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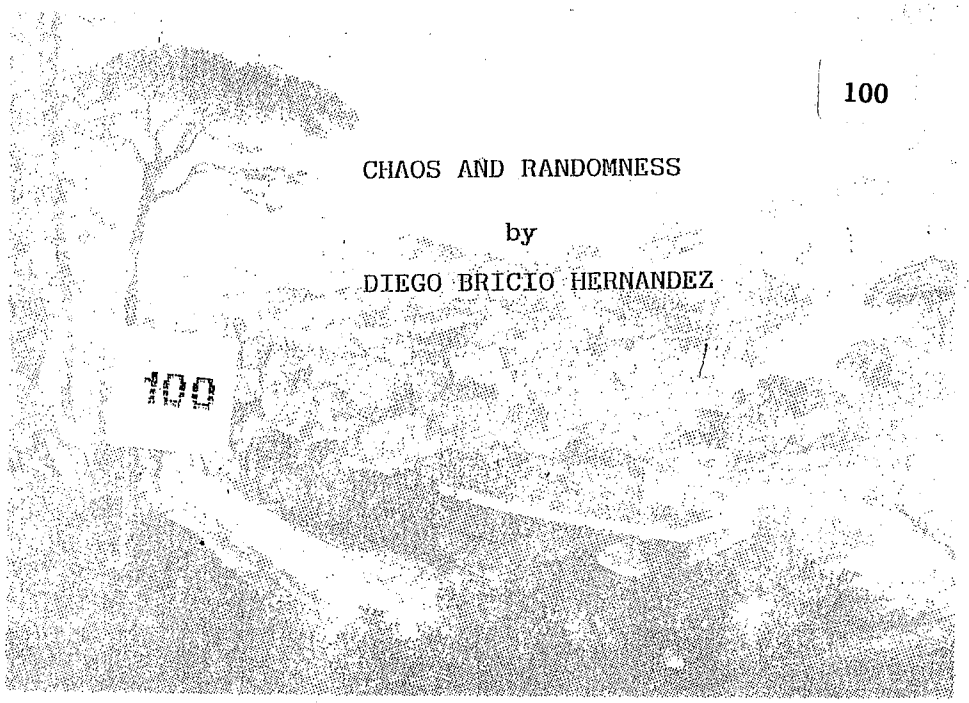

CHAOS AND RANDOMNESS

by

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ABSTRACT

This paper develops the idea that deterministic dynamical systems can exhibit behaviour that passes various tests of randomness. In addition, it purports to explain the origin of probabilistic description in Mechanics, as a means to simplify the enormously complex classical description of very large scale systems. Finally, these two complementary views are integrated in the study of thermal fluctuations, first by means of a Langevin equation, then by means of a purely deterministic dynamical systems exhibiting random behaviour.

1. Introduction

For centuries, Physics stood as the mainstay of determinism, with Classical Mechanics as the clearest cut deterministic paradigm. Mechanical phenomena would be classically described in terms of smooth functions satisfying ordinary differential equations, each such function specifying a trajectory in the phase space Γ (an open subset of some $\mathbb{R}^d$) associated with the particle system under consideration. The deterministic character of such phenomena would be reflected in an existence and uniqueness theorem for the initial value problem associated with the corresponding differential equation in Γ:

$$\dot{x} = f(x) \qquad x(0)=x_0 \tag{1}$$

Let f be Lipschitz continuous in Γ. Then, given $x_0 \in \Gamma$, there is a positive number δ and a unique trajectory $\phi: (-\delta,\delta) \to \Gamma$ solving (1).

This nice deterministic image of the world did not fare very well when systems consisting of a large number of particles began to be studied by Mechanics. Indeed, after being given a first quantitative version by D. Bernoulli in his treatise "Hydrodynamica", published in 1738, kinetic theory was reinvented in the mid 1800's by Maxwell and Boltzmann [Brush, 1976, Book 1]. The purpose of such effort was to explain the macroscopically observed behaviour of substances in terms of the then revolutionary molecular theory of matter, acording to which *matter consists of an enormously large numbers of particles (atoms, molecules) bound together by the interatomic or intermolecular forces, as the case may be; matter manifests as a gas, a liquid or a solid, depending on the intensity of those forces.* Macroscopically, this molecular assembly manifests itself in terms of measurable properties like pressure, volume, temperature and the like. In the kinetic approach to the study of thermal phenomena, the macroscopic observables are obtained from microscopic dynamical quantities by taking averages on the latter. Thus, at least in principle this approach calls for a mechanical study of extremely large assemblies of particles.

Typically, the number of particles that must be considered is of the order of Avogadro's number ($N_{Av} = 6.023 \times 10^{23}$ molecules/g. mole). Note that solving initial value problems like (1) requires knowing the initial phase (position and velocity) of each particle in the asssembly. This is clearly out of the question. Even if such information was available, integrating the equations of motion in order to obtain the corresponding trajectory in phase space would exceed the capabilities of the fastest computers. Statistical mechanics was conceived by Maxwell and Boltzmann in order to circumvent this difficulty. For a discussion on the foundations of the statistical approach in Mechanics, see Chapter 10 of [Brush, 1976], also [Ehrenfest-Ehrenfest, 1959].

On the other hand, the particle systems of kinetic theory are large scale systems in the modern sense of the term [Siljak, 1978], and the corresponding mathematical description (1) could be

conceivably dealt with using any of the various simplification procedures available in the literature on large scale systems.

An often followed simplification strategy [Hernández, 1987] involves splitting the phase x of the particle system into two components: a *slow* component y and *fast* one η, and then rewrite the dynamical equations in the form

$$\dot{y}=g(y,\eta), \qquad \epsilon\dot{\eta}=h(y,\eta), \tag{2}$$

where ϵ is a *small parameter*. Neglecting the fast dynamics amounts to replacing (2) by the simpler description

$$\dot{y}=g(y,\eta), \qquad h(y,\eta)=0 \tag{3}$$

In addition, the decomposition leading to (2) is assumed to be done in such a way that y is of "low"dimension, hance the differential part of (3) consists of "just a few " differential equations.

If, moreover, the additional assumptions are made that a) the fast dynamics can be broken up into two components, say as in

$$g(y,\eta)= b(y) + c(\eta),$$

abd b) the second term on the right hand side is replaced by an *additive noise* term $\sigma\xi$, there results the simpler dynamical description

$$\dot{y}=b(y) + \sigma\xi, \quad y(0)=y_0 \tag{4}$$

Notice that the algebraic part of (3) has been neglected, the effect of the fast components being replaced by the random additive noise term. In (4), σ is a (scalar, vector or matrix) parameter representing the noise intensity, and ξ is some standard time-varying noise term.

Contrary to the fully deterministic picture surrounding the

initial value problems (1) of classical mechanics, initial value problems like (4) offer a much less ambitious description of the world. Indeed, instead of aiming at predicting the value of the slow variables y t seconds after the start of the motion, predictions now take the form *the probability of* y(t) *lying in some set* A *given* y(0) *is so and so.*

Description like (4) have been fruitfully applied in statistical physics [Beck-Roepstorff, 1987], under the denomination of *Langevin equation.* A special case thereof is given ample coverage in section 5 below (*thermal fluctuations*). Thus, large assemblies of particles can indeed be treated by Mechanics, but at the cost of giving up determinism. See [Kac, 1959] for a comprehensive presentation of probabilistic methods and reasoning in the physical sciences; see also section 3 of this paper in connection with the role of probability in particle mechanics.

Section 2 below deals with the concept of dynamical system, as conceived in mathematical system theory [Kalman et al., 1969], centered around the notions of *input, output* and *state.* In particular, the particle systems of classical mechanics are dynamical systems without inputs, and whose state space coincides with Γ, the phase space. An alternative simplification of the large scale description (1) is based on a different splitting of the state vector x, this time into an *observable* part y and a *unobservable* part z. In addition, the assumption is made that a) the dynamics of the observable part follows closely that of z (the new state), and b) the observable component (the new output) does not affect the state dynamics. Thus one gets a simplified version of (1) different form (3), of the form

$$\dot{z}=\phi(z), \qquad y=\psi(z) \tag{5}$$

Now, dynamical systems such as (5) have been devised whose output y behaves in a way clearly resembling the behaviour of nondeterministic systems like tossed coins, see [Taylor, 1987]. See also [Galgani et al., 1988], where macroscopic charged particles in a radiation field (a deterministic dynamical system) are shown to

undergo a highly fluctuating motion, very random in appearance. More classically, deterministic dynamical systems evolving in discrete time are routinely used in order to generate sequences that pass various randomness tests. Indeed, all modern computing systems are provided with such random number generators (see section 4 below), and a whole branch of computational probability is concerned with devising means for transforming those computer generated random sequences into samples from given distributions [Knuth, 1981]. Thus, deterministic dynamical systems can exhibit apparently random behaviour. Section 5 below ends with a presentation of one such system, proposed in [Beck-Roepstorff, 1987] in order to detail the structure of the fluctuating term ξ in the Langevin equation (4), and which qualifies as a *random trajectory generator*.

2. Dynamical Systems

Under deterministic conditions, the purpose of modelling is to provide a means to predict future behaviour in terms of a set of present conditions. The most general form of a deterministic model is a dynamical system [Kalman et al, 1969]

$$<T, U, X, Y, \mathcal{U}, \delta, \lambda>$$

where, typically,

a) T is either an interval (continuous time) or a set of integers (discrete time). The elements of T are known as *instants*.

b) U, X and Y are nonempty sets. The elements of U are known as *input values*, whereas those of X and Y are the *states* and *output values*, respectively.

c) $\mathcal{U}$ is a set of functions form T into U, each $\gamma \in \mathcal{U}$ being referred to as an *admissible input trajectory*. $\delta: \Delta \times X \times U \to X$ is the *state transition map, where* $\Delta := \left\{(s,t) \in T^2 : s \leq t\right\}$. It is such that

$$\delta(s,t,x,\gamma') = \delta(s,t,x,\gamma'') \text{ if } \gamma'|_{[s,t]} = \gamma''|_{[s,t]}. \tag{1a}$$

$$\delta(t,t,x,\gamma) = x \tag{1b}$$

Here, $\delta(s,t,x,\gamma)$ is the state arrived at at time t if the state was x at time s and input trajectory γ was applied over the time interval [s,t].

Note that, given (s,r), $(r,t)\in\Delta$, $\gamma\in\mathcal{U}$, $x\in X$, it necesarily follows that $(s,t)\in\Delta$ and

$$\delta(r,t,\delta(s,r,x,\gamma),\gamma) = \delta(s,t,x,\gamma), \tag{2}$$

lest determinism be violated.

Finally

e) $\lambda: X \to Y$ is the *read - out map*, namely $\lambda(x)$ is the output value observed when the state is x.

To simplify matters somewhat, consider the special case in wich the state is observable ($X = Y$ and λ is the identity map) and $\mathcal{U}$ consists of only one trajectory λ. Then, the dynamical system is identified by simply specifying $<T,X,\delta>$, with $\delta:\Delta\times X \to X$ and (2) specializes into

$$\delta(r,t,\delta(s,r,x)) = \delta(s,t,x) \tag{3}$$

More specifically, consider the situation in which the state transition map is *time homogeneous* i. e. T is closed under addition and

$$\delta(s+h,t+h,x) = \delta(s,t,x) \quad \forall\ h \in T$$

whenever $(s,t) \in \Delta$, $x \in X$. Then,

$$\delta(s,t,x) = \delta(0,t-s,x)$$

and it is convenient to simplify the notation as follows: for each $t \in T$, let $P_t: X \to X$ be defined by

$$P_t x: = \delta(0,t,x)$$

Then (3) transforms into

$$P_s P_t x = P_{s+t} x \quad \forall\, x \in X$$

i. e.

$$P_s P_t = P_{s+t}, \quad s, t \in T \tag{4}$$

and $\{P_t, t \in T\}$ is said to constitute a *semigroup* of operators on X. Note that T is required to be closed under addition in order to write (4). The semigroup property is to be taken as the mathematical counterpart of determinism.

Randomness is said to occur whenever observed behaviour is so erratic that no reproducibility of results can be expected. In other words, observations do not support a deterministic analysis of the situation at hand.

When randomnes is detected, the ultimate aim of the modelling activity has to be lowered somewhat. Instead of predictions like *P_t x is the state arrived at t seconds after starting from state x*, one has to content oneself with weaker predictions of the sort of *the probability of arriving at a state in A t seconds after starting from state x is $\pi_t(x, A)$, a number in [0,1]*, for each set $A \in \mathcal{A}$, with $\mathcal{A}$ a σ-algebra of subsets of X. Thus, X has to be endowed with a measurable structure and a family $\{\pi_t, t \in T\}$ of transition probabilities replaces the semigroup $\{P_t, t \in T\}$.

Note, however, that this family of transition probabilities defines a family $\{P_t,\ t \in T\}$ of operators acting on M, the family of all bounded measurable functions $f: X \to \mathbb{R}$. Thus

$$P_t f(x) := \int_X f(y) \pi_t(x, dy),$$

but it is not always true that $\{P_t, t \in T\}$ constitutes a semigroup. When it does, the semigroup is said to be *Markovian*, and determinism is recovered, at the price of changing the state space (M instead of X). See [Rosenblatt, 1971].

On the other hand, *chaos* is an apparently random behaviour

observed in certain dynamical systems whose orbits exhibit a rather erratic behaviour [Devaney, 1987]. Thus, a deterministic analysis may also lead to apparently random behaviour, of the kind commonly modelled in probabilistic terms.

In connection with chaos, one normally considers discrete time dynamical systems i.e. $T = \{0,1,2,\ldots\}$, so that

$$P_n x = f^{(n)}(x)$$

i.e. the transformations in the semigroup are the iterates of a certain f ($=P_1$). By an abuse of terminology, f itself is commonly refered to as a *dynamical system*. The *orbit* of an element $x \in X$ under a dynamical system f is the sequence of iterates

$$x,\ f(x),\ f^{(2)}(x)\ \ldots$$

Let us examine in what follows the notions of chaos and randomness as encountered in some classical physical systems.

3. Probability in particle mechanics.

Consider a system made up of N point particles of masses $m_1,\ldots,m_n$, interacting according to a potential function V, a smooth function of the 3N spatial coordinales $q_1,\ldots,q_n$ specifying a configuration of this particle systems.

Let $p_1,\ldots,p_N$ be the individual momenta of these particles, whose kinetic energies can then be defined as

$$\frac{\|p_i\|^2}{2m_i}\ (i=1,\ldots,N).$$

Here and in what follows $\|\cdot\|$ denotes the euclidean norm in an appropiate euclidean space. The *hamiltonian* (or *total energy*) function associated with this particle system is defined as

$$H(q,p) = \sum_{i=1}^{N} \frac{\|p_i\|^2}{2m_i} + V(q)$$

where we have put $q:=(q_1,\ldots,q_N)$, $p:=(p_1,\ldots,p_N)$. Each $(q,p)\in\mathbb{R}^{6N}$ will be called a *phase* of the system; let $\Gamma:=\mathbb{R}^{6N}$ denote *phase space*.

According to the Hamiltonian formulation of Classical Mechanics [Arnold, 1978], the physical motions of this particle system correspond to the solutions of the system of 6N ordinary differential quations.

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{1}$$

Let us assume that H is smooth enough that (1) has a unique solution $t \mapsto (\phi(t), \psi(t))$ defined for all $t \in \mathbb{R}$ for each initial phase (q, p). Thus (1) defines a dynamical system with $T = \mathbb{R}$, Γ as state space and associated semigroup given by

$$P_t(q,p) = \big(\phi(t), \psi(t)\big).$$

Note that the semigroup property of $\{P_t, t \in \mathbb{R}\}$ reflects the existence and uniqueness of solutions of the initial value problem associated with (1). Note also that $\{P_t, t \in \mathbb{R}\}$ is in fact a *group*, with $P_t^{-1} = P_{-t}$ for each $t \in \mathbb{R}$, since

$$P_{-t}P_t = P_tP_{-t} = P_o = I.$$

The group property of $\{P_t, t \in \mathbb{R}\}$ embodies the *reversible* character of each of the motions the dynamical system (1) can undergo.

Now, by the chain rule,

$$\frac{d}{dt}H\big(\phi(t), \psi(t)\big) = \frac{\partial H}{\partial q} \cdot \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \cdot \left(-\frac{\partial H}{\partial q}\right) = 0,$$

all partial derivatives on the right hand side being evaluated at $(\phi(t), \psi(t))$. Then, *total energy is conserved along the trajectories of this dynamical system.*

For each Borel subset B of Γ, let P_tB denote its image under P_t, for each $t \in \mathbb{R}$. Liouville's theorem [Arnold, 1978] asserts that

$$|P_tB| = |B| \quad \forall\ B \in B(\Gamma),$$

where $|\cdot|$ stands for Lebesgue measure in $\mathbb{R}^{6N}$. Thus, the flow in Γ induced by (1) is conservative also in this sense: *volume in phase space is conserved by the flow induced by (1).*

Now, by the conservation of total energy, the flow can be restricted to any hypersurface of constant energy like $M_o := \{(q,p)\in\Gamma: H(q,p) = E_o\}$. It is possible to endow M_o with a measure μ (defined on $B(M_o)$, the Borel subsets of M_o in the relative topology as a subset of Γ) which is preserved by the flow in M_o, i.e.

$$\mu(P_t B) = \mu(B) \quad \forall\ B\in B(M_o).$$

Indeed, such measure is given by [Khinchin, 1949].

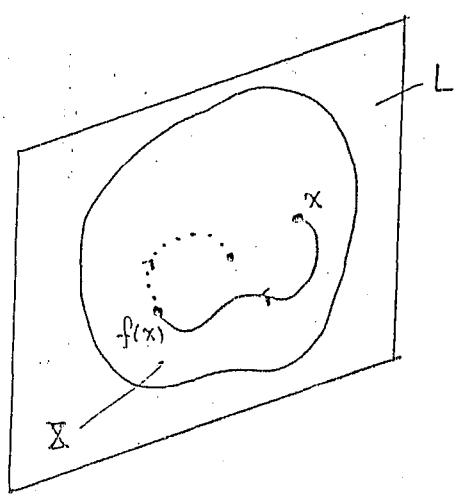
$$\mu(A) = c\int_A \frac{d\sigma}{\|\nabla H\|}$$

where c is a positive normalizing constant i.e. $\mu(M_o) = 1$, μ is a probability measure. It suffices that H goes to infinity with its arguments in order to guarantee that M_o is compact, and therefore $P_t : M_o \to M_o$, is indeed defined for all $t\in\mathbb{R}$. Moreover, it suffices then that $\nabla H\neq 0$ on Γ to quarantee that μ is well defined. In the physical literature [Thompson, 1972] the probability space $(M_o, B(M_o), \mu)$ is refered to as *Gibbs' microcanonical ensemble.*

Classical Mechanics provides other invariants of the motion: the three components of linear momentum, those of angular momentum, etc. Together with total energy, they specify a compact invariant manifold M, with $\dim(M) =: m < 6N$.

For systems of physical interest, m will not be much smaller than 6N, thus precluding the determination of $\{P_t, t\in\mathbb{R}\}$ on the basis of a sufficiently numerous family of invariants of the motion. For instance, in the study of liquids and gases N will be enormously large, of the order of Avogadro's number ($N = 6.023 \times 10^{23}$), then there is no hope to explicitly solve the equations of motion in order to determine the group.

Spurred by this difficulty, Poincaré suggested the use of sections in order to study the motions of such dynamical systems: a *section* is a (low dimensional) linear manifold L, whose intersection with M will be denoted by X. A mapping $f: X \to X$ can then be defined as follows:



For each $x \in X$, consider the corresponding trajectory $t \mapsto P_t x$. Let $\tau(x) := \inf \{t>0: P_t x \in X\}$ and let $f: X \to X$ be given by

$$f(x) = P_{\tau(x)}(x),$$

provided $\tau(x) < +\infty$.

Assume for an instant that

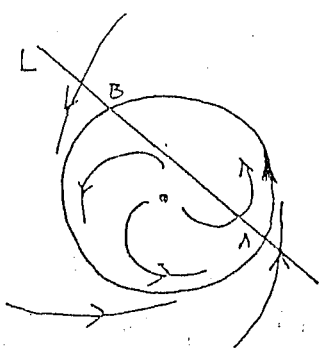
$$\tau(x) < +\infty \quad \forall x \in X, \tag{2}$$

so that f is indeed defined everywhere in X. Then a dynamical system has been defined in X, the orbit of an element $x \in X$ being

$$x,\ f(x),\ f^{(2)}(x),\ \ldots \tag{3}$$

For the sake of illustration, consider the dynamical systems in $\mathbb{R}^2$ specifed by the ordinary differential equations in polar coordinates

$$\dot{r} = r(1-r),\ \dot{\theta} = 1. \tag{4}$$



The phase portait of the system is easily shown to consist of one unstable critical point (the origin) and a stable limit cycle (the unit circle) Let L be any straight line transversal to the unit circle (L is a Poincaré section).

The orbit of an arbitrary $p \in L$ consists of a sequence of points in L, say $\{P_n\}$, with both A and B as limit points.

To apply the method, condition (2) must be guaranteed. The non triviality of this task even in the simplest cases is apparent in the foregoing example (4). In practice this requirement may render this method rather difficult to apply. However, when it can be applied this method may lead to rather suprising results.

For, observe that a Poincaré section L must have low dimension (say n) to allow analysis. The coordinates identifing the points of L are the actual observables, and it is reasonable to expect a substantial loss of information in passing from a description in terms of the phase space (or the invariant manifold M) to a description in terms of the Poincaré sections L or, rather, of its intersection X with M. Be it as it may, the experimenter is constrained to accept this state of affairs.

An observer, unaware of all the underlying dynamical structure given by (1), observes x and indeed its whole orbit (3). In fact, the observer observes only finitety many elements of such orbit (say r of them), but one may conceivably think of observing the whole orbit.

An approach often followed in the physical sciences consists of partitioning X into subsets, generically denoted by A, and then compute the relative frequencies

$$f_r(A) := \frac{\#\{i \le r:\ f^{(i)}x \in A\}}{r}$$

It is an empirical finding that, as r grows beyond bounds, the values of $f_r(A)$ tend to stabilize about a fixed value, depending only on A, and to be denoted by $\pi(A)$. Thus, $\pi(A)$ is *the probability of finding the systems in some phase in A*, i.e.

$$\pi(A) = \lim_{r \to \infty} \frac{\#\{i \leq r : f^{(i)}(x) \in A\}}{r}$$

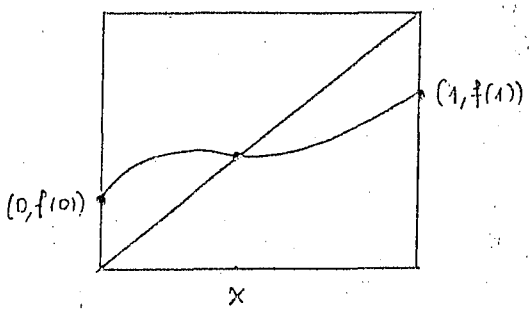
If these measurements are repeated for every set A in a σ-algebra $\mathcal{A}$ of subsets of X, there will result a *probability space*

$$(X, \mathcal{A}, \pi),$$

a nondeterministic model for such a situation. And the experimenter who built this model will remain unaware of the underlying deterministic structure present in the situation under study.

4. Random number generation.

Let X denote the unit interval and let $f: X \to X$ be a dynamical system acting on X.



An elementary topological argument shows that there is at least one $x \in X$ such that $f(x)=x$ if f is continuous. Such x is called a *fixed point* of f.

It is well known that such fixed point is unique if f is a *contraction* of X. i.e. if

$$|f(x)-f(y)| \leq \alpha|x-y|$$

with $0 \leq \alpha < 1$. Moreover, in such a case the unique fixed point is a *global attractor*. In other words, for each $x \in X$, the corresponding orbit $x, f(x), f^{(2)}(x), \ldots$ converges to X.

Clearly such a behaviour is not exactly what one would like to call random i.e. the orbit $\{f^{(n)}(x)\}$ does not constitute a random sequence, whichever concepts of randomnes is chosen. Intuitively, one would like to have dynamical systems f whose orbits fill up the whole of X, if posible in a uniform fashion; such orbits could conceivably be called random sequences.

To make things precise, let's say that sequence $\{x_n\}$ is *equidistributed* in X if

$$\lim_{n\to\infty} \frac{\#\{i\le n: x_i \in [a,b]\}}{n} = b-a,$$

for any subinterval [a,b] of X.

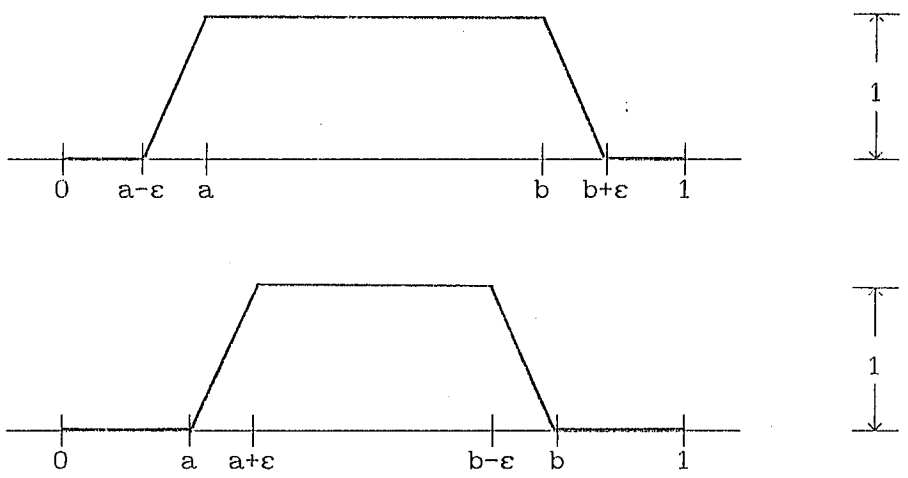
The following result of H. Weyl shows that such sequences do exist. Its proof has been adapted from [Freiberger-Grenander, 1971].

Theorem.

Let a be a positive irrational and let $u_i=\{ia\}$, $i=0,1,2,\ldots$, *where* $\{\cdot\}$ *is the fractional part of the argument. Then* $\{u_i\}$ *is equidistributed in* X.

Proof

Pick an interval $[a,b]\subset X$ and let χ be its characteristic function. Fix $\varepsilon > 0$ such that $0\le a-\varepsilon<a<a+\varepsilon<b-\varepsilon<b<b+\varepsilon\le 1$, and let $\chi_\varepsilon^{\pm}$ be two continuous real functions on X defined by their graphs as follows:



Then

$$\chi_\varepsilon^- \le \chi < \chi_\varepsilon^+ \text{ on } X$$

Note that the equidistributed character of $\{u_i\}$ amounts to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi(u_k) = \int_0^1 \chi(u)du.$$

Note also that if such a relation held for every continuous f, say

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(u_k) = \int_0^1 f(u)du$$

then it would hold for both $\chi_\varepsilon^{\pm}$, and a limiting argument (for $\varepsilon \longrightarrow 0$) would establish it for χ. On the other hand, any such continuons f is the unifrom limit of trigonometric polynomials, [Weiss,1965] hence it suffices to establish the above property for f of the form

$$f(x) = \sum_{k=1}^{m} a_k e^{2\pi i k x}, \quad x \in X.$$

Noting that, for such f,

$$\frac{1}{n} \sum_{j=1}^{n} f(u_j) = \sum_{k=1}^{m} a_k \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i (ja)k} = \sum_{k=1}^{m} a_k \frac{1}{n} \sum_{j=1}^{n} \left[e^{2\pi k i a}\right]^j$$

and recalling that a is irrational, one finally obtains

$$\frac{1}{n} \sum_{j=1}^{n} f(u_j) = \sum_{k=1}^{m} a_k \cdot \frac{1}{n} \frac{1 - e^{2\pi i n k a}}{1 - e^{2\pi i a}} e^{2\pi i k a} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

On the other hand

$$\int_0^1 f(u)du = \sum_{k=1}^{m} a_k \int_0^1 e^{2\pi i k u} du = 0$$

Thus proving the result. ∎

A dynamical system giving rise to the above sequence as orbit is

$$f(x) = x + a \pmod 1;$$

the orbit of $a \in [0,1]$ is precisely $\{\{ia\}, i=0,1,2,\ldots\}$, hence it is a random sequence in the sense that it fills up the unit interval in a uniform fashion i.e. the orbit is equidistributed there.

Note, however, that such erratic behaviour may not be sufficient for randomness, if this last property is understood in the sense of *absence of any systematic behaviour*. To see that consider the following example, taken from [Knuth, 1981]: let $\{U_k\}, \{V_k\}$ be both equidistributed in $[0,1]$ and therefrom construct the new sequence

$$\frac{1}{2}U_0, \frac{1}{2} + \frac{1}{2}V_0, \frac{1}{2}U_1, \frac{1}{2} + \frac{1}{2}V_1, \ldots$$

The subsequence $\{\frac{1}{2}U_k\}$ is equidistributed in $[0, \frac{1}{2}]$ whereas $\{\frac{1}{2} + \frac{1}{2}V_k\}$ is equidistributed in $[\frac{1}{2}, 1]$, hence the combined sequence is equidistributed in $[0,1]$. However, its elements jump back and forth between the left and right subintervals, thus proventing us from considering it "random". Then we want more from a sequence than its being equidistributed in order to call it "random". Following [Knuth, 1981], let us introduce the following concepts:

Definition.

Let $\{U_i\}$ be a sequence in $[0,1]$, $k \geq 1$.

a) $\{U_i\}$ is k-*equidistributed* if

$$\lim_{n\to\infty} \frac{\#\{i \leq n : U_{i+j} \in [a_j, b_j], j=0,1,\ldots,n-1\}}{n} = \prod_{j=0}^{n-1}(b_j - a_j),$$

for all choices of $0 \leq a_j < b_j \leq 1$, $j=0,1,\ldots,n-1$.

b) $\{U_i\}$ is *random* if it is k-equidistributed for each $k \geq 1$.

There are simple dynamical systems which have random orbits, as the following result shows:

Theorem [Franklin, 1963]

For each $\theta>1$, *let* $U_n = \theta^n (mod.1)$. *Then* $\{U_n\}$ *is random* θ - *a.e. with respect to Lebesgue measure. Moreover* $\{U_n\}$ *is random only if* θ *is a transcendental number.*

Thus, *with probability one* the sequences corresponding to $\{\pi^n\}$, etc. are random. There is however no certainty of their random character or absence thereof. An explicit algorithm producing random sequences was given in [Knuth, 1965].

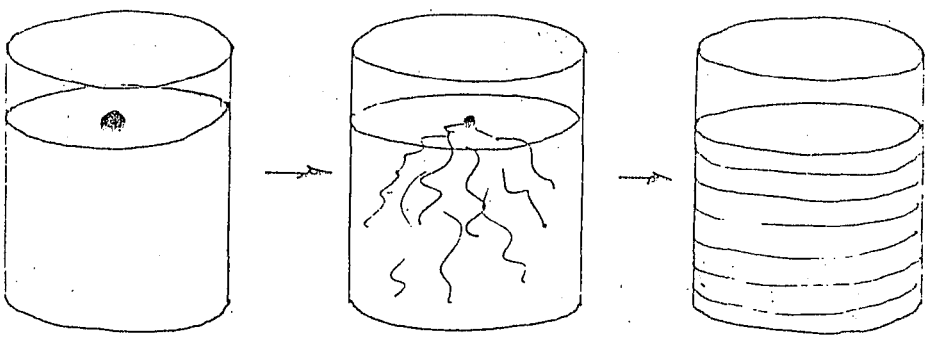
There are many simple, computationally efficient algoritms used in order to generate "random numbers", and they all share the following structure: there is a dynamical system $f: X \longrightarrow X$, and a "random sequence" of any length is generated by simply choosing a "seed" $x \in X$ and computing as many elements of its orbit $\{f^{(n)}(x)\}$ as required. See [Knuth, 1981].

The quotes above reflect the fact that the resulting sequences are not random in the above sense, but they are equidistributed for a suitable $k \geq 1$ (often $k=2$).

The foregoing remarks illustrate the point that chaotic dynamical systems can be used in order to generate randomnes in the real line. The following section will show that they can also be used in order to generate randomness in function space.

5. Thermal Fluctuations.

Let us go back to studying particle systems using the methods of Classical Mechanics, in order to model how a "heavy" particle diffuses within a surrounding fluid.



According to molecular theory, the surrounding fluid consists of an extremely large number of particles (its molecules), which move about constantly and impinge on the diffusing particle: each collision alters the linear momentum of the heavier particle and thereby determines its trajectory. Besides, the bulk fluid acts upon the particle by countering its motion i.e. it manifests itself in the form of a friction force.

To simplify matters, let us decompose the particle's motion into three orthogonal directions and analyze such motion in one of those directions only. Use x to denote the particle's position, v to denote its velocity, and assume $\phi(v)$ is the friction force in the x direction when the particle moves with velocity v. Clearly $\phi(0)=0$.

In the spirit of Statistical Mechanics, let us simplify matters further by replacing the "heat bath" provided by the molecules of the surrounding fluid by a random term, with the properties of gaussian white noise of amplitude σ. Then, in the notation of Ito's calculus [Schuss,1980] Newton's Second Law translates into

$$dv = \phi(v)dt+\sigma dW, \tag{1}$$

where $\{W(t), t\geq 0\}$ is a standard one dimensional Wiener process and we have taken the particle's mass to be unity [compare with eq. (1.4)]. Finally, the kinematics of the situation says that

$$dx = vdt. \tag{2}$$

Let us assume that ϕ is smooth enough to guarantee the existence of a unique solution $\{(X(t),V(t)), 0\leq t\leq T\}$ of (1), (2) corresponding to any initial (nonrandom) phase (x_o, v_o). This is so if, for instance

$$\phi(v) = -\beta v,$$

which is the friction term considered in the original treatment by Langevin: equation (1) transforms into

$$dv = -\beta v dt + \phi dW. \tag{3}$$

The solution to the pair (2), (3) corresponding to an initial phase (x_o, v_o) is referred to as the Ornstein-Uhlenbeck model of brownian motion-briefly, the *O-U process*.

An easy computation shows that such a process is given by

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{\beta}\left(1-e^{-\beta t}\right) \\ 0 & e^{-\beta t} \end{bmatrix} \cdot \begin{bmatrix} x_o \\ v_o \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{\sigma}{\beta}\left(1-e^{-\beta(t-s)}\right) \\ \sigma e^{-\beta(t-s)} \end{bmatrix} dW(s),$$

and the elementary properties of the stochastic integral [Schuss, 1980] show that

i) the O-U process is gaussian

ii) its mean is

$$\begin{bmatrix} x_o + \frac{v_o}{\beta}\left(1-e^{-\beta t}\right) \\ v_o\, e^{-\beta t} \end{bmatrix}$$

iii) its convariance matrix is

$$C(t) = \begin{bmatrix} \frac{\sigma^2}{\beta^2}\left(e^{-\beta t} - 1 + 2\beta t\right) & \sigma^2\left(1-e^{-2\beta t}\right) \\ \sigma^2\left(1-e^{-\beta t}\right) & \sigma^2 \end{bmatrix}$$

See [Doob, 1942], also [Nelson, 1967] for these results, originally derived in [Ornstein-Uhlenbeck, 1930] and other sources as well.

For large t

$$C(t) \sim \begin{bmatrix} \frac{\sigma^2}{\beta} t & 0 \\ 0 & \sigma^2 \end{bmatrix} \tag{4a}$$

Einstein's formula relating the amplitude of the thermal fluctuation to the extent of the dissipation is

$$\sigma^2 = 2\beta kT, \tag{5}$$

where k is Boltzmann's constant and T is absolute temperature. Then, (4) transforms into

$$C(t) \sim \begin{bmatrix} 2kTt & 0 \\ 0 & \sigma^2 \end{bmatrix} \tag{4b}$$

Then, asymptotically, the two components of the O-U process are independent and gaussian with variances 2kTt (position) and σ^2 (velocity). In other words, the limiting velocity distribution is Maxwell's, whereas the limiting position distribution is the one predicted by the simpler Einstein-Smoluchowski theory of brownian motion [Ingarden, 1986], [Einstein, 1956].

For any time instant, the trajectory of the O-U process fills-up the whole phase space of the brownian particle, hence this is a rather random physical system. Let us see that this random system can be realized as the output of a chaotic dynamical system.

For, in the spirit of the discussion leading to (1.4) let us follow [Beck-Roepstorff, 1987] and consider the O-U process as the time evolution of those dynamical variables which vary slowly while there is a much smaller (microscopic) time scale τ giving the evolution of the remaining rapid variables. This microscopic time scale would correspond to the mean collision time between the big particle and the molecules in the heat bath.

Let $\{\xi_t\}$ be a stationary sequence of random variables having zero mean and variance σ^2, defined on a common probability space $(X, \mathcal{A}, \mu)$. Suppose the particle receives at each instant $n\tau$ $(n=0,1,2,\ .\ .\ .)$ an impulse of magnitude $\sqrt{\tau}\,\xi_n$. At time t, $[t/\tau]$ impacts will have ocurred and the total impulse received will be

$$S_\tau(t) = \sqrt{\tau} \sum_{n=0}^{[t/\tau]} \xi_n$$

Newton's Second Law can now be expressed as

$$dv = \phi(v)dt + dS_\tau(t)$$

-compare with the nonlinear Langevin equation (1).

Let $f: X \longrightarrow \mathbb{R}$ be ξ_o and let μ denote its distribution. The stationary character of $\{\xi_n\}$ is equivalent to the invariance of μ under some $\mathcal{A}$-measurable transformation $\sigma: X \longrightarrow X$, i.e.

$$\mu \circ \sigma^{-1} = \mu. \tag{6}$$

In addition, $\xi_{n+1} = \xi_n \circ \sigma$, hence $\xi_n(x) = f(\sigma^n x)$, see [Hida, 1970]. Let $x_n := \sigma^n x_o$, $n = 0,1,2,\ldots$

On the other hand, in between two succesive impacts, say for $n\tau \le t < (n+1)\tau$, $n=0,1,2,\ldots$ the particle's velocity evolves according to

$$\dot{v} = \phi(v)\ , \qquad v(n\tau) = v_n \tag{7}$$

Let $g(t, v_o)$ denote the solution over $[0,\tau]$ of the above initial value problem for $n=0$. Then, the time invariance of the differential equation in (7) allows us to say that

$$\lim_{t \longrightarrow (n+1)\tau} v(t) = g(\tau, v_n)$$

At the end of the n^{th} period, the particle suffers the $(n+1)^{st}$ collision, obtaining therefrom an additional impulse of magnitude $\sqrt{\tau}\, f(x_{n+1})$. Therefore Newton's Second Law translates into

$$v_{n+1} = g(\tau, v_n) + \sqrt{\tau}\, f(x_{n+1}) \tag{8a}$$

where

$$x_{n+1} = \sigma(x_n) \tag{8b}$$

The pair of equations (8) constitutes a dynamical system acting on the space $X \times \mathbb{R}$. Note that there is nothing random in it, and yet the following result holds:

Theorem [Beck-Roepstorff, 1987].

Under some "technical conditions" on σ and f,

$$S(t) := \sqrt{\tau} \sum_{n=0}^{[t/\tau]} f(X_{n+1}) \Rightarrow \sigma W(t)$$

for $t \in [0, t]$, where "$\Rightarrow$" denotes weak convergence and

$$\sigma^2 = E\,\xi_o^2 + 2\sum_{n=1}^{\infty} E(\xi_o \xi_n)$$

This result authorizes us to use the dynamical system (8) in order to generate (approximations to) the trajectories of the stochastic differential equations (1), (2). Thus we have constructed a (deterministic) generator of random functions. To implement it, it suffices to consider any chaotic dynamical system such as

$$f(x) = 2x^2 - 1 \text{ on } [-1, +1],$$

which has an invariant measure μ given by

$$\mu(A) = \frac{1}{\pi} \int_A \frac{dx}{\sqrt{1 - x^2}},$$

assuming it satisfies the "technical conditions" referred to above. See [Beck-Roepstorff, 1987] for the details.

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