt{2}.$$

Thus $s(X_\beta) \ge \sqrt{2}$. By the previous corollary the result follows. □

$X_{\sqrt{2}}$ is also *EIS* for some ε, t and δ, but the proof is very technical and we do not present it here.

Next we will give some results about the behavior of the sum of spaces with the *EIS* property. We remember first the definition of Ramsey's number.

Definition 5. Let $m, l \in \mathbb{N}$. Ramsey's number $R(m, l)$ is the least number of vertices that are needed in order that a complete graph which is colored with two colors A and V, contains either a cycle of color A of length m or a cycle of color V of length l.

Further, we present the following lemma which is part of Proposition 1 in [4]. We give the proof for the sake of completeness.

Lemma 5. *Let X and Y be Banach spaces and $Z = X \bigoplus_\infty Y$. If $f \in S_{X^*}$, $f = g + h$ with $g \in X^*, h \in Y^*, z \in S(f, \delta^2), z = x + y$ with $x \in X, y \in Y$ and $\|g\| \ge \delta$, then $x \in S\left(\frac{g}{\|g\|}, \delta\right)$.*

Proof. If $z \in S(f, \delta^2)$ then

$$1 - \delta^2 \le f(z) = g(x) + h(y) \le g(x) + 1 - \|g\|.$$

Hence

$$\frac{g(x)}{\|g\|} \ge 1 - \frac{\delta^2}{\|g\|} \ge 1 - \delta. \quad \square$$

Similarly one gets:

Lemma 6. *Let X and Y be Banach spaces and $Z = X \bigoplus_p Y$ where $1 < p < \infty$. If $f \in S_{X^*}$, $f = g + h$ with $g \in X^*, h \in Y^*, z \in S(f, \delta^2), z = x + y$ with $x \in X, y \in Y$ and $\|g\|\|x\| \ge \delta$, then $\frac{x}{\|x\|} \in S\left(\frac{g}{\|g\|}, \delta\right)$.*

The following theorem gives conditions under which the l_∞ sum of two *EIS* spaces also enjoys this property.

Theorem 4. *Let X and Y be two Banach spaces and $Z = X \bigoplus_\infty Y$. Let $0 < r_0 < \varepsilon$ and suppose that X and Y are $\frac{\varepsilon - r_0}{2}, t_1$ and $\frac{\varepsilon - r_0}{2}, t_2 - EIS$ spaces respectively with $\frac{\varepsilon - r_0}{2} < \min(s_{t_1}(X^*), s_{t_2}(Y^*))$. If $\varepsilon < s_t(Z^*)$, where $t + 1 = NR(t_1 + 1, t_2 + 1)$ and N is the least integer such that $N\frac{r_0}{2} \ge 1$, then Z is an $\varepsilon, t - EIS$ Banach space.*

Proof. Let $\delta_1, \delta_2 > 0$ be such that if $g_1, \ldots, g_{t_1+1} \in S_{X^*}$ satisfy $\|g_i - g_j\| \geq \frac{\varepsilon - r_0}{2}$ for every $i \neq j, i, j = 1, \ldots, t_1 + 1$, then $S(g_1, \ldots, g_{t_1+1}, \delta_1) = \phi$ and if $h_1, \ldots, h_{t_2+1} \in S_{Y^*}$ satisfy $\|h_i - h_j\| \geq \frac{\varepsilon - r_0}{2}$ for every $i \neq j, i, j = 1, \ldots, t_2 + 1$, then $S(h_1, \ldots, h_{t_2+1}, \delta_2) = \phi$.

Let $\delta = \min\left(\delta_1, \delta_2, \frac{r_0}{2}, \frac{\varepsilon - r_0}{2}\right)$ and let N be the least integer such that $N\frac{r_0}{2} \geq 1$ and $t + 1 = NR(t_1 + 1, t_2 + 1)$.

Suppose that $f_1, \ldots, f_{t+1} \in S_{Z^*}$ are such that $\|f_i - f_j\| \geq \varepsilon$. Then $f_i = g_i + h_i$ with $g_i \in X^*, h_i \in Y^*$ and $1 = \|g_i\| + \|h_i\|$. Assume that $z \in S(f_1, \ldots, f_{t+1}, \delta^2), z = x + y, x \in X, y \in Y$. Let $F = \{1, \ldots, t + 1\}$. Then $F = \bigsqcup_{j=1}^{N} I_j$ where

$$I_j = \left\{ i \in F : (j - 1)\frac{r_0}{2} \leq \|h_i\| \leq j\frac{r_0}{2} \right\}.$$

Thus there exists I_l with $\# I_l \geq R(t_1 + 1, t_2 + 1)$. Hence for every $i, j \in I_l$ with $i \neq j$ either

(a) $\|g_i - g_j\| \geq \frac{\|f_i - f_j\|}{2}$ or

(b) $\|h_i - h_j\| \geq \frac{\|f_i - f_j\|}{2}$.

Let $I_l = \{i_1, \ldots, i_m\}$ and consider the complete graph of m vertices. We color the edge (i, j) with color A if (a) occurs and with color V if (b) but not (a) occurs. Then by Ramsey's theorem there is either a cycle of length $t_1 + 1$ with color A or a cycle of length $t_2 + 1$ of color V.

Case 1. There is $J_1 = \{j_1, \ldots, j_{t_1+1}\} \subset I_l$ such that $\|g_i - g_j\| \geq \frac{\|f_i - f_j\|}{2}$ for $i \neq j, i, j \in J_1$. If $l \neq N$, then $\|g_i\| \geq \frac{r_0}{2}$ for $i \in J_l$ and thus, by Lemma 5, $x \in \bigcap_{i \in J_1} S\left(\frac{g_i}{\|g_i\|}, \delta\right)$. Also, since

$$|\|g_i\| - \|g_j\|| = |\|h_i\| - \|h_j\|| \leq \frac{r_0}{2},$$

by Lemma 3, $\left\| \frac{g_i}{\|g_i\|} - \frac{g_j}{\|g_j\|} \right\| \geq \frac{\|f_i - f_j\|}{2} - \frac{r_0}{2} \geq \frac{\varepsilon}{2} - \frac{r_0}{2}$ and this contradicts the fact that X is $\frac{\varepsilon - r_0}{2}, t_1 - EIS$.

If $l = N$, we have $\frac{r_0}{2} + \min(\|g_i\|, \|g_j\|) \geq \|g_i - g_j\| \geq \frac{\|f_i - f_j\|}{2} \geq \frac{\varepsilon}{2}$. Hence $\min(\|g_i\|, \|g_j\|) \geq \frac{\varepsilon - r_0}{2}$. Therefore $x \in S\left(\frac{g_i}{\|g_i\|}, \delta\right)$ and the rest follows in the same manner as above.

Case 2. There is $J_2 = \{m_1, \ldots, m_{t_2+1}\} \subset I_l$ such that $\|h_i - h_j\| \geq \frac{\|f_i - f_j\|}{2}$ for $i \neq j, i, j \in J_2$. This is done the same as case 1. □

Observe that if X and Y are infinite dimensional spaces then $s_k(X^*) \geq 1$, and $s_k(Y^*) \geq 1$ for every $k \in \mathbb{N}$. Thus if $\varepsilon < 1$ and X and Y are $\varepsilon - r_0, t_1$ and $\varepsilon - r_0, t_2 - EIS$ spaces, then $X \bigoplus_\infty Y$ has *EIS*.

Hence we get the following result.

Corollary 7. *If for $i = 1, \ldots, m$ the Banach space X_i is $\varepsilon, k_i - EIS$ for every $0 < \varepsilon < s(X_i^*)$ and $Z = \sum_{i=1}^{m} \bigoplus_\infty X_i$, then for every $0 < \varepsilon < s(Z^*)$ there exists t such that Z is $\varepsilon, t - EIS$.*

In particular we have the above result if X_i is uniformly smooth for every i.

Unfortunately it is not true that if X and Y are *EIS*, $\sum_{i=1}^{\infty} \bigoplus_\infty X$ is *EIS* as the following lemma and its corollary show.

Lemma 7. *Let X be a Banach space, $2 > \varepsilon > 0$ and let*

$$k(X,\varepsilon) = \begin{array}{l}\min\{k \in \mathbb{N} : \exists \delta > 0 \; s.t. \; \forall \{f_i\}_{i=1}^{k+1} \subset S_{X^*}, \; with \; \|f_i - f_j\| \geq \varepsilon \; for \; i \neq j \\ implies \; \cap_{i=1}^{k+1} S(f_i,\delta) = \emptyset\}.\end{array}$$

Then, if $\{X_i\}_{i=1}^{\infty}$ is a sequence of Banach spaces,

$$k(X_1 \oplus_\infty X_2 \oplus_\infty \cdots \oplus_\infty X_n, \varepsilon) \underset{n\to\infty}{\to} \infty.$$

Proof. Let $x_i \in S_{X_i}$ and $f_i \in S_{X_i^*}$ such that $f_i(x_i) = 1$, $i = 1, 2, \ldots$ and let

$$z_n = x_1 + x_2 + \cdots + x_n \in X_1 \oplus_\infty X_2 \oplus_\infty \cdots \oplus_\infty X_n.$$

Then $\|z_n\| = 1$ and if $h_i = 0 + \cdots + f_i + 0 + \cdots + 0 \in X_1^* \oplus_1 X_2^* \oplus_1 \cdots \oplus_1 X_n^*$ for $i = 1, \ldots, n$ we have that $\|h_i\| = 1$, $\|h_i - h_j\| = 2$, $h_i(z_n) = 1$. Thus $z_n \in \bigcap_{i=1}^n S(h_i,\delta)$ for any $1 \geq \delta > 0$ and $k(X_1 \oplus_\infty X_2 \oplus_\infty \cdots \oplus_\infty X_n, \varepsilon) \geq n - 1$ for every $0 < \varepsilon < 2$. □

Corollary 8. *If $\{X_i\}_i$ is a sequence of Banach spaces $Z = (\sum_i X_i)_{c_0}$ is not EIS.*

Proof. Clearly $s(Z^*) = 2$ and for every $2 \geq \varepsilon > 0$,

$$k(Z,\varepsilon) \geq k(X_1 \oplus_\infty X_2 \oplus_\infty \cdots \oplus_\infty X_n, \varepsilon). \quad \square$$

Similarly to Theorem 4, using Lemma 6 in place of Lemma 5, one can prove the following:

Theorem 5. *Let X and Y be two Banach spaces, suppose that $1 < p < \infty$ and that q is the conjugate exponent of p. Let $Z = X \bigoplus_p Y$ and $0 < r_0 < 2(\frac{\varepsilon}{4})^q$. Assume that X and Y are $\frac{\varepsilon}{2} - (\frac{r_0}{2})^{\frac{1}{q}}, t_1$ and $\frac{\varepsilon}{2} - (\frac{r_0}{2})^{\frac{1}{q}}, t_2$−EIS spaces respectively with $\frac{\varepsilon}{2} - (\frac{r_0}{2})^{\frac{1}{q}} < \min(s_{t_1}(X^*), s_{t_2}(Y^*))$. If $\varepsilon < s_t(Z^*)$, where $t + 1 = 2(N+1)\mathcal{R}(t_1+1, t_2+1)$ and N is the least integer such that $(N+1)\frac{r_0}{2} \geq 1$, then Z is an ε, t − EIS Banach space.*

For the l_1 sum we get the following.

Lemma 8. *Suppose that X is any Banach space and that Y is an infinite dimensional Banach space. Then $Z = X \bigoplus_1 Y$ is not ε, k − EIS for any $\varepsilon < s(Z^*)$, $k \in \mathbb{N}$.*

Proof. Let $\varepsilon < s(Z^*)$ and suppose that $\{h_i\}_{i=1}^{\infty} \subset S_{Z^*}$ is such that $\|h_i - h_j\| \geq \varepsilon$ for $i \neq j$. If $h_i = f_i + g_i$ with $f_i \in X^*$ and $g_i \in Y^*$ then by passing to a subsequence we may assume that either

(1) $f_i \neq f_j$ for every $i \neq j$, $g_i = g_j$ for every i and thus $\|f_i - f_j\| \geq \varepsilon$ for every $i \neq j$.
(2) $g_i \neq g_j$ for every $i \neq j$, $f_i = f_j$ for every i and thus $\|g_i - g_j\| \geq \varepsilon$ for every $i \neq j$.
(3) $f_i \neq f_j$ and $g_i \neq g_j$ for every $i \neq j$. In this case, by Ramsey's theorem, there exists an infinite set $I \subset \mathbb{N}$ such that either $\|g_i - g_j\| \geq \varepsilon$ or $\|f_i - f_j\| \geq \varepsilon$ for every $i, j \in I$ with $i \neq j$.

In the first case we have that $\|f_i\| \geq \frac{\varepsilon}{2}$ for all i except possibly one, because $\|f_i\| < \frac{\varepsilon}{2}$ and $\|f_j\| < \frac{\varepsilon}{2}$ implies $\|f_i - f_j\| < \varepsilon$. Thus we may assume $1 \geq \|f_i\| \geq \frac{\varepsilon}{2}$ for every i. Hence, if $n \in \mathbb{N}$, there exists $0 \leq k \leq n-1$ such that

$$\#\left\{i \in \mathbb{N} : \frac{\varepsilon}{2} + k\frac{(2-\varepsilon)}{2n} \leq \|f_i\| \leq \frac{\varepsilon}{2} + (k+1)\frac{(2-\varepsilon)}{2n}\right\} = \#A_k = \infty.$$

Then for $i, j \in A_k$, $i \neq j$, by Lemma 3

$$\left\| \frac{f_i}{\|f_i\|} - \frac{f_j}{\|f_j\|} \right\| \geq \|f_i - f_j\| - \frac{2-\varepsilon}{n} \geq \varepsilon - \frac{2-\varepsilon}{n}.$$

In the second case similarly we obtain

$$\left\| \frac{g_i}{\|g_i\|} - \frac{g_j}{\|g_j\|} \right\| \geq \|g_i - g_j\| - \frac{2-\varepsilon}{n} \geq \varepsilon - \frac{2-\varepsilon}{n}.$$

In the third case we proceed similarly as in cases one and two. Then, since $\max(s(X^*), s(Y^*)) \leq s(Z^*)$, as a conclusion we have that

$$\max(s(X^*), s(Y^*)) = s(Z^*).$$

Let $k \geq 1 \in N$ and $0 < \varepsilon < s(Z^*)$. Suppose that $s(X^*) = s(Z^*)$. Let $\{f_i\}_{i=1}^{k+1} \subset S_{X^*}$ be such that $\|f_i - f_j\| \geq \varepsilon$ for $i \neq j$ and $i, j = 1, 2, \ldots, k+1$. Let $y_0 \in S_Y$ and $g \in S_{Y^*}$ such that $g(y_0) = 1$. Let $h_i^* \in Z^* = X^* \bigoplus_\infty Y^*$ be given by $h_i^*(x + y) = f_i(x) + g(y)$ for $i = 1, \ldots, k+1$. Then, $z = 0 + y_0 \in Z$, $\|z\| = 1$, $h_i^*(z) = 1$, $\|h_i^*\| = 1$ and if $i \neq j$

$$\|h_i^* - h_j^*\| = \|f_i - f_j\| \geq \varepsilon.$$

Also $z \in S(h_1^*, h_2^*, \ldots, h_{k+1}^*, \delta)$ for any $0 < \delta < 1$ and this proves our claim. $\square$

Remark 1. However, if X and Y are *EIS*, $Z = X \bigoplus_1 Y$ may be $\varepsilon - EIS$ for some $\varepsilon > s(Z^*)$ as the following example shows:

Let H_1 and H_2 be two Hilbert spaces, then $Z = H_1 \bigoplus_1 H_2$ is $\varepsilon, 2{-}EIS$ for every $\sqrt{3} < \varepsilon < 2$.

First we see that if $h_1 = e_1^1 + e_1^2$, $h_2 = e_1^1 - e_1^2$, $h_3 = -e_1^1 + e_1^2$, where $e_1^k \in S_{H_k}$, $k = 1, 2$, then

$$\|h_i - h_j\| = 2$$

for $i \neq j$, $i, j = 1, 2, 3$; so $s_2(Z^*) \geq 3$.

Now let $\sqrt{3} < \varepsilon < 2$ and $f_1, f_2, f_3 \in B_{H_1^*}$ and suppose that $\|f_1 - f_2\| \geq \varepsilon$. It cannot happen that simultaneously $\|f_3 - f_1\| \geq \varepsilon$, and $\|f_3 - f_2\| \geq \varepsilon$, because in this case by (2.1) we would have

$$\|f_1 + f_2 + f_3\|^2 = 3\sum_{i=1}^{3} \|f_i\|^2 - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \|f_i - f_j\|^2 \leq 9 - 3\varepsilon^2 < 0.$$

Also, if $\varepsilon = \sqrt{4-\phi}$ with $1 < \phi < 2$, if $f_1, f_2 \in B_{H_1^*}$ and $\|f_1 - f_2\| \geq \varepsilon$, by the parallelogram law we get:

$$\|f_1 + f_2\|^2 + 4 - \phi \leq \|f_1 + f_2\|^2 + \|f_1 - f_2\|^2 = 2(\|f_1\|^2 + \|f_2\|^2) \leq 4$$

and thus

$$\|f_1 + f_2\|^2 \leq \phi. \tag{2.2}$$

Now let $\{h_i\}_{i=1}^3 \in S_{Z^*}$, $h_i = f_i + g_i$ be an arbitrary sequence such that $\|h_i - h_j\| \geq \varepsilon$. By the above, we may assume that $\|f_1 - f_2\| \geq \varepsilon$, $\|f_1 - f_3\| \geq \varepsilon$ and $\|g_2 - g_3\| \geq \varepsilon$. Let $\delta < \frac{1}{3}(2 - \sqrt{\phi})$ and suppose that there exists $z \in S(h_1, h_2, h_3, \delta)$. Then, since $h_i(z) \geq 1 - \delta$ for $i = 1, 2, 3$, we get that

$$\|h_1 + h_2 + h_3\| \geq 3(1 - \delta).$$

On the other hand, by (2.2), $\|f_1 + f_2\| \leq \sqrt{\phi}$ and $\|g_2 + g_3\| \leq \sqrt{\phi}$. Thus

$$\|h_1 + h_2 + h_3\| = \max(\|f_1 + f_2 + f_3\|, \|g_1 + g_2 + g_3\|) \leq 1 + \sqrt{\phi}$$

and this is a contradiction. Thus $S(h_1, h_2, h_3, \delta) = \emptyset$.

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