Exploration of the mean as a balance point

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Title: EXPLORATION OF THE MEAN AS A BALANCE POINT

Grade levels: 6 - 9

Mathematics concepts, skills, processes

Arithmetic average or mean, distance, midpoint

Science concepts, skills, processes

Lever, arm, weight, balance point, equilibrium

Prerequisite skills

Number line, directed distances, arithmetic average

Objective

Students shall develop the analogy of the mean of a set of numbers as the balance point of a lever with unit weights at those numbers on the number line.

Rationale

The arithmetic average, or mean of a set of data is a number that is used to summarize the information of all data. It is very helpful to deal with a single number that is representative, instead of dealing with many numbers. Providing different physical models of the mean can help students better understand and appreciate some of its properties.

The lever, which has many applications in everyday life, can also be used to give physical meaning to the mean of a set of data. This type of physical thinking can be very fruitful in mathematics, for example, thinking in terms of levers helped Archimedes discover mathematical relationships such as the volume of a sphere and the area of a parabola.

The mean **x** of a set of data $x_1, x_2, x_3, \ldots, x_n$ is the sum of those numbers divided by the total number of data, that is,

$$\mathbf{x} = (x_1 + x_2 + x_3 + \ldots + x_n) / n$$

Students readily learn this computational formula for the mean. However, it is important for students to perceive the mean as a conceptual act as well as a computational act. Several concrete models can be used to develop the concept (Reys, Suydam & Lindquist, 1984). For example, children can use adding machine tape to represent scores, where the length of the strip is determined by the score. To show the average of two scores, tape the two strips of paper and then fold the resulting strip in half. Using columns of blocks and then asking students to "even-out" the blocks as much as possible is another concrete approach that can be taken to find the mean.

The following lesson outline highlights several ways of helping students grasp one physical interpretation of the mean. It begins with the simplest cases and then the model is extended to accomodate more data.

Lesson Outline

1 Finding the mean of two numbers

If two children of equal weight want to balance on a see-saw, they have to sit symmetrically with respect to the position where the see-saw rotates. That is, they sit at the same distance from the point of balance, but in opposite directions (Fig. 1). The point of balance is the midpoint.

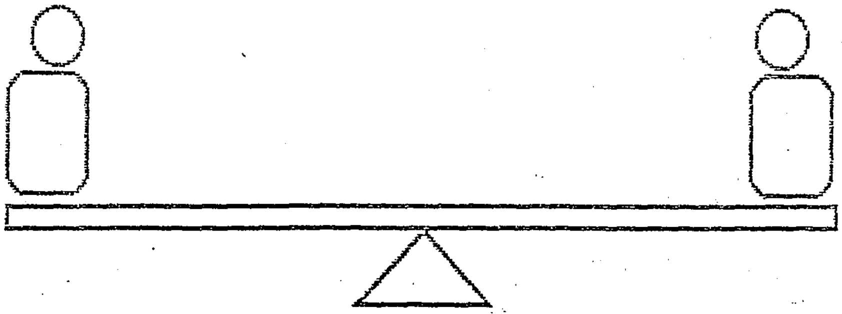

Fig. 1

The mean of two numbers **a** and **b**, is the midpoint of **a** and **b** on the number line (Fig. 2).

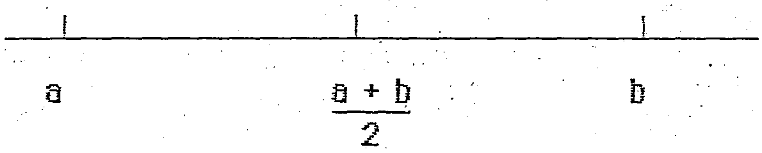

Fig. 2

For example, the mean of 5 and 9 is (5 + 9) / 2 = 7, and 7 is the midpoint of both numbers on the number line (Fig. 3)

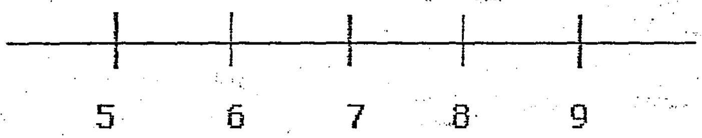

Fig. 3

The midpoint can be thought of as the point of balance for the two numbers in the following sense: imagine the number line as a weightless lever, and put a unit weight on the numbers we want to average. Then, if we hold the lever at the mean, the system is in equilibrium.

For example, the following system is in equilibrium (Fig. 4).

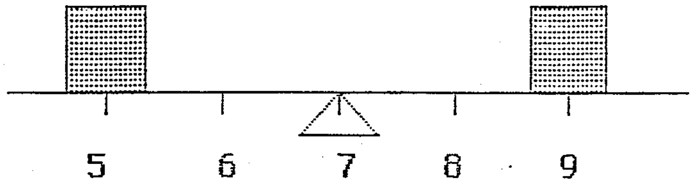

Fig. 4

The mean **x** of two numbers is at the same distance from both. Observe that **a** − **x** is the length of the arm of the lever form the point **x** to the point **a** where the weigth is located. Since both **a** − **x** and **b** − **x** represent distances that are equal in size but in opposite directions, they have different signs and their sum is zero. In the previous example, 9 - 7 = 2, 5 - 7 = -2, hence

$(5 - 7) + (9 - 7) = 2 + (-2) = 0$. The mean **x** has the property that

$(a - x) + (b - x) = 0$, since

$(a - x) + (b - x) = a + b - 2x = a + b - 2(a + b)/2 = a + b - (a + b) = 0$

The sign of **a** − **x** tells us if the weight is to the left or to the right of the mean on th number line. If **a** − **x** is positive, the weight on **a** tends to rotate the system counter clockwise, if **a** − **x** is negative, a weight on **a** tends to rotate it clockwise (Fig. 5). Thus the rotating effects of weights at **a** and **b** tend to cancel each other because **a** − **x** and **b** − **x** have different signs. The system is in equilibrium with respect to the mean because the sum of the rotating effects is zero.

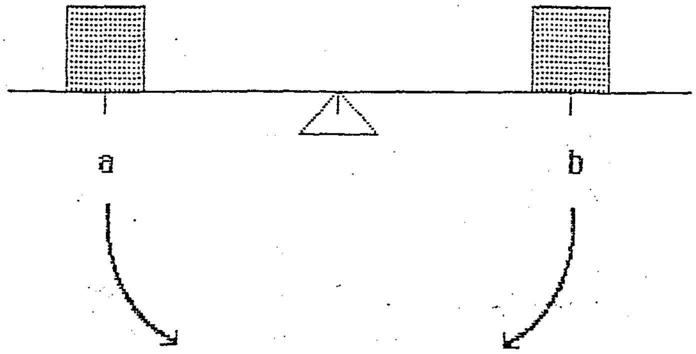

Fig. 5

The lever provides us with another model for additive inverses. If the mean of two numbers is zero, then the numbers are inverses of each other (Fig. 6).

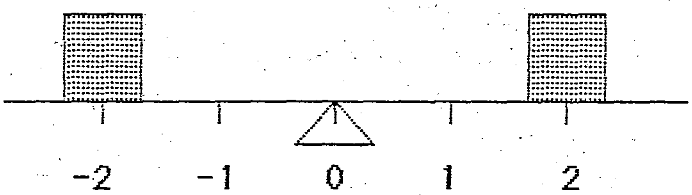

Fig. 6

For two equal numbers, the mean is the same number. In that case the two weights are located on the same point, and the point of balance is the same point.

2 The mean of three numbers

If we work with unit weights, the rotating effects with respect to **x** are given by the arm lengths **a** − **x**, **b** − **x** and **c** − **x**. If we have three equal weights, the system will be in equilibrium at **x**, if the sum of the rotating effects is zero, that is if

$(a - x) + (b - x) + (c - x) = 0$ (Fig. 7)

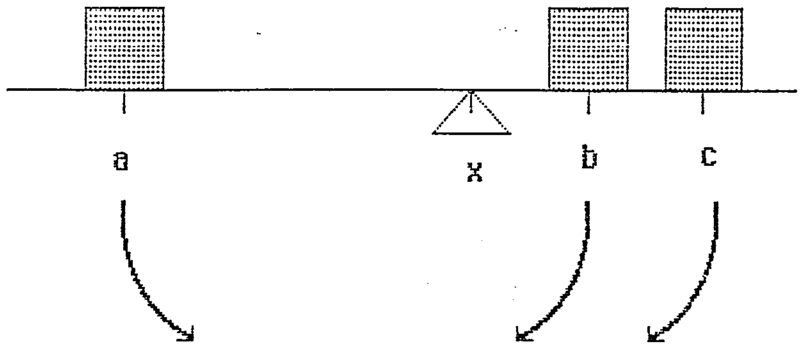

Fig. 7

If we think of the situation as a lever with three unit-weights located on the number line, balanced with respect to the equilibrium point, the mean **x** of three numbers **a**, **b**, and **c** is precisely at the center equilibrium, since

$(a - x) + (b - x) + (c - x) = 0$

This can be verified easily:

Let **x** be the mean of **a**, **b**, and **c**

$x = (a + b + c) / 3$

Then, since $a + b + c = 3x$

$(a - x) + (b - x) + (c - x) = a + b + c - 3x = 0$

This means that the rotating effects of the weights at **a**, **b**, and **c** cancel each other. Thus if we hold the lever at **x**, the system is in equilibrium.

For example, the numbers 2, 6 and 7 have an mean $x = 5$. A weight at 2 tends to rotate the system counter clockwise $(2 - 5 < 0)$, the weights at 6 and 7 tend to rotate it clockwise $(6 - 5) > 0, 7 - 5 > 0)$.

Since $(2 - 5) + (6 - 5) + (7 - 5) = 0$, the system is in equilibrium (Fig. 8).

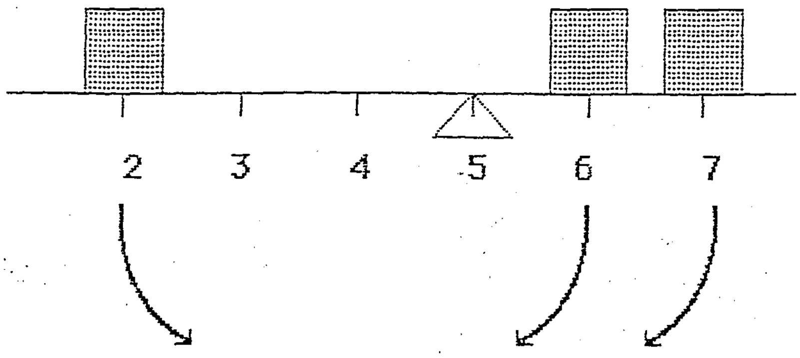

Fig. 8

Two of the numbers may be equal. In this case, two of the weights would be on the same point.

3 The mean in general

In general, if $\mathbf{x}$ is the mean of a set of data $x_1, x_2, x_3, \ldots, x_n$

we have that $x_1 + x_2 + x_3 + \ldots + x_n = n\mathbf{x}$, so that

$$(x_1 - \mathbf{x}) + (x_2 - \mathbf{x}) + (x_3 - \mathbf{x}) + \ldots + (x_n - \mathbf{x}) = x_1 + x_2 + x_3 + \ldots + x_n - n\mathbf{x} = 0$$

Thus, if we think of the system as unit weights located at $x_1, x_2, x_3, \ldots, x_n$, and the number line as a lever balanced at $\mathbf{x}$, then the rotating effects $(x_1 - \mathbf{x}), (x_2 - \mathbf{x}), \ldots, (x_n - \mathbf{x})$ with respect to $\mathbf{x}$ cancel each other and the system is in equilibrium.

4 When is the mean not very representative?

The average of a set of data should be representative of the data. However, in some cases, when there is an extreme value, the mean might not be very representative: the extreme value can greatly affect the value of the mean. With the interpretation of the mean as the balance point of a lever, we can see why a single datum very different from the others has a big effect on the mean: that weight would have a very long lever arm.

For example, if the data are 1, 2, 3, 4, and 15, then the mean $\bar{x} = 5$ is not very representative since it is greater than any of the data except for 15 (Fig. 9).

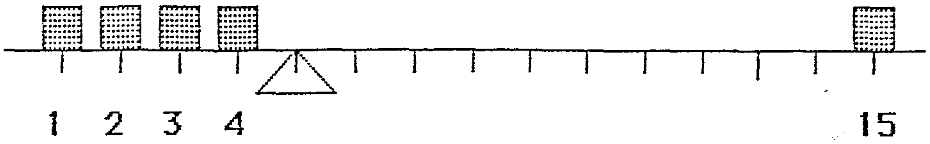

Fig. 9

Notice too, that by dropping the single datum point, 15, the mean changes dramatically. The new mean is 2.5 which is the midpoint of the remaining data.

Because of this, other numbers that represent the data should be given in addition to the mean, for example the median or the mode, and also numbers that describe the dispersion of the data.

Teacher notes:

In the same way that a geometrical meaning can help to give students a better sense for certain numbers, for example areas and volumes for n^2 and n^3, the physical analogy of the mean as the center of equilibrium of the data can give students a better conceptual grasp of the mean. A balanced rod can be used to develop students' physical sense.

Researchers (Pollatsek, Lima and Well, 1981) have found that for many students, dealing with the mean is a computational rather than a conceptual act, and that knowledge of the computational rule does not imply any real understanding of the basic underlying concept. One possible way to develop the concept is by analog knowledge that might involve visual or dynamical images of the mean as a balance point. Strauss and Bichler (1988) list several properties of the mean that are important to understand it as a concept (in parenthesis are the analog properties of the balance point):

a) the mean is located between the extreme values (the balance point is located between the extreme weights)

b) the sum of the deviations from the mean is zero (if the system is in equilibrium, the sum of rotating effects with respect to the balance point is zero)

c) the mean is changed by adding values other than the mean (if a weight is added not on the balance point, the equilibrium will be disturbed, and the new balance point will be different)

d) the mean does not necessarily equal one of the values that was summed (the position of the balance point is not necessarily at one of the weights)

e) the mean is representative of a group of individual values (the balance point is the center of equilibrium of all the weights)

The analogy of the mean as a balance point can contribute to a better grasping of these properties.

For readers interested in learning more about this topic, the books by Polya and Schiffer have nice discussions of levers. The analogy of the balance point can also be used for weighted averages. The analogy can be further extended for the mean of continuous distributions and centers of gravity.

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