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ANALYSIS OF GROWING GRAPHS AND QUANTUM PROBABILITY

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Contents

Introduction

The main topic of this thesis is the spectral analysis of graphs. We are interested particularly in the spectral analysis of large graphs or growing graphs. In this work we focus mainly in the study of two huge families of graphs: distance-k graphs of graph products and distance-regular graphs applying the quantum decomposition method.

The study of deterministic (or random) growing combinatorial objects like partitions, permutations, walks, trees, maps, etc, has been increasing in recent years. In particular, graphs have been studied deeply, motivated by the recent trend of complex network theory.

Non-commutative Probability Theory was iniciated by Hudson and Parthasarathy [28] and began to develop since the 80's in order to settle down mathematical bases for Quantum Physics. It started with the ideas of Von Neumann [49]. From this theory, various concepts of independence have been introduced thanks to various kinds of non-commutative moment relations. It was proved by Muraki [38], that there are essentially four notions of independence: tensorial, free, Boolean and monotone.

Tensorial independence is derivated from the usual independence in classical probability theory. The notion of free independence (freeness) was introduced by Voiculescu [47]. Boolean independence was presented by Speicher and Wourodi [45], and it is implicitly used in Bozejko's work [10]. Finally, the notion of monotone independence was introduced by Muraki [37].

The above mentioned notions of independence also correspond to widely studied graph products. The relation between classical convolution and cartesian product of graphs was observed by Polya [43]. Later, in works by Accardi, Ghorbal and Obata [2], Accardi, Lenczewski and Salapata [3], and others, they found that very well-known graph products are related to convolutions in non-commutative probability. The free product of graphs corresponds to Voiculescu's free convolution [47]. The star product of graphs, studied in Woess [51] corresponds to Boolean convolution, which was studied by Obata [41]. The monotone product of graph is related to monotone convolution, studied by Krishnapur and Peres [30], this last fact was observed by Accardi, Ghorbal and Obata [2]. In this thesis we focus on three of these graph products: cartesian, Boolean and free.

As we said above, one of the objectives of this thesis is the study of distance-k graph of graph products. The distance-k graphs were introduced in 1989 by Brower, Cohen and Neumaier [12], and in particular, the study of distance-k graph of graph products was developed by Kurihara [31], Obata et. al. [23], and others. The spectrum of the distance-k graph of the Cartesian product of graphs was first studied by Kurihara and Hibino [32] where they consider the distance-2 graph of $K_2 \times \cdots \times K_2$ (the n-dimensional hypercube). More recently, in a series of papers [17, 23, 31, 32, 33, 40] the asymptotic spectral distribution of the distance-k graph of the N-fold power of the Cartesian product was studied. These investigations, finally lead to Theorem 4.1.2, which generalizes the central limit theorem for Cartesian product of graphs, and describes the asymptotic spectral distribution of the distance-k graph of the N-fold Cartesian power (as $N \to \infty$). In fact, they found that the distribution (in the normalized trace) of the normalized adjacency matrix of the distance-k graph (for k fixed) of the N-fold Cartesian power converges in moments to the probability distribution of

$$\left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!} \tilde{H}_k(g),$$

where $\tilde{H}_k(g)$ is the monic Hermite polynomial of degree k and g is a random variable obeying the standar normal distribution.

In the same spirit, the author and Arizmendi consider in [6] the analog of Theorem 4.1.2 by changing the Cartesian product by the star product. There, we establish that the asymptotic distribution in the vacuum state of the normalized adjacency matrix of the N-fold Boolean power of a graph converges (as $N \to \infty$) in distribution, to a centered Benoulli distribution. The limit distribution above is universal in the sense that it is independent of the details of a factor G, but also in this case the limit **does not** depend on k. The proof of this theorem is based in a fourth moment lemma for convergence to a centered Bernoulli distribution.

On the other hand, the d-regular tree is the d-fold free product graph of K_2, the complete graph with two vertices. We study the distance-k graph of a d-regular tree for fixed d and k. This is an example where we can find the distribution with respect to the vacuum state in a closed form. Moreover, this example sheds light on the general case of the d-fold free product of graphs, in the same way as the d-dimensional cube was the leading example for investigations of the distance-k graph of the d-fold Cartesian product of graphs (Kurihara [31]).

Then, we consider two related problems which are in the asymptotic regime. On one hand, we show that the asymptotic distributions of distance-k graphs of d-fold free product graphs, as d tends to infinity, are given by the distribution of

$$P_k(s),$$

where s is a semicircular random variable and P_k is the k-th Chebychev polynomial. These polynomials are orthogonal with respect to semicircle distribution (see Chihara [13]). The idea to prove this result is to find a recurrence formula for homogenous tree and notice that the adjacency matrix of the distance-k graph of the free product fullfills the same recurrence formula plus *negligible* matrices. Therefore we calculate mixed moments between these matrices and the adjacency matrix of distance-k graph, and we realize that these mixed moments go to zero.

Apart of this, we find the asymptotic spectral distribution of the distance-k graph of a random d-regular graph of size n, as n tends to infinity. In the original paper by McKay [35], he proved that the asymptotical spectral distributions of d-regular random graph are exactly the distribution of the d-regular tree. Heuristically, the reason is that, locally, large random d-regular graphs look like the d-regular tree and thus asymptotically their spectrum should coincide. This turns out to remain true for the distance-k graph and thus we shall expect to get a similar result. In Section 4.3.4 we formalize this intuition. These results related to distance-k graphs of free product are collected in the published paper by the author and Arizmendi [7].

Although in the above results we use the moments method, it is not always easy (or possible) to compute all the moments. Instead this method, we use *method of quantum decomposition*.

The term of *quantum decomposition* was first introduced by Hashimoto [20] in a study of an adjacency matrix of a large Cayley graph. This idea has been applied also to similar studies for large Hamming graphs [22, 24], Johnson graphs [21, 24, 25], Odd graphs [27], Homogeneous trees [19], and others. Most of these distributions were compute with respect to the vacuum state and deformed vacuum state, except in the case of odd graphs, where only the vacuum state case was studied. A summary of these results can be found in the book by Hora & Obata [26].

The *method of quantum decomposition* describes the distribution of the adjacency matrix of a graph through the three-term recurrence relation and come to the fundamental link with an interacting Fock probability space. This method is effective especially for the asymptotic spectral analysis of growing graphs.

Let us consider a growing family of graphs $\{G^{(k)} = (V^{(k)}, E^{(k)})\}_{k\geq 1}$ and the limit

$$\lim_{k\to\infty} \frac{A_k}{Z_k},$$

where A_k is the adjacency matrix of $G^{(k)}$ and Z_k a normalizing constant. Then we define a stratification: $V^{(k)} = \cup_{n=0}^{\infty} V_n^{(k)}$ on the basis of the natural distance function of $G^{(k)}$ and decompose the adjacency matrix A_k into a sum of quantum components:

$$A_k = A_k^+ + A_k^-.$$

These operators act asymptotically in the Hilbert space $\Gamma(G^{(k)})$ associated with the stratification of $V^{(k)}$. Then, there exists an interacting Fock space $(\Gamma, \{\omega_n\}, B^+, B^-)$ in which the limits

$$\tilde{B}^{\pm} = \lim_{k\to\infty} \frac{A_k^{\pm}}{Z_k},$$

are described, where $\tilde{B}^{\pm}$ is a linear combination of $B^{\pm}$ and a function of the number operator N.

In this sense, Igarashi and Obata [27] studied a growing family of odd graphs and the two-sided Rayleigh distribution appeared in the limit of vacuum spectral distribution of the adjacency matrix. Our aim in Section 3.4.2 is to calculate an explicit probability measure describing the limit distribution of the normalized adjacency matrix of the same growing family as above (odd graphs), but now with respect to deformed vacuum state, using quantum decomposition method.

The thesis is structured as follows. Chapter 1 contains basics on Quantum Probability Theory. We give preliminaries needed for this thesis. In Chapter 2 we introduce the Quantum Decomposition Method and give the framework that we need in order to do spectral analysis of graphs. Chapter 3 is about Distance-Regular Graphs, these are graphs which possess a significant property from the viewpoint of quantum decomposition. In this chapter we also treat the particular case of odd graphs and their spectral distributions in the vacuum and specially, in the deformed vacuum states. Finally, in Chapter 4 we define Distance-k graphs and studies the spectral distribution of distance-k graph of the Cartesian product, Star and free products of graphs.

Chapter 1

Quantum Probability Theory

In this chapter we give some basic definitions and results on Quantum Probabilty. We mainly follow the monograph [26]. We start defining a non-commutative probability space, which is the appropiate framework for Quantum Probability. Next we introduce the interacting Fock space and orthogonal polynomials. Later, the notions of independence and their corresponding central limit theorems. Finally, we define the Cauchy-Stieltjes transform and its continued fraction expansion and we present a 4th moment theorem for the distance defined in equation (1.4.1) to the Bernoulli distribution which appears in [6].

1.1 Non-Commutative Probability Space

Definition 1.1.1 *A C^*-**probability space** is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital C^*-algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a state, i.e. is a positive unital linear functional. The elements of $\mathcal{A}$ are called (non-commutative) **random variables**. An element $a \in \mathcal{A}$ such that $a = a^*$ is called **self-adjoint**.*

The functional φ should be understood as the expectation in classical probability.

For $a_1, \dots, a_k \in \mathcal{A}$, we will refer to the values of $\varphi(a_{i_1} \cdots a_{i_n})$, $1 \leq i_1, ..., i_n \leq k$, $n \geq 1$, as the *mixed moments* of $a_1, \dots, a_k$.

For any self-adjoint element $a \in \mathcal{A}$ there exists a unique probability measure μ_a (its spectral distribution) with the same moments as a, that is,

$$\int_{\mathbb{R}} x^k \mu_a(dx) = \varphi(a^k), \quad \forall k \in \mathbb{N}.$$

We say that a sequence $a_n \in \mathcal{A}_n$ *converges in distribution* to $a \in \mathcal{A}$ if μ_{a_n} converges in distribution to μ_a. In this setting convergence in distribution is replaced by convergence in moments. Let $(\varphi_n, \mathcal{A}_n)$ be a sequence of C^*-probability spaces and let $a \in (\mathcal{A}, \varphi)$ be a selfadjoint random variable. We say that the sequence $a_n \in (\varphi_n, \mathcal{A}_n)$ of selfadjoint random variables converges to *a in moments* if

$$\lim_{n \to \infty} \varphi_n(a_n^k) = \varphi(a^k) \text{ for all } k \in \mathbb{N}.$$

If supp $\{\mu_a\}$ is bounded then convergence in moments implies convergence in distribution. The following proposition is straightforward and will be used frequently in the paper. A sequence of polynomials $\{P_n = \sum_{i=0}^{l} c_{n,i}x^i\}_{n>0}$ of degree at most $l \geq k$ is said to converge to a polynomial $P = \sum_{i=0}^{k} c_i x^i$ of degree k if $c_{i,n} \to c_i$ for $0 \leq i \leq k$ and $c_{i,n} \to 0$ for $k < i \leq l$.

Proposition 1.1.2 *Suppose that the sequence of random variables $\{a_n\}_{n>0}$ converges in moments to a and the sequence of polynomials $\{P_n\}_{n>0}$ converges to P. Then, the random variables $P_n(a_n)$ converge to $P(a)$.*

1.2 Interacting Fock Spaces

Definition 1.2.1 *A real sequence $\{\omega_n\}_{n\geq 1}$ is called a **Jacobi sequence** if*

(i) (infinite type) $\omega_n > 0$ for all $n \geq 1$; or

(ii) (finite type) there exists $m_0 \geq 1$ such that $\omega_1 > 0, \omega_2 > 0, \dots, \omega_{m_0-1} > 0, \omega_{m_0} = \omega_{m_0+1} = \cdots = 0$.

By definition $(0, 0, \dots)$ is a Jacobi sequence ($m_0 = 1$).

Given a Jacobi sequence $\{\omega_n\}$, we consider a Hilbert space Γ as follows: If $\{\omega_n\}$ is of infinite type, let Γ be an infinite dimensional Hilbert space with an orthonormal basis $\{\Phi_0, \Phi_1, \dots\}$. If $\{\omega_n\}$ is of finite type, let Γ be an m_0-dimensional Hilbert space with an orthonormal basis $\{\Phi_0, \Phi_1, \dots, \Phi_{m_0-1}\}$.
We next define linear operators $B^{\pm}$ on Γ by

$$B^+\Phi_n = \sqrt{\omega_{n+1}}\Phi_{n+1}, \quad n = 0, 1, \dots,$$
$$B^-\Phi_0 = 0, \quad B^-\Phi_n = \sqrt{\omega_n}\Phi_{n-1}, \quad n = 1, 2, \dots,$$

where we understand $B^+\Phi_{m_0-1} = 0$ when $\{\omega_n\}$ is of finite type. We call B^- the *annihilation operator* and B^+ the *creation operator*.

Definition 1.2.2 (Jacobi coefficient) *A pair of sequences $(\{\omega_n\}, \{\alpha_n\})$ is called a **Jacobi coefficient** if*

(i) $\{\omega_n\}$ is a Jacobi sequence of infinite type and $\{\alpha_n\}$ is an infinite real sequence; or

(ii) $\{\omega_n\}$ is a Jacobi sequence of finite type with length m_0 and $\{\alpha_1, \alpha_2, \dots, \alpha_{m_0+1}\}$ is a finite real sequence with $m_0 + 1$ terms.

Given a Jacobi parameter $(\{\omega_n\}, \{\alpha_n\})$ we define the Hilbert space Γ with an orthonormal basis $\{\Phi_n\}$, the annihilation operator B^- and the creation operator B^+ as above. In addition we define the *conservation operator* by

$$B^\circ\Phi_n = \alpha_{n+1}\Phi_n, \quad n = 0, 1, 2, \dots.$$

Definition 1.2.3 (Interacting Fock space) *With each Jacobi parameter* $(\{\omega_n\}, \{\alpha_n\})$ *we associate an* ***interacting Fock space***

$$(\Gamma, \{\omega_n\}, B^+, B^-, B^\circ),$$

obtained as above. When $\{\alpha \equiv 0\}$ *is a null sequence, we omit* B° *and* $\{\alpha_n\}$.

Let μ be a probability measure with all moments, that is $m_n(\mu) := \int_{\mathbb{R}} |x^n|\mu(dx) < \infty$. The Jacobi parameters $\gamma_m = \gamma_m(\mu) \geq 0, \alpha_m = \alpha_m(\mu) \in \mathbb{R}$, are defined by the recursion

$$xQ_m(x) = Q_{m+1}(x) + \alpha_m Q_m(x) + \gamma_{m-1}Q_{m-1}(x),$$

where the polynomials $Q_{-1}(x) = 0$, $Q_0(x) = 1$ and $(Q_m)_{m\geq 0}$ is a sequence of orthogonal monic polynomials with respect to μ, that is,

$$\int_{\mathbb{R}} Q_m(x)Q_n(x)\mu(dx) = 0 \qquad \text{if } m \neq n.$$

Example 1.2.4 *The Chebyshev polynomials of the second kind are defined by the recurrence relation*

$$P_0(x) = 1, \quad P_1(x) = x,$$

and

$$xP_n(x) = P_{n+1}(x) + P_{n-1}(x) \quad \forall n \geq 1. \tag{1.2.1}$$

These polynomials are orthogonal with respect to the semicircular law, which is defined by the density

$$\mathbf{d}\mu = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{d}x.$$

The Jacobi parameters of μ *are* $\alpha_m = 0$ *and* $\gamma_m = 1$ *for all* $m \geq 0$.

1.3 Notions of Independence and Central Limit Theorems

In non-commutative probability, in general we have non-commutative algebras. This allow us to define new notions of independence. The independence give us a way to calculate mixed moments of random variables. In this section we define four types of independence (see Muraki [37]), which will help us to describe asymptotic distribution of growing graphs and their central limit theorems (CLT).

Definition 1.3.1 (Tensorial independence) *Let* $(\mathcal{A}, \varphi)$ *a Non-Commutative Probability Space. The random variables* $a, b \in (\mathcal{A}, \varphi)$ *are* ***tensor independent (or classical independent)*** *(with respect to* φ*) if*

$$\varphi\left(a^{m_1}b^{n_1}\cdots a^{m_k}b^{n_k}\right) = \varphi\left(a^{\sum_{i=1}^k m_i}\right)\varphi\left(b^{\sum_{i=1}^k n_i}\right),$$

for all $m_i, n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k$.

Definition 1.3.2 (Free independence) *Let $(\mathcal{A}, \varphi)$ a Non-Commutative Probability Space. The random variables $a, b \in (\mathcal{A}, \varphi)$ are **free independent (or free)** (with respect to φ) if for any polynomials P_i, Q_i, $i = 1, 2, \dots, n$, such that $\varphi(P_i(a)) = 0 = \varphi(Q_j(b))$ for all $i, j = 1, 2, \dots, n$, we have that*

$$\varphi(P_1(a)Q_1(b) \cdots P_n(a)Q_n(b)) = 0.$$

Definition 1.3.3 (Boolean independence) *Let $(\mathcal{A}, \varphi)$ a Non-Commutative Probability Space. The random variables $a, b \in (\mathcal{A}, \varphi)$ are **Boolean independent** (with respect to φ) if*

$$\varphi\left(a^{m_1} b^{n_1} \cdots a^{m_k} b^{n_k}\right) = \prod_{i=1}^{k} \varphi\left(a^{m_i}\right) \varphi\left(b^{n_i}\right)$$

for all $m_i, n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \dots, k$.

Definition 1.3.4 (Monotone independence) *Let $(\mathcal{A}, \varphi)$ a Non-Commutative Probability Space. The random variables $a, b \in (\mathcal{A}, \varphi)$ are **monotone independent** (with respect to φ) if*

$$\varphi\left(a^{m_1} b^{n_1} \cdots a^{m_k} b^{n_k}\right) = \varphi\left(a^{\sum_{i=1}^{k} m_i}\right) \prod_{i=1}^{k} \varphi\left(b^{n_i}\right),$$

for all $m_i, n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \dots, k$.

We can now derive explicit forms of central limit theorems associated with four different notions of independence.

Theorem 1.3.5 (Classical CLT) *Let $\{a_n\}_{n \geq 1} \subset (\mathcal{A}, \varphi)$ be a sequence of non-commutative, classical independent, random variables in a Non-Commutative Probability Space, such that $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$, for all $n \geq 1$, then we have*

$$\lim_{n \to \infty} \varphi\left(\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{\infty} a_i\right)^m\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^m e^{-x^2/2} dx, \qquad m = 1,\ 2,\ \dots,$$

where the r.h.s. are the moments of a standard Gaussian distribution.

Theorem 1.3.6 (Free CLT) *Let $\{a_n\}_{n \geq 1} \subset (\mathcal{A}, \varphi)$ be a sequence of non-commutative, free independent, random variables in a Non-Commutative Probability Space, such that $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$, for all $n \geq 1$, then we have*

$$\lim_{n \to \infty} \varphi\left(\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{\infty} a_i\right)^m\right) = \frac{1}{2\pi} \int_{-2}^{2} x^m \sqrt{4 - x^2} dx, \qquad m = 1,\ 2,\ \dots,$$

where the r.h.s. are the moments of a Wigner semicircular law.

Theorem 1.3.7 (Boolean CLT) *Let $\{a_n\}_{n\geq 1} \subset (\mathcal{A}, \varphi)$ be a sequence of non-commutative, Boolean independent, random variables in a Non-Commutative Probability Space, such that $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$, for all $n \geq 1$, then we have*

$$\lim_{n\to\infty} \varphi\left(\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{\infty} a_i\right)^m\right) = \frac{1}{2}\int_{-\infty}^{\infty} x^m \left(\delta_{-1} + \delta_1\right) dx, \qquad m = 1,\ 2,\ ...,$$

where the r.h.s. are the moments of a Bernoulli distribution.

Theorem 1.3.8 (Monotone CLT) *Let $\{a_n\}_{n\geq 1} \subset (\mathcal{A}, \varphi)$ be a sequence of non-commutative, monotone independent, random variables in a Non-Commutative Probability Space, such that $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$, for all $n \geq 1$, then we have*

$$\lim_{n\to\infty} \varphi\left(\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{\infty} a_i\right)^m\right) = \frac{1}{\pi}\int_{-\sqrt{2}}^{\sqrt{2}} \frac{x^m}{\sqrt{2-x^2}} dx, \qquad m = 1,\ 2,\ ...,$$

where the r.h.s. probability measure is an arcsine law.

1.4 Cauchy-Stieltjes Transform and Continued Fractions

We denote by $\mathcal{M}$ the set of Borel probability measures on $\mathbb{R}$. The upper half-plane and the lower half-plane are respectively denoted as $\mathbb{C}^+$ and $\mathbb{C}^-$.

Definition 1.4.1 *For a measure $\mu \in \mathcal{M}$, the Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is defined by the integral*

$$G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z-x}, \qquad z \in \mathbb{C}^+.$$

The Cauchy transform is an important tool in non-commutative probability. For us, the following relation between weak convergence and the Cauchy Transform will be important.

Proposition 1.4.2 *Let μ_1 and μ_2 be two probability measures on $\mathbb{R}$ and*

$$d_{\mathcal{L}}(\mu_1, \mu_2) = \sup\left\{|G_{\mu_1}(z) - G_{\mu_2}(z)|\, ; \Im(z) \geq 1\right\}. \tag{1.4.1}$$

Then d is a distance which defines a metric for the weak topology of probability measures. Moreover, $|G_\mu(z)|$ is bounded in $\{z : \Im(z) \geq 1\}$ by 1.

In other words, a sequence of probability measures $\{\mu_n\}_{n\geq 1}$ on $\mathbb{R}$ converges weakly to a probability measure μ on $\mathbb{R}$ if and only if for all z with $\Im(z) \geq 1$ we have

$$\lim_{n\to\infty} G_{\mu_n}(z) = G_\mu(z).$$

Definition 1.4.3 *The Hilbert transform Hf of a function f is given by the principal value integral*

$$Hf(x) := \lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{|x-t|\geq\varepsilon} \frac{f(t)}{x-t}dt,$$

whenever the limit exists for a.e. $x \in \mathbb{R}$.

Let μ be a compactly supported probability measure on $\mathbb{R}$. Writing $z = x + iy$ then we have the following decomposition into real and imaginary part of $G_\mu(z)$

$$G_\mu(x+iy) = \int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2+y^2}d\mu(t) - i\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}d\mu(t).$$

Also, note that $t_0 \in \mathbb{R}$ is an isolated point of the support of μ if and only if $z = t_0$ is a simple pole of $G_\mu(z)$. Moreover, $\mu(\{t_0\})$ is the residue of $G_\mu(z)$ at t_0. When μ has a continuous derivative $f = d\mu/dx$, we obtain

$$f(x) = -\frac{1}{\pi}\lim_{y\to+0} \mathrm{Im}G_\mu(x+iy),$$

and

$$Hf(x) = \frac{1}{\pi}\lim_{y\to+0} \mathrm{Re}G_\mu(x+iy),$$

due to properties of the Hilbert transform.

One can observe that the limit of the imaginary part of $G_\mu(x+iy)$ recovers μ up to a factor $-\pi$. This relation is known as *Stieltjes inversion formula*.

The Cauchy transform may be expressed as a continued fraction in terms of the Jacobi parameters, as follows.

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t}\mu(dt) = \cfrac{1}{z-\alpha_0-\cfrac{\omega_0}{z-\alpha_1-\cfrac{\omega_1}{z-\alpha_2-\cdots}}}$$

An important example for this thesis is the Bernoulli distribution $\mathbf{b} = 1/2\delta_1 + 1/2\delta_1$ for which $\alpha_0 = 0, \omega_0 = 1$, and $\alpha_n = \omega_n = 0$ for $n \geq 1$. Thus, the Cauchy transform is given by

$$G_{\mathbf{b}}(z) = \frac{1}{z-1/z}.$$

In the case when μ has $2n+2$-moments we can still make an orthogonalization procedure until the level n. In this case the Cauchy transform has the form

$$G_\mu(z) = \cfrac{1}{z-\alpha_0-\cfrac{\omega_0}{z-\alpha_1-\cfrac{\omega_1}{\ddots\,\cfrac{}{z-\alpha_n-\omega_n G_\nu(z)}}}} \tag{1.4.2}$$

where ν is a probability measure.

The following lemma which shows that the first, second and fourth moments are enough to ensure convergence to a Bernoulli distribution was observed in [4] . We present a proof in terms of Jacobi parameters as in our paper [6].

Lemma 1.4.4 *Let* $\{X_n\}_{n\geq 1} \subset (\mathcal{A}, \varphi)$ *, be a sequence of self-adjoint random variables in some non-commutative probability space, such that* $\varphi(X_n) = 0$ *and* $\varphi(X_n^2) = 1$*. If* $\varphi(X_n^4) \to 1$*, as* $n \to \infty$*, then* μ_{X_n} *converges in distribution to a symmetric Bernoulli random variable* $\mathbf{b}$*.*

Proof. Let $(\{\omega_i(\mu_{X_n})\}, \{\alpha_i(\mu_{X_n})\})$ be the Jacobi parameters of the measures μ_{X_n}. The first moments $\{m_n\}_{n\geq 1}$ are given in terms of the Jacobi Parameters as follows, see [1].

$$\begin{aligned}
m_1 &= \alpha_0 \\
m_2 &= \alpha_0^2 + \omega_0 \\
m_3 &= \alpha_0^3 + 2\alpha_0\omega_0 + \alpha_1\omega_0 \\
m_4 &= \alpha_0^4 + 3\alpha_0^2\alpha_1 + 2\alpha_1\alpha_0\omega_0 + \alpha_1^2\omega_0 + \omega_0^2 + \omega_0\omega_1.
\end{aligned}$$

Since $m_1(\mu_{X_n}) = 0$ and $m_2(\mu_{X_n}) = 1$ we have

$$\alpha_0(\mu_{X_n}) = 0 \quad \text{and} \quad \omega_0(\mu_{X_n}) = 1 \qquad \forall n \geq 1,$$

Hence,

$$m_4(\mu_{X_n}) = \alpha_1^2(\mu_{X_n}) + 1 + \omega_1(\mu_{X_n}). \tag{1.4.3}$$

Now, since $m_4(\mu_{X_n}) \to 1$ and $\omega_1 \geq 0$ we have the convergence

$$\alpha_1(\mu_{X_n}) \underset{n\to\infty}{\to} 0 \quad \text{and} \quad \omega_1(\mu_{X_n}) \underset{n\to\infty}{\to} 0.$$

Let G_{μ_n} be the Cauchy transform of μ_n. By (1.4.2) we can expand G_μ as a continued fraction as follows

$$G_{\mu_n}(z) = \cfrac{1}{z - \cfrac{1}{z - \alpha_1 - \omega_1 G_{\nu_n}(z)}}$$

where ν_n is some probability measure. Now, recall that $|G_{\nu_n}(z)|$ is bounded by 1 in the set $\{z|; \Im(z) \geq 1\}$ and thus, since $\omega_1 \to 0$ and $\alpha_1 \to 0$ we see that $\omega_n G_{\nu_n}(z) \to 0$. This implies the point-wise convergence

$$G_{\mu_n}(z) \to \cfrac{1}{z - \cfrac{1}{z}}$$

in the set $\{z|; \Im(z) \geq 1\}$, which then implies the weakly convergence $\mu_n \to \mathbf{b}$. ■

From the proof of the previous lemma, we can give a quantitative version in terms of the distance given in eq (1.4.1).

Proposition 1.4.5 ([6]) *Let μ be a probability measure such that $m_4 := m_4(\mu)$ is finite. Then*

$$d_{\mathcal{L}}\left(\mu, \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}\right) \le 4\sqrt{m_4 - 1}, \tag{1.4.4}$$

where $d_{\mathcal{L}}$ is defined in (1.4.1).

Proof.

If $m_4 - 1 > 1/16$ then the statement is trivial since $d(\mu, 1/2\delta_1 + 1/2\delta_{-1}) \le 1$ for any measure μ. Thus we may assume that $(m_4 - 1) \le 1/16$.

Denoting by $f(z) = \alpha_1 - \omega_1 G_{\nu_n}(z)$ we have

$$|G_\mu(z) - G_b(z)| = \left| \frac{1}{z - \frac{1}{z}} - \frac{1}{z - \frac{1}{z - f(z)}} \right| = \left| \frac{f(z)}{(z^2 - 1)(z^2 - 1 - f(z)z)} \right|.$$

From (1.4.3) we get the inequalities $\sqrt{m_4 - 1} \ge |\alpha_1|$ and $\sqrt{m_4 - 1} \ge m_4 - 1 \ge \omega_1$. Since, for $Im(z) > 1$, we have that, $|G_\nu(z)| < 1$ we see that $|f(z)| = |\alpha_1 - \gamma_1 G_\nu(z)| \le 2\sqrt{m_4 - 1} \le 1/2.$, from where we can easily obtain the bound $\left|\frac{1}{z^2 - 1 - f(z)z}\right| \le 2$. Also, for $\Im(z) > 0$ we have the bound $\left|\frac{1}{(z^2-1)}\right| < 1$. Thus we have

$$\begin{aligned} |G_\mu(z) - G_b(z)| &= \left| \frac{f(z)}{(z^2 - 1)(z^2 - 1 - f(z)z)} \right| \\ &= |f(z)| \left| \frac{1}{(z^2 - 1)} \right| \left| \frac{1}{z^2 - 1 - f(z)z} \right| \\ &\le 2|f(z)| \le 4\sqrt{m_4 - 1}. \end{aligned}$$

as desired.

■

Another quantitative version of the Boolean central limit theorem is given by Arizmendi & Salazar [8], where instead they use the Lévy distance. However, their estimate is larger compared to (1.4.4).

Definition 1.4.6 *For $\mu, \nu \in \mathcal{M}$ define the Lévy distance between them to be*

$$L(\mu, \nu) := \inf\{\varepsilon > 0 : F(x - \varepsilon) - \varepsilon \le G(x) \le F(x + \varepsilon) + \varepsilon \textit{ for all } x \in \mathbb{R}\},$$

where F and G are the cumulative distribution functions fo μ and ν respectively.

Theorem 1.4.7 (Arizmendi & Salazar [8]) *Let μ be a probability measure with zero mean and unit variance. Then*

$$L(\mu, \mathbf{b}) \le \frac{7}{2}\sqrt[3]{m_4(\mu) - 1}.$$

Chapter 2

Quantum Decomposition Method

In this chapter we present the quantum decomposition method and some basics on Graph Theory. We shall develop the spectral analysis of a graph by regarding the adjacency matrix as an algebraic random variable. The interest in asymptotic aspects of growing combinatorial objects has increased in recent years. In particular, the asymptotic spectral distribution of graphs has been studied from the quantum probabilistic point of view. The term of *quantum decomposition* was first introduced by Hashimoto [20] in a study of an adjacency matrix of a large Cayley graph. This idea has been applied also to similar studies for large Hamming graphs [22, 24], Johnson graphs [21, 24, 25], Odd graphs [27], Homogeneous tree [19], and so on. A summary of these results can be found in the book by Hora & Obata [26].

2.1 Graphs and Adjacency Matrices

Definition 2.1.1 *By a* ***rooted graph*** *we understand a pair* (G, e)*, where* $G = (V, E)$*, is a undirected graph with set of vertices* $V = V(G)$*, and the set of edges* $E = E(G) \subseteq \{(x, x') : x,\ x' \in V,\ x \neq x'\}$ *and* $e \in V$ *is a distinguished vertex called the* ***root****.*

For rooted graphs we will use the notation $V^0 = V \backslash \{e\}$. Two vertices $x, x' \in V$ are called *adjacent* if $(x, x') \in E$, i.e. vertices x, x' are connected with an edge. Then we write $x \sim x'$. Simple graphs have no loops, i.e. $(x, x) \notin E$ for all $x \in V$. A graph is called *finite* if $|V| < \infty$, where $|I|$ stands for the cardinality of I. The *degree* of $x \in V$ is defined by $\kappa(x) = |\{x' \in V : x' \sim x\}|$. A graph is called *locally finite* if $\kappa(x) < \infty$ for every $x \in V$. It is called *uniformly locally finite* if $\sup\{\kappa(x) : x \in V\} < \infty$. Finally, for $x, y \in V$, $\partial_G(x, y)$ denotes the graph distance between x and y, i.e. the length of the shortest walk connecting x and y.

Definition 2.1.2 *The* ***adjacency matrix*** $A = A(G)$ *of* G *is a 0-1 matrix defined by*

$$A_{x,x'} = \begin{cases} 1 & \text{if } x \sim x' \\ 0 & \text{otherwise.} \end{cases} \tag{2.1.1}$$

We identify A with the densely defined symmetric operator on $l^2(V)$ defined by

$$A\delta(x) = \sum_{x \sim x'} \delta(x') \tag{2.1.2}$$

for $x \in V$. Notice that the sum on the right-hand-side is finite since our graph is assumed to be locally finite. It is known that $A(G)$ is bounded iff G is uniformly locally finite. If $A(G)$ is essentially self-adjoint, its closure is called the *adjacency operator* of G and its spectrum is called the spectrum of G.

The unital algebra generated by A, i.e. the algebra of polynomials in A, is called the *adjacency algebra* of G and is denoted by $\mathcal{A}(G)$ or simply $\mathcal{A}$.

2.2 Vacuum and Deformed Vacuum States

Definition 2.2.1 *Let $G = (V, E)$ be a graph and $\mathcal{A}(G)$ its adjacency algebra. The **vacuum state** at a fixed origin $o \in V$ is defined by*

$$\langle a \rangle_o = \langle \delta_o, a\delta_o \rangle, \quad a \in \mathcal{A}(G).$$

It is well known that $\langle A^m \rangle_o$ is the number of m-step walks from $o \in V$ to itself. More generally, we have the following:

$$(A^m)_{xy} = \langle \delta_x, A^m \delta_y \rangle,$$

which coincides with the number of m-step walks connecting y and x.
In this thesis we are also interested in a particular one-parameter deformation of the vacuum state. For $q \in \mathbb{R}$ (one may consider $q \in \mathbb{C}$ though our interesting case happens only when $-1 \leq q \leq 1$, see [11]), we define a matrix $Q = Q_q$, called the *Q-matrix* of a graph $G = (V, E)$, by

$$Q = Q_q = (q^{\partial(x,y)})_{x,y \in V}.$$

For $q = 0$ we understand that $0^0 = 1$ and $Q = 1$ (the identity matrix). Then we have

$$Q\delta_o = \sum_{x \in V} q^{\partial(x,o)} \delta_x.$$

We may define

$$\langle a \rangle_q = \sum_{x \in V} q^{\partial(x,o)} \langle \delta_x, a\delta_o \rangle = \langle Q\delta_o, a\delta_o \rangle, \quad a \in \mathcal{A}(G). \tag{2.2.1}$$

Definition 2.2.2 *A normalized linear function defined in (2.2.1) is called a **deformed vacuum state** on $\mathcal{A}(G)$.*

2.3 Quantum Decomposition of an Adjacency Matrix

Let $G = (V, E)$ be a graph with a fixed origin $o \in V$. The graph is stratified into a disjoint union of strata:

$$V = \bigcup_{n=0}^{\infty} V_n, \quad V_n = \{x \in V : \partial(o, x) = n\}. \tag{2.3.1}$$

This is called the *stratification* (distance partition). For $\epsilon \in \{+, -, \circ\}$ we define A^ϵ by

$$(A^\epsilon)_{xy} = \begin{cases} 1, & \text{if } x \sim y \text{ and } \partial(o, x) - \partial(o, y) = \epsilon, \\ 0, & \text{otherwise,} \end{cases}$$

where ϵ is assigned the numbers $+1, -1, 0$ according as $\epsilon = +, -, \circ$. The adjacency matrix A is decomposed into three parts:

$$A = A^+ + A^- + A^\circ. \tag{2.3.2}$$

Definition 2.3.1 *We call (2.3.2) the* ***quantum decomposition*** *of A associated with the stratification (2.3.1) and A^ϵ, $\epsilon \in \{+, -, \circ\}$ the* ***quantum components****.*

For each $n = 0, 1, 2, \ldots$, we define a unit vector in $l^2(V)$ by

$$\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x, \tag{2.3.3}$$

which is called the *n-th number vector*. In particular, $\Phi_0 = \delta_0$ is called the *vacuum vector*. Let $\Gamma(G)$ denote the closed subspace spanned by $\{\Phi_0, \Phi_1, \ldots\}$. Although $\Gamma(G)$ is not always invariant under the quantum components A^ϵ, the method of quantum decomposition is best effective in some special cases.

Let $\tilde{\mathcal{A}}(G)$ be the $*$-algebra generated by the quantum components A^+, A^-, A° of the adjacency matrix A. Note that $\tilde{\mathcal{A}}(G)$ is non-commutative unless the graph G consists of a single vertex. Except such a trivial case, the quantum decomposition yields a non-commutative extension of $\mathcal{A}(G)$.

Theorem 2.3.2 *Let $G = (V, E)$ be a graph with a fixed origin $o \in V$. Let $A = A^+ + A^- + A^\circ$ be the quantum decomposition of the adjacency matrix and $\Gamma(G)$ the space spanned by $\{\Phi_n\}$ defined in (2.3.3). Then we have*

$$A^+\Phi_n = |V_n|^{-1/2} \sum_{y \in V_{n+1}} \omega_-(y)\delta_y, \tag{2.3.4}$$

$$A^-\Phi_n = |V_n|^{-1/2} \sum_{y \in V_{n-1}} \omega_+(y)\delta_y, \tag{2.3.5}$$

$$A^\circ\Phi_n = |V_n|^{-1/2} \sum_{y \in V_n} \omega_\circ(y)\delta_y. \tag{2.3.6}$$

where $\omega_\epsilon(x) = |\{y \in V; y \sim x, \partial(o, y) = \partial(o, x) + \epsilon\}|$, $\epsilon \in \{+, -, \circ\}$.

It is noted from (2.3.4)-(2.3.6) that $\Gamma(G)$ is not necessarily invariant under the actions of the quantum components of A. The method of quantum decomposition will be best effective when

(i) $\Gamma(G)$ is invariant under the quantum components or

(ii) $\Gamma(G)$ is "asymptotically" invariant under the quantum components.

Proposition 2.3.3 *Notations being as above, $\Gamma(G)$ is invariant under the quantum components A^+, A^-, A° if an only if $\omega_+(y), \omega_-(y), \omega_\circ(y)$ are constant on V_n for all $n = 0, 1, 2, \dots$. In that case $(\Gamma(G), \{\Phi_n\}, A^+, A^-)$ becomes an interacting Fock space and A° a diagonal operator. The associated Jacobi coefficient is given by*

$$\omega_n = \frac{|V_n|}{|V_{n-1}|}\omega_-(y)^2, \quad y \in V_n, \tag{2.3.7}$$

$$\alpha_n = \omega_\circ(y), \quad y \in V_{n-1}, \quad n = 1, 2, \dots. \tag{2.3.8}$$

We note that the vacuum state corresponding to the fixed origin $o \in V$ becomes

$$\langle a \rangle_o = \langle \delta_o, a\delta_o \rangle = \langle \Phi_0, a\Phi_0 \rangle, \quad a \in \mathcal{A}(G).$$

Hence, Proposition 2.3.3 says that the theory of an interacting Fock space is directly applicable to the spectral analysis of $A = A^+ + A^- + A^\circ$ in the vacuum state. Finally, for the case of the deformed vacuum state (Definition 2.2.2), we have an alternative expression:

$$\langle a \rangle_q = \sum_{n=0}^{\infty} q^n |V_n|^{1/2} \langle \Phi_n, a\Phi_0 \rangle, \quad a \in \mathcal{A}(G).$$

Chapter 3

Distance-Regular Graphs

This chapter deals with distance-regular graphs which possess a significant property from the viewpoint of quantum decomposition. We shall establish a general framework for asymptotic spectral distributions for the adjacency matrix and derive the limit distributions in terms of intersection numbers. In particular, in Section 3.4 we study an example of distance-regular graphs: odd graphs, and distribution in vacuum and deformed vacuum states. Most of the results in this chapter are from Hora & Obata [26]. The results related to distribution in deformed vacuum state for odd graph are from the author's manuscript [16].

3.1 Definition and Some Properties

Definition 3.1.1 *Let $G = (V, E)$ be a graph. Let i, j, k be non-negative integers. A graph $G = (V, E)$ is called* ***distance-regular*** *if for any choice of $x, y \in V$ such that $\partial(x, y) = k$, the number*

$$p^k_{ij} = |\{z \in V : \partial(x, z) = i,\ \partial(y, z) = j\}|,$$

does not depend on x and y. These constants are called the ***intersection numbers*** *of $G = (V, E)$.*

Remark 3.1.2 *A distance-regular graph is regular with degree p^0_{11}.*

Let $G = (V, E)$ be a graph, we define the *k-th distance matrix (or k-th adjacency matrix)* A_k by

$$(A_k)_x y = \begin{cases} 1, & \partial(x, y) = k, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.1}$$

We have that, the 0th distance matrix is the identity matrix $A_0 = 1$ and the 1st is the adjacency matrix so that $A_1 = A$. It is noted that A_k is locally finite for all $k = 0, 1, 2, \ldots$. Denoting by J the matrix of which entries are all one, we have

$$\sum_k A_k = J.$$

Moreover, if the graph G is finite, $A_k = 0$ for all $k > \mathrm{diam}(G)$. For a distance-regular graph these matrices are useful. The following propositions are from [26].

Proposition 3.1.3 *The adjacency algebra $\mathcal{A}(G)$ of a distance-regular graph G is a linear space with a linear basis $A_0 = 1$ (the identity matrix), $A_1 = A$ (adjacency matrix), $A_2, \dots$. In particular, if G is finite, we have* $\dim\mathcal{A}(G) = \mathrm{diam}(G) + 1$.

Proposition 3.1.4 *A graph $G = (V, E)$ is distance-regular if and only if for any $k = 0, 1, 2, \dots$, the kth distance matrix A_k is expressible in a polynomial of A of degree k whenever $A_k \neq 0$.*

We give a simple criterion for a graph to be distance-regular. In general, a graph is called *distance-transitive* if for any $x, x', y, y' \in V$ such that $\partial(x, y) = \partial(x', y')$ there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(x) = x',\ \alpha(y) = y'$.

Proposition 3.1.5 *A distance-transitive graph is distance-regular.*

3.2 Spectral Distributions in the Vacuum States

Now, we consider the spectral distribution of A in the vacuum state, i.e., a probability measure $\mu \in \mathcal{M}$ satisfying

$$\langle \delta_o, A^m \delta_o \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \dots,$$

where $o \in V$ is a fixed origin of the graph. We apply quantum decomposition method.

Theorem 3.2.1 *Let G be a distance-regular graph with intersection numbers $\{p^k_{ij}\}$ and A the adjacency matrix. Then $\Gamma(G)$ is invariant under the action of the quantum components A^ϵ, $\epsilon \in \{+, -, \circ\}$. Moreover*

$$A^+\Phi_n = \sqrt{p^{n+1}_{1,n} p^n_{1,n+1}}\,\Phi_{n+1}, \quad n = 0, 1, 2, \dots, \tag{3.2.1}$$

$$A^-\Phi_0 = 0, \quad A^-\Phi_n = \sqrt{p^n_{1,n-1} p^{n-1}_{1,n}}\,\Phi_{n-1}, \quad n = 1, 2, \dots, \tag{3.2.2}$$

$$A^\circ\Phi_n = p^n_{1,n}\Phi_n, \quad n = 0, 1, 2, \dots. \tag{3.2.3}$$

According with last theorem $(\Gamma(G), \{\Phi_n\}, A^+, A^-)$ is an interacting Fock space associated with a Jacobi sequence

$$\omega_n = p^n_{1,n-1} p^{n-1}_{1,n}, \quad n = 1, 2, \dots, \tag{3.2.4}$$

and the quantum component A° is the diagonal operator defined by the sequence

$$\alpha_n = p^{n-1}_{1,n-1}, \quad n = 1, 2, \dots. \tag{3.2.5}$$

Now, we may state the following result.

Theorem 3.2.2 *Let $G = (V, E)$ be a distance-regular graph and A its adjacency matrix. Let μ be a spectral distribution of A in the vacuum state at an origin $o \in V$ fixed arbitrarily. Then the pair of sequences $(\{\omega_n\}, \{\alpha_n\})$ given in (3.2.4) and (3.2.5) is the Jacobi coefficient of μ.*

We see from (3.2.4) and (3.2.5) that

$$\omega_1 = p^0_{11} = \kappa, \quad \alpha_1 = 0.$$

Thus the spectral distribution of A in the vacuum state has mean zero and variance p^0_{11}.

3.3 Spectral Distributions in the Deformed Vacuum States

We next consider the deformed vacuum state defined by

$$\langle a \rangle_q = \langle Q\delta_o, A\delta_o \rangle = \sum_{n=0}^{\infty} q^n |V_n|^{1/2} \langle \Phi_n, a\Phi_0 \rangle, \quad a \in \mathcal{A}(G),$$

where $Q = \left(q^{\partial(x,y)}\right)$ with $q \in \mathbb{R}$. In order to normalize the adjacency matrix in the deformed vacuum state we use the following:

Lemma 3.3.1 *Then mean and the variance of the adjacency matrix A in the deformed vacuum state are respectively given as follows:*

$$\langle A \rangle_q = q\kappa, \tag{3.3.1}$$

$$\Sigma_q^2(A) = \langle (A - \langle A \rangle_q)^2 \rangle_q = \kappa(1-q)(1+q+qp^1_{11}). \tag{3.3.2}$$

Theorem 3.3.2 *Let $G = (V, E)$ be a distance-regular graph. If $Q = \left(q^{\partial(x,y)}\right)$ is a positive definite kernel on V, the deformed vacuum state $\langle \cdot \rangle_q$ is positive (i.e., a state in a strict sense) on the adjacency algebra $\mathcal{A}(G)$.*

Now, let us consider a growing distance-regular graph $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$. Suppose that each $G^{(\nu)}$ is given a deformed vacuum state $\langle \cdot \rangle_q$, where q may depend on ν. The normalized adjacency matrix in which we are interested is given by

$$\frac{A_\nu - \langle A_\nu \rangle_q}{\Sigma_q(A_\nu)}.$$

Taking the quantum decomposition $A_\nu = A^+_\nu + A^-_\nu + A^\circ_\nu$ into account, we obtain

$$\frac{A_\nu - \langle A_\nu \rangle_q}{\Sigma_q(A_\nu)} = \frac{A^+_\nu}{\Sigma_q(A_\nu)} + \frac{A^-_\nu}{\Sigma_q(A_\nu)} + \frac{A^\circ_\nu - q\kappa(\nu)}{\Sigma_q(A_\nu)}.$$

For $n = 1, 2, \dots$ we set

$$\bar{\omega}_n(\nu, q) = \frac{p^n_{1,n-1}(\nu)p^{n-1}_{1,n}(\nu)}{\Sigma_q^2(A_\nu)} \quad \text{and} \quad \bar{\alpha}_n(\nu, q) = \frac{p^{n-1}_{1,n-1}(\nu) - q\kappa(\nu)}{\Sigma_q(A_\nu)}.$$

From Theorem 3.2.1 we have

$$\begin{aligned}
\tfrac{A_\nu^+}{\Sigma_q(A_\nu)}\Phi_n &= \sqrt{\bar{\omega}_{n+1}(\nu, q)}\Phi_{n+1}, \quad n = 0, 1, 2, \dots, \\
\tfrac{A_\nu^-}{\Sigma_q(A_\nu)}\Phi_0 &= 0, \quad \tfrac{A_\nu^-}{\Sigma_q(A_\nu)}\Phi_n = \sqrt{\bar{\omega}_n(\nu, q)}\Phi_{n-1}, \quad n = 1, 2, \dots, \\
\tfrac{A_\nu^\circ - q\kappa(\nu)}{\Sigma_q(A_\nu)}\Phi_n &= \bar{\alpha}_{n+1}(\nu, q)\Phi_n, \quad n = 0, 1, 2, \dots.
\end{aligned}$$

We consider the following limits:

$$\omega_n = \lim_{\nu,q} \bar{\omega}_n(\nu, q) = \lim_{\nu,q} \frac{p^n_{1,n-1}(\nu)p^{n-1}_{1,n}(\nu)}{\Sigma_q^2(A_\nu)}, \tag{3.3.3}$$

$$\alpha_n = \lim_{\nu,q} \bar{\alpha}_n(\nu, q) = \lim_{\nu,q} \frac{p^{n-1}_{1,n-1}(\nu) - q\kappa(\nu)}{\Sigma_q(A_\nu)}, \tag{3.3.4}$$

when they exist under a good scaling balance of ν and q. We consider the condition (DR):

(i) for all $n = 1, 2, \dots$ the limits ω_n and α_n exist with $\omega_n > 0$ or

(ii) there exists $n = 1, 2, \dots$ such that the limits $\omega_1, \dots, \omega_n$ and $\alpha_1, \dots, \alpha_n$ exist with

$$\omega_1 = 1, \omega_2 > 0, \dots, \omega_{n-1} > 0, \omega_n = 0.$$

If (DR) is fulfilled for (3.3.3) and (3.3.4), we obtain a Jacobi coefficient $(\{\omega_n\}, \{\alpha_n\})$. Let $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ be an interacting Fock space associated with $\{\omega_n\}$ and B° the diagonal operator defined by $\{\alpha_n\}$. We set

$$\tilde{A}_\nu^\pm = A_\nu^\pm, \quad \tilde{A}_\nu^\circ = A_\nu^\circ - q\kappa(\nu),$$

and we can now establish the Quantum Cental Limit Theorem for a growing distance-regular graph in the deformed vacuum state.

Theorem 3.3.3 *Let $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing distance-regular graph with A_ν being the adjacency matrix, and each $\mathcal{A}(G^{(\nu)})$ be a given deformed vacuum state $\langle\cdot\rangle_q$. Assume that condition (DR) is fulfilled for (3.3.3) and (3.3.4), and that the limit*

$$c_n = \lim_{\nu,q} q^n |V_n^{(\nu)}|^{1/2} = \lim_{\nu,q} q^n \sqrt{p^0_{nn}(\nu)}, \tag{3.3.5}$$

exists for all n for which $\{\alpha_n\}$ is defined. Let $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ be an interacting Fock space associated with $\{\omega_n\}$, B° the diagonal operator defined by $\{\alpha_n\}$, and Υ the formal sum of vectors defined by

$$\Upsilon = \sum_{n=0}^{\infty} c_n \Psi_n.$$

Then, we have

$$\lim_{\nu,q}\left\langle \frac{\tilde{A}_\nu^{\epsilon_m}}{\Sigma_q(A_\nu)}\cdots\frac{\tilde{A}_\nu^{\epsilon_1}}{\Sigma_q(A_\nu)}\right\rangle_q = \langle \Upsilon, B^{\epsilon_m}\cdots B^{\epsilon_1}\Psi_0\rangle,$$

for any $\epsilon_1,\ldots,\epsilon_m \in \{+,-,\circ\}$ *and* $m = 1,2,\ldots$.

3.4 Odd Graphs

The idea of quantum decomposition has been applied to similar studies for large Hamming graphs [22, 24], Johnson graphs [21, 24, 25], Odd graphs [27], Homogeneous tree [19], and so on. Most of these distributions were computed with respect to vacuum state and deformed vacuum state, except in the case of odd graphs, where only the vacuum state case was studied. In this section we focus on the study of the asymptotic spectral distribution of growing odd graphs in vacuum and deformed vacuum state.

Definition 3.4.1 *Let* $k \geq 2$ *be an integer and set* $S = \{1,2,\ldots,2k-1\}$. *The pair*

$$V = \{x \subset S : |x| = k-1\}, \quad E = \{(x,y) : x,y \in V, x\cap y = \emptyset\},$$

is called the ***odd graph*** *and is denoted by* O_k.

Obviously, O_k is a regular graph of degree k.
The distance between two vertices of an odd graph is characterized by the cardinality of their intersection. Set

$$I_n = \begin{cases} k-1-\frac{n}{2}, & \text{if } n \text{ is even,}\\ \frac{n-1}{2}, & \text{if } n \text{ is odd,}\end{cases}$$

where $n = 0,1,\ldots,k-1$. Then, for a pair of vertices x,y of the odd graph O_k, we have

$$|x\cap y| = I_n \Longleftrightarrow \partial(x,y) = n.$$

As a direct consequence of this fact we have that odd graphs are *distance-transitive*, therefore **distance-regular**.

3.4.1 Distribution in Vacuum States

In order to apply quantum probabilistic techniques to obtain the asymptotic spectral distribution of the adjacency matrix A_k as $k \to \infty$, Igarashi and Obata [27] computed the intersection numbers of O_k.

Proposition 3.4.2 *Let* $\{p_{ij}^h\}$ *be the intersection numbers of the odd graph* O_k, $k \geq 2$. *For* $1 \leq n \leq k-1$,

$$p_{1,n-1}^n = \begin{cases} \frac{n}{2}, & \textit{if } n \textit{ is even,}\\ \frac{n+1}{2}, & \textit{if } n \textit{ is odd.}\end{cases}$$

For $0 \le n \le k-2$,

$$p^n_{1,n+1} = \begin{cases} k - \frac{n}{2}, & \text{if } n \text{ is even}, \\ k - \frac{n+1}{2}, & \text{if } n \text{ is odd}. \end{cases}$$

For $0 \le n \le k-1$,

$$p^n_{1,n} = \begin{cases} 0, & \text{if } 1 \le n \le k-2, \\ \frac{k+1}{2}, & \text{if } n = k-1 \text{ and } k \text{ is odd}, \\ \frac{k}{2}, & \text{if } n = k-1 \text{ and } k \text{ is even}. \end{cases}$$

From the last proposition they obtain the following quantum central limit theorem for odd graphs (w.r.t. vacuum state).

Theorem 3.4.3 *Let A_k be the adjacency matrix of the odd graph O_k and A^ϵ_k its quantum components, $\epsilon \in \{+,-,\circ\}$. Let $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ be the interacting Fock space associated with a Jacobi sequence defined by*

$$\{\omega_n\} = \{1, 1, 2, 2, 3, 3, 4, 4, \dots\}.$$

It then holds that

$$\lim_{k\to\infty} \frac{A^\pm_k}{\sqrt{k}} = B^\pm, \quad \lim_{k\to\infty} \frac{A^\circ_k}{\sqrt{k}} = 0,$$

in the sense of stochastic convergence.

In [27] they proved that there exists a unique Borel probability measure μ on $\mathbb{R}$ such that

$$\langle \Psi_0, (B^+ + B^-)^m \Psi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \dots.$$

Therefore we have the following:

Proposition 3.4.4 *Let A_k be the adjacency matrix of the odd graph O_k. Then there exists a unique probability measure $\mu \in \mathcal{M}$ such that*

$$\lim_{k\to\infty} \left\langle \left(\frac{A_k}{\sqrt{k}} \right)^m \right\rangle_o = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \dots.$$

The Jacobi coefficient of μ is given by

$$\{\omega_n\} = \{1, 1, 2, 2, 3, 3, 4, 4, \dots\}, \quad \{\alpha \equiv 0\}.$$

In particular, μ is symmetric.

After using Stieltjes inversion method they obtained an explicit description of the Borel probability measure μ in Proposition 3.4.3

Theorem 3.4.5 (Igarashi & Obata [27]) *For the adjacency matrix A_k of the odd graph O_k we have*

$$\lim_{k\to\infty} \left\langle \left(\frac{A_k}{\sqrt{k}} \right)^m \right\rangle_o = \int_{-\infty}^{\infty} x^m |x| \exp(-x^2) dx, \quad m = 1, 2, \dots.$$

3.4.2 Distribution in Deformed Vacuum States

Now, we shall compute asymptotic spectral distribution of growing odd graphs in deformed vacuum state. In order to obtain a normalization, we calculate the mean and the variance in the deformed vacuum state of the adjacency matrix A_k:

$$\langle A_k \rangle_q = qk, \quad \Sigma_q^2(A_k) = k(1-q^2),$$

due to the fact that O_k has degree k and $p_{11}^1 = 0$. Then, the normalized adjacency matrix becomes

$$\frac{A_k - \langle A_k \rangle_q}{\Sigma_q(A_k)} = \frac{A_k - qk}{\sqrt{k(1-q^2)}},$$

and we are interested in the asymptotic distribution in deformed vacuum state. In order to obtain that distribution we need three sequences $\{\omega_n\}, \{\alpha_n\}, \{c_n\}$ defined in (3.3.3), (3.3.4) and (3.3.5), respectively.
First to obtain (3.3.3), if n is even, then we have

$$\omega_n = \lim_{k,q} \frac{\binom{n}{2}\left(k - \frac{n}{2}\right)}{k(1-q^2)},$$

and if n is odd

$$\omega_n = \lim_{k,q} \frac{\binom{n+1}{2}\left(k - \frac{n-1}{2}\right)}{k(1-q^2)}.$$

In the case of (3.3.4) we have

$$\alpha_n = \lim_{k,q} \frac{p_{1,n-1}^{n-1}(k) - qk}{\Sigma_q(A_k)} = \lim_{k,q} \frac{-qk}{\sqrt{k(1-q^2)}} = \lim_{k,q} \frac{-q\sqrt{k}}{\sqrt{1-q^2}}.$$

Here we remind that q may depend on k, therefore we need to find a suitable balance between q and k. An appropiate situation is

$$\lim_{k\to\infty} q = 0, \quad \lim_{k\to\infty} q\sqrt{k} = \gamma, \tag{3.4.1}$$

where $\gamma \in \mathbb{R}$ can be arbitrarily chosen. Therefore, under these circunstances we have

$$\{\omega_n\}_{n\geq 1} = \{1, 1, 2, 2, 3, 3, 4, \dots\}, \quad \{\alpha_n\}_{n\geq 1} = \{-\gamma, -\gamma, -\gamma, \dots\}.$$

We still need to compute the limit (3.3.5), whereby we need to compute $p_{nn}^0(k)$. Let n be even, we recall that $\partial(x,y) = n \Leftrightarrow |x \cap y| = \frac{n}{2} - 1 + k$, hence

$$p_{nn}^0 = \left(\begin{array}{c} k-1 \\ k-1-\frac{n}{2} \end{array} \right) \left(\begin{array}{c} k \\ \frac{n}{2} \end{array} \right) = \left(\begin{array}{c} k-1 \\ \frac{n}{2} \end{array} \right) \left(\begin{array}{c} k \\ \frac{n}{2} \end{array} \right),$$

therefore

$$\begin{aligned} c_n &= \lim_{k,q} q^n \sqrt{\begin{pmatrix} k-1 \\ \frac{n}{2} \end{pmatrix}\begin{pmatrix} k \\ \frac{n}{2} \end{pmatrix}} \\ &= \lim_{k,q} \frac{q^n}{\left(\frac{n}{2}\right)!} \sqrt{\left(1-\frac{1}{k}\right)\cdots\left(1-\frac{n/2}{k}\right)(1)\left(1-\frac{\frac{n}{2}+1}{k}\right)k^{\frac{n}{2}+\frac{n}{2}}} \\ &= \frac{\gamma^n}{\left(\frac{n}{2}\right)!}. \end{aligned}$$

If n is odd, we have $\partial(x,y)=n \Leftrightarrow |x\cap y| = \frac{n-1}{2}$, then

$$p^0_{nn}(k) = \begin{pmatrix} k-1 \\ \frac{n-1}{2} \end{pmatrix}\begin{pmatrix} k \\ k-1-\frac{n-1}{2} \end{pmatrix} = \begin{pmatrix} k-1 \\ k-1-\frac{n-1}{2} \end{pmatrix}\begin{pmatrix} k \\ k-1-\frac{n-1}{2} \end{pmatrix},$$

therefore

$$\begin{aligned} c_n &= \lim_{k,q} q^n \sqrt{\begin{pmatrix} k-1 \\ k-1-\frac{n-1}{2} \end{pmatrix}\begin{pmatrix} k \\ k-1-\frac{n-1}{2} \end{pmatrix}} \\ &= \lim_{k,q} q^n \sqrt{\left(1-\frac{1}{k}\right)\cdots\left(1-\frac{n/2}{k}\right)(1)\left(1-\frac{\frac{n}{2}+1}{k}\right)\frac{k^{\frac{n-1}{2}+\frac{n-1}{2}+1}}{\left(\frac{n-1}{2}+1\right)\left(\frac{n-1}{2}\right)!\left(\frac{n-1}{2}\right)!}} \\ &= \frac{\gamma^n}{\sqrt{\frac{n-1}{2}+1}\left(\frac{n-1}{2}\right)!}. \end{aligned}$$

The corresponding (formal) sum of vectors is given by

$$\Omega_\gamma = \sum_{n=0}^{\infty} c_n \Psi_n, \quad \text{where} \quad c_n = \begin{cases} \frac{\gamma^n}{\left(\frac{n}{2}\right)!} & \text{if } n \text{ is even} \\ \frac{\gamma^n}{\sqrt{\frac{n-1}{2}+1}\left(\frac{n-1}{2}\right)!} & \text{if } n \text{ is odd,} \end{cases}$$

which is the coherent state of the Fock space $\Gamma_{\{\omega_n\}}$. Now we are in a position to write a partial description of the asymptotic distribution of odd graphs in the deformed vacuum state.

Theorem 3.4.6 ([16]) *Let A_k be the adjacency matrix of the odd graph O_k, $k \geq 2$. Let $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ be the interacting Fock space associated with $\{\omega_n\} = \{1,1,2,2,3,3,4,\dots\}$. Then taking the limits as in (3.4.1) we have*

$$\lim_{k,q} \frac{A_k^{\pm}}{\Sigma_q(A_k)} = B^{\pm}, \quad \lim_{k,q} \frac{A_k^{\circ} - \langle A_k\rangle_q}{\Sigma_q(A_k)} = -\gamma,$$

in the sense of stochastic convergence, where the left-hand sides are in the deformed vacuum state $\langle\cdot\rangle_q$ and the right-hand sides in the coherent state $\langle\cdot\rangle_{\Omega_\gamma}$. In particular, for $m = 1,2,\dots,$

$$\lim_{k,q} \left\langle \left(\frac{A_k - \langle A_k\rangle_q}{\Sigma_q(A_k)}\right)^m \right\rangle_q = \left\langle \left(B^+ + B^- - \gamma\right)^m \right\rangle_{\Omega_\gamma}. \tag{3.4.2}$$

Calculating the limit measure

In this section our goal is to obtain a probability measure μ such that

$$\left\langle \left(B^{+}+B^{-}-\gamma\right)^{m}\right\rangle_{\Omega_\gamma}=\int_{-\infty}^{\infty} t^m \mu(dt), \quad m=1,2,\ldots.$$

Throughout the rest of this section, let $\Gamma_{\{\omega_n\}}=(\Gamma,\{\Psi_n\},B^{+},B^{-})$ be the interacting Fock space associated with $\{\omega_n\}=\{1,1,2,2,3,3,4,\ldots\}$. We recall that Ω_γ is a coherent state with parameter $\gamma\in\mathbb{R}$, therefore combining Proposition 4.17 in [26] and (3.4.2) we obtain

$$\left\langle \Omega_\gamma,(B^{+}+B^{-}-\gamma)^m\Psi_0\right\rangle=\left\langle \Psi_0,(B^{+}+B^{-}-\gamma B^{+}B^{-})^m\Psi_0\right\rangle.$$

Now, we need to find μ such that

$$\left\langle \Psi_0,(B^{+}+B^{-}-\gamma B^{+}B^{-})^m\Psi_0\right\rangle=\int_{-\infty}^{\infty} t^m \mu(dt), \quad m=1,2,\ldots. \tag{3.4.3}$$

Note that $-\gamma B^{+}B^{-}$ is a diagonal defined by the sequence $\{0,-\gamma,-\gamma,-\gamma,\ldots\}$. Hence the Jacobi coefficient of μ in (3.4.3) is $(\{\omega_n\},\{0,-\gamma,-\gamma,-\gamma,\ldots\})$. The Cauchy transform of μ is given by

$$G_\mu(z)=\int_{-\infty}^{\infty}\frac{\mu(dt)}{z-t}=\cfrac{1}{z-\cfrac{1}{z+\gamma-\cfrac{1}{z+\gamma-\cfrac{2}{z+\gamma-\cdots}}}},$$

where $z\in\{\text{Im } z\neq 0\}$.
Let ν a measure such that $\nu(dt)=|t|e^{-t^2}dt$, which has Jacobi coefficient is $(\{\omega_n\},\{\alpha_n\equiv 0\})$. Then we have

$$G_\mu(z-\gamma)=\cfrac{1}{z-\gamma-\cfrac{1}{z-\cfrac{1}{z-\cfrac{2}{z-\cdots}}}}=\frac{1}{z-\gamma-K_\nu(z)},$$

where $K_\nu(z)=z-1/G_\nu(z)$. We shall apply Stieltjes inversion formula to r.h.s. in last equation, i.e. we need to compute

$$\begin{aligned}-\tfrac{1}{\pi}\lim_{y\to+0}\text{Im } G_\mu(x+iy-\gamma) &= -\tfrac{1}{\pi}\lim_{y\to+0}\tfrac{\text{Im } G_\nu(x+iy)}{\|-\gamma G_\nu(x+iy)+1\|^2}\\ &= \tfrac{f_\nu(x)}{\|-\gamma\lim_{y\to+0}G_\nu(x+iy)+1\|^2},\end{aligned} \tag{3.4.4}$$

where $f_\nu=d\nu/dx$. Due to the fact that ν is symmetric, we have that $G_\nu(z)=zG_\rho(z^2)$ (see [5]), where $\rho(dx)=e^{-x}dx$. Then, we have

$$\begin{aligned}\lim_{y\to+0}G_\nu(z) &= \lim_{y\to+0}zG_\rho(z^2)\\ &= \lim_{y\to+0}\text{Re } zG_\rho(z^2)+i\lim_{y\to+0}\text{Im } zG_\rho(z^2)\\ &= \pi xHf_\rho(x^2)+i\pi|x|f_\rho(x^2),\end{aligned} \tag{3.4.5}$$

where $Hf_\rho(x) = e^{-x}\mathrm{Ei}(x)$ is the Hilbert transform (see Chapter 3 in [18]) of $f_\rho(x) = e^{-x}$ and $\mathrm{Ei}(x)$ is the special function on the complex plane called *exponential integral* defined by

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt.$$

Combining (3.4.4) and (3.4.5) we obtain

$$-\frac{1}{\pi}\lim_{y\to+0} \mathrm{Im}\ G_\mu(x+iy-\gamma) = \frac{|x|e^{-x^2}}{(-\gamma\pi x e^{-x^2}\mathrm{Ei}(x^2)+1)^2 + (\gamma\pi|x|e^{-x^2})^2}.$$

Now we are in position to rephrase Theorem 3.4.6.

Theorem 3.4.7 ([16]) *For the adjacency matrix A_k of the odd graph O_k we have*

$$\lim_{k,q} \left\langle \left(\frac{A_k - \langle A_k\rangle_q}{\Sigma_q(A_k)} \right)^m \right\rangle_q = \int_{-\infty}^{\infty} x^m \mu(dx), \quad m = 1, 2, \ldots,$$

where the explicit form of μ is given as follows:

$$\mu(dx) = \frac{|x-\gamma|e^{-(x-\gamma)^2}}{(-\gamma\pi(x-\gamma)e^{-(x-\gamma)^2}\mathrm{Ei}((x-\gamma)^2)+1)^2 + (\gamma\pi|x-\gamma|e^{-(x-\gamma)^2})^2} dx.$$

In Figure 3.1 we show the density appearing in the above theorem.

Remark 3.4.8 *The case $\gamma = 0$ in Theorem 3.4.7 is the Theorem 6.1 in [27], and corresponds to the two-sided Rayleigh distribution.*

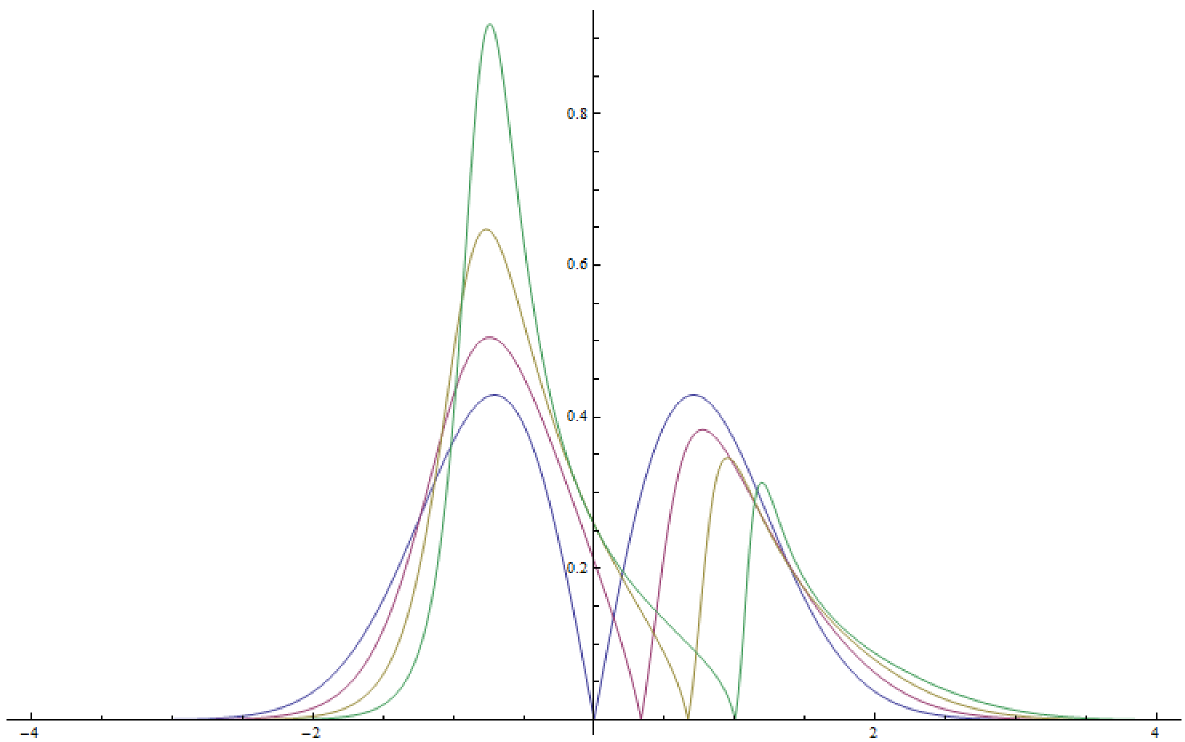

Figure 3.1: $\mu(dx)$ with $\gamma = 0, 1/3, 1/6, 1$ (Theorem 3.4.7)

Chapter 4

Distance-k Graphs

The study of the asymptotic spectral distribution of distance-k graph of product of graphs was iniciated by Kurihara & Hibino [32]. This chapter contains results related to the spectral distribution of distance-k graph of cartesian, Boolean and free product of graphs. The Boolean and free cases were published by the author in joint works with Arizmendi [6, 7].

Definition 4.0.1 *For a given graph $G = (V, E)$, its* ***distance-k graph*** $G^{[k]} = (V, E^{[k]})$ *is defined by*

$$E^{[k]} = \{(x, y) : x, y \in V,\ \partial_G(x, y) = k\}.$$

By definition, the adjacency matrix of $G^{[k]}$ coincides with the k-th distance matrix A_k of G defined in (3.1.1). Clearly, the distance-1 graph $G^{[1]}$ coincides with G itself. Note that the distance-k graph of a connected graph is not necessarily connected.

4.1 Distance-k Graphs of Cartesian Product

The spectrum of the distance-k graph of the Cartesian product of graphs was first studied by Kurihara and Hibino [32] where they consider the distance-2 graph of $K_2 \times \cdots \times K_2$ (the n-dimensional hypercube). More recently, in a series of papers [17, 23, 31, 32, 33, 40] the asymptotic spectral distribution of the distance-k graph of the N-fold power of the Cartesian product was studied. These investigations, finally lead to Theorem 4.1.2 which generalizes the central limit theorem for Cartesian products of graphs.

Definition 4.1.1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graph. The* ***cartesian product graph*** *of G_1 with G_2 is the graph $G_1 \times G_2 = (V_1 \times V_2, E)$ such that for $(v_1, w_1)\,,\ (v_2, w_2) \in V_1 \times V_2$ the edge $e = (v_1, w_1) \sim (v_2, w_2) \in E$ if and only if one of the following holds:*

$$\begin{aligned} 1.\ v_1 &= v_2 \text{ and } w_1 \sim w_2 \\ 2.\ v_1 &\sim v_2 \text{ and } w_1 = w_2. \end{aligned}$$

Theorem 4.1.2 (Hibino, Lee and Obata [23]) *Let $G = (V, E)$ be a finite connected graph with $|V| \geq 2$. For $N \geq 1$ and $k \geq 1$ let $G^{[N,k]}$ be the distance-k graph of $G^N =$*

$G \times \cdots \times G$ (N-fold Cartesian power) and $A^{[N,k]}$ its adjacency matrix. Then, for a fixed $k \geq 1$, the eigenvalue distribution of $N^{-k/2}A^{[N,k]}$ converges in moments as $N \to \infty$ to the probability distribution of

$$\left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!}\tilde{H}_k(g), \tag{4.1.1}$$

where $\tilde{H}_k$ is the monic Hermite polynomial of degree k and g is a random variable obeying the standard normal distribution $\mathcal{N}(0,1)$.

Remark 4.1.3 *It is known that the probability distributions of $\tilde{H}_k(g)$ is the solution to a determinate moment problem for $k = 1, 2$. It is highly expected that the uniqueness does not hold for $k \geq 3$, as is suggested by Berg [9].*

4.2 Distance-k Graphs of Star Product

In this section we study the distribution in the vacuum state of the star product of graphs. That is, we prove the analog of Theorem 4.1.2 by changing the cartesian product by the star product. The results in this section were published by the author in joint work with Arizmendi [6].

Definition 4.2.1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graph with distinguished vertices $o_1 \in V_1$ and $o_2 \in V_2$, the **star product graph** of G_1 with G_2 is the graph $G_1 \star G_2 = (V_1 \times V_2, E)$ such that for $(v_1, w_1)\,,\ (v_2, w_2) \in V_1 \times V_2$ the edge $e = (v_1, w_1) \sim (v_2, w_2) \in E$ if and only if one of the following holds:*

$$\begin{aligned} 1.\ v_1 &= v_2 = o_1 \text{ and } w_1 \sim w_2 \\ 2.\ v_1 &\sim v_2 \text{ and } w_1 = w_2 = o_2. \end{aligned}$$

Theorem 4.2.2 ([6]) *Let $G = (V, E, e)$ be a locally finite connected graph and let $k \in \mathbb{N}$ be such that $G^{[k]}$ is not trivial. For $N \geq 1$ and $k \geq 1$ let $G^{[\star N,k]}$ be the distance-k graph of $G^{\star N} = G \star \cdots \star G$ (N-fold star power) and $A^{[\star N,k]}$ its adjacency matrix. Furthermore, let $\sigma = V_e^{[k]}$ be the number of neighbours of e in the distance-k graph of G, then the distribution with respect to the vacuum state of $(N\sigma)^{-1/2}A^{[\star N,k]}$ converges in distribution as $N \to \infty$ to a centered Bernoulli distribution. That is,*

$$\frac{A^{[\star N,k]}}{\sqrt{N\sigma}} \longrightarrow \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1,$$

weakly.

The limit distribution above is universal in the sense that it is independent of the details of a factor G, but also in this case the limit **does not** depend on k. The proof of Theorem 4.2.2 is based in a fourth moment lemma for convergence to a centered Bernoulli distribution (see Lemma 1.4.4).

Lemma 4.2.3 *Let $G=(V,E,e)$ be a connected finite graph with root e and k such that $G^{[k]}$ is a non-trivial graph. Let ${G^{\star N}}^{[k]}$ be the distance-k graph of the N-th star product of G, then ${G^{\star N}}^{[k]}$ admits a decomposition of the form.*

$$ {G^{\star N}}^{[k]} = (G^{[k]})^{\star N} \cup \hat{G}, $$

where $\hat{G}$ is a graph with same vertex set as G and $\partial_G(z,e)<k$ for all $z\in\hat{G}$.

Proof. Let $G_1,\ G_2,\ \ldots,\ G_N$ be the N copies of G, that form the star product graph $G^{\star N}$ by gluing them at e. For $x,y\in G_i$, the distance between x and y is given by

$$ \partial_{G^{\star N}}\left(x,y\right)=\partial_{G_i}\left(x,y\right)=\partial_{G}\left(x,y\right), $$

hence

$$ (x,y)\in E\left(G_i^{[k]}\right) \text{ if and only if } (x,y)\in E\left(\left(G^{\star N}\right)^{[k]}\right), $$

therefore we have $\left(G^{[k]}\right)^{\star N}\subseteq\left(G^{\star N}\right)^{[k]}$.

Now, if $x\in G_i$ and $y\in G_j$ with $j\neq i$, by definition all the paths in $G^{\star N}$ from x to y must pass throw e, then we have

$$ \partial_{G^{\star N}}\left(x,y\right)=\partial_{G_i}\left(x,e\right)+\partial_{G_j}\left(y,e\right), $$

thus

$$ (x,y)\in E\left(\left(G^{\star N}\right)^{[k]}\right) \text{ if and only if } \partial_{G_i}\left(x,e\right)+\partial_{G_j}\left(y,e\right)=k. $$

Since $\partial_{G_i}\left(x,e\right),\ \partial_{G_j}\left(y,e\right)>0$, we obtain the desired result. ∎

Now, we are in position to prove the main theorem of this section.

Proof of Theorem 4.2.2.

Consider the non-commutative probability space $(\mathcal{A},\phi_1)$ with $\phi_1\left(M\right)=M_{11}$, for $M\in\mathcal{A}$. Then, recall that, if A is an adjacency matrix, $\phi_1\left(A^k\right)$ equals the number of walks of size k starting and ending at the vertex 1.

Since G is a simple graph, it has no loops and then $G^{\star N}$ is also a simple graph. Thus,

$$ \phi_1\left(\frac{A^{[\star N,k]}}{\sqrt{N|V_e^{[k]}|}}\right)=0. $$

Now, observe that since the graph $G^{\star N}$ has no loops, the only walks in G of size 2 which start in e and end in e are of the form (exe), where x is a neighbor of e in $\left(G^{\star N}\right)^{[k]}$. The number of neighbors of e is exactly $N|V_e^{[k]}|$, thus

$$ \begin{aligned} \phi_1\left(\left(\frac{A^{[\star N,k]}}{\sqrt{N|V_e^{[k]}|}}\right)^2\right) &= \frac{1}{N|V_e^{[k]}|}\phi_1\left(\left(A^{[\star N,k]}\right)^2\right) \\ &= \frac{1}{N|V_e^{[k]}|}N|V_e^{[k]}|=1. \end{aligned} $$

Thus we have seen that $\phi(A_N) = 0$ and $\phi(A_N^2) = 1$. Hence, it remains to show that $\phi(A_N^4) \to 1$ as $N \to \infty$.

We are interested in counting the number of walk of size 4 that start and finish at e in $\left(G^{\star N}\right)^{[k]}$. We will divide this walks in two types.

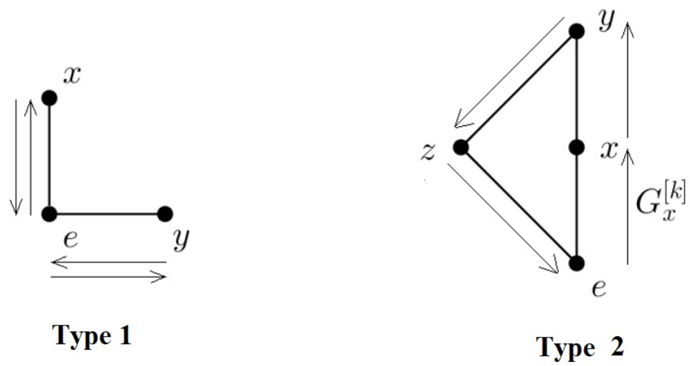

Figure 4.1: Types of walks of size 4

Type 1. The first type of walk is of the form $exeye$. That is, the walk starts at e, then visits a neighbor x of e to then come back to e, this can be done in $N|V_e^{[k]}|$ ways. After this, he again visits a a neighbour y (which could be again x) of e to finally come back to e. Again, this second step can be done in $N|V_e^{[k]}|$ different ways , so there is $\left(N|V_e^{[k]}|\right)^2$ walks of this type.

Type 2. Let $G_x^{[k]}$ be the copy of $G^{[k]}$ in the distance-k graph of the star product $\left(G^{[k]}\right)^{\star N}$ which contains x. The second type of walks is as follows. From e it goes to some $x \in V_e^{[k]}$ (which can be chosen in $N|V_e^{[k]}|$ different ways), and then from x then he goes to some $y \neq e$. This y should belong to $G_x^{[k]}$. Indeed, since $\partial_{(G^{\star N})}(e, x) = k$, if y would be in another copy of $G^{[k]}$ the distance $\partial_{(G^{\star N})}(y, x)$, between y and x would be bigger than k. The number of ways of choosing y is bounded by the number of neighbours of x in $G^{[k]}$.

For the next step of the walk, from y we can only go to a neighbor of e, say $z \in V_e^{[k]}$ (since in the last step it must come back to e). This z indeed must also belong to $G_x^{[k]}$,. If this wouldn't be the case and $z \notin G_x^{[k]}$, then we would have that $\partial_{(G^{\star N})}(e, z) \neq k$, which is a contradiction because of Lemma 4.2.3.

Finally, let $M = \max_{x \in V} |V_x^{[k]}|$. Then, from the above considerations we see that the number of walks of Type 2 is bounded by $M\left(N|V_e^{[k]}|\right)\left(|V_e^{[k]}|\right)$, from where

$$\begin{aligned}
\phi_1\left(\left(\frac{A^{[\star N,k]}}{\sqrt{N|V_e^{[k]}|}}\right)^4\right) &\leq \frac{\left(N|V_e^{[k]}|\right)^2}{\left(N|V_e^{[k]}|\right)^2} + \frac{N|V_e^{[k]}|M|V_e^{[k]}|}{\left(N|V_e^{[k]}|\right)^2} \\
&= 1 + \frac{M}{N} \underset{N\to\infty}{\longrightarrow} 1,
\end{aligned}$$

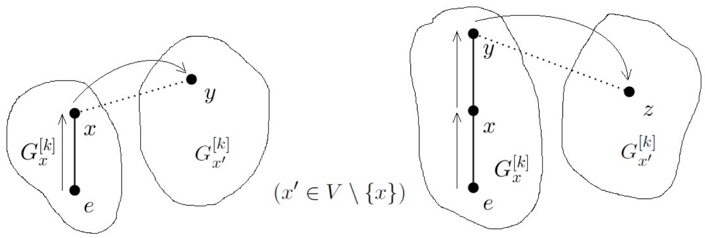

Figure 4.2: Obstructions

since M does not depend on N. Thanks to Lemma 1.4.4 we obtain the desired result. ■

4.3 Distance-k Graphs of Free Product

In this section we consider three problems on the distance-k graphs, which generalize results of Kesten [31] (on random walks on free groups), McKay [35] (on the asymptotic distribution of d-regular graphs) and the free central limit theorem of Voiculescu [47]. The first one is finding for fixed d, the distribution w.r.t. the vacuum state of the distance-k graphs of a d-regular tree. Then we consider two related problems which are in the asymptotic regime. On one hand, we show that the asymptotic distributions of distance-k graphs of d-fold free product graphs, as d tends to infinity, are given by the distribution of $P_k(s)$, where s is a semicircle distribution and P_k is the k-th Chebychev polynomial. On the other hand, we find the asymptotic spectral distribution of the distance-k graph of a random d-regular graph of size n, as n tends to infinity. The results in this section were published by the author in joint work with Arizmendi [7].

We define the *free product* of the rooted vertex sets (V_i, e_i), $i \in I$, where I is a countable set, by the rooted set $(*_{i\in I}V_i, e)$, where

$$*_{i\in I}V_i = \{e\} \cup \{v_1v_2\cdots v_m : v_k \in V_{i_k}^0, \text{ and } i_1 \neq i_2 \neq \cdots \neq i_m,\ m \in \mathbb{N}\},$$

and e is the empty word.

Definition 4.3.1 *The* ***free product of rooted graph*** $(\mathcal{G}_i, e_i)$, $i \in I$, *is defined by the rooted graph* $(*_{i\in I}\mathcal{G}_i, e)$ *with vertex set* $*_{i\in I}V_i$ *and edge set* $*_{i\in I}E_i$, *defined by*

$$*_{i\in I}E_i := \{(vu, v'u) : (v, v') \in \bigcup_{i\in I} E_i \text{ and } u,\ vu,\ v'u \in *_{i\in I}V_i\}.$$

We denote this product by $*_{i\in I}(\mathcal{G}_i, e_i)$ *or* $*_{i\in I}\mathcal{G}$ *if no confusion arises. If* $I = [n]$, *we denote by* $G^{*n} = (*_{i\in I}G, e)$.

Notice that for a fixed word $u = v_1v_2\cdots v_m$ with $j \in I$ with $v_1 \notin V_j$ the subgraph of $(*_{i\in I}\mathcal{G}_i, e)$ induced by the vertex set $\{wu : w \in V_j\}$ is isomorphic to G_j. This motivates the following definition

Definition 4.3.2 *If $x, y \in *_{i\in I} V_i$, we say that x and y are in the same copy of G_i if $x = vu$ and $y = v'u$ for some $u \in *_{i\in I} V_i$ and $v, v' \in V_j$ for some $j \in I$.*

For the rest of this section we define an order which will become handy when estimating vanishing terms in Sections 4.3.2 and 4.3.3.

Definition 4.3.3 *Let A and B be matrices (possibly infinite), we define the order $A \succeq B$ if $A_{ij} \geq B_{ij}$ for all entries ij.*

Remark 4.3.4 *For A, B, C, D with nonnegative entries we have:*
1) $\varphi_1(A^k) \geq \varphi_1(B^k)$ if $A \succeq B$.
2) For G_1 and G_2 graphs with n vertices, G_2 is a subgraph of G_1 iff $A_{G_1} \succeq A_{G_2}$.
3) If $A \succeq B$ and $C \succeq D$ implies $AC \succeq BD$.

4.3.1 Distance-k graph of d-regular trees

As we know, from the free central limit theorem, if we have a sequence of d-regular trees, then the limiting spectral distribution of the sequence, as $d \to \infty$, converges to a semicircular law. However, if d is fixed, and we consider a sequence of d-regular graphs, such that the number of vertices tends to infinity, then the limiting spectral distribution is not semicircular. These limiting spectral distributions, which are known as the Kesten-McKay distributions, were found by McKay [35] while studying properties of d-regular graphs and by Kesten [29] in his works on random walk on (free) groups.

Let $d \geq 2$ be an integer, we define Kesten-McKay distribution, μ_d, by the density

$$\mathbf{d}\mu_d = \frac{d\sqrt{4(d-1) - x^2}}{2\pi(d^2 - x^2)}\mathbf{d}x. \tag{4.3.1}$$

The orthogonal polynomials and the Jacobi parameters of these distributions are well known. More precisely, for $d \geq 2$, the polynomials defined by

$$T_0(x) = 1, \quad T_1(x) = x,$$

and the recurrence formula

$$xT_k(x) = T_{k+1}(x) + (d-1)T_{k-1}(x), \tag{4.3.2}$$

are orthogonal with respect to the distribution μ_d. Thus, it follows that the Jacobi parameters of μ_d are given by

$$\alpha_m = 0, \ \forall m \geq 0 \quad \text{and} \quad \omega_0 = d, \ \omega_n = d - 1 \ \forall n \geq 1.$$

Remark 4.3.5 *If we define the following polynomials*

$$\tilde{T}_k(x) = \begin{cases} 1, & k = 0 \\ \sqrt{\frac{d-1}{d}}P_k(x) - \frac{1}{\sqrt{d(d-1)}}P_{k-2}(x), & k = 1, 2, 3, \ldots, \end{cases}$$

then, $T_k(x) = \tilde{T}_k(x/2\sqrt{d-1})$.

The d-regular tree is the d-fold free product graph of K_2, the complete graph with two vertices. Before we consider asymptotic behavior of the general case of the free product of graphs, we study the distance-k graph of a d-regular tree for fixed d and k. This is an example where we can find the distribution with respect to the vacuum state in a closed form. Moreover, this example sheds light on the general case of the d-fold free product of graphs, in the same way as the d-dimensional cube was the leading example for investigations of the distance-k graph of the d-fold Cartesian product of graphs (Kurihara [31]).

As a warm up and base case, we calculate the distribution of the distance-2 graph with respect to the vacuum state.

For $d \geq 2$, let $A_d^{[k]}$ be the adjacency matrix of distance-k graph of d-regular tree. We will sometimes omit the subindex d in the notation and write $A^{[1]} = A$. Then we have the following equality, which expresses A^2 in terms of the distance-2 graph and the identity matrix:

$$A^2 = A_d^{[2]} + dI. \tag{4.3.3}$$

Since $A_d^{[2]} = A^2 - dI$ then the distribution is given by the law of $x^2 - d$, where x is a random variable obeying the Kesten-McKay distribution of parameter d, μ_d.

For $k \geq 2$ we have the following recurrence formula.

Lemma 4.3.6 *Let $d \geq 1$ fixed, then $A^{[1]} = A$, $A^{[2]} = A^2 - dI$, and*

$$AA^{[k]} = A^{[k+1]} + (d-1)A^{[k-1]} \quad k = 2, \ \ldots, d-1. \tag{4.3.4}$$

Proof. Let i and j be vertices of the d-regular tree, Y_d. We have the following three cases.
Case 1. If $\partial(i,j) = k+1$ then $(A^{[k]}A)_{ij} = 1$, that is because, in this case, there is only one way to get from vertex j to vertex i. Indeed, since this Y_d is a tree there is only one walk from i to j of size $k+1$ in Y_d. Thus, there is exactly one neighbor l of j at distance k from i and thus the only way to go across the distance-k graph and after across Y_d to reach j is trough l.
Case 2. When we have $\partial(i,j) = k-1$, then $(A^{[k]}A)_{ij} = d-1$. In fact, for the vertex i there are $d-1$ ways to arrive to j from a neighbor of j at distance k from i. Thus, if we are in vertex i, there are $d-1$ ways to travel across the distance-k graph and finally go down one level in the d-regular tree to vertex j, .
Case 3. Suppose $|\partial(i,j) - k| \neq 1$, then $(A^{[k]}A)_{ij} = 0$. To see this, we just note that, in the d-regular tree we can go up one-level or go down one-level, after going across the distances-k graph, this means that the distance between i and j would be $k-1$ or $k+1$, which is a contradiction. Therefore if $|\partial(i,j) - k| \neq 1$, there is no way to go from the vertex i to the vertex j, going across the distance-k graph and after, across the d-regular tree in one step.
Thanks to the above, we obtain the following recurrence formula

$$A^{[k]}A = A^{[k+1]} + (d-1)A^{[k-1]}. \tag{4.3.5}$$

Since all involved matrices in 4.3.5 are symmetric, by taking the adjoint we can rewrite equation (4.3.5) in the more convenient way as follows

$$AA^{[k]} = A^{[k+1]} + (d-1)A^{[k-1]},$$

as we desired. ■

Now we can calculate the distribution of the distance-k graph of the d-regular tree, for d fixed.

Theorem 4.3.7 *For $d \geq 2$, $k \geq 1$, let $A_d^{[k]}$ be the adjacency matrix of distance-k graph of the d-regular tree. Then the distribution with respect to the vacuum state of $A_d^{[k]}$ is given by the probability distribution of*

$$T_k(b) = \sqrt{\frac{d-1}{d}} P_k\left(\frac{b}{2\sqrt{d-1}}\right) - \frac{1}{\sqrt{d(d-1)}} P_{k-2}\left(\frac{b}{2\sqrt{d-1}}\right) \quad ,$$

where P_k is the Chebyshev polynomial of order k and b is a random variable with Kesten-McKay distribution, μ_d.

Proof. From equation (4.3.4) we can see that $A_d^{[k]}$ fulfills the same recurrence formula than T_k in (4.3.2). Since A is distributed as the Kesten-McKay distribution μ_d, we arrive to the conclusion. ■

To end this section we observe that from the considerations above, by letting d approach infinity, we may find the asymptotic behavior of the distribution fo the distance-k graph of the d-regular tree. The same behavior is expected when changing the d-regular tree with the d-fold free product of any finite graph. We will prove this in Section 4.3.3.

Theorem 4.3.8 *For $d \geq 2$, let $A_d^{[k]}$ be the adjacency matrix of the distance-k graph of the d-regular tree. Then the distribution with respect to the vacuum state of $d^{-k/2}A_d^{[k]}$ converges in moments as $d \to \infty$ to the probability distribution of*

$$P_k(s), \tag{4.3.6}$$

where $P_k(s)$ is the Chebychev polynomial of degree k and s is a random variable obeying the semicircle law.

Proof. If we divide the equation (4.3.4) by $d^{(k+1)/2}$ we obtain

$$\frac{A_d}{d^{1/2}} \frac{A_d^{[k]}}{d^{k/2}} = \frac{A_d^{[k+1]}}{d^{(k+1)/2}} + \frac{A_d^{[k-1]}}{d^{(k-1)/2}} - \frac{1}{d} \frac{A_d^{[k-1]}}{d^{(k-1)/2}}$$

We write $X = \frac{A_d}{d^{1/2}}$, then we have

$$P^{(1)}(X) = X, \quad P^{(2)}(X) = X^2 - I,$$

and the recurrence

$$XP^{(k)}(X) = P^{(k+1)}(X) + P^{(k-1)}(X) - \frac{1}{d}P^{(k-1)}(X),$$

which when $d \to \infty$ becomes the recurrence formula

$$XP^{(k)}(X) = P^{(k+1)}(X) + P^{(k-1)}(X).$$

Thus $P^{(k)}(x)$ and $P_k(x)$ satisfy the same recurrence formula asymptotically and thanks to the free central limit theorem for graphs we have the convergence, $X \xrightarrow{m} s$. Consequently, combining these two observations and using Proposition 1.1.2 we obtain the proof. ■

4.3.2 Distance-2 graph of free products

In this subsection we derive the asymptotic spectral distribution of the distance-2 graph of the n-free power of a graph when n goes to infinity.

In order to analyze the distance-2 graphs we give a simple, but useful, decomposition of the square of the adjacency matrix.

Lemma 4.3.9 *Let G be a simple graph with adjacency matrix A, we have the following decomposition of A^2:*

$$A^2 = \tilde{A}^{[2]} + D + \Delta, \tag{4.3.7}$$

where D is diagonal with $(D)_{ii} = deg(i)$, $(\Delta)_{ij} = |triangles\ in\ G\ with\ one\ side\ (i,j)|$ and $(\tilde{A}^{[2]})_{ij} = |paths\ of\ size\ 2\ from\ i\ to\ j|$, whenever $(A^{[2]})_{ij} = 1$.

Proof. Indeed $(A^2)_{ij}$ is zero if the distance between i and j is greater than 2. So $(A^2)_{ij} > 0$ implies that $\partial(i,j) = 0, 1$ or 2. If $\partial(i,j) = 0$ then $i = j$ and since $(A^2)_{ii} = deg(i)$ we get D, a diagonal matrix with $(D)_{ii} = deg(i)$. Next, if $\partial(i,j) = 1$ then each path of size 2 which forms a triangle with side (i,j) will contribute to $(A^2)_{ij} = (\Delta)_{ij}$ where $(\Delta)_{ij} = |\text{triangles in } G \text{ with one side } (i,j)|$. Finally if $\partial(i,j) = 2$ then $(A^2)_{ij}$ equals the number of paths of size 2 from i to j, which is non-zero exactly when $(\tilde{A}^{[2]})_{ij} > 0$. ∎

Remark 4.3.10 *Notice in Lemma 4.3.9, that when G is a tree then $\Delta = 0$, $\tilde{A}^{[2]} = A^{[2]}$, therefore $A^{[2]} = A^2 - D$.*

Let $G = (V, E, e)$ be a rooted graph, $A_n = A_{G^{*N}}$ and define D_n and Δ_n by the decomposition (4.3.9) applied to $G^{*N} = G * \cdots * G$, i.e.

$$A_n^2 = \tilde{A}_n^{[2]} + D_n + \Delta_n. \tag{4.3.8}$$

We will describe the asymptotic behavior of each of these matrices. First, we consider the diagonal matrix D_n.

Lemma 4.3.11 *$D_n/n \to I deg(e)$ entrywise and in distribution w.r.t. the vacuum state.*

Proof. For any $i \in G_n$ $(D_n)_{ii} = deg_{G_n}(i) = c_i + (n-1)deg(e)$ for some $0 < c_i < maxdeg(G)$. Thus,

$$\frac{(D_n)_{ii}}{n} = \frac{c_i}{n} + \frac{(n-1)deg(e)}{n} \to deg(e).$$

∎

In order to consider the other matrices in the decomposition we will use the order $\succeq$ from Definition 4.3.3.

Lemma 4.3.12 *The mixed moments of A_n^2/n and Δ_n/n asymptotically vanish.*

Proof. Note that the free product does not generate new triangles other than the ones in copies of the original graph. Thus there exist $c \geq 0$ not depending on n such that $cA_n \succeq \Delta_n$. Hence, for $m_1, m_2, \ldots, m_s, l_1, l_2, \ldots, l_s \in \mathbb{N}\backslash\{0\}$ from Remark 4.3.4, we have that

$$\begin{aligned}&\varphi_1\left[\left(\frac{A_n^2}{n}\right)^{m_1}\left(\frac{\Delta_n}{n}\right)^{l_1}\cdots\left(\frac{A^2}{n}\right)^{m_s}\left(\frac{\Delta_n}{n}\right)^{l_s}\right]\\ &\leq c^{\sum_i l_i}\varphi_1\left[\left(\frac{A_n^2}{n}\right)^{m_1}\left(\frac{A}{n}\right)^{l_1}\cdots\left(\frac{A^2}{n}\right)^{m_s}\left(\frac{A}{n}\right)^{l_s}\right],\end{aligned}$$

from free central limit theorem for graphs we have that A^2/n and $A/n^{1/2}$ converge, then the right hand side of the preceding inequality converges to zero as n goes to infinity. ∎

Since $\tilde{A}_n^{[2]}$ and D_n are subgraphs of A_n^2 we have the following.

Corollary 4.3.13 *The mixed moments of the pairs* $(\tilde{A}_n^{[2]}/n,\ \Delta/n,)$ *and* $(D_n/n,\ \Delta/n,)$ *asymptotically vanish.*

Finally, we consider the matrix $\tilde{A}^{[2]}$.

Lemma 4.3.14 $\tilde{A}_n^{[2]}$ *converges to* $A_n^{[2]}$ *as* n *goes to infinity.*

Proof. Observe that we can write $A_n^{[2]}$ as

$$\tilde{A}_n^{[2]} = A_n^{[2]} + \Box_n,$$

where for (i,j) at distance 2 in G^{*n}, the entry $(\Box_n)_{ij}$ exceeds in one the number of vertices k such that $i \sim k$ and $k \sim j$.

We will extend G in the following way. For each (i,j) such that $\Box_{ij}$ is positive we put the edge ij and call this new graph $G(ext)$. Now notice that, by construction, $\Delta_{G(ext)^{*n}} \succeq \Box$ and $A_{G(ext)^{*n}} \succeq A_{G^{*n}}$. Finally, by the previous lemma the mixed moments of $\Delta_{G(ext)^{*n}}$ and $A^2_{G(ext)^{*n}}$ asymptotically vanish. But $A^2_{G(ext)^{*n}} \succeq A_n^{[2]}$, so the mixed moments of $A_n^{[2]}$ and $\Box_n$ also vanish in the limit. This of course means that $\tilde{A}_n^{[2]}$ and $A_n^{[2]}$ are asymptotically equal in distribution.

∎

We have shown that asymptotically D_n/n approximates I, $\tilde{A}_n^{[2]}$ approximates $A_n^{[2]}$ and that the joint moments between $\tilde{A}_n^{[2]}$ or D_n and Δ_n vanish. Thus, we arrive to the following theorem.

Theorem 4.3.15 *The asymptotic distributions of distance-*2 *graph of the* n*-fold free product graph, as* n *tends to infinity, is given by the distribution of* $s^2 - 1$*, where* s *is a semicircle.*

Proof. From the equation (4.3.8), and thanks to Lemmas 4.3.11, 4.3.14, Corolary 4.3.13, free central limit theorem for graph and Proposition 1.1.2 we have

$$A_n^{[2]} \xrightarrow{D} \tilde{A}_n^{[2]} \xrightarrow{D} A_n^2 - D_n - \Delta_n \xrightarrow{D} A_n^2 - I \xrightarrow{D} s^2 - 1.$$

∎

4.3.3 Distance-k graphs of free products

This subsection contains a result describing the asymptotic behavior of the distance-k graph of the d-fold free power of graphs. We will show that the adjacency matrix satisfies in the limit the recurence formula (1.2.1). We want to point out that, while the strategy of proving this theorem is similar to the one used bye Hibino, Lee and Obata [23] for the cartesian product, new technical difficulties appear since in this case the state φ_1 is not tracial (i.e. not necessarily $\varphi_1(ab) = \varphi_1(ba)$, for all a, b). In particular, the main tool to deal with the estimates, Proposition 2.1 from [23], does not apply here.
We start by showing a decomposition similar to the one seen above for d-regular trees which plays the role of Lemma 4.3.9 in the last section.

Theorem 4.3.16 *Let G be a simple finite graph, let $N, k \in \mathbb{N}$ with $N \geq 2$ and $k \geq 3$ and let $A = A_N$ denote the adjacency matrix of G^{*N}. Then, we have de following recurrence formula*

$$A^{[k]}A = \tilde{A}^{[k+1]} + (N-1)\deg(e)A^{[k-1]} + D_N^{[k-1]} + \Delta_N^{[k]}, \tag{4.3.9}$$

where $(\tilde{A}^{[k+1]})_{ij} = |\{l \sim j : \partial(i,l) = k\}|$ *whenever* $\partial(i,j) = k+1$, $(D_N^{[k-1]})_{ij} = |\{l \sim j : \partial(i,l) = k, \text{ and } l \text{ is in the same copy of } G \text{ that } j\}|$ *if* $\partial(i,j) = k-1$ *and* $(\Delta_N^{[k]})_{ij} = |\{l \sim j : \partial(i,l) = k\}|$ *when* $\partial(i,j) = k$.

Proof. It's easy to see that $(A^{[k]}A)_{ij}$ is zero if $|\partial(i,j) - k| \geq 2$. So $(A^{[k]}A)_{ij} > 0$ implies that $\partial(i,j) = k-1,\ k$ or $k+1$.

Notice that for each neighbor l of j at distance k from i, there is one edge from i to l in $A^{[k]}$ and one from l to j in A. Thus each of these neighbors adds 1 to $(A^{[k]}A)_{ij}$ and there is no further contribution.

First, if $\partial(i,j) = k-1$ there are two types of neighbors l at distance k in G^{*N}. The first ones come from the $(N-1)$ copies of G in G^{*N} which have j as a root and contribute to the matrix $A^{[k-1]}$ by $(N-1)\deg(e)$ and the second ones in which j is in the same copy that l, which contribute to $D_N^{[k-1]}$.

Secondly, if $\partial(i,j) = k$ and $(A^{[k]}A)_{ij} > 0$ is the number of neighbors of j which are at distance k from i, then we get $\Delta_N^{[k]}$.

Finally, if we have $\partial(i,j) = k+1$, so there exists at least one minimal path from i to j, which contains itself a neighbor of j which is at distance k from i, therefore this path contributes to $\tilde{A}^{[k+1]}$. Hence we obtain the claim.

∎

We are now in position to establish the main theorem of this section.

Theorem 4.3.17 ([7]) *Let $G = (V, E, e)$ be a finite connected graph and let $k \in \mathbb{N}$. For $N \geq 1$ and $k \geq 1$ let $G^{[*N,k]}$ be the distance-k graph of $G^{*N} = G * \cdots * G$ (N-fold free power) and $A^{[*N,k]}$ its adjacency matrix. Furthermore, let σ be the number of neighbors of e in the graph G. Then the distribution with respect to the vacuum state of $(N\sigma)^{-k/2}A^{[*N,k]}$ converges in moments (and then weakly) as $N \to \infty$ to the probability distribution of*

$$P_k(s), \tag{4.3.10}$$

where P_k is the Chebychev polynomial of order k and s is a random variable obeying the semicircle law.

Now in order to prove Theorem 4.3.17 we proceed in various steps. While the steps are very similar as the one for the case $k = 2$ there are some non trivial modifications to be done for the general case.
We will use induction over k. First, observe that for $k = 2$, we obtained the conclusion in the last section. Now, suppose that the fact holds for all $l \leq k$. In order to complete the proof we need the following Lemmas and Corollaries.

Lemma 4.3.18 *The mixed moments of $A^{[k]}A/N^{\frac{k+1}{2}}$ and $\Delta_N^{[k]}/N^{\frac{k+1}{2}}$ asymptotically vanish.*

Proof. By definition we have that

$$\Delta_N^{[k]} \preceq \max \deg(G) A^{[k]},$$

then we have that

$$\begin{aligned}
&\varphi_1 \left(\left(\frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)^{m_1} \left(\frac{\Delta_N^{[k]}}{N^{\frac{k+1}{2}}} \right)^{n_1} \cdots \left(\frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)^{m_l} \left(\frac{\Delta_N^{[k]}}{N^{\frac{k+1}{2}}} \right)^{n_l} \right) \\
&\leq (\max \deg)^{\sum_i n_i} \varphi_1 \left(\left(\frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)^{m_1} \left(\frac{A^{[k]}}{N^{\frac{k+1}{2}}} \right)^{n_1} \cdots \left(\frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)^{m_l} \left(\frac{A^{[k]}}{N^{\frac{k+1}{2}}} \right)^{n_l} \right),
\end{aligned}$$

by the induction hypothesis $\left(\frac{A^{[k]}A}{N^{\frac{k+1}{2}}} \right)$ and $\left(\frac{A^{[k]}}{N^{\frac{k}{2}}} \right)$ converge. Hence, the right hand side of the last inequality goes to zero. ■

Since $\tilde{A}^{[k+1]}$ and $D_N^{[k-1]}$ are also subgraphs of $A^{[k]}A$, we obtain the following result as a consequence of the previous lemma.

Corollary 4.3.19 *The mixed moments of $\left(\tilde{A}^{[k+1]}/N^{\frac{k+1}{2}},\ \Delta_N^{[k]}/N^{\frac{k+1}{2}} \right)$ and $\left(\Delta_N^{[k]}/N^{\frac{k+1}{2}}, D_N^{[k-1]}/N^{\frac{k+1}{2}} \right)$ asymptotically vanish.*

Corollary 4.3.20 *The matrices $\Delta_N^{[k]}/N^{\frac{k+1}{2}}$ and $D_N^{[k-1]}/N^{\frac{k+1}{2}}$ converge to zero as N tends to infinity.*

Proof. In the proof of Lemma 4.3.18 we proved the conclusion for $\Delta_N^{[k]}/N^{\frac{k+1}{2}}$. Analogously, using $A^{[k-1]}$ instead $A^{[k]}$ we obtain the same result for $D^{[k-1]}/N^{\frac{k+1}{2}}$. ■

In the proof of the next lemma, we shall use the following extension of a graph G. For $k \geq 2$ we define $G_{ext}(k)$ as the graph which contains the graph G, and if G has a cycle of even length smaller than $2k$, we add all the possible edges between the vertices of this cycle. It is important to notice the fact that

$$(G_{ext}(k))^{*N} = (G^{*N})_{ext}(k).$$

Otherwise, note that if $k = 2$, so we have $G_{ext}(2) = G_{ext}$.

Lemma 4.3.21 *Let $k \geq 2$, then* $\lim_{N\to\infty} \frac{\tilde{A}^{[k+1]}}{N^{\frac{k+1}{2}}} = \lim_{N\to\infty} \frac{A^{[k+1]}}{N^{\frac{k+1}{2}}}$.

Proof. Let $i, j \in \underset{s\in[N]}{*} V$ be such that $\left(\tilde{A}^{[k+1]}\right)_{ij} > 0$. Let

$$C_{ij}^{k+1} = \{\text{cycles of even length in a path of length } k+1 \text{ from } i \text{ to } j\},$$

notice that

$$|C_{ij}^{k+1}| \leq (\max \deg(G))^{k+1}.$$

Here, is important to observe that the right side bound does not depend on i, j neither N, because the free product of graph does not produce new cycles. Then we can write

$$\tilde{A}^{[k+1]} - A^{[k+1]} \preceq (\max \deg(G))^{k+1} \left(A^{[k]}_{G_{ext}(k+1)} + A^{[k-1]}_{G_{ext}(k+1)} + \cdots + A_{G_{ext}(k+1)} \right). \tag{4.3.11}$$

Then, we obtain from (4.3.11)

$$\begin{aligned}
&\left(\frac{\tilde{A}^{[k+1]} - A^{[k+1]}}{N^{(k+1)/2}}\right) \\
&\preceq (\max \deg(G))^{k+1} \left(\frac{A^{[k]}_{G_{ext}(k+1)}}{N^{(k+1)/2}} + \frac{A^{[k-1]}_{G_{ext}(k+1)}}{N^{(k+1)/2}} + \cdots + \frac{A_{G_{ext}(k+1)}}{N^{(k+1)/2}} \right) \\
&= (\max \deg(G))^{k+1} \left(\frac{A^{[k]}_{G_{ext}(k+1)}}{N^{\frac{k}{2}}} \frac{1}{N^{1/2}} + \frac{A^{[k-1]}_{G_{ext}(k+1)}}{N^{\frac{k-1}{2}}} \frac{1}{N} + \cdots + \frac{A_{G_{ext}(k+1)}}{N^{\frac{1}{2}}} \frac{1}{N^{\frac{k}{2}}} \right).
\end{aligned}$$

By the induction hypothesis we have that $\left(A^{[i]}_{G_{ext}(k)}/N^{i/2}\right)$ converges for all $i \leq k$. Therefore we have

$$\left(\frac{\tilde{A}^{[k+1]} - A^{[k+1]}}{N^{(k+1)/2}}\right) \underset{N\to\infty}{\longrightarrow} 0,$$

which completes the proof. ■

Finally, from (4.3.9) we have that

$$\frac{A_N^{[k+1]}}{(\deg(e)N)^{\frac{k+1}{2}}} = \frac{A_N^{[k]} A_N}{(\deg(e)N)^{\frac{k+1}{2}}} - \frac{A_N^{[k-1]}}{(\deg(e)N)^{\frac{k-1}{2}}} - C(N, k+1), \tag{4.3.12}$$

where

$$C(N, k+1) = \frac{\deg(e) A_N^{[k-1]} + \Delta_N^{[k]} + D_N^{[k-1]} - \left(\tilde{A}^{[k+1]} - A^{[k-1]}\right)}{(\deg(e)N)^{\frac{k+1}{2}}}.$$

Due to the induction hypothesis we have, $\deg(e)A_N^{[k-1]}/(\deg(e)N)^{\frac{k+1}{2}}$ converging to zero, furthermore by Corollary 4.3.20 and Lemma 4.3.21

$$\frac{\Delta_N^{[k]} + D_N^{[k-1]} - \left(\tilde{A}^{[k+1]} - A^{[k-1]}\right)}{(\deg(e)N)^{\frac{k+1}{2}}},$$

converges to zero. Hence

$$C(N, k+1) \longrightarrow 0. \tag{4.3.13}$$

Otherwise, using the induction hypothesis we can see that

$$\frac{A_N^{[k]}A_N}{(\deg(e)N)^{\frac{k+1}{2}}} - \frac{A_N^{[k-1]}}{(\deg(e)N)^{\frac{k-1}{2}}} \longrightarrow P_k(s)s - P_{k-1}(s) = P_{k+1}(s), \tag{4.3.14}$$

where the last equality is given by (1.2.1). Thus, mixing (4.3.12) with (4.3.13) and (4.3.14) we obtain that

$$\frac{A_N^{[k+1]}}{(\deg(e)N)^{\frac{k+1}{2}}} \longrightarrow P_{k+1}(s).$$

4.3.4 d-Regular Random Graphs

Apart from the Erdos-Renyi models [14, 15], possibly, the random d-regular graphs are possibly the most studied and well understood random graphs.

In the original paper by McKay [35], he proved that the asymptotical spectral distributions of d-regular random graph are exactly the ones appearing in (4.3.1). Heuristically, the reason is that, locally, large random d-regular graphs look like the d-regular tree and thus asymptotically their spectrum should coincide. This turns out to remain true for the distance-k graph and thus we shall expect to get a similar result. In this section we formalize this intuition.

Let X be a d-regular graph with vertex set $\{1, 2, \ldots, n(X)\}$. For each $i \geq 3$ let $c_i(X)$ be the number of cycles of length i. Let $A^{[k]}(X)$ be the adjacency matrix of the distance-k graph of X. The following is a generalization of the main theorem in McKay [35].

Theorem 4.3.22 ([7]) *For $d \geq 2$ fixed, let $X_1, X_2, \ldots$ be a sequence of d-regular graphs such that $n(X_i) \to \infty$ and $c_j(X_i)/n(X_i) \to 0$ as $i \to \infty$ for each $j \geq 3$. Then the distribution with respect to the normalized trace of $A^{[k]}(X_i)$ converges in moments, as $i \to \infty$, to the distribution of $A_d^{[k]}$ with respect to the vacuum state.*

Proof. We follow the original proof of McKay[35] with simple modifications. Let $n_r(X_i)$ denote the number of vertices v of X_i such that the subgraph of X_i induced by the vertices at distance at most $r = mk$, where $m \in \mathbb{N}$, from v is acyclic. By hypothesis we have that $n_r(X_i)/n(X_i) \to 1$ as $i \to \infty$. The number of closed walks of length m in the distance-k graph of the d-regular graph starting at each such vertex is $\varphi\left((A_d^{[k]})^m\right)$. For each of the remaining vertices the number of closed walks of length m is certainly less than d^r. Then, for each m, there are numbers $\hat{\varphi}_m(X_i)$ such that $0 \leq \hat{\varphi}_m(X_i) \leq d^r$, and

$$\begin{aligned}\varphi_{tr}\left(\left(A^{[k]}(X_i)\right)^m\right) &= \frac{\varphi_1\left((A_d^{[k]})^m\right) n_r(X_i)}{n(X_i)} + \frac{(n(X_i) - n_r(X_i))\hat{\varphi}_m(X_i)}{n(X_i)} \\ &\longrightarrow \varphi_1\left((A_d^{[k]})^m\right) \qquad \text{as } i \to \infty.\end{aligned}$$

■

Fix $d > 0$. Let $s_1 < s_2, < ..$ be the sequence of possible cardinalities of regular graphs with degree d. For each n define R_n to be the set of all labeled regular graphs with degree n and order s_i.

In order to consider the d-regular uniform random graphs we use the following lemma of Wormald [52].

Lemma 4.3.23 *For each $k > 3$ define $c_{k,n}$ to be the average number of k-cycles in the members of R_n. Then for each k, $c_{k,n} \to (d-1)^k/2k$ as $n \to \infty$.*

Theorem 4.3.24 *Let d, k be fixed integers and, for each n, let $F_n(x)$ be the expected eigenvalue distribution of the distance-k graph of a random regular graph with degree d and order $2n$. Then, as n tends to infinity, $F_n(x)$ converges to the distribution of $A_d^{[k]}$ with respect to the vacuum state, described in Theorem 4.3.7.*

Proof. Consider a graph G_n which consist a disjoint union of the all the labelled graphs of size s_n. The eigenvalue distribution of G_n coincides with the expected eigenvalue distribution of R_n. Now by Lemma 4.3.23, G satisfies the assumptions of Theorem 4.3.22 and thus we arrive to the theorem. ■

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