On the intersection of two planar polygons

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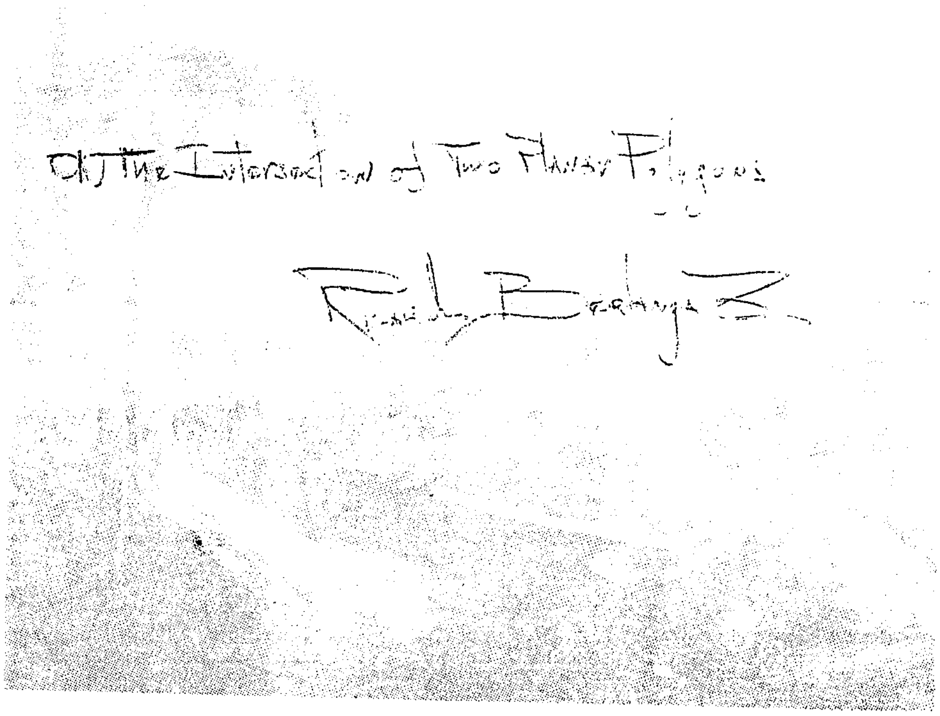

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Abstract __ Polygons have proved to be important elements for representing and manipulating visual data. Hence, a knowledge of the geometric and the combinatorial nature of the intersection of planar polygons may serve to tackle such problems as the removal of hidden surfaces. A complete classifying scheme for the intersection of stable pairs of polygons and an explicit enumeration for the case of two triangles is given.

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Stability and Equivalence.

We begin the present exposition in the general setting of simple closed curves, and then restrict ourselves to the class of (convex) polygons. Afterwards, the discussion specializes to the concrete situation of triangles.

A **simple closed curve** Γ is the continuous, injective image of a given circle into the euclidean plane $\mathbb{E}^2$. That is, Γ is just a curve which starts and ends at the same point and does not intersect with itself. It is not difficult to see that for any compact subset K of Γ there is a curve Λ with $\Gamma \cap \Lambda = K$. This somehow finishes the classification problem of intersecting simple curves, unless one is concerned with such things as fractals.

No matter how bizarre Γ might be, the "Jordan-Shönflies Theorem" states that $\mathbb{E}^2$ can be deformed continuously and bijectively in such a way that Γ is mapped onto the standard unit circle. Therefore we may speak of the **inner region** bounded by Γ. There are two ways of circling Γ: One leaving its interior always on the left (the counter-clockwise or positive direction), and other having the interior of Γ on the right hand side. An **orientation** of Γ is choosing a way of circling it.

We say that the simple closed curve Γ is **polygonal** if it can be represented as the finite union of line segments (i.e. **edges**) in such a way that each two distinct segments are either disjoint or meet at most at a common **vertex**. Now let Λ be another polygon. Then Γ intersects Λ **stably** if no vertex of one of the curves lies on an edge of the other. This implies, in particular, that Γ and Λ have no common vertices and that the intersection of one edge of Γ with one of Λ is either empty or a unique point contained in the **interior** of both edges. To put this in another way: The intersection of edges is either empty or "honest" (transversal). In this situation, the pair (Γ,Λ) is called a **stable pair** of polygons.

Let (Γ,Λ) and (Γ',Λ') be two stable pairs of polygons. They are defined to be **equivalent** if there exists an orientation preserving continuous bijection from $\mathbb{E}^2$ onto itself sending Γ (resp. Λ) onto Γ' (resp. Λ'), and the vertices of Γ (resp. Λ) onto the vertices of Γ' (resp. Λ').

With this notion, it is now true that if the pair (Γ,Λ) is stable, then "small", but independent, perturbations of Γ and Λ do not change the equivalence class of the pair. Think of Γ and Λ as two elastic strings (of different color) sitting over the table. We are asserting that by shaking one string a little, and then the other, the "nature" of $\Gamma \cap \Lambda$ is not affected. From the arithmetical point of view, "stability" says that minor computer errors do not change "the geometry".

The Intersection Scheme.

By a **directed polygon** we mean a polygon Γ together with an orientation and a fixed vertex v_o called its **origin**. With this structure, we can give a listing $(v_o, v_1, v_2, \ldots, v_{m-1}, v_m = v_o)$ of all the vertices of Γ with the property that, for $0 \le i < m$, each pair (v_i, v_{i+1}) determines an edge of Γ, denoted by $[v_i, v_{i+1}]$, in such a way that the orientation induced by the ordering of the pair agrees with the one given by the global orientation of Γ. Similarly, <u>any</u> finite set of points in Γ can be numbered in a unique compatible way.

Let Γ, Λ be directed polygons. The pair (Γ, Λ) is itself directed if Γ is positively oriented and Λ is oriented in the negative direction (in what follows, any other convention would work as well, but one has to be chosen).

Let (Γ,Λ) be a directed, stable pair of polygons with non-empty intersection. Note that the number k of points in $\Gamma\cap\Lambda$ is always even. The points in $\Gamma\cap\Lambda$ are given two different numberings, hence inducing a permutation ρ on the set $\{0,1,\dots,k-1\}$: If $x \in \Gamma\cap\Lambda$, $x=x_s$ with respect to Γ, and $x=x_t$ with respect to Λ, then $\rho(s)=t$.

Define a function γ from $\{0,1,\dots,k-1\}$ to $\{0,1,\dots m-1\}$ such that $\gamma(s)=i$ if $x_s\in[v_i,v_{i+1}]$, where $(v_i)_{i=0}^{m}$ is the vertex listing for Γ and $\Gamma\cap\Lambda = (x_s)_s$ is numbered according to Γ. Define a function λ from $\{0,1,\dots,k-1\}$ to $\{0,1,\dots,n-1\}$ such that $\lambda(t)=j$ if $x_t\in[w_j,w_{j+1}]$, where $(w_j)_{j=0}^{n}$ is the vertex listing for Λ and $\Gamma\cap\Lambda = (x_t)_t$ is numbered according to Λ.

It is much easier to visualize the permutation ρ, and the functions γ and λ by constructing the following **intersection scheme**:

Draw a line segment. Think of it as the interval of real numbers [0,m] and label each integer coordinate. Think of the marked subintervals as a representation of the edges of Γ.

Within each subinterval [i,i+1], mark as many small crosses as points are in $[v_i,v_{i+1}]\cap\Lambda$. There will be exactly k crosses in all. Its natural ordering and position corresponds to γ. Above the s-th crossing, write

$\rho(s)$. Below, write the number j if the intersection it represents lies on $[w_j, w_{j+1}]$. This represents λ.

We need to add one more bit of information to this scheme: Whether the origin of Γ lies inside or outside Λ.

It is amusing to note that only two essentially different, simple closed curves can now be drawn intersecting the line containing $[0,m]$ precisely in the marked crosses and following the sequence written on top (hence imposing an orientation on each of these curves). As an example, the reader might try the sequence (23,22,11,10,9,8,1,6,3,4,5,2,7,16,17,18,19,0,33,20,15,14,13,12,21, 24,29,28,25,26,27,32,31,30). One curve will look as a "reflection" of the other. Let the line containing the interval $[0,m]$ represent Γ, and let the upper half plane represent the interior of Γ. In this way, Λ corresponds to the curve whose right hand side region contains the set $\{0,m\}$ (or for that matter, the infinity) if and only if the origin of Γ lies inside Λ.

Given a pair (Γ,Λ) with empty intersection, define its intersection scheme just as the information of whether the origin of Γ is inside or outside Λ.

Let (Γ,Λ), (Γ',Λ') be directed, stable pairs of polygons. After rather a few drawings, it should become clear that a sufficient condition for the pairs to be equivalent is the equality of their respective intersection schemes. Conversely, if the given pairs are equivalent, then their intersection schemes will coincide after possibly changing the origins of Γ' or Λ'.

The Convex Case.

A set X in the euclidean plane is **convex** if for any two points in X the whole straight segment joining them is also contained in X. A polygon Λ is **convex** if it has a convex interior.

Let (Γ,Λ) be a directed, stable pair of polygons such that Λ is convex. As before, let $(v_i)_{i=0}^{m}$ and $(w_j)_{j=0}^{n}$ be the vertices of Γ and Λ respectively. Suppose that $\Gamma \cap \Lambda$ is not empty, and that it contains k elements. Then, by convexity, for any given edge $[v_i,v_{i+1}]$ of Γ, the intersection $[v_i,v_{i+1}] \cap \Lambda$ contains at most two points, hence $k \leq 2m$. Note in particular, that if $[v_i,v_{i+1}] \cap \Lambda$ consists of exactly two points, then v_i and v_{i+1} must lie outside Λ. Also, if v_i and v_{i+1} lie inside Λ then obviously $[v_i,v_{i+1}]$ is all inside Λ. If Γ is convex as well, the permutation ρ on $\{0,1,..,k-1\}$, induced by the two different numberings of $\Gamma \cap \Lambda$ is just a cyclic variation of

$$(*) \quad \begin{vmatrix} 0 & 1 & 2 & \dots & k-2 & k-1 \\ 0 & k-1 & k-2 & & 2 & 1 \end{vmatrix}.$$

This is because the intersection of the interiors of Γ and Λ is convex. Even more, it is always possible to choose the origins of Γ and Λ such that ρ is exactly the permutation (*). Indeed, let $[v_i,v_{i+1}]$ and $[w_j,w_{j+1}]$ be two intersecting edges of Γ and Λ, such that w_j is outside Γ. If it happens that $[w_j,w_{j+1}]$ has two intersections with Γ, assume further that $[w_j,w_{j\pm1}] \cap [v_i,v_{i+1}]$

is the first one when traveling from w_j to w_{j+1}. Then v_i must be outside Λ because, by the convexity of Λ, its interior is all to the right hand side of $[w_j, w_{j+1}]$. By choosing v_i and w_j as new origins we get the desired permutation (the sequence $(k-1, k-2, \ldots, 1, 0)$ is obtainable only if Λ has at least one vertex inside Γ).

The Intersection Of Two Triangles.

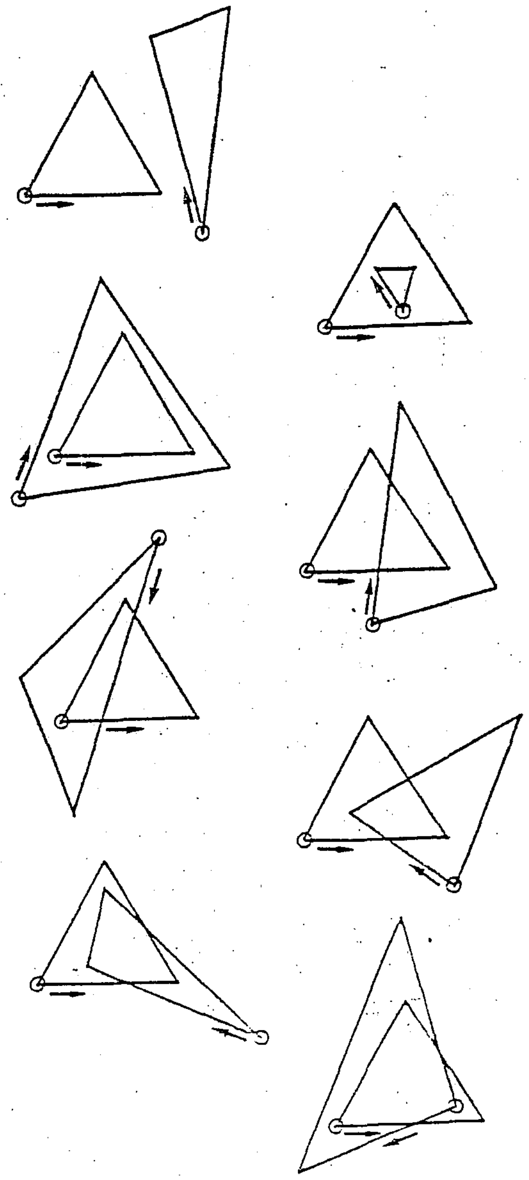
By a **triangle** we mean a polygon with only three vertices. Since a polygon is a simple closed curve then, by definition, a triangle is never degenerate. Also, triangles are always convex.

Let (Γ, Λ) be a directed,stable pair of triangles. The intersection $\Gamma \cap \Lambda$ consists of cero, two, four, or six points. If $\Gamma \cap \Lambda$ is empty, then the pair is disjoint, or one triangle is contained in the other. If there are two intersections, then these can lie on a single edge of Γ or not. In any event, the complement $\Lambda \backslash (\Gamma \cap \Lambda)$ consists of two (open) arcs where the vertices of Λ must be distributed. In the first case, the distribution of the vertices of Γ is fixed, but in the second, either one or two vertices of Γ lie inside Λ. We have seven possibilities in all. If there are four intersections, then these can be distributed in the following way in Γ: Two sides with two intersections each, or one side with two intersections and the other sides with just one. In any event, $\Lambda \backslash (\Gamma \cap \Lambda)$ consists of four arcs where the vertices of

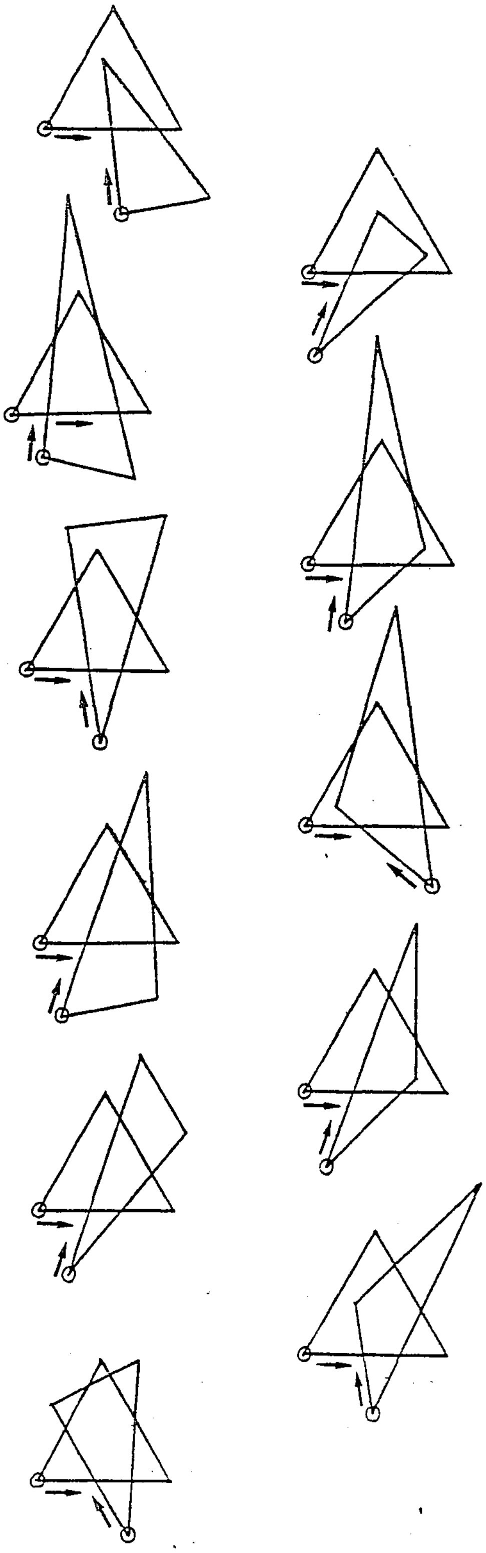
Λ must be distributed. There are eight cases in all. Finally, there is only one situation with six intersections.

The table given below summarizes all nineteen possibilities. In the first column, the triad ‹abc› says how many intersections lie on each edge of Γ. In the second, the intersection scheme is written in different notation. The pair (x,y) in the third column exhibits how many vertices of Γ lie inside Λ and how many of Λ lie inside Γ.

1)	‹000›	$--\vert--\vert--_{out}$	(0,0)
2)	‹000›	$--\vert--\vert--_{out}$	(0,3)
3)	‹000›	$--\vert--\vert--_{in}$	(3,0)
4)	‹110›	$0-\vert0-\vert--_{out}$	(1,0)
5)	‹110›	$0-\vert0-\vert--_{in}$	(2,0)
6)	‹110›	$0-\vert1-\vert--_{out}$	(1,1)
7)	‹110›	$0-\vert2-\vert--_{out}$	(1,2)
8)	‹110›	$0-\vert2-\vert--_{in}$	(2,1)



9)	‹200›	$01\vert{-}{-}\vert{-}{-}_{out}$	(0, 1)
10)	‹200›	$02\vert{-}{-}\vert{-}{-}_{out}$	(0, 2)
11)	‹211›	$01\vert 1{-}\vert 0{-}_{out}$	(1, 0)
12)	‹211›	$02\vert 1{-}\vert 0{-}_{out}$	(1, 1)
13)	‹211›	$02\vert 2{-}\vert 0{-}_{out}$	(1, 0)
14)	‹211›	$02\vert 2{-}\vert 1{-}_{out}$	(1, 1)
15)	‹220›	$01\vert 10\vert{-}{-}_{out}$	(0, 0)
16)	‹220›	$02\vert 10\vert{-}{-}_{out}$	(0, 1)
17)	‹220›	$02\vert 20\vert{-}{-}_{out}$	(0, 0)
18)	‹220›	$02\vert 21\vert{-}{-}_{out}$	(0, 1)
19)	‹222›	$02\vert 21\vert 10_{out}$	(0, 0)



Reference.

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