Centro de Investigación en Matemáticas, A.C.

RIGIDITY FOR COMPLEX ACTIONS AND ANTI-KÄHLER MANIFOLDS

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Emilio Salcedo Martínez

Director de Tesis:

Dr. Raúl Quiroga Barranco

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Contents

Chapter 1

Introduction

It is not an easy task to speak historically about the collection of ideas, theorems, books, and articles that lead one to a motivation for a work like this one.

Practically speaking, the story begins with Daniel Mostow. His research on rigidity played an essential role in the work of three Fields medalists, namely Grigori Margulis, William Thurston, and Grigori Perelman.

In chronological order, Michael Gromov was born twenty years after Mostow. His contributions are vast and important, we use his generalization of geometric structures in this thesis. It is written in Section 2.3 of the next chapter.

Two years after Gromov, Grigory Margulis came to the world. Famous for his (super)rigidity theorem, he was a pioneer in the mixing of three huge areas of mathematics: ergodic theory, dynamics and Lie groups. Inspired by Mostow, his work influenced our fourth historical character.

Robert J. Zimmer, only a year younger than Margulis, improved and generalized the results of Margulis in a broader sense. Margulis has agreed that his later work has been influenced by that of Zimmer.

Raúl Quiroga-Barranco, Ph. D. student of Robert Zimmer, has given partial answers to Zimmer's Program through many articles, using innovating techniques. Motivated by all these important people this work, with the same flavor from Quiroga-Barranco's articles, tries to plant a grain of sand in the Zimmer's Program.

1.1 Rigidity theorems

As is usual in the mathematical world, the first results in rigidity were done for particular cases, as we can see in the work of Selberg, Calabi, and Weil.

However, there was a big gap in the theory when Mostow proved the following global theorem on compact quotients, mixing different tools from topology, differential geometry, group theory and harmonic analysis.

Theorem 1.1 (Mostow [22]). *Let X and Y be compact quotients of symmetric spaces of non-compact type with the same fundamental group. If X has no closed two dimensional local factor, then X and Y are isometric up to normalizing constants.*

Going from the particular to the general, as history has shown that modern mathematics is done, Margulis extended this result, assuming semisimplicty and other necessary conditions for a stronger theorem.

Theorem 1.2 (Margulis' superrigidity Theorem)**.** *Let Γ be an irreducible lattice in a connected semisimple Lie group G of real rank at least 2, trivial center, and without compact factors. Suppose $\mathbb{K}$ is a local field. Then any homomorphism π of Γ into a non-compact $\mathbb{K}$-simple group over $\mathbb{K}$ with Zariski dense image either has precompact image or π extends to a homomorphism of the ambient group G.*

In the context of Lie group actions, Zimmer extended Margulis' Superrigidity to a cocycle superrigidity which has shown to be very useful in the study of actions of semisimple Lie groups without compact factors.

Margulis' theorem, classifying all linear representations, leads one to believe that it is possible to classify all homomorphisms to other interesting classes of topological groups.

Inspired by Margulis' superrigidity theorem, in the early 1980's Zimmer proved a superrigidity theorem for cocycles from which he proved results about orbit equivalence of higher-rank group actions Motivated by earlier results in the rigidity of linear representations and the cocycle superrigidity theorem, Zimmer proposed studying non-linear representations of lattices in higher-rank simple Lie groups. That is, given a lattice $\Gamma \subset G$, rather than studying linear representations $\rho\colon \Gamma \to \mathrm{GL}(n, \mathbb{R})$, Zimmer proposed studying representations $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ where M is a compact Riemannian manifold. The main objective of the Zimmer program is to show that all such non-linear representations α are of an "algebraic origin". In particular, the Zimmer conjecture states that if the dimension of M is sufficiently small (relative to data associated to G) then any action $\alpha\colon \Gamma \to \mathrm{Diff}(M)$ should preserve the Riemannian metric or factor through the action of a finite group.

Let G be a connected non-compact simple Lie group acting isometrically on a connected analytic manifold M with a finite-volume pseudo-Riemannian metric. Following Zimmer's program, it has been shown that such actions are rigid in the sense of having distinguished properties that restrict the possibilities for M (see for example [10], [31], [33]). The general belief is that any such action, with some additional non-triviality conditions, must essentially be an algebraic double coset of the form $K\backslash H/\Gamma$. More precisely, such coset is given by some Lie group H together with a homomorphism $G \to H$, a lattice $\Gamma \subset H$ and a compact subgroup $K \subset H$ centralizing the image of G in H. The G-action is then given by the natural left action on $K\backslash H/\Gamma$. We note that when H is semisimple these G-actions are isometric for a metric induced by the Killing form of the Lie algebra of H. Some results have already been obtained by Quiroga-Barranco in [26], [4], [28], [24], proving that suitable geometric conditions imply that such G-actions are of the double coset type.

1.2 Anti-Kähler manifolds

Among the tensor structures on a smooth manifold, one of the most studied is *almost-complex structure*, i.e. an endomorphism of the tangent bundle whose square, at each point, is minus the identity. The manifold must be even-dimensional. Usually it is equipped with a Hermitian metric, which is a metric such that the almost-complex structure acts as an isometry with respect to the metric. The associated $(0,2)$-tensor of the Hermitian metric is a 2-form and hence one sees the relationship with symplectic geometry.

A relevant counterpart is the case when the almost-complex structure acts as an anti-isometry regarding a pseudo-Riemannian metric. Such a metric is known as an anti-Hermitian metric or Norden metric, first studied by an named after A.P. Norden [23]. These metrics appear naturally in complex Lie groups.

We should note that one of this work's objective is to study a structure that the complex Lie algebra of a complex Lie group has: the Killing form. We recall that a complex Lie group is a complex manifold with group structure such that its operation of product and inverse are holomorphic maps. As a consequence, its Lie algebra is complex.

We observe the properties of the complex Killing form and we abstract them in order to define a more general structure on almost-complex manifolds. This results in a metric different from the extremely well-studied Hermitian metric.

The first step in order to abstract the properties mentioned above is to define it in a real vector space with complex structure. The first surprise is that this metric is not Hermitian, but is rather a symmetric, non-degenerate complex bilinear form. If we take the real part, then it is an (m, m)-bilinear real form, where $2m$ is the dimension of the complex Lie group as a manifold. Moreover, the complex structure acts as an anti-isometry. That is, the real part is an anti-Hermitian metric!

However we give a different (but equivalent) definition that suits our work better. We define an anti-Hermitian metric to be not the real part, but rather the symmetric non-degenerate complex bilinear form that we can obtain from it. This kind of metrics has been studied under many names: “Norden”, “pure”, and “B-metrics”.

The choise of the name (we could have chosen Norden metric instead) is because we want to reflect the ‘opposite’ properties of Hermitian metrics. Later on, we realize that we need a stronger ingredient and the notion of anti-Kähler manifold comes naturally. It is indeed ‘the same’ property as Kähler manifolds: we ask the almost-complex structure to be parallel with respect to the Levi-Civita connection of the real part of the anti-Hermitian metric. The results in previous works show that this is sufficient for our manifold to be complex.

1.3 Main Theorem

One notices that the lemmas, propositions and corollaries from [4], [24] and [28] are almost the same except for the fact that we need them to be true for complex maps, functions, spaces, and without much effort we prove this for our setup.

We recall that a connected pseudo-Riemannian manifold is weakly irreducible if the tangent space at some (and hence any) point has no proper non-degenerate invariant subspaces under the restricted holonomy group at that point. In particular, a weakly irreducible pseudo-Riemannian manifold cannot have a non-trivial product as universal covering space.

Let G be a non-compact simple complex Lie group with Lie algebra $\mathfrak{g}$. Let us denote by $m(\mathfrak{g})$ the real dimension of the smallest non-trivial $\mathfrak{g}$-module with an invariant non-degenerate symmetric real bilinear form. Analogously, let $m_{\mathbb{C}}(\mathfrak{g})$ be the complex dimension of the smallest non-trivial $\mathfrak{g}$-module with an invariant non-degenerate symmetric complex bilinear form.

We find this theorem in [4].

Theorem 1.3. *Let M be a connected analytic pseudo-Riemannian manifold. Suppose that M is complete, weakly irreducible, has finite volume and admits an analytic and isometric $\widetilde{\mathrm{SO}}_0(p,q)$-action with a dense orbit, for some integers p, q such that p, $q \geq 1$ and $n = p+q \geq 5$. In this case we have $m(\mathfrak{so}(p,q)) = n$. If the equality:*

$$\dim(M) = \dim(\widetilde{\mathrm{SO}}_0(p,q) + m(\mathfrak{so}(p,q)) = \frac{n(n+1)}{2},$$

holds, then for H equal to either $\widetilde{\mathrm{SO}}_0(p,q+1)$ or $\widetilde{\mathrm{SO}}(p+q,q)$ there exist:

(1) a lattice $\Gamma \subset H$, and

(2) an analytic finite covering map $\varphi\colon H/\Gamma \to M$,

such that φ is $\widetilde{\mathrm{SO}}_0(p,q)$-equivariant, where the $\widetilde{\mathrm{SO}}_0(p,q)$-action on H/Γ is induced by some non-trivial homomorphism $\widetilde{\mathrm{SO}}_0(p,q) \to H$. Furthermore, we can rescale the metric on M along the $\widetilde{\mathrm{SO}}_0(p,q)$-orbits and their normal bundle to assume that φ is a local isometry for the bi-invariant pseudo-Riemannian metric on H given by the Killing form of its Lie algebra. The result holds for the case $(p,q) = (3,1)$ as well if we further assume that $X^* \perp Y^*$ on M for all $X \in \mathfrak{su}(2)$ and $Y \in i\mathfrak{su}(2)$ under the identification $\mathfrak{so}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C})$.

The above result is very similar to our main theorem:

Theorem 1.4. *Let (M,J,g) be a connected anti-Kähler manifold. Let $h = \mathrm{Re}(g)$. Suppose that (M,h) is complete, weakly irreducible, has finite volume and admits a holomorphic $\mathrm{Spin}(n,\mathbb{C})$-action by isometries of g with a dense orbit, for some integer $n \geq 3$, $n \neq 4$. Them, in this case we have that $m_{\mathbb{C}}(\mathfrak{so}(n,\mathbb{C})) = n$. If the inequality*

$$\dim(M) \leq \dim(\mathrm{Spin}(n,\mathbb{C})) + n = \frac{n(n+1)}{2}$$

holds, then for $H = \mathrm{Spin}(n+1,\mathbb{C})$ there exist:

(1) a lattice $\Gamma \subset H$, and

(2) a holomorphic finite covering map $\varphi : H/\Gamma \to M$,

such that φ is $\mathrm{Spin}(n,\mathbb{C})$-equivariant, where the $\mathrm{Spin}(n,\mathbb{C})$-action on H/Γ is induced by some non-trivial homomorphism $\mathrm{Spin}(n,\mathbb{C}) \to H$. Furthermore, we can rescale the metric on M along the $\mathrm{Spin}(n,\mathbb{C})$-orbits and their normal bundle to assume that φ is a local isometry for the bi-invariant anti-Hermitian metric on H given by the Killing form of its Lie algebra.

The main theorem provides a rigidity result for $\mathrm{Spin}(n,\mathbb{C})$-actions on anti-Kähler manifolds. Note that there is no $\mathbb{R}$-rank restriction.

The proof is broken into various lemmas in the last chapter of this work and is based on the application of representation theory to the Killing vector fields centralizing the G-action, where the latter are as found in Gromov-Zimmer's machinery.

Chapter 2

Geometric structures and anti-Hermitian metrics on manifolds

We begin this work by discussing anti-Hermitian metrics on vector spaces. Section 2.1 is about this issue. However, treating these metrics from a different perspective provides a new powerful tool for rigidity almost-complex manifolds. This is done on Section 2.2.

Next, in Section 2.3, we remember the basic concepts of geometric structures, including the concept of rigidity defined by Gromov. We use the same notation used in [5] throughout this work.

Section 2.4 deals with the product of geometric structures and gives a useful method to determine when certain types of geometric structures are rigid. Section 2.5 contains a generalization of a proposition found in [27] with the transversality condition dropped. Finally, Section 2.6 is about the analogous version of Kähler manifolds in the context of anti-Hermitian manifolds, which we call anti-Kähler manifolds. It also contains the complex version of the last proposition but in the category of complex manifolds.

2.1 Preliminaries of anti-Hermitian metrics

In this section we will develop the algebraic tools needed to define anti-Hermitian metrics on manifolds. The algebraic results of the present section will be applied to tangent spaces of manifolds later on.

We shall begin with the definition of an anti-Hermitian metric in a vector space, and its relation to a particular pseudo-Riemannian metric. This is not the more familiar holomorphic metric that uses the complexified tangent space. This will be clarified in the next section. Without further preambles we begin with the basics.

Let V be a finite-dimensional real vector space. Recall that a complex structure on V is a linear endomorphism J of V such that $J^2 = -I$, where I stands for the identity transformation of V. A real vector space V with a complex structure J can be turned into a complex vector space by defining scalar multiplication by complex numbers as follows:

$$(a+ib)X = aX + bJX \text{ for } x \in V \text{ and } a, b \in \mathbb{R}.$$

It is clear that the real dimension m of V must be even and $\frac{1}{2}m$ is the complex dimension of V. Conversely, given a complex vector space V of complex dimension n, let J be the linear endomorphism of V defined by

$$JX = iX \text{ for } X \in V.$$

If we consider V as a real vector space of real dimension $2n$, then J is a complex structure of V.

Let $\mathbb{C}^n$ be the vector space of n-tuples of complex numbers $z = (z^1, \ldots, z^n)$. If we set

$$z^k = x^k + iy^k, \quad x^k, y^k \in \mathbb{R}, \quad k = 1, \ldots, n,$$

then $\mathbb{C}^n$ can be identified with the real vector space $\mathbb{R}^{2n}$: given the canonical basis of $\mathbb{C}^n$, $\{e_1, \ldots, e_n\}$, the ordered real basis $\{e_1, \ldots, e_n, ie_1, \ldots, ie_n\}$ identifies z with $(x^1, \ldots, x^n, y^1, \ldots, y^n) \in \mathbb{R}^{2n}$.

From now on, the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ will be as above. The complex structure of $\mathbb{R}^{2n}$ induced from $\mathbb{C}^n$ (multiplication by i) maps $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ into $(-y^1, \ldots, -y^n, x^1, \ldots, x^n)$ and is called the *canonical complex structure* of $\mathbb{R}^{2n}$. It is given by the matrix

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n denotes the identity matrix of degree n.

To motivate the definition of an anti-Hermitian metric on vector spaces and subsequently on manifolds, let us begin with the case $V = \mathbb{R}^{2n}$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ as above. We consider $H(z, w) = \sum_{j=1}^{n} z_j \overline{w}_j$ and $g_0(z, w) = \sum_{j=1}^{n} z_j w_j$.

The function H is the Hermitian metric, and its properties are well known: it is nondegenerate, linear in the first slot, anti-linear in the second one, and positive definite but not symmetric.

In contrast, g_0 is complex bilinear, symmetric, and nondegenerate. With this metric we gain complex linearity and symmetry but we lose positive definiteness. However, we are interested precisely in the properties that g_0 has.

Given a symmetric nondegenerate $\mathbb{F}$-bilinear form B on V, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and an $\mathbb{F}$-basis e_1, …, e_n, the matrix representation of B in that basis is $(B_{\alpha\beta})$ where $B_{\alpha\beta} = B(e_\alpha, e_\beta)$. The matrix will be symmetric and non singular.

For our case, we have that this matrix is the complex matrix $(g_0(e_j, e_k)) = I$, thus $g_0(z, w) = z^T I w = z^T w$. If $z^j = x^j + iy^j$, $w^j = u^j + iv^j$, then $g_0(z, w) = \sum_{j=1}^{n}(x^j u^j - y^j v^j) + \sum_{j=1}^{n} i(x^j v^j + y^j u^j)$. If we take the real and the imaginary parts, then in the real basis $\{e_1, \ldots, e_n, ie_1, \ldots, ie_n\}$ we get

$$\mathrm{Re}(g_0) = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \quad \mathrm{Im}(g_0) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Let us denote these matrices by $I_{n,n}$ and S respectively. It is important to note that $S = -I_{n,n} J_0$. Thus given $\mathrm{Re}(g_0)$, and the complex structure J_0, we can recover g_0 again by making $g_0(z,w) = \mathrm{Re}(g_0)(z,w) - i\,\mathrm{Re}(g_0)(J_0 z, w)$. It must be pointed out that the real part is a pseudo-Riemannian metric in $\mathbb{R}^{2n}$ of signature (n,n) that is compatible with the complex structure, i.e. $\mathrm{Re}(g_0)(iz,w) = \mathrm{Re}(g_0)(J_0 z, w) = \mathrm{Re}(g_0)(z, J_0 w) = \mathrm{Re}(g_0)(z, iw)$. After this analysis we are ready to generalize these ideas.

Definition 2.1. Let V be a real vector space with complex structure J. An *anti-Hermitian metric* on V, compatible with J, is a symmetric nondegenerate bilinear form $g\colon V \times V \to \mathbb{C}$ such that

$$g(JX,Y) = g(X,JY) = ig(X,Y) \qquad \text{for all } X,Y \in V.$$

In other words, if V is considered as a complex vector space via J as above, then g is a symmetric nondegenerate bilinear form over $\mathbb{C}$.

Remark. The complex bilinear form g_0 of $\mathbb{C}^n$ compatible with the canonical complex structure J_0 will be called the *canonical anti-Hermitian metric of* $\mathbb{C}^n$ from now on.

Let us point out that the compatibility of a complex Riemannian metric is equivalent to saying that

$$g(JX,JX) = -g(X,Y)$$

in other words, the complex structure J can be thought of as an anti-isometry of the metric g.

For the purpose of reference, we state the well known result of Sylvester's law of inertia for the real and complex cases.

Proposition 2.2. *Let V a finite-dimensional vector space over the field of complex numbers. Let f be a symmetric bilinear form on V which has rank r. Then there is an ordered basis $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ for V such that the matrix of f in the ordered basis is diagonal and*

$$f(\beta_j, \beta_j) = \begin{cases} 1, & j = 1, \ldots, r \\ 0, & j > r. \end{cases}$$

Proposition 2.3. *Let V an n-dimensional vector space over the field of real numbers. Let f be a symmetric bilinear form on V which has rank r. Then there is an ordered basis $\mathcal{B} = \{\beta_1, \ldots, \beta_n\}$ for V in which the matrix of f is diagonal and such that*

$$f(\beta_j, \beta_j) = \begin{cases} \pm 1, & j = 1, \ldots, r \\ 0, & j > r. \end{cases}$$

Furthermore, the number of basis vectors β_j for which $f(\beta_j, \beta_j) = 1$ is independent of the choice of basis.

Considering V as a complex vector space, as we mentioned g is a symmetric nondegenarate bilinear form, so it is clear from Proposition 2.2 that g is equivalent to the identity matrix. However, we have an interesting property for the real and imaginary parts.

Proposition 2.4. *Let* $h = \dfrac{1}{2}(g + \bar{g}) = \operatorname{Re}(g)$. *Then* h *is a real, symmetric, nondegenerate bilinear form of* V *of signature* (n, n) *such that*

$$h(u, Jv) = h(Ju, v) \text{ for all } u, v \in V. \tag{2.1}$$

Conversely, if h *is a real symmetric nondegenerate bilinear form on* V *that satisfies equation* (2.1)*, then there exists a unique anti-Hermitian metric* g *such that* $h = \operatorname{Re}(g)$. *In particular,* h *has signature* (n, n) *and rather one should have* $\operatorname{Im}(g)(u, v) = -ih(Ju, v)$ *for all* u, $v \in V$.

Proof. It is clear that g and $\bar{g}$ are symmetric, nondegenereate bilinear *real* forms, so it is h. Moreover,

$$\begin{aligned} h(Ju, v) &= \frac{1}{2}(g(Ju, v) + \overline{g(Ju, v)}) \\ &= \frac{1}{2}(g(u, Jv) + \overline{g(u, Jv)}) \\ &= h(u, Jv) \end{aligned}$$

Let us note that for such h, if $\{e_1, \dots, e_{2n}\}$ is a real basis as in Proposition 2.3, then $\{Je_1, \dots, Je_{2n}\}$ is another real basis for V and $h(Je_j, Je_k) = -h(e_j, e_k)$, which shows that the signature of h must be (n, n). The latter follows from equation (2.1). Now, let us suppose that such g exists, so that $g = h + i \operatorname{Im}(g)$. Then

$$ig = ih - \operatorname{Im}(g).$$

But by Definition 2.1,

$$ig(u, v) = g(Ju, v) = h(Ju, v) + i \operatorname{Im}(g)(Ju, v),$$

so we conclude that $\operatorname{Im}(g)(u, v) = -h(Ju, v)$, and then

$$g(u, v) = h(u, v) - ih(Ju, v).$$

Then g is completely determined by h and J. □

Remark. Modifying the proof of Proposition 2.4 we get that $\operatorname{Im}(g)$ is a symmetric nondegenerate bilinear real form compatible with J, so it has signature (n, n), and $\operatorname{Re}(g)(u, v) = \operatorname{Im}(g)(Ju, v)$ for all u, $v \in V$. Also, by the compatibility, J plays the role of an anti-isometry for both $\operatorname{Re}(g)$ and $\operatorname{Im}(g)$.

$\operatorname{Re}(g)$ and $\operatorname{Im}(g)$ are known as *twin metrics*. However, even though they have been studied, we lack information for our setup which in turn provides a potential area for future research although we will not develop it in this thesis.

Now, if we have a linear transformation $T \colon V \to V'$ between two vector spaces V, V' with complex structures J, J' respectively, we might wonder when it is a linear transformation of complex vector spaces.

Proposition 2.5. *A real-linear map* $T \colon V \to V'$ *is a linear transformation between complex vector spaces if and only if* $J' \circ T = T \circ J$.

Proof. The result follows from the identities

$$J'(T(u)) = iT(u) = T(iu) = T(Ju).$$

This only shows the "only if" part, although the "if" part is similar. □

So far, we have not considered other structures on V. Since studying representations of Lie algebras is one of our main tools it worths to analyze it. If V has a real Lie algebra structure, then it is well known that V is a complex Lie algebra via J if and only if

$$[X, JY] = J[X, Y]; \text{ for all } X,\, Y \in V.$$

By the antisymmetry of the Lie bracket we have $[JX, Y] = -[Y, JX] = -J[Y, X] = J[X, Y]$ for all X, $Y \in V$. If $\mathfrak{gl}(V, \mathbb{C})$ denotes the set of complex linear maps from V to V with Lie bracket defined by the commutator $[A, B] = AB - BA$, then it has the structure of a complex Lie algebra via $J \in \mathfrak{gl}(V, \mathbb{C})$,

$$\begin{aligned} J \colon \mathfrak{gl}(V, \mathbb{C}) &\to \mathfrak{gl}(V, \mathbb{C}) \\ A &\to J \circ A. \end{aligned}$$

Then $\mathrm{ad} : V \to \mathfrak{gl}(V, \mathbb{C})$ is complex linear since $\mathrm{ad}(JX)(Y) = [JX, Y] = J[X, Y] = J\,\mathrm{ad}(X)(Y)$. Furthermore, if V is semisimple, its Killing form K is nondegenerate, symmetric and satisfies

$$\begin{aligned} K(JX, Y) &= \mathrm{tr}(\mathrm{ad}(JX) \circ \mathrm{ad}(Y) \\ &= \mathrm{tr}(J \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y)) \\ &= \mathrm{tr}(\mathrm{ad}(X) \circ J \circ \mathrm{ad}(Y)) \\ &= \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(JY)) \\ &= K(X, JY). \end{aligned}$$

Thus it is an anti-Hermitian metric and we have proved the following result:

Proposition 2.6. *If (V, J) is a semisimple complex Lie algebra, then its Killing form is an anti-Hermitian metric.*

We see that anti-Hermitian metrics arise naturally on complex semisimple Lie algebras.

For this work, the isometries of anti-Hermitian metrics are more important that the metrics themselves. What follows is the definition of these metrics isometries.

Definition 2.7. Let be V, V' real vector spaces with complex structures J, J' and anti-Hermitian metrics g, g' compatible with J and J', respectively. An *isometry* between V and V' is a $\mathbb{R}$-linear isomorphism $T \colon V \to V'$ such that $g'(Tu, Tv) = g(u, v)$ for all u, $v \in V$. When $V' = V$, $J' = J$ and $g' = g$, the set of all isometries of V will be denoted by $\mathbf{O}(g)$.

Lemma 2.8. *With the notation of the above definition, if $T \colon V \to V'$ is an isometry, then T is $\mathbb{C}$-linear.*

Proof. Given u, $v \in V$ we have that

$$\begin{aligned} g'(J'Tu, Tv) &= ig'(Tu, Tv) \\ &= ig(u, v) \\ &= g(Ju, v) \\ &= g'(TJu, Tv). \end{aligned}$$

Since T is an isomorphism, its range is all V'. Furthermore, g' is nondegenerate, so we conclude that $J'Tu = TJu$. But u was taken arbitrarily, so $J'T = TJ$, and by Proposition 2.5 T is complex linear. $\square$

It is well known that the unitary group can be expressed as $\mathbf{U}(n) = \mathbf{O}(2n, \mathbb{R}) \cap \mathbf{Sp}(2n, \mathbb{R}) \cap \mathrm{GL}(n, \mathbb{C})$, and we have a similar result for anti-Hermitian metrics.

Proposition 2.9. *Let be V a real vector space with complex structure J and an anti-Hermitian metric g compatible with J. Let $\mathrm{GL}(V, J) = \{T \in \mathrm{GL}(V) \mid JT = TJ\}$, and $\mathbf{O}(h, \mathbb{R}) = \{T \in \mathrm{GL}(V, \mathbb{R}) \mid h(Tu, Tv) = h(u, v) \,\forall\, u,\, v \in V\}$ for any real symmetric nondegenerate bilinear form h, then*

$$\mathbf{O}(g) = \mathrm{GL}(V, J) \cap \mathbf{O}(\mathrm{Re}(g), \mathbb{R}) \cap \mathbf{O}(\mathrm{Im}(g), \mathbb{R}).$$

Furthermore, it is the intersection of any two of these three. Thus a compatible pseudo-Riemannian metric and a complex structure give the third one, and so forth.

In particular, if g_0 is the usual anti-Hermitian metric on $\mathbb{C}^n$, then we get

$$\mathbf{O}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C}) \cap \mathbf{O}(n, n) \cap \mathbf{O}(S, \mathbb{R}),$$

where $\mathbf{O}(S, \mathbb{R}) = \left\{T \in \mathrm{GL}(2n, \mathbb{R}) \mid T^T S T = S\right\}$, and $S = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$.

At the level of equations

$$\begin{array}{r|l} \textit{Complex} & TJ_0 = J_0 T \\ \textit{Real part} & T^T I_{n,n} T = I_{n,n} \\ \textit{Imaginary part} & T^T S T = S \end{array}$$

Proof. We take $T \in \mathbf{O}(g)$, by the lemma above, $T \in \mathrm{GL}(V, J)$. Given $u,\, v \in V$,

$$\begin{aligned} \mathrm{Re}(g)(Tu, Tv) &= \frac{1}{2}(g(Tu, Tv) + \overline{g(Tu, Tv)}) \\ &= \frac{1}{2}(g(u, v) + \overline{g(u, v)}) \\ &= \mathrm{Re}(g)(u, v) \end{aligned}$$

and the proof is analogous for $\mathrm{Im}(g)$, so we have one inclusion.

If $T \in \mathrm{GL}(V, J) \cap \mathbf{O}(\mathrm{Re}(g), \mathbb{R}) \cap \mathbf{O}(\mathrm{Im}(g), \mathbb{R})$, then

$$\begin{aligned} g(Tu, Tv) &= \mathrm{Re}(g)(Tu, Tv) + i\,\mathrm{Im}(g)(Tu, Tv) \\ &= \mathrm{Re}(g)(u, v) + i\,\mathrm{Im}(g)(u, v) \\ &= g(u, v). \end{aligned} \tag{2.2}$$

It only remains to prove that the intersection of any two implies the third one.

If $T \in \mathrm{GL}(V, J) \cap \mathbf{O}(\mathrm{Re}(g), \mathbb{R})$, then using Proposition 2.4,

$$\begin{aligned} \mathrm{Im}(g)(Tu, Tv) &= -\mathrm{Re}(g)(JTu, Tv) \\ &= -\mathrm{Re}(g)(TJu, Tv) \\ &= -\mathrm{Re}(g)(Ju, v) \\ &= \mathrm{Im}(g)(u, v), \end{aligned}$$

thus $T \in \mathbf{O}(\mathrm{Im}(g), \mathbb{R})$.

If $T \in \mathrm{GL}(V, J) \cap \mathbf{O}(\mathrm{Im}(g), \mathbb{R})$, we first note that by Proposition 2.4 we can conclude that $\mathrm{Re}(g)(u, v) = \mathrm{Im}(g)(Ju, v)$, so

$$\begin{aligned}\mathrm{Re}(g)(Tu, Tv) &= \mathrm{Im}(g)(JTu, Tv) \\ &= \mathrm{Im}(g)(TJu, Tv) \\ &= \mathrm{Im}(g)(Ju, v) \\ &= \mathrm{Re}(g)(u, v),\end{aligned}$$

thus $T \in \mathbf{O}(\mathrm{Re}(g), \mathbb{R})$.

If $T \in \mathbf{O}(\mathrm{Re}(g), \mathbb{R}) \cap \mathbf{O}(\mathrm{Im}(g), \mathbb{R})$, then by equation (2.2) $T \in \mathbf{O}(g)$, so by the lemma above, $T \in \mathrm{GL}(V, J)$.

The last statement comes easily from the relation $J_0 I_{n,n} = S$. □

We end this section an analogous proposition for skew-symmetric linear maps that will be useful in the last sections of this chapter, but first, we give some notation. In the rest of this work, for a complex (real) vector space W, and a complex (real) symmetric bilinear form g, we will denote with $\mathfrak{so}(W, g)$ the real (complex) Lie algebra of linear maps on W that are skew-symmetric with respect to g.

Proposition 2.10. *Let V be a vector space with complex structure J and g an anti-Hermitian metric compatible with J. Then $f \in \mathfrak{so}(V, g)$ if and only if $J \circ f = f \circ J$ and $f \in \mathfrak{so}(V, h)$, where $h = \mathrm{Re}(g)$.*

Proof. If $f \in \mathfrak{so}(V, g)$ then f is complex linear and so, by Proposition 2.5, it must commute with J. Let u, $v \in V$. Then

$$h(fu, v) = \mathrm{Re}(g(fu, v)) = \mathrm{Re}(-g(u, fv)) = -h(u, fv).$$

Conversely, if $f \in \mathfrak{so}(V, h)$ and commutes with J, then, using Proposition 2.4,

$$\begin{aligned}g(fu, v) &= h(fu, v) - ih(Jfu, v) \\ &= -h(u, fv) - ih(fJu, v) \\ &= -h(u, fv) + ih(Ju, fv) \\ &= -g(u, fv)\end{aligned}$$

□

2.2 Anti-Hermitian metrics on manifolds

Let us remember the concept of an almost-complex structure. An *almost-complex structure* on a real manifold M is a $(1, 1)$-tensor field J which is, at every point $x \in M$, an endomorphism of T_xM such that $J_x^2 = -I$. A fixed pair (M, J) is called an almost-complex manifold. It is well known that every almost complex manifold is even dimensional and orientable.

Let (M, J) be an almost-complex manifold. Let us recall that J equips TM with a complex vector bundle structure. In this work we will refer to $TM_{\mathbb{C}}$ as the complex vector bundle and let TM refer to the same set considered as a real vector bundle.

At the risk of being overly pedantic, let us point out that here $TM_{\mathbb{C}}$ is not the complexification of TM, which we denote by $TM \otimes \mathbb{C}$, because the fibers of $TM \otimes \mathbb{C}$ are $T_xM \otimes \mathbb{C}$ and the fibers of $TM_{\mathbb{C}}$ are T_xM considered as complex vector spaces via J_x. Thus, if M has a complex structure J, then $TM_{\mathbb{C}}^*$ is the complex dual, that is, each fiber is the complex vector space of complex linear functions $T_xM \to \mathbb{C}$.

Let M be a smooth manifold. A Riemannian metric g on M is classically defined as a nondegenerate, symmetric, covariant 2-tensor which assigns an inner product to each tangent space. This definition is equivalent to saying that a Riemannian metric is a section g of $\mathrm{Sym}^2(TM^*)$ which restricts to each fiber as a positive-definite symmetric quadratic form.

So, if M is an almost-complex manifold, our objective is to find a definition of a 'complex metric' that resembles the bilinear and symmetric properties of the Riemannian metric. But thanks to the discussion of the previous section this is now an easy task.

Definition 2.11. An anti-Hermitian metric g on an almost-complex manifold (M, J) with almost-complex structure J is a smooth section of the complex vector bundle $\mathrm{Sym}^2(TM_{\mathbb{C}}^*)$ that restricts to each fiber as an anti-Hermitian metric.

We use the notation (M, J, g) or just M when no confusion arises.

Remark. By Proposition 2.4 we could define an anti-Kähler metric in terms of its real part, i.e. we could say that an anti-Kähler metric is a pseudo-Riemannian metric h such that

$$h(JX, Y) = h(X, JY)$$

for all X, $Y \in \mathfrak{X}(M)$, and by the same proposition h must has signature (n, n). But the definitions are equivalent and the one we give here is best suited for our purposes.

The definition of an anti-Hermitian metric is not related with the more popular Hermitian metrics. This is because there is an identification between the usual definition of Hermitian metrics and sections of the vector bundle $\overline{TM}^* \otimes TM^*$ that restricts to each fiber as an Hermitian product.

Let us point out that anti-Hermitian metrics do not have a signature. This is consequence of the Sylvester's law of inertia, as seen in Proposition 2.2, and thus there is no difference between the Riemannian and the pseudo-Riemannian cases. However, we are interested in taking the real part of a complex metric.

As we saw in the remark above, if (M, J) is an almost-complex manifold of complex dimension n with an anti-Hermitian metric g, then $h = \mathrm{Re}(g)$ gives M the structure of a pseudo-Riemannian metric (with signature (n, n) as a real manifold that) satisfies

$$h(JX, Y) = h(X, JY) \tag{2.3}$$

for each X, $Y \in \mathfrak{X}(M)$. We say that h is compatible with J when equality (2.3) holds.

Conversely, given a pseudo-Riemannian metric h compatible with an almost-complex structure J, we obtain a complex Riemannian metric by setting

$$g(X, Y) = h(X, Y) - ih(JX, Y).$$

Since we can define an anti-Hermitian metric from a pseudo-Riemannian metric of signature (n, n) that is compatible with J, we have the following proposition.

Proposition 2.12. *Every Lie group G with a left-invariant complex structure J, admits a left-invariant anti-Hermitian metric.*

Proof. We can always find a left-invariant complex structure by defining a complex structure J_e on the Lie algebra $\mathfrak{g}$ and extending it by translations. Let $e \in G$ be the identity element. It is enough to find a symmetric nondegenerate form of signature (n, n) compatible with J_e on $\mathfrak{g}$, because we can extend it to TM by translations. We obtain a pseudo-Riemannian metric compatible with J_e by taking a complex base $e_1, \ldots, e_n$ as in Proposition 2.2, and considering the real basis $e_1, \ldots, e_n, J_e e_1, \ldots, J_e e_n$, and define $B(e_i, e_j) = \delta_{ij}, B(e_i, Je_j) = B(Je_i, e_j) = 0$, $B(J_e e_i, J_e e_j) = -\delta_{ij}$, and extend by bilinearity. The result is a left invariant pseudo-Riemannian metric h on M of signature (n, n). Finally, for $X, Y \in \mathfrak{X}(M)$ we define

$$g(X, Y) = h(X, Y) - ih(JX, Y),$$

and we are done. $\square$

Corollary 2.13. *Every semisimple complex Lie group admits an anti-Hermitian metricvia the Killing form.*

Proof. This is a consequence of the previous proposition and Proposition 2.6. $\square$

After defining an object, the following step is to define the isomorphisms between those objects.

Definition 2.14. An *isometry* between anti-Hermitian manifolds (M, J, g) and (M', J', g') is a diffeomorphism

$$f \colon M \to M'$$

such that $g_x(X, Y) = g'_{f(x)}(df_x X, df_x Y)$ for all $x \in M$ and $X, Y \in T_x M$.

A demanding reader should have noticed that we are not asking that f be a complex mapping. However, this comes for free thanks to Proposition 2.9.

Corollary 2.15. *If f is an isometry of anti-Hermitian manifolds, then f is an almost-complex mapping. Furthermore, f is an isometry for both the real and imaginary parts of g.*

2.2.1 Comparison with holomorphic metrics

In the literature there exists a similar concept that we now discuss. Let us briefly review the results of complex manifolds as in [18], again without proofs. If M is a complex manifold, then, by definition, the coordinate charts have range in $\mathbb{C}^n$ and overlap holomorphically. This is not necessarily true for an almost-complex manifolds in general. A complex manifold has a natural almost-complex structure. To show this we consider $(z^1, \ldots, z^n) \in \mathbb{C}^n$ with $z^j = x^j + iy^j$,

$j = 1, \ldots, n$. With respect to the coordinate system $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ we define an almost-complex structure J on $\mathbb{C}^n$ by

$$J(\partial/\partial x^j) = \partial/\partial y^j, \quad J(\partial/\partial y^j) = -\partial/\partial x^j, \quad j = 1, \ldots, n.$$

The holomorphic overlapping property provides a complex structure on M, via the coordinate charts and independent of them.

For each $x \in M$, if we extend J_x to the complexification of T_xM it will have two eigenvalues i and $-i$ and $T_xM \otimes \mathbb{C}$ will split into two eigenspaces $T_xM \otimes \mathbb{C} = T_x^{(1,0)}M \oplus T_x^{(0,1)}M$. The i-eigenspace $T_x^{(1,0)}M$ is the holomorphic tangent space with basis $\dfrac{\partial}{\partial z^j} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^j} - i\dfrac{\partial}{\partial y^j}\right)$, $j = 1, \ldots, n$ and $T^{(0,1)}M$ is the anti-holomorphic tangent space with basis $\dfrac{\partial}{\partial \bar{z}^j} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^j} + i\dfrac{\partial}{\partial y^j}\right)$, $j = 1, \ldots, n$,

A *holomorphic vector field* Z on a complex manifold (M, J) is a complex vector field of type $(1,0)$, i.e. $Z_x \in T_x^{(1,0)}M$ for all $x \in M$, such that Zf is holomorphic for every locally-defined holomorphic function $f\colon U \subset M \to \mathbb{C}$. If we write

$$Z = \sum_{j=1}^{n} f^j(\partial/\partial z^j)$$

in terms of $z^1, \ldots, z^n$, then Z is holomorphic if and only if the components f^j are all holomorphic functions.

A *holomorphic metric* on a complex manifold (M, J) can be defined either as a holomorphic section $g\colon M \to \mathrm{Sym}^2(T^{(1,0)}M)$ or a holomorphic assignation $x \mapsto g_x$ where g_x is a symmetric nondegenerate complex bilinear form on $T_x^{(1,0)}M$. Locally this kind of metric looks like

$$g = \sum g^{jk} dz^j \otimes dz^k, \tag{2.4}$$

where $g^{jk} = g\left(\dfrac{\partial}{\partial z^j}, \dfrac{\partial}{\partial z^k}\right)$ are $\mathbb{C}$-valued holomorphic functions and are the entries of a non-singular, symmetric matrix. It is worth mentioning that holomorphic metrics are defined on complex manifolds via the almost-complex structure, but use the full power as complex manifold (the Cauchy-Riemann equations).

Anti-Hermitian metrics generalize this idea in the sense that they are defined for almost-complex manifolds and 'almost coincide' with the holomorphic metrics. We now make this precise.

Let h be an the real part of anti-Hermitian metric g on (M, J), as we have seen above, it is a pseudo-Riemannian metric that satisfies:

$$h(JX, JY) = -h(X, Y)$$

or equivalently:

$$h(JX, Y) = h(X, JY)$$

Then, by Section 2.1 the metric h necessarily has a neutral (Kleinian) signature (n, n). We recall that $g = h - ih(J\cdot, \cdot)$ is the anti-Hermitian metric induced by h. Let be $X, Y \in \mathfrak{X}(M)$ and $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ a coordinate system as above, thus locally, $\partial/\partial x^1, \ldots, \partial/\partial x^n, \partial/\partial y^1, \ldots, \partial/\partial y^n$ form a *real* basis for the tangent space at each point. If $X^j = \partial/\partial x^j$ and $Y^j = \partial/\partial y^j$ then, using the Einstein notation, we write

$$\begin{aligned} g &= g(X^j, X^k) dx^j \otimes dx^k + g(X^j, Y^k) dx^j \otimes dy^k \\ &\quad + g(Y^j, X^k) dy^j \otimes dx^k + g(Y^j, Y^k) dy^j \otimes dy^k \\ &= g\left(\frac{1}{2}(X^j - iY^j), \frac{1}{2}(X^k - iY^k)\right)(dx^j + idy^j) \otimes (dx^k + idy^k) \\ &= g\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right) dz^j \otimes dz^k. \end{aligned}$$

In the last expression, $g\left(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}\right)$ are complex-valued smooth functions, in contrast with the holomorphic metrics where the functions must necessarily be holomorphic.

The previous calculations seem to be tricky since we are not using the complexified vector bundle $TM \otimes \mathbb{C}$ and complex vector fields, but rather TM considered as a complex vector bundle via J and real vector fields. Let us analyze the complexified case.

Let (M, J) be an almost-complex manifold, h and g as above. We extend h in the well-known way by $\mathbb{C}$-linearity to the complexification of the tangent bundle $TM \otimes \mathbb{C}$. Let us fix a (real) basis $\{X^1, \ldots, X^n, JX^1, \ldots, JX^n\}$ in each tangent space T_xM. Then the set $\left\{Z^j, Z^{\bar{j}}\right\}$, where $Z^j = \frac{1}{2}(X^j - iY^j)$, $Z^{\bar{j}} = \frac{1}{2}(X^j + iY^j)$, forms a basis for each complexified tangent space $T_xM \oplus \mathbb{C}$. We set $h^{jk} = h(Z^j, Z^k)$. Then the following holds:

Proposition 2.16. *Let (M, J) be an almost-complex manifold and h the real part of an anti-Hermitian metric g on M. Then the complex extended pseudo-Riemannian metric h (in the complex basis introduced above) satisfies the following conditions*

$$h^{j\bar{k}} = h^{\bar{j}k} = 0 \tag{2.5}$$

$$h^{\bar{j}\bar{k}} = \overline{h^{jk}}. \tag{2.6}$$

Conversely, if the complex extended metric h satisfies (2.5)-(2.6) *then the initial metric must be the real part of an anti-Hermitian metric.*

Proof. In the basis $\left\{Z^j, Z^{\bar{j}}\right\}$

$$\begin{aligned} 4h(Z^j, Z^{\bar{k}}) &= h(X^j - iJX^j, X^k + iJX^k) \\ &= h(X^j, X^k) + h(JX^j, JX^k) + i(h(X^j, JX^k) - h(JX^j, X^k)) \\ &= h(X^j, X^k) - h(X^j, X^k) + i(h(JX^j, X^k) - h(JX^j, X^k)) \\ &= 0. \end{aligned}$$

In the third equality we are using the compatibility between h and J. The proof for $h(Z^{\bar{j}}, Z^k)$ is analogous.

$$
\begin{aligned}
4h(Z^{\bar{j}}, Z^{\bar{k}}) &= h(X^j + iJX^j, X^k + iJX^k) \\
&= h(X^j, X^k) - h(JX^j, JX^k) + i(h(X^j, JX^k) + h(JY^j, X^k)) \\
&= \overline{h(X^j, X^k) - h(JX^j, JX^k) - i(h(X^j, JX^k) + h(JY^j, X^k))} \\
&= \overline{h(X^j - iJX^j, X^k - iJX^k)} \\
&= \overline{4h(Z^j, Z^k)}
\end{aligned}
$$

Conversely, we can always express h as

$$h = h^{jk} dz^j \otimes dz^k + h^{j\bar{k}} dz^j d\bar{z}^k + h^{\bar{j}k} d\bar{z}^j \otimes dz^k + h^{\bar{j}\bar{k}} d\bar{z}^j \otimes d\bar{z}^k.$$

Hence if the complex extended metric h satisfies (2.5)-(2.6) and h^{jk} are the entries of nondegenerate symmetric matrix, we get

$$h = h^{jk} dz^j \otimes dz^k + \overline{h^{jk}} d\bar{z}^j \otimes d\bar{z}^k. \tag{2.7}$$

It is straightforward to prove the symmetry, nondegeneracy and bilinearity. To check the compatibility with J, let X, Y be complex vector fields. Then

$$
\begin{aligned}
X &= a^j Z^j + b^j Z^{\bar{j}}, \\
Y &= c^k Z^k + d^k Z^{\bar{k}},
\end{aligned}
$$

where a^j, b^j, c^j, d^j are local smooth complex functions. One has

$$
\begin{aligned}
JZ^j &= \frac{J}{2}(X^j - iJX^j) \\
&\frac{1}{2}(JX^j - iJ^2 X^j) \\
&\frac{1}{2}(iX^j + JX^j) \\
&= \frac{1}{1}(iX^j - i^2 JX^j) \\
&= iZ^j,
\end{aligned}
$$

and similarly $JZ^{\bar{j}} = -iZ^{\bar{j}}$. It is enough to prove the result for $dz^j \otimes dz^k$ and $d\bar{z}^j \otimes d\bar{z}^k$:

$$
\begin{aligned}
d\bar{z}^j \otimes d\bar{z}^k(JX, Y) &= d\bar{z}^j \otimes d\bar{z}^k(J(a^j Z^j + b^j Z^{\bar{j}}), c^k Z^k + d^k Z^{\bar{k}}) \\
&= d\bar{z}^j \otimes d\bar{z}^k(ia^j Z^j - ib^j Z^{\bar{j}}), c^k Z^k + d^k Z^{\bar{k}}) \\
&= -ib^j d^k \\
&= d\bar{z}^j \otimes d\bar{z}^k(a^j Z^j + b^j Z^{\bar{j}}), ic^k Z^k - id^k Z^{\bar{k}}) \\
&= d\bar{z}^j \otimes d\bar{z}^k(a^j Z^j + b^j Z^{\bar{j}}, J(c^k Z^k + d^k Z^{\bar{k}})) \\
&= d\bar{z}^j \otimes d\bar{z}^k(X, JY)
\end{aligned}
$$

For $dz^j \otimes dz^k$ is similar. $\square$

To end this subsection, we emphasize the difference between the expressions (2.4) and (2.7):

$$g = \sum g^{jk} dz^j \otimes dz^k \quad \text{and} \quad h = h^{jk} dz^j \otimes dz^k + \overline{h^{jk}} d\bar{z}^j \otimes d\bar{z}^k.$$

2.3 Jet bundles and geometric structures

Let us review the basic concepts of jets and geometric structures. Propositions and assertions are stated without proof because they are contained in [13].

The concept of the germ of a function is extensively used in the mathematical world. A notion of 'differential' germs of functions rises naturally in the category of smooth manifolds and depending on the degree of the derivative we called them r-jets.

Definition 2.17. Let M and Q be smooth manifolds and $f, g\colon M \to Q$ smooth maps. We say that f and g define the same *r-jet at* x if $f(x) = g(x)$ and they have the same partial derivatives up to order r at x with respect to some choice of smooth coordinates around x and $f(x)$. The equivalence relation determined by f at x is called *the r-jet of* f and does not depend on the choice of coordinates. We denote it by $j^r_x(f)$.

Let $J^r_n(Q)$ denote the set of r-jets at the origin of smooth maps $f\colon \mathbb{R}^n \to Q$. For simplicity in notation the r-jet of a smooth map $f\colon \mathbb{R}^n \to Q$ at the origin $0 \in \mathbb{R}^n$ will be denoted simply by $j^r(f)$ when no confusion can arise.

There is a natural smooth manifold structure on $J^r_n(Q)$ given by the following identification: if $\phi\colon U \to \mathbb{R}^m$ is a diffeomorphism from an open subset U of Q, then $\tilde{\phi}\colon J^r_n(U) \to J^r_n(\mathbb{R}^m)$ is a homeomorphism induced by ϕ, where $J^r_n(\mathbb{R}^m)$ is canonically isomorphic to $\prod\limits_{k=0}^{r} S_k(\mathbb{R}^n;\mathbb{R}^m)$. Here $S_k(\mathbb{R}^n;\mathbb{R}^m)$ stands for the vector space of symmetric $\mathbb{R}^m$-valued k-multi-linear transformations on $\mathbb{R}^n$.

It is important to note that if Q is a Lie group, then $J^r_n(Q)$ inherits a group structure defined by $j^r(g_1)j^r(g_2) = j^r(g_1g_2)$.

Let $\mathrm{Gl}^{(k)}(n)$ denote the group of k-jets at 0 of diffeomorphisms of $\mathbb{R}^n$ that fix 0. As a manifold:

$$\mathrm{Gl}^{(k)}(n) = \{(A, L_2, \ldots, L_k) \mid A \in \mathrm{GL}(n),\ L_j \in S_j(\mathbb{R}^n;\mathbb{R}^n),\ \text{for every } j \geq 2\}\,.$$

$\mathrm{Gl}^{(k)}(n)$ is in fact a Lie group. $\mathrm{Gl}^{(1)}(n)$ is the general linear group $\mathrm{GL}(n,\mathbb{R})$ and, for any pair of integers $k \leq l$, there is a canonical homomorphism

$$\pi^l_k : \mathrm{Gl}^{(l)}(n) \to \mathrm{Gl}^{(k)}(n).$$

Let $\mathfrak{gl}^{(k)}(n)$ denote the space of k-jets at 0 of vector fields on $\mathbb{R}^n$ that vanish at 0. The bracket of two elements $j^k(X)$, $j^k(Y) \in \mathfrak{gl}^{(k)}(n)$ is defined by

$$\left[j^k(X), j^k(Y)\right] = -j^k([X,Y]).$$

This provides a Lie algebra structure on $\mathfrak{gl}^{(k)}(n)$. It is not surprising that $\mathfrak{gl}^{(k)}(n)$ is the Lie algebra of $\mathrm{Gl}^{(k)}(n)$.

To define a geometric structure on a manifold we need an important ingredient. We start with a smooth manifold M of dimension n and let $L^{(k)}(M)$ denote the kth-order frame bundle of M, that is the collection of k-jets at the origin $0 \in \mathbb{R}^n$ of diffeomorphisms from a neighborhood of $0 \in \mathbb{R}^n$ into M (frames). It has a natural structure of a smooth manifold as a submanifold of $J^r_n(M)$. $L^{(k)}(M)$ has an additional structure: it is in fact a locally-trivial principal fiber bundle over M with structure group $\mathrm{Gl}^{(k)}(n)$. The bundle map

$\pi\colon L^{(k)}(M) \to M$ is the obvious base point projection, that is, given $j^k(\varphi)$ we associate to it the point $\varphi(0)$. The natural right action of $\mathrm{Gl}^{(k)}(n)$ is given by

$$(j^k(\varphi), j^k(f)) \mapsto j^k(\varphi) \circ j^k(f) = j^k(\varphi \circ f).$$

Remark. Let us note that $L(M) = L^{(1)}(M)$ is isomorphic to the frame bundle.

Definition 2.18. Let Q be a manifold on which $\mathrm{Gl}^{(k)}(n)$ acts smoothly on the left. *A geometric structure of order* k and type Q on M is a $\mathrm{Gl}^{(k)}(n)$-equivariant map $\sigma\colon L^{(k)}(M) \to Q$.

Remark. By the same arguments as in [17], there exists a natural correspondence of $\mathrm{Gl}^{(k)}(n)$-equivariant maps $L^{(k)}(M) \to Q$ and sections of the fiber bundle $Q^k(M)$ over M, which is the quotient of $L^{(k)}(M) \times Q$ by the action of $\mathrm{Gl}^{(k)}(n)$ given by $(\alpha, q)g = (\alpha g, g^{-1}q)$. However we will only take into consideration the above definition.

A geometric structure will be called of algebraic type Q, or just of algebraic type, if Q is a real algebraic variety and the action of $\mathrm{Gl}^{(k)}(n)$ is algebraic.

Before defining rigidity as Gromov did, we need some of the concepts provided below. One of them is the prolongation of a geometric structure. For this we will require the homomorphic embedding of $\mathrm{Gl}^{(k+r)}(n)$ into $\mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$.

Let $a\colon \mathrm{Gl}^{(k+r)}(n) \to J^r_n(\mathrm{Gl}^{(k)}(n))$ be defined as follows. If $g \in \mathrm{Gl}^{(k+r)}(n)$ is of the form $g = j^{k+r}(f)$, let $f_k : \mathbb{R}^n \to \mathrm{Gl}^{(k)}(n)$ be the map given by

$$f_k(x) = j^k(\tau_{-x} \circ f \circ \tau_{f^{-1}(x)}),$$

where $\tau_v(y) = y + v$ is the translation by v in $\mathbb{R}^n$, and set $a(g) = j^r(f_k)$. The map a satisfies $a(g_1 g_2) = a(g_1)a(g_2) \circ \pi^{k+r}_r(g_1^{-1})$ where π^{k+r}_r is the natural projection as mentioned above. Let $\mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$ be the semi-direct product with group multiplication $(g, h)(g', h') = (gg', h(h' \circ g^{-1}))$. Then the map

$$(\pi^{k+r}_r, a) : \mathrm{Gl}^{(k+r)}(n) \to \mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n))$$

is a homomorphism of Lie groups.

Therefore, if Q is a smooth manifold which admits a smooth left action of $\mathrm{Gl}^{(k)}(n)$, then $J^r(Q)$ admits a natural smooth action of $\mathrm{Gl}^{(r)}(n) \ltimes J^r_n(\mathrm{Gl}^{(k)}(n)$ given by $(g, h)q = h{\cdot}(q{\circ}g^{-1})$, where the dot product is defined by $j^r(f_1){\cdot}j^r(f_2) = j^r(f_1 f_2)$. With the above discussion of a, this induces a canonical action

$$\mathrm{Gl}^{(k+r)}(n) \times J^r_n(Q) \to J^r_n(Q)$$

given by $gq = a(g) \cdot (q \circ \pi^{k+r}_r(g^{-1}))$.

From now on, let $\sigma\colon L^{(k)}(M) \to Q$ be the $\mathrm{Gl}^{(k)}(n)$-equivariant map that defines a geometric structure on a smooth manifold M. Then the rth prolongation of σ is the geometric structure whose associated $\mathrm{Gl}^{(k+r)}(n)$-equivariant map is given by

$$\begin{aligned}\sigma^r\colon L^{(k+r)}(M) &\to J^r_n(Q)\\ j^{k+r}(\varphi) &\mapsto j^r(\sigma(j^k(\varphi \circ \tau_\bullet))),\end{aligned}$$

where $\sigma^r(j^k(\varphi \circ \tau_\bullet))$ denotes the map $\mathbb{R}^n \ni v \mapsto \sigma(j^k(\varphi \circ \tau_v)) \in Q$.

A smooth diffeomorphism f of the manifold M induces a bundle diffeomorphism $f_{(k)}$ on $L^{(k)}(M)$ by setting $f_{(k)}(j^r(\varphi)) = j^r(f \circ \varphi)$. If $g \in \mathrm{Gl}^{(k)}(n)$ and $\alpha \in L^{(k)}(M)$ then $f_{(k)}(\alpha g) = f_{(k)}(\alpha)g$, so that $f_{(k)}$ is an automorphism of the principal $\mathrm{Gl}^{(k)}(n)$-bundle $L^{(k)}(M)$.

Definition 2.19. A (local) diffeomorphism $f\colon M \to M$ is called a (local) automorphism of σ if it (locally) satisfies $\sigma \circ f_{(k)} = \sigma$. If this happens, we say that f preserves σ. The group of diffeomorphisms of M that preserve σ is denoted by $\mathrm{Aut}(\sigma)$ and is called the group of automorphisms (or isometries) of σ. Similarly, $\mathrm{Aut}^{\mathrm{loc}}(\sigma)$ denotes the pseudogroup of local diffeomorphisms of M which preserve σ.

For x, $y \in M$, let $D^{(k)}_{x,y}(M)$ denote the space of k-jets at x of diffeomorphisms from a neighborhood of $x \in M$ into M and which send x to y. The group $D^{(k)}_x(M) = D^{(k)}_{x,x}(M)$ has a Lie group structure under which it is isomorphic to $\mathrm{Gl}^{(k)}(n)$. Its Lie algebra is $\mathcal{D}^{(k)}_x(M)$, the space of k-jets of vector fields on M vanishing at x.

If $j^k_x(f) \in D^{(k)}_{x,y}(M)$, then $f_{(k)}$ maps the fiber of $L^{(k)}(M)$ over x onto the fiber over y so that the $\mathrm{Gl}^{(k)}(n)$-equivariant map which it defines on such fiber depends only on the jet $j^k_x(f)$. In particular, the Lie group $D^{(k)}_x(M)$ acts transitively on the fiber of $L^{(k)}(M)$ over x, which we denote by $L^{(k)}(M)_x$, and this action commutes with the action of $\mathrm{Gl}^{(k)}(n)$.

Even though we have the concept of local automorphism we need an infinitesimal version for defining rigidity, essentially motivated by the discussion of the previous paragraph.

Definition 2.20. Let σ be a geometric structure of order k and type Q on M. For x, $y \in M$ and $j^{k+r}_x(f) \in D^{(k+r)}_{x,y}(M)$, we say that f *preserves σ up to order r* if $\sigma^r(f_{k+r}(\alpha)) = \sigma^r(\alpha)$ for every α in the fiber of $L^{(k+r)}(M)$ above x. The set

$$\mathrm{Aut}^{k+r}(\sigma, x, y) = \left\{ j^{k+r}_x(f) \in D^{(k+r)}_{x,y}(M) \,|\, f \text{ preserves } \sigma \text{ up to order } r \right\}$$

is called the set of infinitesimal automorphisms of σ of order $k + r$ taking x to y. For simplicity, let $\mathrm{Aut}^{k+r}(\sigma, x)$ denote $\mathrm{Aut}^{k+r}(\sigma, x, x)$.

Definition 2.21. Let r be a nonnegative integer. A geometric structure σ of order k on M is said to be *r-rigid* if, for every $x \in M$, the canonical projection

$$\pi^{k+r+1}_{k+r}\colon \mathrm{Aut}^{k+r+1}(\sigma, x) \to \mathrm{Aut}^{k+r}(\sigma, x)$$

is injective.

It is worth mentioning that most of the classical geometric structures that we know are generalized by these concepts, in particular the ones we use in this work, i.e. pseudo-Riemannian metrics (0-rigid geometric structures of order 1 of algebraic type [5]) and almost-complex structures, which are geometric structures (in the above sense) of order 1, but not rigid.

Next we make a precise identification of both, letting aside the technical proofs for the properties (as smoothness) of the objects and maps involved.

First, let us remember that a pseudo-Riemannian metric h on M is a smooth assignation $x \mapsto h_x$ where $h_x\colon T_xM \times T_xM \to \mathbb{R}$ is a symmetric bilinear nondegenerate form of constant signature.

Proposition 2.22. *Let*

$$\mathfrak{p}_{p,q} = \left\{ B\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \;\middle|\; \begin{array}{l} B \textit{ is a symmetric, nondegenerate} \\ \textit{bilinear form of order } p,q \end{array} \right\}.$$

If the action of $\mathrm{GL}(n,\mathbb{R})$ *on* $\mathfrak{p}$ *is given by* $A \cdot B = B(A^{-1}, A^{-1})$*, then there exists a correspondence between pseudo-Riemannian metrics* h *on* M *of signature* (p,q) *and* $\mathrm{GL}(n,\mathbb{R})$*-equivariant maps* $\sigma\colon L(M) \to \mathfrak{p}_{p,q}$*.*

Let us recall that $L(M) = L^{(1)}(M)$ is isomorphic to the frame bundle, if the reader finds it more convenient. Given an element $\alpha \in L(M)$ we can think of it as a linear isomorphism $\alpha\colon \mathbb{R}^n \to T_xM$ that depends on $\pi(\alpha) = x$. If h is a pseudo-Riemannian metric on M, we define $\sigma_h\colon L(M) \to \mathfrak{p}_{p,q}$ in the following fashion: given $\alpha \in L(M)$ we set $\sigma_h(\alpha)(u,v) := h_{\pi(\alpha)}(\alpha(u), \alpha(v))$. This result in an algebraic $\mathrm{GL}(n,\mathbb{R})$-equivariant map. Conversely, if $\sigma\colon L(M) \to \mathfrak{p}_{p,q}$ is a $\mathrm{GL}(n,\mathbb{R})$-equivariant map of constant signature, then we define $h^\sigma_x(X,Y) = \sigma(\alpha)(\alpha^{-1}X, \alpha^{-1}Y)$ for any α such that $\pi(\alpha) = x$, which does not depend of the election. Then we get a pseudo-Riemannian metric. It is worth mentioning that $\mathrm{Iso}(M,h) = \mathrm{Aut}(M,\sigma_h)$.

Now we do the same analysis for almost-complex manifolds. Let (M,J) be an almost-complex manifold of complex dimension n. Let $\mathfrak{J}$ be the space of linear automorphisms T of $\mathbb{R}^n$ such that $T^2 = -I$ (this forces n to be even), and the $\mathrm{GL}(n,\mathbb{R})$-action on $\mathfrak{J}$ given by $A \cdot T = ATA^{-1}$. Then there exists a correspondence between almost-complex structures J on M and $\mathrm{GL}(n,\mathbb{R})$-equivariant maps $\sigma\colon L(M) \to \mathfrak{J}$. We give the correspondence now. Fixing J, we obtain a geometric structure $\sigma_J\colon L(M) \to \mathfrak{J}$ by doing $\sigma_J(\alpha) = \alpha^{-1}J_{\pi(\alpha)}\alpha$. Conversely, given a geometric structure $\sigma\colon L(M) \to \mathfrak{J}$, we define $J^\sigma_x = \alpha\sigma(\alpha)\alpha^{-1}$ where α is such that $\pi(\alpha) = x$. Again, we have that $\mathrm{Aut}(M,\sigma_J)$ is the set of almost-complex diffeomorphisms of (M,J).

Complete parallelisms, linear connections, volume forms, (r,s)-tensor fields (in particular pseudo-Riemannian metrics and complex structures), symplectic structures, contact structures, A-geometric structures (in the sense of Cartan) are all geometric structures in the sense we defined above If clarification is needed, we will refer it as 'the sense of Gromov'.

2.4 Product of geometric structures

All the material comes from [7], except the last part, which will be clear when we arrive there.

Remark. Given an action of a closed subgroup $H < L$ on a space X, we can induce an action of L. The space acted upon is $(L \times X)/H$ where the H-action we quotient by is given by $(l,x)h = (lh, h^{-1}x)$. The L-action on the space is defined by the left L action on the first factor, which is well-defined on the quotient since it commutes with the H-action defined above. Note that this definition only works for left actions of H on X: analogous definitions allow us to induce right actions to right actions. If the action of H on X is algebraic and H is an algebraic subgroup of an algebraic group L, then the induced action is an algebraic action on an algebraic variety. Also, if $F\colon X \to Y$ is an H-equivariant map, then $F'\colon (L \times X)/H \to (L \times Y)/H$ is an L-equivariant map.

Let M and N be differentiable manifolds of dimensions m and n respectively, Q_1, Q_2 algebraic varieties acted upon by $\mathrm{Gl}^{(k_1)}(m)$ and $\mathrm{Gl}^{(k_2)}(n)$, and $\sigma_1\colon L^{(k_1)}(M) \to Q_1$ and $\sigma_2\colon L^{(k_2)}(N) \to Q_2$ geometric structures. If we consider prolongations of the structures, we may suppose that $k = k_1 = k_2$. To define a geometric structure on $M \times N$, we begin with the map

$$\sigma_1 \times \sigma_2\colon L^{(k)}(M) \times L^{(k)}(N) \to Q_1 \times Q_2$$

and, using the inclusion $\mathrm{Gl}^{(k)}(n) \times \mathrm{Gl}^{(k)}(m) < \mathrm{Gl}^{(k)}(n+m)$, induce to a map

$$(\sigma_1 \times \sigma_2)'\colon (\mathrm{Gl}^{(k)}(n+m) \times L^{(k)}(M) \times L^{(k)}(N))/(\mathrm{Gl}^{(k)}(n) \times \mathrm{Gl}^{(k)}(m)) \to V$$

where $V = (\mathrm{Gl}^{(k)}(n+m) \times Q_1 \times Q_2)/(\mathrm{Gl}^{(k)}(n) \times \mathrm{Gl}^{(k)}(m))$ is an algebraic variety provided that Q_1 and Q_2 are algebraic. We are using the above remark as follows: $L = \mathrm{Gl}^{(k)}(n+m)$, $H = \mathrm{Gl}^{(k)}(n) \times \mathrm{Gl}^{(k)}(m)$, $X = L^{(k)}(M) \times L^{(k)}(N)$ and $Y = Q_1 \times Q_2$.

Let us note that $(\mathrm{Gl}^{(k)}(n+m) \times L^{(k)}(M) \times L^{(k)}(N))/(\mathrm{Gl}^{(k)}(n) \times \mathrm{Gl}^{(k)}(m))$ may be canonically identified with $L^{(k)}(M \times N)$, if we define

$$\begin{aligned} \mathrm{Gl}^{(k)}(n+m) \times L^{(k)}(M) \times L^{(k)}(N) &\to L^{(k)}(M \times N) \\ \big(j^k(\varphi), j^k(f), j^k(g)\big) &\mapsto j^k((f \times g) \circ \varphi), \end{aligned}$$

where $f \times g\colon \mathbb{R}^m \times \mathbb{R}^n \to M \times N$ is the map $(x_1, x_2) \mapsto (f(x_1), g(x_2))$, and finally pass to the quotient (it is constant on the fibers).

About the rigidity of the product, we have the next result that we state without proof because is in the reference mentioned above.

Proposition 2.23. *If $\sigma_1\colon L^{(k)}(M) \to Q$ and $\sigma_2\colon L^{(l)}(N) \to Q_2$ are r-rigid geometric structures, then the product structure $(\sigma_1 \times \sigma_2)'\colon L^{(\max(k,l))}(M \times N) \to V$ is also r-rigid.*

Now, if we have two different geometric structures on the same manifold M, $\sigma_j\colon L^{(k_j)}(M) \to Q_j$, $j = 1$, 2, then we can consider the product as above, and then restrict to the diagonal ($M \hookrightarrow M \times M$) or try to make them work as a single one in the next fashion. As above, taking into account prolongations, we may suppose $k = \max(k_1, k_2)$, and then we consider the map

$$\sigma_1 \times \sigma_2\colon L^{(k)}(M) \to Q_1 \times Q_2$$

In this case, we do not need to go through the quotient and we have a better condition for rigidity.

Proposition 2.24. *If $\sigma_j\colon L^{(k_j)}(M) \to Q_j$, $j = 1$, 2, are geometric structures, and σ_1 is r-rigid, then the product structure*

$$\sigma_1 \times \sigma_2\colon L^{(\max(k_1,k_2))}(M) \to Q_1 \times Q_2$$

is also r-rigid.

Proof. By passing to a prolongation of one structure, it suffices to consider $k = \max(k_1, k_2)$. A k-jet of a diffeomorphism of M fixing a point x leaves $\sigma_1 \times \sigma_2$ invariant if and only if it leaves invariant σ_1 and σ_2. We conclude that

$$\mathrm{Aut}^k(\sigma_1 \times \sigma_2, x) = \mathrm{Aut}^k(\sigma_1, x) \cap \mathrm{Aut}^k(\sigma_2, x), \tag{2.8}$$

so if σ_1 is r-rigid, that is $\mathrm{Aut}^{k+r+1}(\sigma_1, x) \to \mathrm{Aut}^{k+r}(\sigma_1, x)$ is injective for all $x \in M$, then $\mathrm{Aut}^{k+r+1}(\sigma_1 \times \sigma_2, x) \to \mathrm{Aut}^{k+r}(\sigma_1 \times \sigma_2, x)$ is injective for all $x \in M$. $\square$

Returning to the anti-Hermitian metric, let (M, J) be an almost-complex manifold of real dimension $2n$. By abuse of notation J would mean both the complex structure and the geometric structure induced, as was seen in Section 2.3. Let us suppose that M accepts an anti-Hermitian metric g. Let $\mathfrak{C}$ denote the set of $B\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{C}$ such that B is bilinear, symmetric, nondegenerate, and compatible with some complex structure on $\mathbb{R}^{2n}$, i.e. an anti-Hermitian metric on $\mathbb{R}^{2n}$. Then we can think of g as a geometric structure $\sigma\colon L(M) \to \mathfrak{C}$ by defining $\sigma(\alpha)\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{C}$ as $\sigma(\alpha)(X, Y) = g_{\pi(\alpha)}(\alpha(X), \alpha(Y))$, And conversely, given a geometric structure $\sigma\colon L(M) \to \mathfrak{C}$, it is the same as asking for an anti-Hermitian metric on $\mathbb{R}^{2n}$ and a structure $J(\alpha)$ on $\mathbb{R}^{2n}$, namely the one that is compatible with $g(\alpha)$. We define $g_x(X, Y) = \sigma(\alpha)(\alpha^{-1}X, \alpha^{-1}Y)$ where $\alpha \in L(M)$ is any element such that $\pi(\alpha) = x$. This establishes a one-to-one correspondence between anti-Hermitian metrics and smooth geometric structures $\sigma\colon L(M) \to \mathfrak{C}$. It is worth mentioning that the automorphisms of σ are precisely the isometries of g so again by abuse of notation we make $\sigma = g$ and use them interchangeably.

If we take the real part of g, we saw at the end of Section 2.2 that $h = \mathrm{Re}(g)$ gives M the structure of a pseudo-Riemannian manifold. If $\mathfrak{p} = \{B\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R} \mid B \text{ is bilinear, symmetric and nondegenerate}\}$ and the action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathfrak{p}$ is given by $A \cdot B = B(A^{-1}, A^{-1})$ then h can be thought of as a $\mathrm{GL}(n, \mathbb{R})$ equivariant map $h\colon L(M) \to \mathfrak{p}$. We can recover g from h pretty much as in Section 2.2: $g(\alpha)(\cdot, \cdot) = h(\alpha)(\cdot, \cdot) - ih(\alpha)(J(\alpha)\cdot, \cdot)$

But we are interested in the automorphisms of g. The next corollary shows an important relation.

Corollary 2.25. *A diffeomorphism $f\colon M \to M$ is an automorphism of g, if and only if is an automorphism of h and J (and $\mathrm{Im}(g)$).*

Moreover, we have that $\mathrm{Aut}(g) = \mathrm{Aut}(h) \cap \mathrm{Aut}(J)$, $\mathrm{Aut}^{loc}(g) = \mathrm{Aut}^{loc}(h) \cap \mathrm{Aut}^{loc}(J)$ and $\mathrm{Aut}^{1+r}(g, x) = \mathrm{Aut}^{1+r}(h, x) \cap \mathrm{Aut}^{1+r}(J, x)$.

Proof. This a consequence of Proposition 2.9. $\square$

Then by equation (2.8) we conclude that study of the automorphisms of g is the same as the study of the automorphisms of $h \times J$ (with an extra condition: the compatibility of h and J.) By Proposition 2.24 we have the next corollary.

Corollary 2.26. *The anti-Hermitian metric g is rigid because h is.*

2.5 Killing fields of manifolds

Let us leave the anti-Hermitian metrics aside for a moment. The content of this section is based on the results obtained in [5] and [27].

For the sake of completeness we write the definitions and properties of Killing fields from [5].

Definition 2.27. Let M be a manifold and σ a geometric structure on M. A *(local) Killing field of σ on M* is a smooth vector field on M whose (local) flow acts on M by (local) automorphisms of σ. The space of Killing fields an local Killing fields of a geometric structure σ are denoted by $\mathrm{Kill}(\sigma)$ and $\mathrm{Kill}^{\mathrm{loc}}(\sigma)$, respectively.

If X is a vector field on M, its local flow lifts to a local flow on the bundle $L^{(k)}(M)$, and this defines a vector field $X_{(k)}$ on $L^{(k)}(M)$, which is called the lift of X to $L^{(k)}(M)$. But if we also take X to be the Killing field of a geometric structure σ, an easy-to-prove property is that, $d\sigma_\alpha(X_{(k)}(\alpha)) = 0$ for all $\alpha \in L^{(k)}(M)$, which suggests the next definition.

Definition 2.28. Let σ be a geometric structure of order k on M. An *infinitesimal Killing field $X \in \mathfrak{X}(M)$ of order $k+r$ at x for σ* is a $(k+r)$-jet $j_x^{k+r}(X)$ of a germ at x of a vector field X so that $d\sigma_\alpha^r(X_{(k+r)}) = 0$ for every $\alpha \in L^{(k+r)}(M)$ that lies in the fiber of x. Let $\mathrm{Kill}^{k+r}(\sigma, x)$ denote the space of infinitesimal Killing fields for σ of order $k+r$ at x, and let $\mathrm{Kill}_0^{k+r}(\sigma, x)$ denote the subspace consisting of those vanishing at x.

We use the special notation $\mathrm{Kill}(\sigma, x)$ for the Lie algebra of germs at x of local vector Killing fields defined in a neighborhood of x and $\mathrm{Kill}_0(\sigma, x)$ for the Lie subalgebra consisting of those vanishing at x.

The above definition does not depend on the choice of the vector field because for a given vector field X, the value of $d\sigma_\alpha(X_{(k+r)}(\alpha))$ depends only on $j_x^{k+r}(X)$.

Just as we have the generalization of germs of functions, we have tangents bundles of higher order. Let M be any manifold and

$$T_x^{(k)}M = \left\{ j_x^{k-1}(X) \mid X \in \mathfrak{X}(M) \right\}$$

which is the vector space of $(k-1)$-jets of vector fields at $x \in M$. Then the set $T_x^{(k)}M = \bigcup_{x \in M} T_x^{(k)}M$ has the structure of a smooth vector bundle over M, and we call it the tangent bundle of order k of M.

We recall that $D_x^{(k)}(M)$ denotes the group of k-jets at x of local diffeomorphisms fixing x, whose group structure is induced by the composition of maps. Any jet $j_0^k(\varphi) \in L^{(k)}(M)$ defines a linear isomorphism $T_0^{(k)}\mathbb{R}^n \to T_{\varphi(0)}^{(k)}M$. With respect to vector fields, we denote by $\mathcal{D}_x^{(k)}(M)$ the space of k-jets at x of vector fields vanishing at x, and we use the special notation $\mathfrak{gl}^{(k)}(n)$ when $M = \mathbb{R}^n$ and $x = 0$. If M carries a geometric structure σ, the next result provides a natural representation of $D_x^{(k)}(M)$ from which the Lie algebra of this group and the Lie algebra of $\mathrm{Aut}^k(\sigma, x)$ are described in terms of $\mathcal{D}_x^{(k)}(M)$ and $\mathrm{Kill}_0^k(\sigma, x)$, respectively; the proof is elementary, but it is detailed in Sections 2 and 4 of [5].

Lemma 2.29. *For a smooth manifold M and any given point $x \in M$ the following properties hold for every $k \geq 1$:*

(1) The map

$$\Theta_x \colon D_x^{(k)}(M) \to \mathrm{GL}(T_x^{(k)}M)$$
$$\Theta_x(j_x^k(\varphi))(j_x^{k-1}(X)) = j_x^{k-1}(d\varphi(X))$$

is a Lie group monomorphism.

(2) The assignment $[j_x^k(X), j_x^k(Y)]^k = -j_x^k([X, Y])$ yields a well defined Lie *algebra structure on $\mathcal{D}_x^{(k)}(M)$.*

(3) The map:

$$\theta_x\colon \mathcal{D}_x^{(k)}(M) \to \mathfrak{gl}(T_x^{(k)}M)$$
$$\theta_x(j_x^k(X))(j_x^{k-1}(Y)) = -j_x^{k-1}([X,Y])$$

is a Lie algebra monomorphism for the Lie algebra structure on $\mathcal{D}_x(M)$ given by $[\cdot,\cdot]^k$. Furthermore, $\theta_x(\mathcal{D}_x^{(k)}(M)) = \mathrm{Lie}(\Theta_x(D_x^{(k)}(M)))$.

(4) If M has a rigid geometric structure σ, then we have

$$\theta_x(\mathrm{Kill}_0^k(\sigma,x)) = \mathrm{Lie}(\Theta_x(\mathrm{Aut}^k(\sigma,x))).$$

In particular, with respect to the homomorphisms Θ_x and θ_x, the Lie algebra of $\mathrm{Aut}^k(\sigma,x)$ is realized by $\mathrm{Kill}_0^k(\sigma,x)$ with the Lie algebra structure given by $[\cdot,\cdot]^k$.

For the rest of this work we need a notion of a Zariski measure that we define now.

Definition 2.30. A measure on an analytic manifold M is called a Zariski measure if every proper analytic variety of a connected open subset of M is a null set.

Now, let G be a connected noncompact simple Lie group acting on M. We recall that an action is locally free if its stabilizers are discrete. When the manifold and the action are analytic, if the action is not trivial and leaves invariant a finite Zariski measure, then, by Proposition 3.8 of [5], it is locally free on a conull, open, dense subset and so the orbits define a foliation that we denote with $\mathcal{O}$. From [27] and the subsequent articles [4] and [24], we observe that is crucial to understand the properties of the transverse to the G-orbits.

If $\mathfrak{g}$ is the Lie algebra of G and $X \in \mathfrak{g}$ then the vector field induced by the one-parameter group of diffeomorphisms $\exp(tX)\cdot x$ will be denoted by X^*, i.e.

$$\mathfrak{g} \to \mathfrak{X}(M)$$
$$X \mapsto X^*$$

and if $\mathrm{ev}_x\colon G \to M$ denotes the map $\mathrm{ev}_x(a) = a\cdot x$ for all $a \in G$ (evaluate the diffeomorphism a on x), then

$$\begin{aligned} X_x^* &= \left.\frac{d}{dt}\right|_{t=0} (\exp(tX)\cdot x) \\ &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{ev}_x\,(\exp(tX)) \\ &= (\mathrm{ev}_x)_*\left(\left.\frac{d}{dt}\right|_{t=0} \exp(tX)\right) \\ &= (\mathrm{ev}_x)_*(X) \end{aligned} \tag{2.9}$$

It is well known that the bundle $T\mathcal{O}$ tangent to the foliation $\mathcal{O}$ is a trivial vector bundle isomorphic to $M \times \mathfrak{g}$, under the isomorphism given by:

$$M \times \mathfrak{g} \to T\mathcal{O}$$
$$(x,X) \mapsto X_x^*$$

for every $x \in M$. This induces an isomorphism between the fiber $T_x\mathcal{O}$ and $\mathfrak{g}$, which we will refer to as the natural isomorphism. We denote by $L_a\colon M \to M$ the diffeomorphism $x \mapsto a\cdot x$ induced by the action of G. The following two well-known results are of fundamental importance when one considers the isotropy of left actions of groups on manifolds.

Lemma 2.31. *$(L_a)_*(X^*) = (\mathrm{Ad}_a X)^*_{L_a}$ for all $X \in \mathfrak{g}$ and $a \in G$.*

Lemma 2.32. *$[X^*, Y^*] = -[X, Y]^*$ for all X, $Y \in \mathfrak{g}$.*

The next result is a generalization found in [27].

Proposition 2.33. *Let G be a connected non-compact simple Lie group, σ a geometric structure on M of order k and type Q. Suppose that Q is algebraic, M and σ are analytic and G acts on M by automorphisms of σ. Assume that the G-action on M has a dense orbit and preserves a finite Zariski measure. Consider the $\widetilde{G}$-action on $\widetilde{M}$ lifted from the G-action on M and $\widetilde{\sigma}$ the lifted geometric structure. Then, there exists a dense subset $S \subset \widetilde{M}$ such that for every $x \in S$ the following properties are satisfied.*

(1) There is a homomorphism of Lie algebras $\rho_x\colon \mathfrak{g} \to \mathrm{Kill}(\widetilde{\sigma}, x)$ which is an isomorphism onto its image $\rho_x(\mathfrak{g}) = \mathfrak{g}(x)$.

(2) $\mathfrak{g}(x) \subset \mathrm{Kill}_0(\widetilde{\sigma}, x)$, i.e. every element of $\mathfrak{g}(x)$ vanishes at x.

(3) For every $X, Y \in \mathfrak{g}$ we have:

$$[\rho_x(X), Y^*] = [X, Y]^* = -[X^*, Y^*]$$

in a neighborhood of x.

Proof. For every $k \geq 1$, let us denote with $\sigma^k\colon L^{(k)}(M) \to Q_k$ the $\mathrm{Gl}^{(k)}(n)$-equivariant map that defines the kth order extension of the geometric structure σ. By Proposition 3.8 of [5], there exists a G-invariant dense open subset U of M such that the action of G is locally free on U. Consider the set:

$$\mathcal{G}^k = \left\{ j_x^{k-1}(X^*) : X \in \mathfrak{g}, x \in U \right\}$$

which, by the local freeness of the G-action, is a smooth subbundle of $T^{(k)}U$. In fact, we have $\mathcal{G} = \mathcal{G}^1 = T\mathcal{O}$, and as with this bundle there is a trivialization given by:

$$\begin{aligned} U \times \mathfrak{g} &\to \mathcal{G}^k \\ (x, X) &\mapsto j_x^{k-1}(X^*). \end{aligned}$$

The corresponding trivialization of the frame bundle of $\mathcal{G}^k$ is given by:

$$\begin{aligned} U \times \mathrm{GL}(\mathfrak{g}) &\to L(\mathcal{G}^k) \\ (x, A) &\mapsto A_x. \end{aligned}$$

where $A_x(X) = j_x^{k-1}((AX)^*)$. Note that we have taken $\mathfrak{g}$ as the standard fiber of the bundle $\mathcal{G}^k$.

Choose a subspace $\mathcal{G}_0$ of $T_0^{(k)}\mathbb{R}^n$ isomorphic to $\mathfrak{g}$. We will fix such a subspace as well as an isomorphism with $\mathfrak{g}$ through which we will identify these two spaces. Let us now consider:

$$L^{(k)}(U, \mathcal{G}^k) = \left\{ \alpha \in L^{(k)}(U) \colon \alpha(\mathcal{G}_0) = \mathcal{G}^k_x \text{ if } \alpha \in L^{(k)}(U)_x \right\}$$

where $L^{(k)}(U)_x$ stands for the fiber over x. $L^{(k)}(U, \mathcal{G}^k)$ is a smooth reduction of $L^{(k)}(U)$ to the subgroup of $\mathrm{GL}^{(k)}(n)$ that preserves the subspace $\mathcal{G}_0$; we will denote this subgroup by $\mathrm{GL}^{(k)}(n, \mathcal{G}_0)$. Recall from the remarks preceding Lemma 2.29 that for every $j_0^k(\varphi) \in L^{(k)}(U)$ we obtain a linear isomorphism:

$$\begin{aligned} T_0^{(k)}\mathbb{R}^n &\to T^k_{\varphi(0)}U \\ j_0^{k-1}(X) &\mapsto j^{k-1}_{\varphi(0)}(d\varphi(X)). \end{aligned}$$

In particular, if we let:

$$\begin{aligned} f_k \colon L^{(k)}(U, \mathcal{G}^k) &\to L(\mathcal{G}^k) \\ j_0^k(\varphi) &\mapsto j_0^k(\varphi)\mid_{\mathcal{G}_0}, \end{aligned}$$

then, by the identification between $\mathcal{G}_0$ and $\mathfrak{g}$, we can consider f_k as a well-defined smooth principal bundle morphism that covers the identity. The associated homomorphism of structure groups for f_k is given by:

$$\begin{aligned} \pi_k \colon \mathrm{GL}^{(k)}(U, \mathcal{G}_0) &\to \mathrm{GL}(\mathfrak{g}) \\ j_0^k(\varphi) &\mapsto j_0^k(\varphi)\mid_{\mathcal{G}_0}, \end{aligned}$$

which is clearly surjective. Note that we have used again our identification between $\mathfrak{g}$ and $\mathcal{G}_0$.

Fix μ an arbitrary ergodic component for the G-action on U for the volume. Then, there is a measurable reduction P of $L^{(k)}(U, \mathcal{G}^k)$ so that $\sigma^k(P)$ is (μ-a.e. over U) a single point $q_0 \in Q_k$. Furthermore, the structure group of P is the subgroup of $\mathrm{GL}^{(k)}(n, \mathcal{G}_0)$ that stabilizes q_0, and in particular it is algebraic. This claim is a consequence of the fact that the $\mathrm{GL}^{(k)}(n, \mathcal{G}_0)$-action on Q_k is algebraic; we refer to Section 4 and the proof of Proposition 8.4 of [5] for further details.

On the other hand, since π_k is a surjection and f_k is G-equivariant, by Proposition 8.2 of [5], there exist reductions Q_1 and Q_2 of $L^{(k)}(U, \mathcal{G}^k)$ and $L(\mathcal{G}^k)$, respectively, to subgroups $L_1 \subset \mathrm{GL}^{(k)}(n, \mathcal{G}_0)$ and $\overline{\mathrm{Ad}(G)}^Z \subset \mathrm{GL}(\mathfrak{g})$, such that $f_k(Q_1) \subset Q_2$ (μ-a.e. over U) and such that $\pi_k(L_1)$ is a finite index subgroup of $\overline{\mathrm{Ad}(G)}^Z$. Here L_1 can be chosen to be the algebraic hull of $L^{(k)}(U, \mathcal{G}^k)$ for the G-action on U with respect to the ergodic measure μ. This claim uses the well known fact that $\overline{\mathrm{Ad}(G)}^Z$ is the algebraic hull of $U \times \mathrm{GL}(g)$ for the product action.

It is not difficult to see that this can be chosen so that $Q_2 = U \times \overline{\mathrm{Ad}(G)}^Z$ (μ-a.e. over U) with respect to the above identification $U \times \mathrm{GL}(\mathfrak{g}) \cong L(\mathcal{G}^k)$. We can also assume that $Q_1 \subset P$, μ-a.e. over U, because the reduction Q_1 is the smallest one to an algebraic subgroup. The above discussion ensures that for μ-a.e. $x \in U$, we have the following relations:

$$\begin{aligned}
&f_k((Q_1)_x) \subset (Q_2)_x = \{x\} \times \overline{\mathrm{Ad}(G)}^Z \\
&(Q_1)_x \subset (P)_x \subset L^{(k)}(U, \mathcal{G}^k)_x \\
&\sigma^k((P)_x) = \{q_0\}\,.
\end{aligned}$$

Let us now fix a point x such that these conditions hold. Choose $\alpha_x \in (Q_1)_x$ and let $f_k(\alpha_x) = (x, k_x)$ where $k_x \in \overline{\mathrm{Ad}(G)}^Z$. Since π_k is surjective, there exists $\widehat{k}_x \in \mathrm{GL}^{(k)}(n, \mathcal{G}_0)$ such that $\pi_k(\widehat{k}_x) = k_x$. In particular, by the π_k-equivariance of f_k we have $f_k(\alpha_x \widehat{k}_x^{-1}) = (x, e)$. We also have by the same reason:

$$f_k(\alpha_x g \widehat{k}_x^{-1}) = f_k(\alpha_x \widehat{k}_x^{-1} \widehat{k}_x g \widehat{k}_x^{-1}) = (x, k_x \pi_k(g) k_x^{-1}),$$

for every $g \in L_1$. Also, the inclusion $(Q_1)_x \subset L^{(k)}(M, \mathcal{G}^k)_x$ implies that, for every $g \in L_1$, the k-jets of diffeomorphisms $\alpha_x \widehat{k}_x^{-1}$, $\alpha_x g \widehat{k}_x^{-1}$ considered as linear isomorphisms $T_0^{(k)}\mathbb{R}^n \to T_x^{(k)}M$ map $\mathcal{G}_0$ onto G_x^k. Hence, from the definition of f_k it follows that $\alpha_x g \alpha_x^{-1} = (\alpha_x g \widehat{k}_x^{-1})(\alpha_x \widehat{k}_x^{-1})^{-1}$ is a k-jet of local diffeomorphism of M at x whose associated isomorphism $T_x^{(k)}M \to T_x^{(k)}M$ maps $\mathcal{G}_x$ onto itself by the assignment:

$$j_x^{k-1}(X^*) \mapsto j_x^{k-1}((k_x \pi_k(g) k_x^{-1} X)^*).$$

for which we have used the above trivialization $U \times \mathrm{GL}(\mathfrak{g}) \cong L(\mathcal{G}^k)$. Since $\pi_k(L_1)$ has finite index in $\overline{\mathrm{Ad}(G)}^Z$ it contains the identity component $\mathrm{Ad}(G)$, and because $k_x \in \overline{\mathrm{Ad}(G)}^Z$ the group $k_x \pi_k(L_1) k_x^{-1}$ also contains $\mathrm{Ad}(G)$. It follows that $\alpha_x L_1 \alpha_x^{-1}$ is a subgroup of $D_x^{(k)}(U)$ for which the homomorphism from Lemma 2.29 (1) induces a homomorphism:

$$\begin{aligned}
H_x : \alpha_x L_1 \alpha_x^{-1} &\to \mathrm{GL}(\mathcal{G}_x^k) \\
\alpha_x g \alpha_x^{-1} &\mapsto \Theta_x(\alpha_x g \alpha_x^{-1})|_{\mathcal{G}_x^k}
\end{aligned}$$

whose image contains $\mathrm{Ad}(G) \subset \mathrm{GL}(\mathfrak{g})$ with respect to the identification between $\mathfrak{g}$ and $\mathcal{G}_x^k$ given by the above isomorphism $U \times \mathfrak{g} \cong \mathcal{G}_x^k$. This implies that the induced Lie algebra homomorphism:

$$h_x \colon \mathrm{Lie}(\alpha_x L_1 \alpha_x^{-1}) \to \mathfrak{gl}(\mathcal{G}_x^k)$$

has image $\mathrm{ad}(\mathfrak{g})$, again with respect to the referred identification between $\mathfrak{g}$ and $\mathcal{G}_x^k$.

On the other hand, we have for every $g \in L_1$:

$$\sigma^k((\alpha_x g \alpha_x^{-1})\alpha_x) = \sigma^k(\alpha_x g) = \sigma^k(\alpha_x)$$

because $\sigma^k((Q_1)_x) \subset \sigma^k((P)_x) = \{q_0\}$ is a single point. But this identity proves that every such k-jet $\alpha_x g \alpha_x^{-1}$ preserves the volume up order k (see [5]). In other words, $\alpha_x L_1 \alpha_x^{-1}$ is a subgroup of $\mathrm{Aut}^k(\sigma, x)$ and by Lemma 2.29 we also have that $\mathrm{Lie}(\alpha_x L_1 \alpha_x^{-1})$ is a Lie subalgebra of $\mathrm{Kill}_0^k(\sigma, x)$.

From the above remarks, it follows that there is a Lie algebra homomorphism $\widehat{\rho}_x^k \colon \mathfrak{g} \to \mathrm{Kill}_0^k(\sigma, x)$ such that:

$$\Theta_x(\widehat{\rho}_x^k(X))(j_x^{k-1}(Y^*)) = j_x^{k-1}([X, Y]^*) \text{ for every } X, Y \in \mathfrak{g}. \tag{2.10}$$

For k fixed, the existence of the homomorphism $\widehat{\rho}_x^k$ has been established for μ-a.e. $x \in U$, where μ is an arbitrary ergodic component of the volume of U. Thus, for k fixed, it follows that the homomorphism $\widehat{\rho}_x^k$ exists for every $x \in S_k$, where S_k is some subset of U which is conull with respect to the volume of U. Finally, if we let $S_0 = \bigcap_{n=1}^{\infty} S_k$, then S_0 is conull with respect to the volume and for every $x \in S_0$ and every $k \geq 1$ there exist a homomorphism $\widehat{\rho}_x^k \colon \mathfrak{g} \to \mathrm{Kill}_0^k(\sigma, x)$ satisfying (2.10).

By Theorem 2, Proposition 6 and Proposition 7 of [3], we can conclude that for $x \in U$ there is some integer $k(x) \geq 1$ so that, for $k \geq k(x)$, every element of $\mathrm{Kill}_0^k(\sigma, x)$ extends uniquely to an element of $\mathrm{Kill}_0(\sigma, x)$. This property use the analyticity assumption.

The upshot of these remarks is that for every $x \in U$, there is some $k(x) \geq 1$ so that the map:

$$\begin{aligned} J_x^k \colon \mathrm{Kill}_0(\sigma, x) &\to \mathrm{Kill}_0^k(\sigma, x) \\ X &\mapsto j_x^k(X), \end{aligned}$$

is a linear isomorphism for every $k \geq k(x)$. Note that in this case, for the usual brackets in $\mathrm{Kill}_0(\sigma, x)$ and the brackets $[\cdot,\cdot]^k$ in $\mathrm{Kill}_0^k(\sigma, x)$ considered above, the map J_x^k is a Lie algebran anti-isomorphism. For S_0 and U as above, consider the dense subset $S = S_0 \cap U \subset M$. Next choose $x \in S$ and $k \geq \max(k(x), 2)$. Then, the map J_x^k is a Lie algebran anti-isomorphism, and there exists a Lie algebra homomorphism $\rho_x^k \colon \mathfrak{g} \to \mathrm{Kill}_0^k(M, x)$ satisfying (2.10). If we let $\rho_x = -(J_x^k)^{-1} \circ \rho_x^k \colon \mathfrak{g} \to \mathrm{Kill}_0(\sigma, x)$, then ρ_x defines a Lie algebra homomorphism such that: $j_x^{k-1}([\rho_x(X), Y^*]) = j_x^{k-1}([X, Y]^*)$, for every $X, Y \in \mathfrak{g}$. For this, we have used (2.10) and the definition of θ_x from Lemma 2.29. Since $k - 1 \geq 1$ and because germs of Killing fields are determined by any jet of order at least 1, we conclude that, at our chosen point x, ρ_x satisfies (from our statement) (1), (2) and the identity in (3) in a neighborhood of x with M replaced with $\widetilde{M}$. It is important to mention that in statement (1) we are using the analyticity of M and σ and that $\widetilde{M}$ is simply connected, because Proposition 7 of [3] asserts this. The identity in (3) now proves that every element of $\mathfrak{g}(x)$ preserves the tangent bundle to $\mathcal{O}$ in a neighborhood of x: i.e. the corresponding Lie derivatives map sections of $T\mathcal{O}$ into sections of $T\mathcal{O}$. This completes the proof of our statement for the dense subset $S \subset M$ and for M replaced with $\widetilde{M}$ in (1)–(3). Finally, this yields the statement for M for the dense subset which is the inverse image of S under the covering map since such map is a local isometry.

□

2.6 Anti-Kähler manifolds and their Killing fields

2.6.1 Anti-Kähler manifolds

We have the analogous definition of Kähler manifold but in the context of anti-Hermitian manifolds.

Definition 2.34. Let (M, J, g) be an anti-Hermitian manifold and let ∇ be the Levi-Civita connection of $h = \mathrm{Re}(g)$. We say that M is an anti-Kähler manifold if J is parallel with respect to ∇, that is, $\nabla J = 0$.

As in the Hermitian case, by [2] an anti-Hermitian manifold is anti-Kähler if and only J is integrable. J being parallel will give us a strong conclusion on the set of Killing fields.

Let us remember that a Killing vector field of g is a vector field X such that its 1-parameter group φ_t acts by local isometries of g. By Corollary 2.25 this occurs if and only if φ_t acts by local isometries of both $h = \mathrm{Re}(g)$ and J, so that X is a Killing vector field of h and J. We conclude that $\mathrm{Kill}(g) = \mathrm{Kill}(h) \cap \mathrm{Kill}(J)$ and $\mathrm{Kill}^{\mathrm{loc}}(g) = \mathrm{Kill}(h)^{\mathrm{loc}} \cap \mathrm{Kill}^{\mathrm{loc}}(J)$.

Remark. We recall from [25] or [17] that

$$\nabla_Y \circ J = \nabla_Y J + J \circ \nabla_Y,$$

so M is an anti-Kähler manifold if and only if

$$\nabla_Y \circ J = J \circ \nabla_Y. \tag{2.11}$$

Proposition 2.35. *If M is an anti-Kähler manifold, then* $\mathrm{Kill}(g)$*,* $\mathrm{Kill}^{loc}(g)$ *and* $\mathrm{Kill}_0(g, x)$ *are* complex *Lie algebras.*

Proof. It is enough to prove that $\mathrm{Kill}(g)$ is a complex vector space. Let $X \in \mathrm{Kill}(g)$ be a Killing field of g. By the discussion above, $X \in \mathrm{Kill}(J)$ and $X \in \mathrm{Kill}(h)$. Then the following equations hold:

$$[X, JY] = J\,[X, Y] \tag{2.12}$$

$$h(\nabla_Y X, Z) + h(Y, \nabla_Z X) = 0 \tag{2.13}$$

for all Y, $Z \in \mathfrak{X}(M)$ (See [25]).

Since M is a complex manifold, $JX \in \mathrm{Kill}(J)$ (See [18]), and furthermore by the torsion-free property of the Levi-Civita connection, and the commutativity of J and ∇,

$$\begin{aligned}
\nabla_{JY} X &= \nabla_X JY - [X, JY] && \text{Torsion-free property}\\
&= J\nabla_X Y - J\,[X, Y] && \text{(2.11) and (2.12)}\\
&= J(\nabla_X Y - [X, Y]) &&\\
&= J(\nabla_Y X) && \text{Torsion-free property}
\end{aligned} \tag{2.14}$$

for all $Y \in \mathfrak{X}(M)$. Thus

$$\begin{aligned}
h(\nabla_Y JX, Z) + h(Y, \nabla_Z JX) &= h(J\nabla_Y X, Z) + h(Y, J\nabla_Z X) && \text{by (2.11)}\\
&= h(\nabla_{JY} X, Z) + h(JY, \nabla_Z X) && \text{by (2.14) and (2.3)}\\
&= 0 && \text{by (2.13).}
\end{aligned}$$

for all Y, $Z \in \mathfrak{X}(M)$, and thus $JX \in \mathrm{Kill}(h)$ and by Corolarry 2.25, $JX \in \mathrm{Kill}(g)$. This proves that $\mathrm{Kill}(g)$ is a complex vector space. □

We just defined a new object, and it is important because it lead us to a whole new category of things, but we need some examples to help us understand the importance of these concepts. Instead of giving only one, we give many examples through the following results.

Proposition 2.36. *Every complex Lie group is an anti-Kähler manifold.*

Proof. Let be G a complex Lie group with complex Lie algebra $\mathfrak{g}$. If g_e is the Killing form on $\mathfrak{g}$ and we extend it by translations, the resulting pseudo-Riemannian metric g is a bi-invariant anti-Hermitian metric for G. The Levi-Civita connection is

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

(See [11]) so $\nabla J = 0$. Thus G is an anti-Kähler manifold. $\square$

Corollary 2.37. *Every quotient of a complex Lie group over a discrete subgroup is an anti-Kähler manifold.*

A natural question arise motivated for what happen in the real case, that is: Does the isometires (automorphisms) of an anti-Hermitian manifold form a complex Lie group? The answer in general is uncertain, and with high probability for being negative, but by the Lemma above we can answer this question for anti-Kähler manifolds.

Proposition 2.38. *Let (N, J, g) be a connected complete anti-Kähler manifold (all the Killing fields of g are complete). The set of isometries of g, $\mathrm{Iso}(N, g) = \mathrm{Aut}(g)$ has the structure of complex Lie group. If $\mathfrak{Iso}(N, g)$ denotes the Lie algebra of $\mathrm{Iso}(N, g)$, then the map:*

$$\mathfrak{Iso}(N, g) \to \mathrm{Kill}(g), \quad X \mapsto X^*,$$

is an anti-isomorphism of complex Lie algebras. In particular, $[X, Y]^* = -[X^*, Y^*]$ for every X, $Y \in \mathfrak{Iso}(N, g)$. Furthermore, the isometry group $\mathrm{Iso}(N, g)$ acts holomorphically on N.

Proof. Similarly to [25] we conclude that the Lie algebra $\mathfrak{Iso}(N, g)$ is anti-isomorphic to the subalgebra of complete Killing fields of g via the anti-isomorphism mentioned in the statement of this proposition. and from Proposition 2.35 we know that the latter is a complex Lie subalgebra. Then by Theorem 3.1 of [16], the group of diffeomorphisms of N that are isometries of the anti-Hermitian metric has a unique structure of real Lie group, but in [18] we find that a real Lie group is complex if and only if its Lie algebra is complex. We conclude that $\mathrm{Iso}(N, g)$ is a complex Lie group. The action is holomorphic because the isomorphism $X \mapsto -X^*$ is complex. $\square$

The above implies that on a complete manifold every Lie algebra of Killing fields can be realized from an isometric right action. The details can be found in [4] or [19].

Lemma 2.39. *Let (N, J, g) be a complete anti-Kähler manifold and H a* simply-connected *complex Lie group with complex Lie algebra $\mathfrak{h}$. If $\psi\colon \mathfrak{h} \to \mathrm{Kill}(N, g)$ is a homomorphism of complex Lie algebras, then there exists an isometric right H-action $N \times H \to N$ such that $\psi(X) = X^*$, for every $X \in \mathfrak{h}$. Furthermore, the H-action is holomorphic.*

The next lemma will be very useful throughout this work.

Lemma 2.40. *Let N be an anti-Kähler manifold with an almost-complex structure J, an anti-Hermitian metric g and $x \in N$. Then the map:*

$$\begin{gathered} \lambda_x\colon\ \mathrm{Kill}_0(g, x) \to \mathfrak{so}(T_x N, g_x) \\ \lambda_x(Z)(v) = [Z, V]_x, \end{gathered}$$

where V is any vector field such that $V_x = v$, is a well defined homomorphism of complex Lie algebras.

Proof. By Proposition 2.35 $\mathrm{Kill}_0(g,x)$ has the structure of a complex Lie algebra. Let Z be in $\mathrm{Kill}_0(g,x)$; by the above discussion, $Z \in \mathrm{Kill}_0(h,x)$, and thus $\lambda_x(Z) \in \mathfrak{so}(T_xN, h_x)$ (See [27]). Since $Z \in \mathrm{Kill}_0(J,x)$ as well, for every vector field V we have

$$J[Z,V]_x = [Z,JV]_x$$

by equation (2.12), which implies that $\lambda_x(Z)$ commutes with J_x, so by Proposition 2.10 we conclude that $\lambda_x(Z) \in \mathrm{Kill}_0(g,x)$. $\square$

The objective is to study a complex Lie group acting smoothly on an anti-Kähler manifold by isometries of a anti-Hermitian metric. But we need some compatibility between the complex structures of G and M. By placing some conditions on G, M, and the action we get a surprising result.

This will we done by requiring the map ev_x, from the previous section, to be a complex map between G and M. If $J_{\mathfrak{g}}$ stands for the complex structure on $\mathfrak{g}$ that comes from the one in G, asking ev_x to be complex for each $x \in M$, equation (2.9) implies

$$(J_{\mathfrak{g}}X)^*_x = (\mathrm{ev}_x)_*(J_{\mathfrak{g}}X) = J_x(\mathrm{ev}_x)_*(X) = J_xX^*_x. \tag{2.15}$$

Moreover, the orbits would be complex submanifolds of M. Since G acts by holomorphic automorphisms of M, the whole action $G \times M \to M$ would be holomorphic. Assuming either of the above, by equation (2.15) we conclude that $J_xX^*_x \in T_x\mathcal{O}$ for every $X \in \mathfrak{g}$ and $x \in M$. We will assume the action to be holomorphic from now on.

As in [29], we will prove some interesting facts. Let be X, $Y \in \mathfrak{g}$, we define $\Phi_1 \colon M \to \mathfrak{g}^* \otimes \mathfrak{g}^*$ by $\Phi_1(x)(X,Y) = h_x(X^*,Y^*)$, that is, $\Phi_1(x) = \mathrm{ev}^*_x(h)$. In the article mentioned above, it is proved that Φ_1 is G-equivariant via the coadjoint action in $\mathfrak{g}^* \otimes \mathfrak{g}^*$, which is defined for all $\phi \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ by $a \cdot \phi(X,Y) = \phi(\mathrm{Ad}_{a^{-1}} X, \mathrm{Ad}_{a^{-1}} Y)$; afterwards, it is proved that if G is semisimple without compact factors acting by isometries on a connected finite-volume pseudo-Riemannian manifold M and preserving this volume, then Φ_1 is G-invariant. This is done by pushing forward the metric of M to $\mathfrak{g}^* \otimes \mathfrak{g}^*$ and using the next well-known lemma.

Lemma 2.41. *Let G be a connected semisimple Lie group without compact factors and let V be a real finite-dimensional representation of G. If ν is a G-invariant Borel probability measure on V, then $\mathrm{supp}(\nu) \subset V^G$, where V^G stands for the points fixed by the action of G.*

Finally, it is concluded that Φ_1 is constant assuming that the action has a dense orbit. In other words, we can pull back the pseudo-Riemannian metric of M to a nondegenerate bilinear form in $\mathfrak{g}$.

We will repeat these ideas for the complex structure J. As we discussed in Section 2.5, there is an isomorphism

$$\begin{aligned} \alpha_x \colon T_x\mathcal{O} &\to \mathfrak{g} \\ \alpha_x(X^*_x) &= X \end{aligned}$$

We use the notation α_x following that of [28]. Now, as in [29] we define a map $\Phi_2\colon M \to \mathfrak{g}^* \otimes \mathfrak{g}$ by $\Phi_2(x)(X) = \alpha_x(J_x X_x^*)$. Since by hypothesis $J_x X_x^* \in T_x\mathcal{O}$, and α_x is an isomorphism, Φ_2 is well defined. We note that

$$\begin{aligned}\Phi_2(x)(\Phi_2(x)(X)) &= \alpha_x J_x(\alpha_x J_x X_x^*)_x^* \\ &= \alpha_x J_x J_x X_x^* \\ &= -\alpha_x X_x^* = -X\end{aligned}$$

for all $X \in \mathfrak{g}$, i.e. $\Phi_2^2(x) = -1$, and this defines a complex structure on $\mathfrak{g}$ for each $x \in M$. In addition to its left action on M, the group G acts on $\mathfrak{g}^* \otimes \mathfrak{g}$ from the left via the adjoint action, which is defined for all $a \in G$, $X \in \mathfrak{g}$ and $\phi \in \mathfrak{g}^* \otimes \mathfrak{g}$ by $a \cdot \phi(X) = \operatorname{Ad}_a \phi(\operatorname{Ad}_{a^{-1}} X)$. The following Lemma shows that the map Φ_2 is equivariant for this action.

Lemma 2.42. *For all $a \in G$ and $x \in M$, $\Phi_2(a \cdot x) = a \cdot \Phi_2(x)$.*

Proof. Since G acts by automorphisms of J, then $J_{ax} \circ (L_a)_* = (L_a)_* \circ J_x$ for all $a \in G$, $x \in M$. From Lemma 2.31 we know $(L_a)_*(X_x^*) = (\operatorname{Ad}_a X)_{ax}^* \in T_{ax}\mathcal{O}$ and so

$$\begin{aligned}\alpha_{ax} \circ (L_a)_*(X_x^*) &= \alpha_{ax}(\operatorname{Ad}_a X)_{ax}^* \\ &= \operatorname{Ad}_a X \\ &= \operatorname{Ad}_a \alpha_x(X_x^*).\end{aligned}$$

Thus, $\alpha_{ax} \circ (L_a)_* = \operatorname{Ad}_a \circ \alpha_x$. Finally we have

$$\begin{aligned}\Phi_2(a \cdot x)(X) &= \alpha_{ax}(J_{ax} X_{ax}^*) \\ &= \alpha_{ax}(J_{ax}(\operatorname{Ad}_a \operatorname{Ad}_{a^{-1}} X)_{ax}^*) \\ &= \alpha_{ax}(J_{ax}(L_a)_*(\operatorname{Ad}_{a^{-1}} X)_x^*) \\ &= \alpha_{ax}((L_a)_* J_x(\operatorname{Ad}_{a^{-1}} X)_x^*) \\ &= \operatorname{Ad}_a(\alpha_x(J_x(\operatorname{Ad}_{a^{-1}} X)_x^*)) \\ &= \operatorname{Ad}_a \Phi_2(x)(\operatorname{Ad}_{a^{-1}} X) \\ &= a \cdot \Phi_2(x)(X).\end{aligned}$$

□

Now we consider the Borel measure ψ on M with $\operatorname{supp}(\psi) = M$ induced by the pseudo-Riemannian metric h. By assumption, ψ is G-invariant and $\psi(M) < +\infty$. By multiplying the metric by a constant if necessary, we may assume that ψ is a probability measure.

Proposition 2.43. *Let be G a connected semisimple complex Lie group without compact factors acting holomorphically by isometries of an anti-Hermitian manifold M with finite volume. Then for all $a \in G$ and $x \in M$, $\Phi_2(ax) = \Phi_2(x)$.*

Proof. Consider the measure $\nu = (\Phi_2)_*(\psi)$. Since ψ is a G-invariant probability measure and Φ_2 is G-equivariant by Lemma 2.42, ν is a G-invariant probability measure. By Lemma 2.41, $\operatorname{supp}(\nu) \subset (\mathfrak{g}^* \otimes \mathfrak{g})^G$. It follows that $\Phi_2(a \cdot x) = \Phi_2(x)$ for all $x \in M$ and $a \in G$. □

Proposition 2.44. *Let G and M as in the above Proposition. If the G-action has a dense orbit, then the map Φ_2 is constant.*

Proof. By the above Proposition, Φ_2 is constant on G orbits. Since Φ_2 is continuous and M is connected, Φ_2 must be constant on M. $\square$

By equation (2.15), we conclude that $\Phi_2 \equiv J_{\mathfrak{g}}$.

Remark. The proofs were made only assuming that G was acting by isometries of h and J and that $J_x X_x^* \in T_x\mathcal{O}$ for (almost) all $x \in M$. So, if A is a $(1,s)$-tensor, h a pseudo-Riemannian metric, $A_x((X_1)_x^*, \ldots, (X_s)_x^*) \in T_x\mathcal{O}$ and G acts by automorphisms of h and A, it can be proved that if G is connected, semisimple, without compact factors, and has a dense orbit then a Φ defined in the same fashion as Φ_2, is constant. In other words, we can pull back the structure of M to G via Φ.

2.6.2 Killing fields of anti-Kähler manifolds

For this subsection it is necessary to assume the integrability of a complex structure. From now on to the end we ask (M, J) to be a complex manifold. The induced complex structure (as we saw in section 2.2) will be called J, as usual.

We repeat the definitions of previous sections in the context of complex manifolds because we need new notation. The proof of all the assertions are omitted but they are essentially the same as in the real case.

Let $\mathrm{Gl}^{(k)}(n,\mathbb{C})$ denote the group of k-jets at 0 of biholomorphisms of $\mathbb{C}^n$ that fix 0. As a manifold:

$$\mathrm{Gl}^{(k)}(n,\mathbb{C}) = \left\{ (A, L_2, \ldots, L_k) \;\middle|\; \begin{array}{l} A \in \mathrm{GL}(n,\mathbb{C}), \\ L_j \in S_j(\mathbb{C}^n;\mathbb{C}^n), \text{ for every } j \geq 2 \end{array} \right\}.$$

$\mathrm{Gl}^{(k)}(n,\mathbb{C})$ is in fact a complex Lie group. $\mathrm{Gl}^{(1)}(n,\mathbb{C})$ is the general linear group $\mathrm{GL}(n,\mathbb{C})$ and, for any pair of integers $k \leq l$, there is a canonical homomorphism

$$\pi_k^l : \mathrm{Gl}^{(l)}(n,\mathbb{C}) \to \mathrm{Gl}^{(k)}(n,\mathbb{C}).$$

Let $\mathfrak{gl}^{(k)}(n,\mathbb{C})$ denote the space of k-jets at 0 of holomorphic vector fields on $\mathbb{C}^n$ that vanish at 0. The bracket of two elements $j^k(X)$, $j^k(Y) \in \mathfrak{gl}^{(k)}(n,\mathbb{C})$ is defined the same as in the real case. Defining $Jj^k(X) = j^k(J_0X)$ provides a complex Lie algebra structure on $\mathfrak{gl}^{(k)}(n)$. Moreover, $\mathfrak{gl}^{(k)}(n,\mathbb{C})$ is the Lie algebra of $\mathrm{Gl}^{(k)}(n,\mathbb{C})$.

Let (M, J) be a complex manifold of dimension n and let $L_J^{(k)}(M)$ denote the kth order complex frame bundle of M. This is the collection of k-jets at the origin $0 \in \mathbb{C}^n$ of biholomorphisms at a neighborhood of $0 \in \mathbb{C}^n$ into M, and it has a natural structure of a complex manifold. $L_J^{(k)}(M)$ has an additional structure; it is indeed a locally-trivial complex principal fiber bundle over M with structure group $\mathrm{Gl}^{(k)}(n,\mathbb{C})$. The bundle map $\pi\colon L_J^{(k)}(M) \to M$ is the obvious base point projection. The natural right action of $\mathrm{Gl}^{(k)}(n,\mathbb{C})$ is as in Section 2.3.

T_xM is a complex vector space via J_x, as is $T_x^{(k)}M$, by defining a complex structure J_x by $J_x(j_x^k(X)) = j_x^k(JX)$. It is well defined because just depends on the k-jets of X since locally J is just a shuffle of the partial derivatives. Then $T^{(k)}M$ has a structure of complex bundle.

Remark. We note that the $\mathrm{Gl}^{(k)}(n,\mathbb{C})$-principal fiber bundle $L_J^{(k)}(M)$ is a reduction of the $\mathrm{Gl}^{(k)}(n)$-principal fiber bundle $L^{(k)}(M)$.

From now on to the end of this section, (M,J) is a complex manifold and g an anti-Hermitian metric on M. We assume that there is no confusion with the election of J, so we write $L_{\mathbb{C}}^{(k)}(M) = L_J^{(k)}(M)$. As a geometric structure in the sense of Gromov, thanks to the holomorphicity we can improve the definition of g by restricting it to $L_{\mathbb{C}}(M)$ and considering the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. We take $\mathfrak{C}_0 = \{B\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}\} \subset \mathfrak{C}$, where B is compatible with the complex structure J_0. We recall that $\mathfrak{C}$ is the set of $B\colon \mathbb{R}^{2n}\times\mathbb{R}^{2n} \to \mathbb{C}$ bilinear, symmetric and nondegenerate, compatible with some complex structure on $\mathbb{R}^{2n}$. So, g as a geometric structure is a $\mathrm{Gl}^{(k)}(n,\mathbb{C})$-equivariant map

$$g\colon L_{\mathbb{C}}^{(k)}(M) \to \mathfrak{C}_0.$$

It is worth mentioning that $\mathfrak{C}$ and $\mathfrak{C}_0$ are algebraic.

To end this section, we want to adapt the ideas found in [4] and [27] to the complex case, in particular, to give a complex version of Proposition 2.33.

If M is an anti-Kähler manifold, then M is complex and the vector space of Killing vector fields of J, $\mathrm{Kill}(J)$, has the structure of a complex Lie algebra via J itself ([18]). As we saw in the previous section, $\mathrm{Kill}(g)$ has the structure of a complex Lie algebra.

Proposition 2.45. *Let G be a connected noncompact complex simple Lie group acting isometrically and with a dense orbit on a connected finite volume anti-Kähler manifold (M,J,g). Consider the $\widetilde{G}$-action on $\widetilde{M}$ lifted from the G-action on M, $\widetilde{J}$ and $\widetilde{g}$ the lifted complex structure and anti-Hermitian metric, respectively. Assume that the G-action on M is holomorphic. Then there exists a conull subset $S \subset \widetilde{M}$ such that for every $x \in S$ the following properties are satisfied:*

(1) There is a homomorphism of complex *Lie algebras $\rho_x : \mathfrak{g} \to \mathrm{Kill}(\widetilde{g},x)$ which is an isomorphism onto its image $\rho_x(\mathfrak{g}) = \mathfrak{g}(x)$.*

(2) $\mathfrak{g}(x) \subset \mathrm{Kill}_0(\widetilde{g},x)$, i.e. every element of $\mathfrak{g}(x)$ vanishes at x.

(3) For every $X,Y \in \mathfrak{g}$ we have:

$$[\rho_x(X), Y^*] = [X,Y]^* = -[X^*,Y^*]$$

in a neighborhood of x.

In particular, the elements in $\mathfrak{g}(x)$ and their corresponding local flows preserve both $\mathcal{O}$ and $T\mathcal{O}^\perp$ in a neighborhood of x.

(4) The homomorphism of complex Lie algebras $\lambda_x \circ \rho_x : \mathfrak{g} \to \mathfrak{so}(T_x\widetilde{M}, \widetilde{g}_x)$ induces a $\mathfrak{g}$-module structure on $T_x\widetilde{M}$ for which the subspaces $T_x\mathcal{O}$ and $T_x\mathcal{O}^\perp$ are $\mathfrak{g}$-submodules

Proof. The proof of (1) - (3) is similar to that in Proposition 2.33, so we write just the differences. First, we consider the restriction of $g\colon L_{\mathbb{C}}(M) \to \mathfrak{C}_0$ and $g^k\colon L_{\mathbb{C}}^{(k)}(M) \to Q_k$ the $\mathrm{Gl}^{(k)}(n,\mathbb{C})$-equivariant map that defines the kth order extension of g. Then we take $\mathcal{G}_0$ complex isomorphic to $\mathfrak{g}$ and

$$L_{\mathbb{C}}^{(k)}(U,\mathcal{G}^k) = \left\{\alpha \in L_{\mathbb{C}}^{(k)}(U)\colon \alpha(\mathcal{G}_0) = \mathcal{G}_x^k \text{ if } \alpha \in L_{\mathbb{C}}^{(k)}(U)_x\right\}$$

that is not empty because M is a complex manifold.

Second, by [14] we can adapt what is found in [30] and choose P to be a complex reduction of $L_{\mathbb{C}}^{(k)}(U, \mathcal{G}^k)$.

Third, since M is an anti-Kähler manifold every subspace and function involved turns out to be complex, and $\overline{\mathrm{Ad}(G)}^Z$ results in the algebraic complex hull of $G \times \mathrm{GL}(\mathfrak{g})$. It ends with ρ_x being a complex map.

For (4), the identity in (3) proves that every element of $\mathfrak{g}(x)$ preserves the tangent bundle to $\mathcal{O}$ in a neighborhood of x: i.e. the corresponding Lie derivatives map sections of $T\mathcal{O}$ into sections of $T\mathcal{O}$. By Proposition 2.2 of [21] we conclude that the local flows of the elements of $\mathfrak{g}(x)$ preserve $\mathcal{O}$ as well in a neighborhood of x. Since the elements of $\mathfrak{g}(x)$ are Killing fields, we conclude that they (and their local flows) also preserve the normal bundle $T\mathcal{O}^{\perp}$ in a neighborhood of x. □

Chapter 3

The centralizer of isometric actions

We continue considering G and M satisfying the conditions of Section 2.6. From now on $\mathcal{H}$ will be the subalgebra of $\mathrm{Kill}(\widetilde{g})$ consisting of the fields that centralize the $\widetilde{G}$-action on $\widetilde{M}$, that is, the global Killing fields X on $\widetilde{M}$ such that

$$[X, Y^*] = 0$$

for all $Y \in \mathfrak{g}$ (here $\mathfrak{g}$ is the Lie algebra of $\widetilde{G}$ which is the same as that of G). We recall that Y^* refers to the vector field on M such that its flow on each point $y \in X$ is given by $(\exp(tY)y)_t$ through the G-action on M.

It is important to mention that $\mathcal{H}$ is a complex Lie algebra via J. Since M is a complex manifold, given $X \in \mathcal{H}$ and $Y \in \mathfrak{g}$, we have

$$[JX, Y^*] = [X, JY^*] = J[X, Y^*] = 0.$$

This chapter is dedicated to analyzing the properties of $\mathcal{H}$ and it is organized in a similar fashion as that of [28].

3.1 The $\mathfrak{g}$-module structure of $\mathcal{H}$

We start with a well-known local homogeneity result which is a particular case of Gromov's open dense orbit theorem. For its proof we refer to [10] and [32] (See also [1], [20]).

Proposition 3.1. *For G and M satisfying the above conditions, there is an open dense conull subset $U \subset \widetilde{M}$ such that for every $x \in U$ the evaluation map*

$$\begin{aligned} ev_x \colon \mathcal{H} &\to T_x\widetilde{M} \\ Z &\mapsto Z_x \end{aligned}$$

is surjective.

Remark. Thanks to the complex structures of the vector spaces above, ev_x is a complex Lie homomorphism. Indeed

$$ev_x(JX) = (JX)_x = J_x X_x = J_x ev_x(X).$$

Remark. The map $\mathrm{ev}_x : G \to M$ of Section 2.6 and $ev_x : \mathfrak{X}(M) \to T_x\widetilde{M}$ are different maps and we distinguish them by using a different typography.

The next result provides an embedding of $\mathfrak{g}$ into $\mathcal{H}$ that allows us to apply representation theory to study the structure of $\mathcal{H}$. We observe that this statement is at the core of Gromov-Zimmer's machinery on the study of actions preserving geometric structures (see [10, 33]).

Lemma 3.2. *Let S be as in Proposition 2.45. Then, for every $x \in S$ and for ρ_x given as in Proposition 2.45, the map $\widehat{\rho}_x \colon \mathfrak{g} \to \mathrm{Kill}(\widetilde{M}, \widetilde{g})$ given by:*

$$\widehat{\rho}_x(X) = \rho_x(X) + X^*,$$

is an injective homomorphism of complex Lie algebras whose image $\mathcal{G}(x)$ lies in $\mathcal{H}$. In particular, $\mathcal{G}(x) \cong \mathfrak{g}$ as complex Lie algebras and the Lie brackets of $\mathcal{H}$ turn it into a $\mathfrak{g}$-module.

Proof. First, observe that the identity in Proposition 2.45 (3) is easily seen to imply that the image of $\widehat{\rho}_x$ lies in $\mathcal{H}$.

To prove that $\widehat{\rho}_x$ is a homomorphism of Lie algebras we apply Proposition 2.45(3) as follows for X, $Y \in \mathfrak{g}$:

$$\begin{aligned}
[\widehat{\rho}_x(X), \widehat{\rho}_x(Y)] &= [\rho_x(X) + X^*, \rho_x(Y) + Y^*] \\
&= [\rho_x(X), \rho_x(Y)] + [\rho_x(X), Y^*] + [X^*, \rho_x(Y)] + [X^*, Y^*] \\
&= \rho_x([X,Y]) + [X,Y]^* + [X,Y]^* + [X^*, Y^*] \\
&= \rho_x([X,Y]) + [X,Y]^* \\
&= \widehat{\rho}_x([X,Y]).
\end{aligned}$$

For the injectivity of $\widehat{\rho}_x$ we observe that $\widehat{\rho}_x(X) = 0$ implies $X^*_x = (\rho_x(X) + X^*)_x = 0$, which in turns yields $X = 0$ because the G-action is locally free. The last claim is now clear.

To see that is complex, we just notice that it is a sum of complex linear mappings. ☐

Proposition 3.1 allows us to define a $\mathcal{G}(x)$-module structure on $T_x\widetilde{M}$. Furthermore, this can be done so that de natural evaluation map intertwines the $\mathcal{G}(x)$-module structure on $\mathcal{H}$ and $T_x\widetilde{M}$. Note that by Lemma 3.2 the map $\widehat{\rho}_x(X)$ provides a particular realization of the isomorphism $\mathcal{G}(x) \cong \mathfrak{g}$. The latter allows us to describe the isomorphism types of $\mathcal{G}(x)$-modules in therm of known $\mathfrak{g}$-modules. We will make use of this in the rest of the work.

Some of the constructions have already been developed in Lemma 2.40, [24] and [28]. However, we present the proofs in our set up because we lack a proof for the complex part.

Lemma 3.3. *For G and M as above, let S and U be as in Proposition 2.45 and 3.1, respectively. Fix some point $x \in S \cap U$. Then, the following properties hold.*

(1) The map $\lambda_x \colon \mathcal{G}(x) \to \mathfrak{so}(T_x\widetilde{M}, \widetilde{g}_x)$ given by $\lambda_x(Z)(v) = [Z, V]_x$, where $V \in \mathcal{H}$ is such that $V_x = v$, is a well defined homomorphism of complex Lie algebras.

(2) The evaluation map $ev_x\colon \mathcal{H} \to T_x\widetilde{M}$ is a homomorphism of complex $\mathcal{G}(x)$-modules, and it satisfies $ev_x(\mathcal{G}(x)) = T_x\mathcal{O}$. In particular, $T_x\mathcal{O}$ is a complex $\mathcal{G}(x)$-module isomorphic to the complex $\mathfrak{g}$-module $\mathfrak{g}$.

(3) The complex subspace $T_x\mathcal{O}^\perp$ is a complex $\mathcal{G}(x)$-submodule of $T_x\widetilde{M}$.

Proof. By the choice of x, for every $v \in T_x\widetilde{M}$ there exists $V \in \mathcal{H}$ such that $V_x = v$. If $Z \in \mathcal{G}(x)$ is given, then there are some $X \in \mathfrak{g}$ such that $Z = \rho_x(X) + X^*$. With these choices we have

$$[Z, V] = [\rho_x(X) + X^*, V] = [\rho_x(X), V],$$

where the second identity follows from the fact that V centralizes the G-action. Since $\rho_x(X)$ vanishes at x, this shows that the dependence of $[Z, V]_x$ on V is only on $V_x = v$. In particular, the map given above is well defined. That λ_x is a homomorphism of Lie algebras into $\mathfrak{gl}(T_x\widetilde{M})$ follows from the Jacobi identity and the fact $\mathcal{H}$ is a $\mathcal{G}(x)$-module.

Next, for $\widetilde{h}$ the real part of the anti-Hermitian metric of $\widetilde{M}$ and $\widetilde{J}$ the complex structure, for every $X \in \mathfrak{g}$, we have

$$\begin{aligned}
&\widetilde{h}_x([\rho_x(X), V]_x, V'_x) + \widetilde{h}_x(V_x, [\rho_x(X), V']_x) = 0,\\
&\left[\rho_x(X), \widetilde{J}V\right]_x = \widetilde{J}_x\,[\rho_x(X), V]_x
\end{aligned}$$

for every pair of vector fields V, $V' \in \mathcal{H}$. This is a consequence of the fact that $\rho_x(X)$ is a Killing vector field of $\widetilde{h}$ and $\widetilde{J}$ that vanishes at x. Hence, for V, $V' \in \mathcal{H}$ and $X \in \mathfrak{g}$, the previous computations show that

$$\begin{aligned}
&\widetilde{h}_x([\rho_x(X) + X^*, V]_x), V'_x) + \widetilde{h}_x(V_x, [\rho_x(X) + X^*, V']_x) = 0,\\
&\left[\rho_x(X) + X^*, \widetilde{J}V\right]_x = \widetilde{J}_x\,[\rho_x(X) + X^*, V]_x
\end{aligned}$$

thus proving that for every $Z \in \mathcal{G}(x)$ and every v, $v' \in T_x\widetilde{M}$ we have

$$\begin{aligned}
&\widetilde{h}_x(\lambda_x(Z)(v), v') + \widetilde{h}_x(v, \lambda_x(Z)(v')) = 0,\\
&\lambda_x(Z)(\widetilde{J}_x v) = \widetilde{J}_x\lambda_x(Z)(v).
\end{aligned}$$

We conclude that $\lambda_x(Z) \in \mathfrak{so}(T_x\widetilde{M}, \widetilde{g}_x)$ for every $Z \in \mathcal{G}(x)$, thus completing the proof of (1).

On the other hand, from the definitions involved, it is clear that ev_x is a complex homomorphism of $\mathcal{G}(x)$-modules and that $ev_x(\mathcal{G}(x)) = T_x\mathcal{O}$. That $T_x\mathcal{O}$ is isomorphic to $\mathfrak{g}$ as $\mathfrak{g}$-module is a consequence of the above expressions and of Proposition2.45(3). This yields (2). Finally, that $T_x\mathcal{O}^\perp$ is a complex $\mathcal{G}(x)$-submodule now follows from (1) and (2). □

3.2 The Lie algebra structure of $\mathcal{H}$

For $x \in S \cap U$, in the rest of this work we consider $\mathcal{H}$ and $T_x\widetilde{M}$ endowed with the $\mathcal{G}(x)$-module structures defined in Lemmas 3.2 and 3.3, respectively. We now introduce a Lie subalgebra of $\mathcal{H}$ that is very useful to study the structure of $\mathcal{H}$.

Lemma 3.4. *For G and M as above, let S and U be as in Propositions 2.45 and 3.1, respectively. Fix some point $x \in S \cap U$. Then, the subspace $\mathcal{H}_0(x) = \ker(ev_x)$ is a complex Lie subalgebra of both $\mathcal{H}$ and $\mathrm{Kill}_0(\widetilde{M}, x)$, as well as a complex $\mathcal{G}(x)$-submodule of $\mathcal{H}$. Furthermore, the sum $\mathcal{G}(x) + \mathcal{H}_0(x)$ is direct and it is a complex Lie subalgebra of $\mathcal{H}$ that contains $\mathcal{H}_0(x)$ as an ideal. In particular, $\mathcal{H}$ is a complex $\mathcal{G}(x) + \mathcal{H}_0(x)$-module.*

Proof. In fact, we have $\mathcal{H}_0(x) = \mathcal{H} \cap \mathrm{Kill}_0(\widetilde{M}, x)$, which implies that it is a complex Lie subalgebra of $\mathcal{H}$. That $\mathcal{H}_0(x)$ is a $\mathcal{G}(x)$-submodule follows from the fact that ev_x is a complex homomorphism of $\mathcal{G}(x)$-modules.

Next assume that $Z = \rho_x(X) + X^* \in \mathcal{G}(x)$ vanishes at x, where $X \in \mathfrak{g}$. Hence, we have $X^*_x = 0$ and the local freeness of the G-action implies that $X = 0$. Hence, we conclude that $\mathcal{G}(x) \cap \mathcal{H}_0(x) = 0$ and the sum $\mathcal{G}(x) + \mathcal{H}_0(x)$ is direct. The rest of the statement follows directly from the properties proved so far. □

The constructions considered up to this point yield the following module structure over $\mathcal{G}(x) + \mathcal{H}_0(x)$ together with some useful properties.

Proposition 3.5. *For G and M as above, let S and U be as in Propositions 2.45 and 3.1, respectively. For a fixed point $x \in S \cap U$, let $\mathcal{G}(x)$ and $\mathcal{H}_0(x)$ be the complex Lie subalgebras of $\mathcal{H}$ defined in Lemmas 3.2 and 3.4, respectively. Consider the map defined by*

$$\begin{aligned} \lambda_x \colon \mathcal{G}(x) + \mathcal{H}_0(x) &\to \mathfrak{so}(T_x\widetilde{M}, \widetilde{g}_x) \\ \lambda_x(Z)(v) &= [Z, V]_x \end{aligned}$$

where for a given $v \in T_x\widetilde{M}$ we choose $V \in \mathcal{H}$ such that $V_x = v$. Then, the following properties are satisfied.

1. *The map λ_x is a well defined homomorphism of complex Lie algebras. In particular, $T_x\widetilde{M}$ is a $\mathcal{G}(x) + \mathcal{H}_0(x)$-module.*
2. *The evaluation map $ev_x \colon \mathcal{H} \to T_x\widetilde{M}$ is a homomorphism of complex $\mathcal{G}(x) + \mathcal{H}_0(x)$-modules for the module structures on $\mathcal{H}$ and $T_x\widetilde{M}$ defined by Lemma 3.4 and (1), respectively.*
3. *The subspaces $T_x\mathcal{O}$ and $T_x\mathcal{O}^\perp$ are complex $\mathcal{G}(x) + \mathcal{H}_0(x)$-submodules of $T_x\widetilde{M}$.*

Proof. Claim (1) is proved with arguments similar to those used in the proof of Lemma 3.3. Hence, (2) is an immediate consequence of the definition of λ_x. By (1) and Lemma 3.3 (2) to prove (3) it is enough to show that $\mathcal{H}_0(x)$ leaves invariant $T_x\mathcal{O}$. For this we observe that, from the previous results we have $T_x\mathcal{O} = ev_x(\mathcal{G}(x))$, $[\mathcal{H}_0(x), \mathcal{G}(x)] \subset H_0(x)$ and ev_x is a complex homomorphism of $\mathcal{H}_0(x)$-modules; these imply that $\lambda_x(\mathcal{H}_0(x))(T_x\mathcal{O}) = 0$. In particular, $T_x\mathcal{O}$ is a trivial $\mathcal{H}_0(x)$-module. □

Using Proposition 3.5, we define the homomorphism of complex Lie algebras

$$\begin{aligned} \lambda_x^\perp \colon \mathcal{G}(x) + \mathcal{H}_0(x) &\to \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x) \\ \lambda_x^\perp(Z) &= \lambda_x(Z)|_{T_x\mathcal{O}^\perp}. \end{aligned}$$

The following result proves that $\mathcal{H}_0(x)$ is completely determined by the representation $\lambda_x^\perp$ when $T_x\mathcal{O} \cap T_x\mathcal{O}^\perp = 0$.

Proposition 3.6. *For G and M as above, let S and U be as in Propositions 2.45 and 3.1, respectively. For a fixed point $x \in S \cap U$ assume that $T_x\mathcal{O} \cap T_x\mathcal{O}^\perp = 0$. Then, the homomorphism of complex Lie algebras*

$$\lambda_x^\perp \colon \mathcal{H}_0(x) \to \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$$

is injective. Furthermore, $\lambda_x^\perp(\mathcal{H}_0(x))$ is a complex Lie subalgebra and a $\lambda_x^\perp(\mathcal{G}(x))$-submodule of $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$.

Proof. We recall that every Killing field is completely determined by its 1-jet at x. If we fix $Z \in \mathcal{H}_0(x)$, then $Z_x = 0$, and so Z is completely determined by $[Z, V]_x$ where V varies in a set of vector fields $\mathcal{A}$ such that $ev_x(\mathcal{A}) = T_x\widetilde{M}$. From the above, we already know that $ev_x(\mathcal{G}(x)) = T_x\mathcal{O}$ and $[Z, V]_x = 0$ for every $V \in \mathcal{G}(x)$. Hence, given the condition $T_x\mathcal{O} \cap T_x\mathcal{O}^\perp = 0$, we further have that every $Z \in \mathcal{H}_0(x)$ is completely determined by $[Z, V]_x$ where V varies in a set of vector fields $\mathcal{A}$ such that $ev_x(A) = T_x\mathcal{O}^\perp$, which implies the injectivity of $\lambda_x^\perp$ on $\mathcal{H}_0(x)$. The rest of the claims now follow easily using that $\lambda_x^\perp$ is a complex homomorphism and that $\mathcal{H}_0(x)$ is an ideal in $\mathcal{G}(x) + \mathcal{H}_0(x)$. □

With the above Lie subalgebras of $\mathcal{H}$ we now provide a first description of the structure of $\mathcal{H}$.

Proposition 3.7. *For G and M as above, let S and U be as in Propositions 2.45 and 3.1, respectively. For a fixed point $x \in S \cap U$ assume that $T_x\mathcal{O} \cap T_x\mathcal{O}^\perp = 0$. Then there exists a $\mathcal{G}(x)$-submodule $\mathcal{V}(x)$ of $\mathcal{H}$ such that*

$$\begin{aligned} \mathcal{H} &= \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x) \\ T_x\mathcal{O}^\perp &= ev_x(\mathcal{V}(x)). \end{aligned}$$

Proof. We know that $ev_x(\mathcal{G}(x) + \mathcal{H}_0(x)) = T_x\mathcal{O}$. Hence, the simplicity of $\mathcal{G}(x)$ and Proposition 3.5 (2) and (3) imply the existence of a $\mathcal{G}(x)$-submodule $\mathcal{V}(x)$ of $\mathcal{H}$ with the required properties. □

We will now consider the integrability of the normal bundle $T\mathcal{O}^\perp$ for the case where $T_x\mathcal{O} \cap T_x\mathcal{O}^\perp = 0$ at every point x. The next result provides a necessary condition for this to hold. It is a consequence of Lemma 2.7 from [27] (see also Lemma 1.4 from [4]).

Lemma 3.8. *Let G and M be as above. If $\dim_{\mathbb{R}}(M) < 2\dim_{\mathbb{R}} G$, then $T\mathcal{O}$ and $T\mathcal{O}^\perp$ have non-degenerate fibers with respect to the metric of M. In particular, we have $TM = T\mathcal{O} \oplus T\mathcal{O}^\perp$, a sum of analytic vector subbundles.*

3.3 Integrability conditions for $T\mathcal{O}^\perp$

Assume from now on that $T_x\mathcal{O} \cap T_x\mathcal{O}^\perp = 0$ at every point x, in other words, that we have $TM = T\mathcal{O} \oplus T\mathcal{O}^\perp$ as well as the corresponding property for $\widetilde{M}$. As a consequence we obtain an analytic map of vector bundles

$$\overline{\omega} \colon T\widetilde{M} \to T\mathcal{O}$$

given by the orthogonal projection onto $T\mathcal{O}$. Note that by our assumptions this map is complex linear. We also recall from 2.6 that there is an isomorphism

$$\alpha_x \colon T_x\mathcal{O} \to \mathfrak{g}$$
$$\alpha_x(X_x^*) \mapsto X$$

that varies analytically with respect to x. This yields the analytic $\mathfrak{g}$-valued 1-form ω on $\widetilde{M}$ defined by the expression

$$\omega_x = \alpha_x \circ \overline{\omega}_x$$

where $x \in \widetilde{M}$. This is complex too. Next, we consider the analytic $\mathfrak{g}$-valued 2-form Ω defined by

$$\Omega_x = d\omega_x|_{\wedge^2 T_x\mathcal{O}^\perp},$$

for every $x \in \widetilde{M}$. If X, Y are smooth sections of $T\mathcal{O}^\perp$, then $\omega(X) = \omega(Y) = 0$ and from Proposition 3.11 of [17] we have

$$2\Omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) = -\omega([X,Y]),$$

which implies the following result (see [10, 27]).

Lemma 3.9. *For G, M and S as above, assume that $T\widetilde{M} = T\mathcal{O} \oplus T\mathcal{O}^\perp$. Then,*

(1) for every $x \in S$ the maps ω_x and Ω_x are both homomorphisms of complex $\mathfrak{g}$-modules, for the $\mathfrak{g}$-module structure from Proposition 2.45.

(2) The normal bundle $T\mathcal{O}^\perp$ is integrable if and only if $\Omega \equiv 0$.

Let us continue to assume that $T\widetilde{M} = T\mathcal{O} \oplus T\mathcal{O}^\perp$, which implies that for every $x \in \widetilde{M}$ the subspace $T_x\mathcal{O}^\perp$ is non-degenerate with respect to the scalar product of $T_x\widetilde{M}$.

By the next lemma, we obtain from the linear map $\Omega_x \colon \wedge^2 T_x\mathcal{O}^\perp \to \mathfrak{g}$ a corresponding map $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x) \to \mathfrak{g}$ given by

$$\Omega_x \circ \varphi_x^{-1}$$

where $\varphi_x \colon \wedge^2 T_x\mathcal{O}^\perp \to \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ is the isomorphism defined by the following elementary property.

Lemma 3.10. *Let E be a finite dimensional (complex) real vector space with (complex) real scalar product (anti-Hermitian metric) $\langle\cdot,\cdot\rangle$ Then, the assignment*

$$u \wedge v \mapsto \langle\cdot, u\rangle v - \langle\cdot, v\rangle u,$$

defines an isomorphism $\varphi\colon \wedge^2 E \to \mathfrak{so}(E, \langle,\rangle)$ of $\mathfrak{so}(E, \langle,\rangle)$-modules. In particular, φ yields an isomorphism of (complex) $\mathfrak{g}$-modules for every Lie subalgebra $\mathfrak{g}$ of $\mathfrak{so}(E, \langle,\rangle)$.

This does not change the $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$-module structure on the domain. Hence, we will denote with the same symbol Ω_x the linear map given by the 2-form Ω when considered as a map $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x) \to \mathfrak{g}$. We notice that the map is complex.

It turns out that the forms ω_x and Ω_x have intertwining properties with respect to the module structure over $\mathcal{G}(x) + \mathcal{H}_0(x)$.

Proposition 3.11. *For G and M as above, let S and U be as in Propositions 2.45 and 3.1, respectively. Assume that $T\widetilde{M} = T\mathcal{O} \oplus T\mathcal{O}^\perp$. For a fixed point $x \in S \cap U$, the following properties hold.*

(1) For every $X \in \mathfrak{g}$ and $Y \in \mathfrak{X}(\widetilde{M})$ we have

$$\omega_x([\rho_x(X), Y]_x) = [X, \omega_x(Y)].$$

(2) The linear map $\Omega_x\colon \wedge^2 T_x\mathcal{O}^\perp \to \mathfrak{g}$ intertwines the homomorphism of complex Lie algebras $\widehat{\rho}_x\colon \mathfrak{g} \to \mathcal{G}(x)$ for the actions of $\mathfrak{g}$ on $\mathfrak{g}$ and of $\mathcal{G}(x)$ on $T_x\mathcal{O}^\perp$ via $\lambda_x^\perp$. More precisely, we have

$$[X, \Omega_x(u \wedge v)] = \Omega_x(\lambda_x^\perp(\widehat{\rho}_x(X))(u \wedge v))$$

for every $X \in \mathfrak{g}$ and u, $v \in T_x\mathcal{O}^\perp$.

(3) The complex linear map $\Omega_x\colon \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x) \to \mathfrak{g}$ is $\mathcal{H}_0(x)$-invariant via $\lambda_x^\perp$. More precisely, we have

$$\Omega_x([\lambda_x^\perp(Z), T]) = 0,$$

for every $Z \in \mathcal{H}_0(x)$ and $T \in \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$. In other words, we have

$$[\lambda_x^\perp(\mathcal{H}_0(x)), \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)] \subset \ker(\Omega_x).$$

Proof. In what follows, for any vector field Y we will denote with $Y^\top$ and $Y^\perp$ the sections of $T\mathcal{O}$ and $T\mathcal{O}^\perp$, respectively, such that $Y = Y^\top + Y^\perp$. To prove (1), fix $x \in \mathfrak{g}$ and $Y \in \mathfrak{X}(\widetilde{M})$. Since $\rho_x(X)$ preserves both $T\mathcal{O}$ and $T\mathcal{O}^\perp$, it follows that $[\rho_x(X), Y^\perp]$ and $[\rho_x(X), Y]$ are sections of $T\mathcal{O}$ and $T\mathcal{O}^\perp$, respectively. We also note that $Y_x^\top = \omega_x(Y)_x^*$. On the other hand, $\rho_x(X)$ vanishes at x and the dependence of $[\rho_x(X), Y^\top]_x$ with respect to $Y^\top$ is only on $Y_x^\top$. Hence, we have the following identities

$$\begin{aligned}
\omega_x([\rho_x(X), Y]_x) &= \omega_x([\rho_x(X), Y^\top]_x) \\
&= \omega_x([\rho_x(X), \omega_x(Y)^*]_x) \\
&= \omega_x([X, \omega_x(Y)]_x^*) \\
&= [X, \omega_x(Y)],
\end{aligned}$$

where we have used in the third equality Proposition 2.45(3).

For (2), we consider the interpretation of Ω_x as a bilinear form and prove that $[X, \Omega_x(u, v)] = \Omega_x(\lambda_x^\perp(\widehat{\rho}_(X))(u), v) + \Omega_x(u, \lambda_x^\perp(\widehat{\rho}_x(X))(v))$ for every $X \in \mathfrak{g}$ and u,$v \in T_x\mathcal{O}^\perp$. Let Y_1, $Y_2 \in \mathcal{H}$ such that $Y_1(x) = u$ and $Y_2(x) = v$. Then, by definition we have

$$\lambda_x^\perp(\widehat{\rho}_x(X))(u) = [\overline{\rho}_x(X), Y_1]_x = [\rho_x(X) + X^*, Y_1]_x = [\rho_x(X), Y_1]_x$$

and similarly we have

$$\lambda^\perp(\widehat{\rho}_x(X))(v) = [\rho_x(X), Y_2]_x\,.$$

We now choose $\widehat{Y}_1$, $\widehat{Y}_2$ sections of $T\mathcal{O}^\perp$ such that $\widehat{Y}_1(x) = u$ and $\widehat{Y}_2(x) = v$. As remarked above, since $\rho_x(X)$ vanishes at x we have

$$\begin{aligned}
\lambda_x^\perp(\widehat{\rho}_x(X))(u) &= [\rho_x(X), Y_1]_x = [\rho_x(X), \widehat{Y}_1]_x, \\
\lambda_x^\perp(\widehat{\rho}_x(X))(v) &= [\rho_x(X), Y_2]_x = [\rho x(X), \widehat{Y}_2]_x.
\end{aligned}$$

Using the above we now compute

$$\begin{aligned}
&\Omega_x(\lambda_x^\perp(\widehat{\rho}_x(X))(u), v) + \Omega_x(u, \lambda_x^\perp(\widehat{\rho}_x(X))(v)) \\
&\qquad\qquad = \Omega_x([\rho_x(X), \widehat{Y}_1]_x, \widehat{Y}_2(x)) + \Omega_x(\widehat{Y}_1(x), [\rho_x(X), \widehat{Y}_2]_x) \\
&\qquad\qquad = -\omega_x([[\rho_x(X), \widehat{Y}_1], \widehat{Y}_2]_x) - \omega_x([\widehat{Y}_1, [\rho_x(X), \widehat{Y}_2]]_x) \\
&\qquad\qquad = -\omega_x([\omega_x(X), [\widehat{Y}_1, \widehat{Y}_2]]_x) \\
&\qquad\qquad = -[X, \omega_x([\widehat{Y}_1, \widehat{Y}_2])] \\
&\qquad\qquad = [X, \Omega_x(\widehat{Y}_1(x), \widehat{Y}_2(x))] \\
&\qquad\qquad = [X, \Omega_x(u, v)],
\end{aligned}$$

where we have used (1) in the fourth equality.

To prove (3) we observe that it is enough to show that

$$\Omega_x(\lambda_x^\perp(Z)(u), v) + \Omega_x(\lambda_x^\perp(Z)(v)) = 0,$$

for any given $Z \in \mathcal{H}_0(x)$, $u,v \in T_x\mathcal{O}^\perp$, i.e. we can consider Ω_x as a linear map $\wedge^2 T_x\mathcal{O}^\perp \to \mathfrak{g}$. This is the case by the above remarks on Lemma 3.10, which imply that $\varphi_x\colon \wedge^2 T_x\mathcal{O}^\perp \to \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ is an isomorphism of $\mathcal{H}_0(x)$-modules via the representation $\lambda_x^\perp\colon \mathcal{G}(x) + \mathcal{H}_0(x) \to \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$. Given $Z \in \mathcal{H}_0(x)$ and $u,v \in T_x\mathcal{O}^\perp$, we start by choosing vector fields Y_1, Y_2, $\widehat{Y}_1$, $\widehat{Y}_2$ as above: Y_1, Y_2 belong to $\mathcal{H}$, $\widehat{Y}_1$ $\widehat{Y}_2$ are sections of $T\mathcal{O}^\perp$, $Y_1(x) = \widehat{Y}_1(x) = u$ and $Y_2(x) = \widehat{Y}_2(x) = v$. As in the proof of (2), since $Z_x = 0$, we have

$$\begin{aligned}
\lambda_x^\perp(Z)(u) &= [Z, Y_1]_x = [Z, \widehat{Y}_1]_x, \\
\lambda_x^\perp(Z)(v) &= [Z, Y_2]_x = [Z, \widehat{Y}_2]_x,
\end{aligned}$$

and more generally, for any pair of vector fields $\widehat{W}$ and W whose value at x are the same we have $[Z, \widehat{W}]_x = [Z, W]_x$.

Next, we observe that for any vector field $\widehat{W} \in \mathfrak{X}(\widetilde{M})$ if we let $W \in \mathcal{H}$ be such that $W_x = \widehat{W}^\top$, then

$$\begin{aligned}
\overline{\omega}_x([Z, \widehat{W}]_x) &= \overline{\omega}_x([Z, \widehat{W}^\top]_x) \\
&= \overline{\omega}_x([Z, W]x) \\
&= \overline{\omega}_x(\lambda_x(Z)(W_x)) \\
&= \lambda_x(Z)(\overline{\omega}_x(W_x)) = 0,
\end{aligned}$$

where we have used that $\overline{\omega}_x$ is a homomorphism of $H_0(x)$-modules and that $T_x\mathcal{O}$ is a trivial $H_0(x)$-module. This relation in the case $\widehat{W} = [\widehat{Y}_1, \widehat{Y}_2]$ implies that

$$0 = \omega_x\left([Z, [\widehat{Y}_1, \widehat{Y}_2]]_x\right) = \omega_x\left([[Z, \widehat{Y}_1], \widehat{Y}_2]_x\right) + \omega_x\left([\widehat{Y}_1, [Z, \widehat{Y}_2]]_x\right)$$

and applying α_x we obtain

$$\begin{aligned}
0 &= \omega_x\left([[Z, \widehat{Y}_1], \widehat{Y}_2]_x\right) + \omega_x\left([\widehat{Y}_1, [Z, \widehat{Y}_2]]_x\right) \\
&= -\omega_x\left([Z, \widehat{Y}_1]_x, \widehat{Y}_2(x)\right) - \omega_x\left(\widehat{Y}_1(x), [Z, \widehat{Y}_2]_x\right) \\
&= -\omega_x\left(\lambda_x^\perp(Z)(u), v\right) - \omega_x\left((Z)(v)\right),
\end{aligned}$$

thus proving our last claim. Note that we have used in the second equality that, for $i = 1,2$, the vector fields $\widehat{Y}_i$ and $[Z, \widehat{Y}_i]$ are sections of $T\mathcal{O}^\perp$, and in the third identity the above formulas for $\lambda_x(Z)$ applied to u,v. $\square$

For our subsequent analysis, we will consider the two cases given by the following result.

Proposition 3.12. *Let G and M be as above, and assume that $T\widetilde{M} = T\mathcal{O} \oplus T\mathcal{O}^\perp$. Then, one of the following conditions is satisfied.*

(1) The normal bundle $T\mathcal{O}^\perp$ is integrable.

(2) There is a dense conull subset $S_0 \subset \widetilde{M}$ contained in $S \cap U$, where S and U are given by Propositions 2.45 and 3.1, respectively, such that for every $x \in S_0$ the following properties are satisfied.

  (a) The linear map $\Omega_x \colon \wedge^2 T_x\mathcal{O}^\perp \to \mathfrak{g}$ is surjective.

  (b) The $\mathcal{G}(x)$-module structure on $T_x\mathcal{O}^\perp$ is non-trivial.

  (c) The homomorphism of Lie algebras $\lambda_x^\perp \colon \mathcal{H}_0 x) \to \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ is injective. Furthermore, $\lambda_x^\perp(\mathcal{H}_0(x))$ is a $\lambda_x^\perp(\mathcal{G}(x))$-submodule and a Lie subalgebra of $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ that satisfies

$$\left[\lambda_x^\perp(\mathcal{H}_0(x)), \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)\right] \subset \ker(\Omega_x).$$

In particular, if (2) holds, then $T\mathcal{O}^\perp$ is not integrable.

Proof. Let us assume that $T\mathcal{O}^\perp$ is not integrable. By Lemma 3.9 we have $\Omega \neq 0$, and since Ω is analytic, the set S' of points $x \in \widetilde{M}$ where $\Omega_x \neq 0$ is the complement of a proper analytic subset. In particular, S' is an open dense conull subset of $\widetilde{M}$. Then, we take $S_0 = S' \cap S \cap U$. Hence, (a) follows from Proposition3.11(2) and the fact that $\mathfrak{g}$ is simple. Now (b) follows from (a). Also, (c) is a restatement of Proposition 3.6 and Proposition 3.11(3). Finally, if (2) holds, then from its part (a) it follows that $\Omega \neq 0$ for every $x \in S_0$ and so that $T\mathcal{O}^\perp$ is not integrable. $\square$

Case (1) of Proposition 3.12 has already been considered in [27]. With this respect, the following is a consequence of Theorem 1.1 of [27].

Proposition 3.13. *Let G and M be as above such that $T\widetilde{M} = T\mathcal{O} \oplus T\mathcal{O}^\perp$, and assume that M is geodesically complete. If case (1) of Proposition 3.12 holds, then there exist*

(1) an isometric finite covering map $\widehat{M} \to M$ to which the G-action lifts,

(2) a simply connected pseudo-Riemannian manifold N,

(3) and a discrete subgroup $\Gamma \subset G \times Iso(N)$, such that $\widehat{M}$ is G-equivariantly isometric to $(G \times N)/\Gamma$ for a product metric on $G \times N$ where G carries a bi-invariant metric.

Hence, to complete the study of the structure of M it remains to consider case (2) of Proposition 3.12.

Chapter 4

Actions of $\mathrm{Spin}(n, \mathbb{C})$

We want to apply the results of the previous chapters when we have in particular the Lie group $G = \mathrm{SO}(n, \mathbb{C})$ acting holomorphically on an anti-Kähler manifold M by isometries of some anti-Hermitian metric. Then $\widetilde{G} = \mathrm{Spin}(n, \mathbb{C})$ and we know some properties about this Lie group.

Instead of working directly with $\mathrm{SO}(n, \mathbb{C})$ and $\mathrm{Spin}(n, \mathbb{C})$ we first investigate the algebraic properties of $\mathfrak{so}(n, \mathbb{C})$, mainly its irreducible representations.

As in [8] and [15], we recall that $\mathrm{SO}(n, \mathbb{C})$ is the Lie group of matrices $A \in \mathrm{M_n}(\mathbb{C})$ such that $S = A^T S A$ where

$$S = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \text{ if } n \text{ is even or } S = \begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } n \text{ is odd.}$$

and $\mathfrak{so}(n, \mathbb{C})$ is the correspondingly Lie algebra of matrices $X \in \mathrm{M_n}(\mathbb{C})$ such that $X^T S + SX = 0$.

Some topological properties about the Lie group $\mathrm{SO}(n, \mathbb{C})$ are well known, such as that it is connected (indeed it is the connected component of the identity matrix), it is non-compact for $n \geq 2$ and is not simply connected.

4.1 Some facts about $\mathfrak{so}(n, \mathbb{C})$.

We state some well known properties about the complex Lie algebra $\mathfrak{so}(n, \mathbb{C})$, which can be found in the classical literature such as [8], [11] or [12].

The first one is that $\mathfrak{so}(n, \mathbb{C})$ is simple for $n \geq 3$, except for $n = 4$ which is semisimple. Second, we establish some isomorphisms in low dimension.

$$\begin{aligned} &\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{sp}(2, \mathbb{C}) \\ &\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \\ &\mathfrak{so}(5, \mathbb{C}) \cong \mathfrak{sp}(4, \mathbb{C}) \\ &\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C}) \end{aligned}$$

We recall that the dimension of $\mathfrak{so}(n, \mathbb{C})$ is $n(n-1)/2$ over $\mathbb{C}$. Since $\mathfrak{so}(n, \mathbb{C})$ is semisimple, any finite dimensional representation decomposes as a sum of irreducible representations, so it is of particular interest to study the irreducible representations of $\mathfrak{so}(n,\mathbb{C})$. We make a difference between even $n = 2m$ and odd $n = 2m+1$ representations. The following results are consequence of the theory developed in [9] and [15].

Lemma 4.1. *Let $\varpi_1, \dots, \varpi_m$ be the fundamental weights of $\mathfrak{so}(2m+1,\mathbb{C})$.*

1. *The representation of $\mathfrak{so}(2m+1,\mathbb{C})$ on $\bigwedge^l \mathbb{C}^{2m+1}$ is irreducible for $l < m$ with highest weight ϖ_l, and the dimension of the irreducible representation is $\binom{2m+1}{l}$.*
2. *The irreducible representation attached to ϖ_m is the spin representation with dimension 2^m.*

Lemma 4.2. *Let $\varpi_1, \dots, \varpi_m$ be the fundamental weights of $\mathfrak{so}(2m,\mathbb{C})$.*

1. *The representation of $\mathfrak{so}(2m,\mathbb{C})$ on $\bigwedge^l \mathbb{C}^{2m}$ is irreducible for $l < m-1$ with highest weight ϖ_l and the dimension of the irreducible representation is $\binom{2m}{l}$.*
2. *The irreducible representations attached to ϖ_{m-1} and ϖ_m are the spin representations with dimension 2^{m-1}.*

However, we know (from [15] or [8]) that not every representation of $\mathfrak{so}(n,\mathbb{C})$ can be "integrated" to one of the Lie group $\mathrm{SO}(n,\mathbb{C})$ as the following lemmas state.

Lemma 4.3. *Let $\varpi_1, \dots, \varpi_m$ be the fundamental weights of $\mathfrak{so}(2m+1,\mathbb{C})$. Then the fundamental weights of $\mathrm{SO}(m,\mathbb{C})$ are $\varpi_1, \dots, \varpi_{m-1}, 2\varpi_m$.*

1. *The representation of $\mathrm{SO}(2m+1,\mathbb{C})$ on $\bigwedge^l \mathbb{C}^{2m+1}$ is irreducible for $l < m$ with highest weight ϖ_l, and the dimension of the irreducible representation is $\binom{2m+1}{l}$.*
2. *The irreducible representation attached to $2\varpi_m$ is the spin representation with dimension 2^m.*

Lemma 4.4. *Let $\varpi_1, \dots, \varpi_m$ be the fundamental weights of $\mathfrak{so}(2m,\mathbb{C})$. Then, the fundamental weights of $\mathrm{SO}(2m,\mathbb{C})$ are $\varpi_1, \dots, \varpi_{m-2}, 2\varpi_{m-1}, 2\varpi_m$.*

1. *The representation of $\mathfrak{so}(2m,\mathbb{C})$ on $\bigwedge^l \mathbb{C}^{2m}$ is irreducible for $l < m-1$ with highest weight ϖ_l and the dimension of the irreducible representation is $\binom{2m}{l}$.*
2. *The irreducible representations attached to $2\varpi_{m-1}$ and $2\varpi_m$ are the spin representations with dimension 2^{m-1}.*

Since we are interested to work with the universal covering we can avoid possible problems using the Spin groups. So, for $\mathrm{Spin}(n,\mathbb{C})$, the fundamental weights are the same of $\mathfrak{so}(n,\mathbb{C})$.

Let us denote by $m_{\mathbb{C}}(\mathfrak{g})$ the lowest complex dimension of a non-trivial complex $\mathfrak{g}$-module with an invariant anti-Hermitian metric. Using the information of the previous lemmas and that the Weyl dimension formula is non decreasing with respect to the fundamental weights, we conclude that, for large n, the first fundamental weight has the lowest dimension. For n small, what is found in [9] and the discussion of 2.1 we state a more precise result.

Lemma 4.5. *Let $n \geq 3$, $n \neq 4$, then $m_{\mathbb{C}}(\mathfrak{so}(n,\mathbb{C})) = n$, i.e. there is no non-trivial $\mathfrak{so}(n,\mathbb{C})$-module with dimension less than n and carrying an invariant anti-Hermitian metric. Moreover, the only non-trivial irreducible $\mathfrak{so}(n,\mathbb{C})$-module of dimension $\leq n$ is $\mathbb{C}^n$, except for the complex Lie algebras $\mathfrak{so}(3,\mathbb{C})$, $\mathfrak{so}(5,\mathbb{C})$, $\mathfrak{so}(6,\mathbb{C})$ and $\mathfrak{so}(8,\mathbb{C})$. For these last complex Lie algebras, there also exist the following irreducible modules:*

- $\mathbb{C}^2$ *corresponding to* $\mathfrak{so}(3,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})$.
- $\mathbb{C}^4$ *corresponding to* $\mathfrak{so}(5,\mathbb{C}) \simeq \mathfrak{sp}(4,\mathbb{C})$.
- $\mathbb{C}^4$ *corresponding to* $\mathfrak{so}(6,\mathbb{C}) \simeq \mathfrak{sl}(4,\mathbb{C})$.
- $\mathfrak{so}(8,\mathbb{C})$*-invariant forms of the half spin representations of* $\mathfrak{so}(8,\mathbb{C})$*, both 8 dimensional.*

As a consequence of Lemma 3.10 but rewritten in our setup, i.e. $E = \mathbb{C}^n$ and $\langle,\rangle = g_0$ we have

Lemma 4.6. *Let $n \geq 3$, $n \neq 4$. Then for every $c \in \mathbb{C}$, the map $T_c \colon \wedge^2 \mathbb{C}^n \to \mathfrak{so}(n,\mathbb{C})$ given by*

$$T_c(u \wedge v) = cg_0(\cdot, u)v - cg_0(\cdot, v)u$$

for every u, $v \in \mathbb{C}^n$, is a well defined homomorphism of $\mathfrak{so}(n,\mathbb{C})$-modules. Also, T_c is an isomorphism of $\mathfrak{so}(n,\mathbb{C})$-modules if and only if $c \neq 0$. If $n \neq 4$ then these maps exhaust all the $\mathfrak{so}(n,\mathbb{C})$-module homomorphisms $\wedge^2\mathbb{C}^n \to \mathfrak{so}(n,\mathbb{C})$.

The only non trivial part is the last statement, but the proof becomes easy using the simplicity of $\mathfrak{so}(n,\mathbb{C})$ ($n \neq 4$) and Schur's Lemma. For further details we refer to [27].

From Theorem 1.2 of [6] we conclude the maximality of $\mathfrak{so}(n,\mathbb{C})$ in $\mathfrak{so}(n+1,\mathbb{C})$.

Theorem 4.7. *Assume that $n \geq 3$ and $n \neq 4$, and let $\mathfrak{g} = \mathfrak{so}(n+1,\mathbb{C})$. Suppose that $\rho\colon \mathfrak{so}(n,\mathbb{C}) \hookrightarrow \mathfrak{g}$ is an injective Lie algebra homomorphism and let $\mathfrak{h} = \rho(\mathfrak{so}(n,\mathbb{C}))$. If $\mathfrak{g}$, $\mathfrak{h} \not\simeq \mathfrak{so}(2,\mathbb{C}) \times \mathfrak{so}(2,\mathbb{C})$, then $\mathfrak{h}$ is a maximal subalgebra of $\mathfrak{g}$.*

Next, we establish a decomposition of $\mathfrak{so}(n+1,\mathbb{C})$ as a $\mathfrak{so}(n,\mathbb{C})$-module.

Lemma 4.8. *For $n \geq 3$, $n \neq 4$ let*

$$[\cdot,\cdot]_c : \mathfrak{so}(n,\mathbb{C}) \oplus \mathbb{C}^n \times \mathfrak{so}(n,\mathbb{C}) \oplus \mathbb{C}^n \to \mathfrak{so}(n,\mathbb{C}) \oplus \mathbb{C}^n$$

be given by:

- $[X,Y]_c = XY - YX$ *for* $X,\ Y \in \mathfrak{so}(n,\mathbb{C})$.
- $[X,u]_c = -[u,X]_c = X(u)$ *for* $X \in \mathfrak{so}(n,\mathbb{C})$ *and* $u \in \mathbb{C}^n$.
- $[u,v]_c = T_c(u \wedge v)$ *for* $u,\ v \in \mathbb{C}^n$*, where* T_c *is the map defined in Lemma 4.6.*

Then, $[\cdot,\cdot]_c$ *defines a Lie algebra structure on* $\mathfrak{so}(n,\mathbb{C}) \oplus \mathbb{C}^n$*, which satisfies:*

$$(\mathfrak{so}(n,\mathbb{C}) \oplus \mathbb{C}^n, [\cdot,\cdot]_c) \simeq \mathfrak{so}(n+1,\mathbb{C})$$

as $\mathfrak{so}(n,\mathbb{C})$*-modules.*

Proof. Using Lemma 4.6 it is an easy exercise to prove that $[\cdot,\cdot]$ defines a Lie algebra structure. An isomorphism is easily seen to be given by:

$$(X,u) \to \begin{pmatrix} X & u \\ u^* & 0 \end{pmatrix}$$

where $u \in \mathbb{C}^n$ is considered as a column vector and $u^* = -u^T S$. $\square$

We state a uniqueness property for $\mathfrak{so}(n,\mathbb{C})$-invariant anti-Hermitian metric related to the constructions of the previous lemma. Its proof follows easily from Schur's Lemma and the uniqueness (up to a multiple) of the Killing form of complex simple Lie algebras.

Lemma 4.9. *Assume that* $n \geq 3$ *and* $n \neq 4$*. Let* $\langle\cdot,\cdot\rangle_1$ *and* $\langle\cdot,\cdot\rangle_2$ *be some anti-Hermitian metrics on* $\mathfrak{so}(n,\mathbb{C})$ *and* $\mathbb{C}^n$*, respectively. Assume that* $\langle\cdot,\cdot\rangle_1$ *and* $\langle\cdot,\cdot\rangle_2$ *are* $\mathfrak{so}(n,\mathbb{C})$*-invariant, in other words:*

- $\langle [X,Y], Z\rangle_1 = -\langle Y, [X,Z]\rangle_1$ *for every* $X,\ Y,\ Z \in \mathfrak{so}(n,\mathbb{C})$*, and*
- $\langle X(u), v\rangle_2 = -\langle u, X(v)\rangle_2$ *for every* $X \in \mathfrak{so}(n,\mathbb{C})$ *and* $u,\ v \in \mathbb{C}^n$.

If $c \in \mathbb{C} \setminus \{0\}$ *is given, then there exist* $a_1,\ a_2 \in \mathbb{C}$ *such that* $a_1\langle\cdot,\cdot\rangle_1 + a_2\langle\cdot,\cdot\rangle_2$ *is the Killing form of* $(\mathfrak{so}(n,\mathbb{C}) \oplus \mathbb{C}^n, [\cdot,\cdot]_c)$.

So far, we have established some facts about the Lie algebra $\mathfrak{so}(n,\mathbb{C})$.

To end this section, we consider the following lemma about the center of $\mathrm{SO}(n,\mathbb{C})$. Its proof is essentially the same of that in [4].

Lemma 4.10. *Suppose that* G *is a connected Lie group locally isomorphic to* $\mathrm{SO}(n,\mathbb{C})$*, where* $n \geq 3$ *and* $n \neq 4$*, and consider* $\rho\colon \mathrm{SO}(n,\mathbb{C}) \to G$ *a non-trivial homomorphism of Lie groups. Assume that* $\mathfrak{so}(n,\mathbb{C})$ *satisfy the same conditions as in Theorem 4.7. Then, the centralizer* $Z_G(\rho(\mathrm{SO}(n,\mathbb{C})))$ *of* $\rho(SO(n,\mathbb{C}))$ *in* G *contains* $Z(G)$ *(the center of* G*) as a finite index subgroup.*

4.2 The centralizer of isometric $\mathrm{Spin}(n,\mathbb{C})$-actions

In the rest of this work we will assume that $G = \mathrm{SO}(n,\mathbb{C})$. More precisely, we assume that we are given an holomorphic isometric $\mathrm{SO}(n,\mathbb{C})$-action on a finite volume, complete anti-Kähler manifold M with a dense $\mathrm{SO}(n,\mathbb{C})$-orbit. Hence, all the result from Chapter 3 hold. In particular, the results of Section 2.6 apply for this set up. We will also assume that $\dim_{\mathbb{C}} M \leq n(n+1)/2$ and $n \geq 3$, $n \neq 4$. Observe that our bound is precisely the complex dimension of $\mathrm{SO}(n+1,\mathbb{C})$.

Given the above assumptions, we observe that by Lemma 3.8 we have a fiberwise orthogonal decomposition $TM = TO \oplus TO^\perp$. On the other hand, if case (1) from Proposition 3.12 holds, then Proposition 3.13 describes the structure of the manifold M. Hence, we can assume that case (2) from Proposition 3.12 is satisfied.

We note that for our setup we have $\mathcal{G}(x) \simeq \mathfrak{so}(n,\mathbb{C})$, and so we can describe $\mathcal{G}(x)$-modules in terms of $\mathfrak{so}(n,\mathbb{C})$-modules. We denote by C^+ and C^- the $\mathfrak{so}(8,\mathbb{C})$-modules given by the two half spin representations. We start with a lemma that tell us the structure of $T_x\mathcal{O}^\perp$.

Lemma 4.11. *For a* $\mathrm{SO}(n,\mathbb{C})$*-action on* M *as above, assume that case (2) from of Proposition 3.12 holds, let* $S_0 \subset \widetilde{M}$ *be given by such case and* $n \geq 3$*,* $n \neq 4$*. We choose a fixed* $x \in S_0$*. Consider* $T_x\mathcal{O}^\perp$ *endowed with the* $\mathfrak{so}(n,\mathbb{C})$*-module structure given by Proposition 2.45(4). Then, for every* $x \in S_0$*:*

(1) If $n \neq 8$ *then* $\dim_{\mathbb{C}}(M) = n(n+1)/2$ *and* $\mathcal{V}(x) \simeq \mathbb{C}^n$ *as* $\mathfrak{so}(n,\mathbb{C})$*-modules. In particular,* $T_x\mathcal{O}^\perp \simeq \mathbb{C}^n$ *as* $\mathfrak{so}(n,\mathbb{C})$*-modules and* $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ *is isomorphic to* $\mathfrak{so}(n,\mathbb{C})$ *as a Lie algebra*

(2) If $n = 8$*, the* $\mathfrak{so}(n,\mathbb{C})$*-module* $T_x\mathcal{O}^\perp$ *is isomorphic to either* $\mathbb{C}^8$*,* C^+ *or* C^-*.*

Proof. Let us choose and fix $x \in S_0$ such that $\Omega_x \neq 0$. Since case (2) from Proposition 3.12 holds for x, this can be done and the map $\Omega_x \colon \wedge^2 T_x\mathcal{O}^\perp \to \mathfrak{so}(n,\mathbb{C})$ is a homomorphism of $\mathfrak{so}(n,\mathbb{C})$-modules, we conclude that $T_x\mathcal{O}^\perp$ is a non-trivial $\mathcal{G}(x)$-module whose dimension is at most n. Since $\dim_{\mathbb{C}}(T_x\mathcal{O}^\perp) \leq n$ then $\mathfrak{so}(n,\mathbb{C})$ is an irreducible module, it follows that Ω_x is an isomorphism. Then, the irreducibility of $\mathfrak{so}(n,\mathbb{C})$ implies that $T_x\mathcal{O}^\perp$ is irreducible as well. Lemma 4.5 implies that $T_x\mathcal{O}^\perp$ must be n-dimensional and isomorphic to $\mathbb{C}^n$ as $\mathfrak{so}(n,\mathbb{C})$-module except for the case given by the Lie algebra $\mathfrak{so}(8,\mathbb{C})$. For this Lie algebra the other possibilities are $\mathbb{C}^8$ and forms C^+ and C^- of the two half spin representations of $\mathfrak{so}(8,\mathbb{C})$.

Finally, the construction of $\mathcal{V}(x)$ in Proposition 3.7 implies that ev_x restricted to $\mathcal{G}(x) \oplus \mathcal{V}(x)$ is injective and $T_x\mathcal{O}^\perp = ev(\mathcal{V}(x))$. □

The previous results allow us to obtain the following conclusion about $\mathcal{H}_0(x)$ that helps realizing the description of the centralizer $\mathcal{H}$ as a Lie algebra.

Lemma 4.12. *Let* S_0 *be as in Propositions 2.45, 3.1 and 3.12. Then, for every* $x \in S_0$*,* $n \geq 3$*,* $n \neq 4$*,* $\mathcal{H}_0(x)$ *is either* 0 *or a Lie subalgebra of* $\mathcal{H}$ *isomorphic to* $\mathfrak{so}(n,\mathbb{C})$*. In the latter case,* $\mathcal{H}_0(x)$ *is also isomorphic to* $\mathfrak{so}(n,\mathbb{C})$ *as a* $\mathfrak{so}(n,\mathbb{C})$*-module.*

Proof. In our setup, $\mathfrak{g}(x) = \mathfrak{so}(n,\mathbb{C})(x)$. We claim that

$$\lambda_x^\perp(\mathfrak{so}(n,\mathbb{C}))(x) = \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x).$$

By Proposition 2.45 (4) the vector space $T_x\mathcal{O}^\perp$ has a $\mathfrak{so}(n,\mathbb{C})$-module structure induced from the homomorphism $\lambda_x^\perp \circ \rho_x$. By our choice of x and Lemma 4.11 such module structure is in fact non-trivial. Hence, $\lambda_x^\perp \circ \rho_x \colon \mathfrak{so}(n,\mathbb{C}) \to \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ is non-trivial as well and so it is injective. But then it has to be surjective because the domain and target have the same dimensions.

Let $Z \in \mathcal{H}_0(x)$ and $T \in \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ be given. Then, by the above claim there is some $X \in \mathfrak{so}(n, \mathbb{C})$ such that $T = \lambda_x^\perp(\rho_x(X))$. For every local vector field V such that $V_x \in T_x\mathcal{O}$ we have:

$$\begin{aligned}\left[T, \lambda_x^\perp(Z)\right](V_x) &= \left[\lambda_x^\perp(\rho_x(X)), \lambda_x^\perp(Z)\right](V_x)\\ &= [\rho_x(X), [Z, V]]_x - [Z, [\rho_x(X), V]]_x\\ &= [[\rho_x(X), Z], V]_x\\ &= [[\rho_x(X) + X^*, Z], V]_x\\ &= [[\widehat{\rho}_x(X), Z], V]_x .\end{aligned}$$

Since the $\mathfrak{so}(n, \mathbb{C})$-module structure on $\mathcal{H}$ is defined by $\widehat{\rho}_x$ and $\mathcal{H}_0(x)$ is a submodule of such structure, we have $[\widehat{\rho}_x(X), Z] \in \mathcal{H}_0(x)$, and so the last formula proves that $\left[T, \lambda_x^\perp(Z)\right] = \lambda_x^\perp([\widehat{\rho}_x(X), Z])$, thus showing that $\lambda_x^\perp(\mathcal{H}_0(x))$ is an ideal in $\mathfrak{so}(T_x\mathcal{O}^\perp)$.

Proposition 3.6 shows that $\mathcal{H}_0(x)$ is a Lie algebra isomorphic to its image in $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ under $\lambda_x^\perp$. Such image is, by above, an ideal of $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$. By our choice of x and Lemma 4.11, the Lie algebra $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ is isomorphic to $\mathfrak{so}(n, \mathbb{C})$, which is simple since $n \geq 3$ and $n \neq 4$. This implies that $\mathcal{H}_0(x)$ is either 0 or isomorphic to $\mathfrak{so}(n, \mathbb{C})$ as a Lie subalgebra of $\mathcal{H}$.

On the other hand, for $X \in \mathfrak{so}(n, \mathbb{C})$ and $Z \in \mathcal{H}_0(x)$, considering the definitions of the $\mathfrak{so}(n, \mathbb{C})$-module structures involved we have:

$$\lambda_x^\perp(X \cdot Z) = \lambda_x^\perp([\widehat{\rho}_x(X), Z]) = \lambda_x^\perp(\left[\rho_x(X), \lambda_x^\perp(Z)\right]) = X \cdot \lambda_x^\perp(Z),$$

where the second identity holds by the definiton of $\widehat{\rho}_x$ in terms of ρ_x and because $\mathcal{H}_0(x)$ centralizes the $\mathrm{Spin}(n, \mathbb{C})$-action. But this of last relation shows that $\lambda_x^\perp$ restricted to $\mathcal{H}_0(x)$ is a homomorphism of $\mathfrak{so}(n, \mathbb{C})$-modules. By Lemma 4.11 we conclude that $\mathcal{H}_0(x)$ is either 0 or isomorphic to $\mathfrak{so}(n, \mathbb{C})$ as a $\mathfrak{so}(n, \mathbb{C})$-module. $\square$

Finally, we rule out one of the possibilities for $\mathcal{H}_0(x)$ and we obtain a description of the Lie algebra structure of the centralizer $\mathcal{H}$.

Lemma 4.13. *Let S_0 as in Propositions 2.45, 3.1 and 3.12. With the notation from 3.7, for $x \in S_0$ we have that $\mathcal{H}_0(x) = \{0\}$ and $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ is isomorphic as a complex Lie algebra to $\mathfrak{so}(n+1, \mathbb{C})$.*

Proof. By Propositions 3.6 and 3.11, the Lie subalgebra $\mathcal{H}_0(x)$ is completely determined by its image $\lambda_x^\perp$ so that $\lambda_x^\perp(\mathcal{H}_0(x))$ is a Lie subalgebra and a $\lambda_x^\perp(\mathcal{G}(x))$-submodule of $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ so that

$$\left[\lambda_x^\perp(\mathcal{H}_0(x)), \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)\right] \subset \ker(\Omega_x).$$

On the other hand, by Lemma 4.11 we know that $\mathcal{V}(x) \simeq \mathbb{C}^n$, as $\mathcal{G}(x)$-modules, and by Lemma 4.6 the scalar product on $ev_x(\mathcal{V}(x))$ inherited from $T_x\mathcal{O}^\perp$ is (a nonzero multiple of) the canonical complex Riemannian product in $\mathbb{C}^n$.

Furthermore, since this conclusion is obtained from the $\mathcal{G}(x)$-module structure, it also shows that we can assume that the embedding $\lambda_x^\perp : \mathcal{G}(x) \to \mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$ is equivalent to the restriction to $\mathfrak{so}(n, \mathbb{C})$ of one of the embeddings in Lemma 4.8.

Hence, Lemma 4.8 provides the $\mathcal{G}(x)$-module and Lie algebra structure of $\mathfrak{so}(T_x\mathcal{O}^\perp, \widetilde{g}_x)$. From this we see that $\ker(\Omega_x)$ is the sum of the submodules complementary to $\lambda_x^\perp(\mathcal{G}(x))$. In particular, we have

$$\ker(\Omega_x) \simeq \mathbb{C}^n.$$

Considering the above restrictions on $\lambda_x^\perp(\mathcal{H}_0(x))$ and the bracket identities from Lemma 4.8, we conclude that the only possibility for $\mathcal{H}_0(x)$ is to be 0.

□

4.3 Proof of the main theorem

In this last section we will assume the hypotheses of the Main Theorem 1.4 and use the notation from the previous section. More precisely, we assume that M is a connected holomorphic anti-Kähler manifold which is complete weakly irreducible and has finite volume. We also assume that M admits a holomorphic and isometric $\mathrm{Spin}(n, \mathbb{C})$-action with a dense orbit for some integer $n \geq 3$, $n \neq 4$. Finally we are assuming that $\dim_{\mathbb{C}}(M) \leq n(n+1)/2$.

Instead of proving the theorem directly, we have broken the proof in many lemmas to make the readability easier.

Lemma 4.14. *There is an isomorphism*

$$\psi\colon \mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}^n \to \mathcal{H} = \mathcal{G}(x_0) \oplus \mathcal{V}(x_0)$$

of Lie algebras that preserves the summands in that order, where the domain has the Lie algebra structure given by $[\cdot,\cdot]_c$ for some $c \neq 0$ as defined in Lemma 4.8. In particular, ψ is an isomorphism of $\mathfrak{so}(n, \mathbb{C})$-modules as well.

Proof. The result follows from the arguments in the second to last paragraph in the proof of Lemma 4.13 when $\mathcal{V}(x_0) \simeq \mathbb{C}^n$ as $\mathfrak{so}(n, \mathbb{C})$-modules and $n \geq 3$, $n \neq 4$. Hence, by Lemma 4.11 we can assume that $n = 8$ in the rest of the proof.

By Lemma 4.13 there is an isomorphism $\psi\colon \mathfrak{h} \to \mathcal{H}$, for $\mathfrak{h} = \mathfrak{so}(9, \mathbb{C})$, The restriction of this homomorphism to $\psi^{-1}(\mathcal{G}(x_0))$ yields a representation of $G(x_0) \simeq \mathfrak{so}(8, \mathbb{C})$ on the 9-dimensional space $\mathbb{C}^9$. Since $\mathfrak{so}(8, \mathbb{C})$ is split and using Weyl's dimension formula we find that $\mathfrak{so}(8, \mathbb{C})$ does not admit 9-dimensional irreducible representations. We conclude the existence of a line $L \subset V$ which is a $\mathcal{G}(x_0)$-submodule.

This yields an orthogonal decomposition $V = L \oplus L^\perp$ into non-degenerate subspaces which is clearly a decomposition into $\mathcal{G}(x_0)$-submodules. Hence, ψ induces an isomorphism $\mathfrak{so}(L^\perp, \widetilde{g}_{x_0}) \to \mathcal{G}(x_0)$. In particular, $\mathfrak{so}(L^\perp, \widetilde{g}_{x_0}) \simeq \mathfrak{so}(n, \mathbb{C})$ as Lie algebras under ψ. With respect to the corresponding $\mathfrak{so}(n, \mathbb{C})$-module structure, it is easily seen that $\mathfrak{so}(L^\perp, \widetilde{g}_{x_0})$ has a complementary module in $\mathfrak{h}$ isomorphic to $\mathbb{C}^n$. This provides an isomorphism $h \simeq \mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}^n$ so that the Lie algebra structure on $\mathfrak{h}$ corresponds to the one given by $[\cdot,\cdot]_c$ on $\mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}^n$ for some $c \neq 0$. Hence, under the identification $\mathfrak{h} \simeq \mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}^n$ of Lie algebras, ψ is the required isomorphism. □

Let us fix an isomorphism of Lie algebras $\psi : \mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}^n \to \mathcal{H} = G(x_0) \oplus \mathcal{V}(x_0)$ as in Lemma 4.14. We will identify $\mathfrak{h} = \mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}^n$ with $\mathfrak{so}(n+1, \mathbb{C})$

through the appropriate isomorphism as considered in Lemma 4.8. Also, we will denote with H the Lie group $\mathrm{Spin}(n+1,\mathbb{C})$, chosen so that $\mathrm{Lie}(H) = \mathfrak{h}$.

By Lemma 2.39, there is a holomorphic isometric right H-action on $\widetilde{M}$ such that $\psi(X) = X^*$ for every $X \in h$. As in the previous lemmas, we now consider the orbit map:

$$f\colon H \to \widetilde{M}, \quad h \mapsto x_0 h$$

which satisfies $df_I(X) = X^*_{x_0} = \psi(X)_{x_0}$ for every $X \in \mathfrak{h}$. By the choice of ψ and Lemma 4.12 it follows that df_I is an isomorphism that maps $\mathfrak{so}(n,\mathbb{C})$ onto $T_{x_0}\mathcal{O}$ and $\mathbb{C}^n$ onto $T_{x_0}\mathcal{O}^\perp$. Since f is H-equivariant for the right action on its domain, we conclude that it is an holomorphic local diffeomorphism.

Lemma 4.15. *Let $\overline{g}$ be the metric on $h = \mathfrak{so}(n,\mathbb{C}) \oplus \mathbb{C}^n$ defined as the pullback under df_I of the metri g_{x_0} on $T_{x_0}\widetilde{M}$. Then, $\overline{g}$ is $\mathfrak{so}(n,\mathbb{C})$-invariant.*

Proof. By the above expression of df_I and since ψ is an isomorphism of Lie algebras with $\psi(\mathfrak{so}(n,\mathbb{C})) = \mathcal{G}(x_0)$, it is enough to show that the metric on $\mathcal{H}$ defined as the pullback of g_{x_0} with respect to the evaluation map:

$$H \to T_{x_0}\widetilde{M}, \quad X \mapsto X_{x_0},$$

is $\mathcal{G}(x_0)$-invariant. For simplicity, will denote with $\overline{g}$ such metric on $\mathcal{H}$. Let X, Y, $Z \in \mathcal{H}$ be given with $X \in \mathcal{G}(x_0)$. In particular, there exist $X_0 \in \mathfrak{so}(n,\mathbb{C})$ such that $X = \rho_{x_0}(X_0) + X_0^*$, where ρ_{x_0} is the homomorphism from Proposition 2.45 and X_0^* is the vector field on $\widetilde{M}$ induced by the left $\mathrm{Spin}(n,\mathbb{C})$-action. Then, the following proves the required invariance:

$$\begin{aligned}
\overline{g}([X,Y],Z) &= g_{x_0}([X,Y]_{x_0}, Z_{x_0}) = g([X,Y],Z)|_{x_0} \\
&= g([\rho_{x_0}(X_0) + X_0^*, Y], Z)|_{x_0} = g([\rho_{x_0}(X_0), Y]Z)|_{x_0} \\
&= \rho_{x_0}(X_0)(g(Y,Z))|_{x_0} - g(Y, [\rho_{x_0}(X_0), Z])|_{x_0} \\
&= -g(Y, [\rho_{x_0}(X_0), Z])|_{x_0} = -g(Y, [\rho_{x_0}(X_0) + X_0, Z])|_{x_0} \\
&= -g(Y, [X,Z])|_{x_0} = -\overline{g}(Y, [X,Z]).
\end{aligned}$$

We have used in lines 2 and 4 that H centralizes X_0^*. To obtain the third line we used that $\rho_{x_0}(X_0)$ is a Killing field for the metric g. And the first identity in line 4 uses the fact that $\rho_{x_0}(X_0)$ vanishes at x_0. □

From the previous result and Lemma 4.9, for $n \geq 3$, $n \neq 4$, we can rescale the metric along the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$ in M so that the new metric $\widehat{g}$ on $\widetilde{M}$ satisfies $(df_I)^*(\widehat{g}_{x_0}) = K$, the Killing form on $\mathfrak{h}$.

Note that since the elements of $\mathcal{H}$ preserve the decomposition $T\mathcal{O} \oplus T\mathcal{O}^\perp$, then $\mathcal{H} \subset \mathrm{Kill}(\widetilde{M}, g)$. In other words, the elements of $\mathcal{H}$ are Killing vector fields for the metric $\widehat{g}$ rescaled as above. In particular, $\widehat{g}$ is invariant under the right H-action. Similarly, the left $\mathrm{Spin}(n,\mathbb{C})$-action on $\widetilde{M}$, from the hypotheses of the Main Theorem, preserves the rescaled metric $\widehat{g}$. Also note that the metric $\widehat{g}$ is the lift of a correspondingly rescaled metric $\widehat{g}$ in M.

Consider the bi-invariant metric on H induced by the Killing form K, which we will denote with the same symbol. The previous discussion implies that the local diffeomorphism $f\colon (H,K) \to (\widetilde{M},\widehat{g})$ is a local isometry. Then, Corollary 29 in page 202 of [25], the completeness of (H,K) and the simple connectedness of $\widetilde{M}$ imply that f is an isometry.

Hence, from the previous discussion we obtain the following result.

Lemma 4.16. *Let M be as in the statement of the Main Theorem 1.4 If $\dim_{\mathbb{C}}(M) = n(n+1)/2$, then for $H = \mathrm{Spin}(n,\mathbb{C})$, there exists an holomorphic diffeomorphism $f\colon H \to \widetilde{M}$ and a holomorphic isometric right H-action on $\widetilde{M}$ such that:*

(1) on $\widetilde{M}$, the left $\mathrm{Spin}(n,\mathbb{C})$-action and the right H-action commute with each other,

(2) f is H-equivariant for the right H-action on its domain,

(3) for an anti-Hermitian metric $\widehat{g}$ in M obtained by rescaling the original one on the summands of $TM = T\mathcal{O} \oplus T\mathcal{O}^{\perp}$, the map $f\colon (H,K) \to (\widetilde{M},\widehat{g})$ is an isometry where K is the bi-invariant metric on H induced from the Killing form of its Lie algebra.

If we consider H endowed with the bi-invariant pseudo-Riemannian metric K induced by the Killing form of its Lie algebra, then Lemma 4.16 allows to consider (H,K) as the isometric universal covering space of $(M,\widehat{g})$. We will use this identification in the rest of the arguments.

The isometry group $\mathrm{Iso}(H)$ for the pseudo-Riemannian manifold (H,K) has finitely many connected components (see for example Section 4 of [26]). Furthermore, the connected component of the identity is given as $\mathrm{Iso}_0(H) = L(H)R(H)$, the subgroup generated by $L(H)$ and $R(H)$, the left and right translations, respectively.

Let $\rho : \mathrm{Spin}(n,\mathbb{C}) \to \mathrm{Iso}(H)$ be the homomorphism induced by isometric left $\mathrm{Spin}(n,\mathbb{C})$-action on H. With respect to the natural covering $H \times H \to L(H)R(H)$, this yields homomorphisms ρ_1, $\rho_2\colon \mathrm{Spin}(n,\mathbb{C}) \to H$ such that:

$$\rho(g) = L_{\rho_1(g)} \circ R_{\rho_2(g)^{-1}},$$

for every $g \in \mathrm{Spin}(n,\mathbb{C})$. By Lemma 4.16 we have $\rho(g) \circ R_h = R_h \circ \rho(g)$ for every $g \in \mathrm{Spin}(n,\mathbb{C})$ and $h \in H$. In particular, $\rho_2(\mathrm{Spin}(n,\mathbb{C}))$ lies in the center $Z(H)$ and so (being connected) it is trivial. We conclude that $\rho = L_{\rho_1}$ which implies that the $\mathrm{Spin}(n,\mathbb{C})$-action on H is induced by the homomorphism $\rho_1\colon \mathrm{Spin}(n,\mathbb{C}) \to H$ and the left action of H itself. Note that ρ_1 is necessarily non-trivial.

By Lemma 4.16, we have $\pi_1(M) \subset \mathrm{Iso}(H)$, and from the above remarks $\Gamma_1 = \pi_1(M) \cap \mathrm{Iso}(H)$ is a finite index subgroup of $\pi_1(M)$. In particular, every $\gamma \in \Gamma_1$ can be written as $\gamma = L_{h_1} \circ R_{h_2}$ for some h_1, $h_2 \in H$.

On the other hand, since the left $\mathrm{Spin}(n,\mathbb{C})$ on H is the lift of an action on M, it follows that it commutes with the Γ_1-action. Applying this property to $\gamma = L_{h_1} \circ R_{h_2}$ we conclude that $L_{h_1} \circ L_{\rho_1(g)} = L_{\rho_1(g)} \circ L_{h_1}$, for every $g \in \mathrm{Spin}(n,\mathbb{C})$, which implies $\Gamma_1 \subset L(Z)R(H)$, where Z is the centralizer of $\rho_1(\mathrm{Spin}(n,\mathbb{C}))$ in H. By Lemma 4.10, the center of $Z(H)$ has finite index in Z, which implies that $R(H)$ has finite index in $L(Z)R(H)$. In particular, $\Gamma = \Gamma_1 \cap R(H)$ is a finite index subgroup of Γ_1, and so has finite index in $\pi_1(M)$ as well.

Hence, the natural identification $R(H) = H$ realizes Γ as a discrete subgroup of H such that H/Γ is a finite covering space of M. Furthermore, if $\varphi\colon H/\Gamma \to M$ is the corresponding covering map, and for the left $\mathrm{Spin}(n,\mathbb{C})$-action on H/Γ given by the homomorphism $\rho_1\colon \mathrm{Spin}(n,\mathbb{C})$, then the above constructions show that φ is $\mathrm{Spin}(n,\mathbb{C})$-equivariant. We also note that φ is a local isometry for the metric $\widehat{g}$ on M considered in Lemma 4.16.

To complete the proof in the Main Theorem it only remains to show that Γ is a lattice in H. For this it is enough to prove that M has finite volume in the metric $\widehat{g}$. The following result provides proofs of these facts since we are assuming that M has finite volume in its original metric.

Lemma 4.17. *Let us denote with* vol *and* $\mathrm{vol}_{\widehat{g}}$ *the volume elements on M for the constant original on M and the rescaled metric $\widehat{g}$, respectively. Then, there is some constant $C > 0$ such that* $\mathrm{vol}_{\widehat{g}} = C\,\mathrm{vol}$.

Proof. Clearly, it suffices to verify this locally, so we consider some coordinates $(x^1, \dots, x^m)$ of M in a neighborhood U of a given point such that $(x^1, \dots, x^r)$ defines a set of coordinates of the leaves of the foliation $\mathcal{O}$ in such neighborhood. For the original metric g on M, consider as above the orthogonal bundle $T\mathcal{O}^\perp$ and a set of 1-forms $\theta^1, \dots, \theta^{m-r}$ that define a basis for its dual $(T\mathcal{O}^\perp)^*$ at every point in U. Hence, in U the metric g has an expression of the form:

$$g = \sum_{i,j=1}^{r} h_{ij} dx^i \otimes dx^j + \sum_{i,j=1}^{m-r} k_{ij} \theta^i \otimes \theta^j.$$

From this and the definition of the volume element as an m-form, its is easy to see that:

$$\mathrm{vol} = \sqrt{|\det(h_{ij}) \det(k_{ij})|}\, dx^1 \wedge \cdots \wedge dx^r \wedge \theta^1 \wedge \cdots \wedge \theta^{m-r}.$$

On the other hand, the metric $\widehat{g}$ is obtained by rescaling g along the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp$, and so it has an expression of the form:

$$\widehat{g} = \sum_{i,j=1}^{r} c_1 h_{ij} dx^i \otimes dx^j + \sum_{i,j=1}^{m-r} c_2 k_{ij} \theta^i \otimes \theta^j,$$

for some constants c_1, $c_2 \neq 0$. Hence, the volume element of $\widehat{g}$ satisfies:

$$\begin{aligned} \mathrm{vol}_{\widehat{g}} &= \sqrt{|\det(c_1 h_{ij}) \det(c_2 k_{ij})|}\, dx^1 \wedge \cdots \wedge dx^r \wedge \theta^1 \wedge \cdots \wedge \theta^{m-r} \\ &= \sqrt{|c_1^r c_2^{m-r}|}\, \mathrm{vol}\,. \end{aligned}$$

$\square$

Chapter 5

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