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A mapping theorem for topological sigma-compact manifolds

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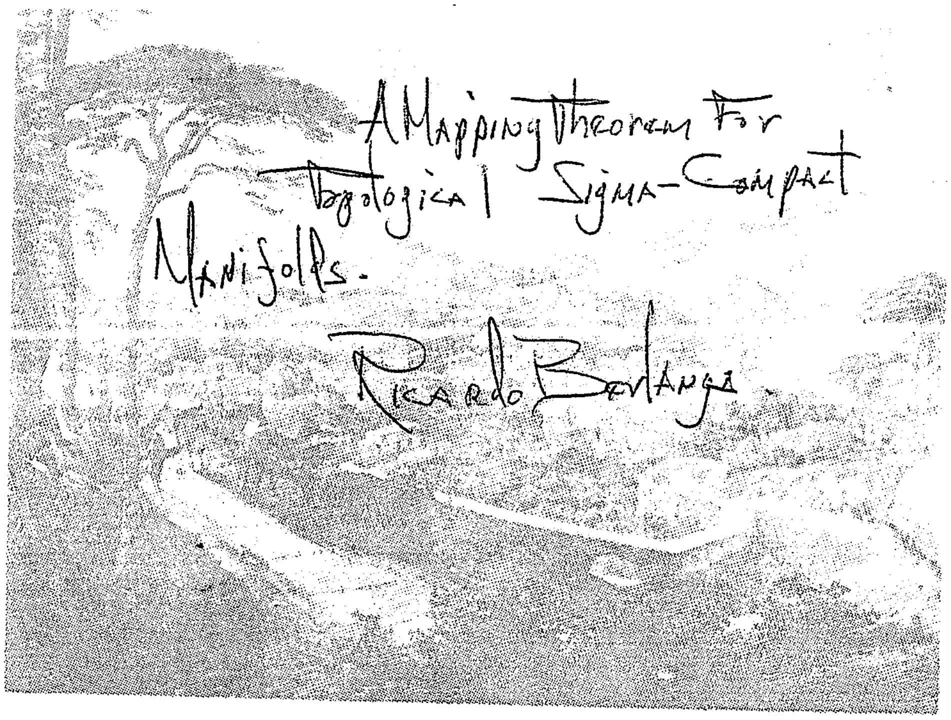

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A Mapping Theorem For Topological Sigma-Compact Manifolds.

It is the purpose of this paper to prove a generalization to σ-compact manifolds of a well known result due to M.Brown (see [4]), which asserts the existence of a special kind of continuous, "non-pathological" surjections from the unit n-dimensional cube onto a given compact connected manifold M^n.

In the more general setting when M^n is σ-compact, the space $\mathcal{E}(M)$ of ends of M^n plays an important role: Since $\mathcal{E}(M)$ is a totally disconnected, compact, metrizable space, a set E contained in the boundary of the unit cube I^n can be constructed in such a way that E is homeomorphic to $\mathcal{E}(M)$. Now $I^n \setminus E$ and M^n are two manifolds with the same set of ends. Broadly speaking, our result states that M^n is the identification space obtained from $I^n \setminus E$ by identifying points within the boundary of $I^n \setminus E$ alone.

The set $E(M)$ is empty exactly when M is compact. In this case, the arguments are reduced to those given by M. Brown for compact manifolds. Some applications are mentioned afterwards.

§ 1.- The Set of Ends.

The concept of the set of ends of a space is due to Freudenthal. Here we recall some basic notions.

Let X be a locally compact, Hausdorff space. Denote by $K(X)$ the set of all compact subsets of X partially ordered by inclusion. If $K \in K(X)$, denote by $C(X\backslash K)$ the set of connected components of $X\backslash K$ considered as a discrete topological space.

If $K, L \in K(X)$ with $K \subset L$, then there is a well defined continuous function

$$\rho_K{}^L : C(X\backslash L) \longrightarrow C(X\backslash K)$$

such that for each $V \in C(X\backslash L)$, $\rho_K{}^L(V)$ is the unique component of $X\backslash K$ containing V. In this manner, the collection

$$\{ C(X\setminus K), \rho_K{}^L \mid K, L \in \mathcal{K}(X) \text{ and } K \subset L \}$$

constitutes an inverse system of topological spaces indexed over the directed set $\mathcal{K}(X)$.

An end of X is, by definition, a point in the inverse limit space of this system. In other words, an end of X is a function e which assigns to each compact set K of X a non-empty connected component e(K) of X\K, in such a way that $K_1 \subset K_2$ implies $e(K_2) \subset e(K_1)$. Let $E(X)$ be the set of all ends. There is a topology on $X \cup E(X)$ having as a basis of neighbourhoods of $e_0 \in E(X)$ the $N_K(e_0) = e_0(K) \cup \{ \text{ends } e \mid e(K) = e_0(K) \}$, $K \in \mathcal{K}(X)$. With this topology $X \cup E(X)$ is a Hausdorff space containing $E(X)$, with its inverse limit topology, as a closed (nowhere dense) subspace.

If $f : X \to Y$ is a continuous proper function (i.e. $F \subset Y$ compact implies $f^{-1}(F)$ compact), then f is extended uniquely and continuously to a function

$$f \cup f_\epsilon \; : \; X \cup E(X) \longrightarrow Y \cup E(Y)$$

such that for $e \in E(X)$ and $F \subset Y$ compact $f_{\epsilon}(e)F$ is the (unique) component of $Y \setminus F$ containing $f(e(f^{-1}(F)))$.

Let X be a space, and let $K \in K(X)$. A connected component V of $X \setminus K$ is said to be bounded if its closure is compact, and otherwise we say that V is unbounded. Define

$$\hat{K} = X \setminus \cup \{ V \in C(X \setminus K) \mid V \text{ is unbounded} \} .$$

The proof of the following lemma may be found in Berlanga and Epstein [2].

1.1 Lemma.

Let X be a connected, locally connected, locally compact, Hausdorff space and let $K \in K(X)$. Then $X \setminus K$ has only finitely many unbounded components and $\hat{K}$ is compact.

1.2 Remark.

It follows that $E(X)$ is compact since $\hat{K}(X) = \{ \hat{K} \mid K \in K(X) \}$.

is cofinal in $K(X)$ and each $C(X\setminus\hat{K})$ is finite. It is also known that $X \cup E(X)$ is compact and that $E(X)$ is totally disconnected. Also if X is metric $X \cup E(X)$ is metrizable.

§ 2.- *Definitions.*

Let X be a subset of a topological space Y. We define $\overset{o}{X}$ and ClX to be, respectively, the topological interior and the topological closure of X in Y. Call X a (closed) n-cell if X is homeomorphic to the unit n-cube $I^n = [0, 1]^n$. For a subset X of a manifold M we define IntX to be $(M\setminus\partial M) \cap \overset{o}{X}$, where ∂M denotes the boundary of M.

§ 3.- *The Main Theorem.*

Let M^n be a connected, second countable manifold of dimension n. Then there exists a compact set $E \subset \partial I^n$ and a continuous proper surjection $\psi : I^n\setminus E \longrightarrow M$ such that

(1) $\psi|_{\text{Int } I^n} : \text{Int } I^n \longrightarrow \psi(\text{Int } I^n)$ is a homeomorphism;

(2) $\psi(\text{Int } I^n) \cap \psi(\partial I^n \setminus E) = \emptyset$;

(3) ψ extends naturally to $\tilde{\psi} : I^n \longrightarrow M \cup \mathcal{E}(M)$ in such a way that $\tilde{\psi}|_E$ is a homeomorphism from E onto $\mathcal{E}(M)$.

Furthermore, if $n \geq 2$ then E can be chosen to be contained in $[1/3, 2/3] \times \{(1/2, 1/2, \ldots, 1/2, 1)\}$.

§ 4 .- *Definitions, Lemmas and Proof of the Main Theorem.*

4.1 Definitions.

An (n-1)-dimensional submanifold B of an n-manifold M is bicollared in M if there is a homeomorphism P of $B \times \langle -1, 1 \rangle$ onto a neighbourhood of B in M such that $P(b,0) = b$, for all $b \in B$. If B is closed in M we require also that P can be extended to a closed embedding of $B \times [1, 1]$ into M.

If B is the boundary of an n-dimensional submanifold C of M, then $B \times \langle -1, 0]$ and $B \times [0, 1 \rangle$ denote the inner and outer collars of B. In general, we will not distinguish $(b,t) \in B \times \langle -1, 1 \rangle$ from $P((b,t))$.

Define $H(M)$ to be the group of homeomorphisms of M onto itself. If $h : M \longrightarrow M$ is a homeomorphism, then supp h denotes the support of h, that is, the closure of the set of points of M which are actually moved by h.

The following result, proved in Appendix 1 below, is just a straightforward generalization of lemma 2 in M. Brown [4] (or lemma 6 in Berlanga and Epstein [2]).

4.2 Lemma.

Let M^n be a manifold with $n \geq 3$ and let d be a metric on M. Let C^n be a closed n-dimensional manifold with bicollared boundary ∂C in M.

Let $\epsilon > 0$ be given and suppose $\Lambda = \{ D_j \}_{j \in J}$ is a locally finite family of sets in M such that each D_j is a closed n-cell of diameter less than $\epsilon/2$ whose interior intersects C. Let $X = \{ x_i \}_{i \in L}$ be a locally finite set of points in $\bigcup_j \operatorname{Int} D_j$.

Suppose that $0 < \gamma < 1$. Then there is an ϵ-homeomorphism f in $H(M)$ such that $f(C) \supset f(\overset{\circ}{C}) \supset C \cup X$ and

$\operatorname{supp} f \subset (\bigcup_j \operatorname{Int} D_j \setminus C) \cup \partial C \times \langle -\gamma , \gamma \rangle$. In particular, f fixes pointwise the inner n-manifold bounded by $\partial C \times \{ -\gamma \}$.

4.3 Lemma.

Let M be a connected, second countable, n-dimensional manifold with $n \geq 3$ and let $X \subset \operatorname{Int} M$ be a locally finite set of points. Then there exists a compact set
$E \subset [1/3 , 2/3] \times \{ (1/2 , 1/2 , \dots , 1/2 , 1) \} \subset \partial I^n$, and a proper embedding $\psi_\star : I^n \setminus E \longrightarrow M$ with bicollared boundary such that $\psi_\star(\operatorname{Int} I^n) \supset X$ and $\psi_\star$ extends naturally to $\tilde{\psi}_\star : I^n \longrightarrow M \cup \mathcal{E}(M)$ in such a way that $\tilde{\psi}_\star|_E$ is a homeomorphism from E onto $\mathcal{E}(M)$.

Proof. Define a clean (closed) n-cube in I^n to be a cube C of the form $[0 , \beta]^n + \nu$, for some $\beta > 0$ and $\nu \in \mathbb{R}^n$, such that $C \subset I^n$ and $C \cap \partial I^n = ([0 , \beta]^{n-1} \times \{ \beta \}) + \nu$.

Observe that if $C_1, \dots , C_k$ is a disjoint collection of clean cubes then $\operatorname{Cl}(I^n \setminus \bigcup_i C_i)$ is homeomorphic to I^n. We divide the proof in three steps.

Step 1 . Let $\{K_i\}_{i\in \mathbb{N}}$ be any collection of $K_i \in \mathcal{K}(M)$ such that $M = \bigcup_i K_i$ and $K_i \subset \overset{\circ}{K}_{i+1}$; further properties of the K_i will be specified in Step 2. It is not difficult now to define a sequence $\{L_i\}_{i\in\mathbb{N}}$ of n-cells in I^n with $L_i \subset \overset{\circ}{L}_{i+1}$ and such that

(a) The complement of $\overset{\circ}{L}_i$ is the finite disjoint union of clean cubes of diameter less or equal $1/2^i$, and such that, for each $A \in \mathcal{C}(I^n\setminus L_i)$, we have,

$A \cap [1/3 , 2/3] \times \{(1/2 , 1/2 , \ldots , 1/2 , 1)\} \neq \emptyset$.

Hence, $E = \bigcap_i I^n \setminus L_i$ is contained in

$[1/3 , 2/3] \times \{(1/2 , 1/2 , \ldots , 1/2 , 1)\}$;

(b) For each $i \in \mathbb{N}$ there exists a bijection

$\lambda_i : C(I^n\setminus L_i) \longrightarrow C(M\setminus\hat{K}_i)$ such that the diagrams

$$\begin{array}{ccc} C(I^n\setminus L_j) & \xrightarrow{\lambda_j} & C(M\setminus\hat{K}_j) \\ \rho_i^j \big\downarrow & & \big\downarrow \rho_i^j \\ C(I^n\setminus L_i) & \xrightarrow{\lambda_i} & C(M\setminus\hat{K}_i) \end{array}$$

commute ($i < j$).

The reader can readily verify the following assertion:

Assertion :

$E = \cap_i I^n \setminus L_i$, $\mathcal{E}(I^n \setminus E)$ and $\mathcal{E}(M)$ are homeomorphic. Furthermore, the identity map $I^n \setminus E \longrightarrow I^n \setminus E$ extends naturally to a homeomorphism of $I^n = (I^n \setminus E) \cup E$ onto $(I^n \setminus E) \cup \mathcal{E}(I^n \setminus E)$.

Before proceeding into Step 2 of this lemma, the reader may refer to **Appendix 2** for a preliminary discussion.

Step 2. Let the K_i be constructed as to satisfy also the following properties:

$\overset{o}{K}_i$ is connected;

$M \setminus \hat{K}_i$ has exactly the same number of components as $\overset{o}{K}_{i+1} \setminus \hat{K}_i$.

If $\psi_0 : I^n \longrightarrow \overset{o}{K}_0$ is an embedding with bicollared boundary, then there exists a homeomorphism h_1 of M with compact support such that

(1) supp $h_1 \cap \psi_0(L_0) = \emptyset$;

(2) supp $h_1 \subset \mathring{K}_1$;

(3) If $A \in C(I^n \setminus L_1)$ then

(a) $h_1(\psi_0(A)) \subset \lambda_0(\rho_0^{1}(A))$;

(b) $h_1(\psi_0(A))$ and $\lambda_1(A)$ are not separated

in M by $h_1(\psi_0(I^n \setminus A)) \cup \hat{K}_0$, (that is, $h_1(\psi_0(A))$ and $\lambda_1(A)$ lie in the same connected component of $M \setminus (h_1(\psi_0(I^n \setminus A)) \cup \hat{K}_0)$.

Proof. Let $A_1, A_2, \ldots, A_k$ be the components of $I^n \setminus L_1$. It is not difficult to construct a family of disjoint arcs, say $\{\gamma_i : [0, 2] \to M \mid 1 \le i \le k\}$ and a family $\{U_i \mid 1 \le i \le k\}$ of disjoint connected open sets in $\mathring{K}_1$ such that, for each i,

$U_i \cap \psi_0(I^n) \subset \psi_0(A_i)$;

$\gamma_i([0, 1]) \subset U_i$;

$\gamma_i([1, 2]) \subset \mathring{K}_2 \setminus \hat{K}_0$;

$\gamma_i(0) \in \psi_0(A_i)$;

$\gamma_i(1) \in \lambda_0(\rho_0^{1}(A_i))$;

$\gamma_i(2) \in \lambda_1(A_i)$.

This can be done because $M \setminus \psi_0 (I^n)$ is connected and an n-dimensional manifold cannot be disconnected by a set of dimension n-2 (see Hurewicz and Wallman [5]).

Since the group of compactly supported homeomorphisms of a connected manifold acts transitively on interior points, we can find, for each i, a homeomorphism $h_{1,i}$ compactly supported on U_i which sends $\gamma_i(0)$ to $\gamma_i(1)$.

For each $i = 1, 2, \ldots, k$, let $\tau_i \in [1,2]$ be the last parameter such that its image under γ_i lies in $h_{1,i} (\psi_0 (I^n))$.

Consequently, there is a unique x_i in $\partial I^n \cap A_i$ with $\gamma_i (\tau_i) = h_{1,i} (\psi_0 (x_i))$. Now choose a clean closed cube B_i such that $x_i \in B_i \subset A_i$ and $h_{1,i}(\psi_0 (B_i)) \subset \lambda_0 (\rho_0^1 (A_i))$.

With a homeomorphism of M sending $\psi_0 (I^n)$ onto itself and supported in a small neighbourhood of $\psi_0 (\mathrm{Cl}\, A_i)$ we can shrink $\psi_0 (\mathrm{Cl}\, A_i)$ onto $\psi_0 (B_i)$ before applying $h_{1,i}$. Therefore, without loss of generality we can assume that $A_i = B_i$ and that $\mathrm{supp}\, h_{1,i} \cap \mathrm{supp}\, h_{1,j} = \emptyset$ for $i \neq j$. Hence,

$h_{1,i}(\psi_0(A_i)) \subset \lambda_0(\rho_0^1(A_i))$ and $h_{1,i}(\psi_0(A_i))$, $\lambda_1(A_i)$ are not separated in M by $h_{1,i}(\psi_0(I^n \setminus A_i)) \cup \hat{K}_0$.

Finally, the homeomorphism $h_1 = h_{1,1} \circ h_{1,2} \circ \ldots \circ h_{1,k}$ has the required properties.

Step 3 . By induction, we can construct a sequence $\{h_i\}_{i \in \mathbb{N}}$ of homeomorphisms with compact support such that, for each i,

(1) $\operatorname{supp} h_{i+1} \cap (h_i \circ h_{i-1} \circ \ldots \circ h_1 \circ \psi_0(L_i)) = \emptyset$;

(2) $\operatorname{supp} h_{i+1} \subset \overset{o}{K}_{i+1}$;

(3) $\operatorname{supp} h_{i+1} \cap \hat{K}_{i-1} = \emptyset$;

(4) If $A \in C(I^n \setminus L_{i+1})$ then

(a) $h_{i+1} \circ h_i \circ \ldots \circ h_1 \circ \psi_0(A) \subset \lambda_i(\rho_i^{i+1}(A))$;

(b) $h_{i+1} \circ h_i \circ \ldots \circ h_1 \circ \psi_0(A)$ and $\lambda_{i+1}(A)$ are

not separated in M by $h_{i+1} \circ h_i \circ \ldots \circ h_1 \circ \psi_0(I^n \setminus A) \cup \hat{K}_i$.

Define $\psi_i = h_i \circ \ldots \circ h_1 \circ \psi_0$, $i \in \mathbb{N}$. Therefore, the following properties hold:

(5) $\psi_i \mid_{L_i} = \psi_{i+k} \mid_{L_i}$ for all $i, k \in \mathbb{N}$;

(6) $\psi_{i+k}(A) \subset \lambda_i(\rho_i^{i+1}(A))$ for all $i \in \mathbb{N}$,

$k \in \mathbb{N}\setminus\{0\}$, and all $A \in C(I^n \setminus L_{i+1})$. It follows that $\lim_{i\to\infty} \psi_i = \psi_\star$ exists in $\bigcup_i L_i$ and is such that

(7) $\psi_\star|_{L_i} = \psi_i|_{L_i}$ for all $i \in \mathbb{N}$;

(8) $\psi_\star(A) \subset \lambda_i(\rho_i^{i+1}(A))$ for all $i \in \mathbb{N}$ and all $A \in C((I^n \setminus E) \setminus L_{i+1})$;

(9) $\psi_\star^{-1}(\hat{K}_i) \subset L_{i+1}$.

Property (7) says that $\psi_\star$ is continuous and injective. Property (9) (which follows from (8)) tells us that $\psi_\star : I^n \setminus E \longrightarrow M$ is proper, and therefore induces a map

$\psi_\star \cup \psi_\epsilon : (I^n \setminus E) \cup \mathcal{E}(I^n \setminus E) = I^n \longrightarrow M \cup \mathcal{E}(M)$ such that if e is an end of $I^n \setminus E$, $\psi_\epsilon(e)\ \hat{K}_i$ is the component of $M \setminus \hat{K}_i$ containing $\psi_\star(e(\psi_\star^{-1}(\hat{K}_i)))$, hence, by (9), it is equal to the component of $M \setminus \hat{K}_i$ containing $\psi_\star(e(L_{i+1}))$, but, by (8), this is just $\lambda_i(\rho_i^{i+1}(e(L_{i+1}))) = \lambda_i(e(L_i))$. That is, we have proved that the following diagram commutes:

$$\begin{array}{ccc} E(I^n \setminus E) & \xrightarrow{\Pi_i} & C(I^n \setminus L_i) \\ \Big\downarrow \psi_\epsilon & & \Big\downarrow \lambda_i \\ E(M) & \xrightarrow{\Pi_i} & C(M \setminus \hat{K}_i) \end{array}$$

Since each λ_i is bijective, ψ_ϵ must be a homeomorphism. Therefore, we have constructed a proper embedding $\psi_\star : I^n \setminus E \longrightarrow M$ inducing a homeomorphism on ends.

In order to complete the proof of lemma 4.3 we need to produce a bicollar of $\psi_\star(\partial I^n \setminus E)$ and we need to "expand" the image C of $I^n \setminus E$ in M as to contain X in its interior.

Let E' be the projection of E into I^{n-1}, so $E = E' \times \{1\}$. It is not difficult to see that the spaces $W = [-1,2]^n \setminus (E' \times [1/2,2])$ and $T = W \cap I^n$ are homeomorphic to $I^n \setminus E$ and that the inclusion map $T \longrightarrow W$ is a proper map inducing a homeomorphism on ends.

Therefore without loss of generality, we can assume that the domain of the map $\psi_\star$ is W. But now $\psi_\star|_T$ has the same properties of $\psi_\star$ with the advantage that ∂T has a natural bicollar contained in M.

It now only remains to "expand" the image of $\psi_\star$. To this purpose we can construct a locally finite family $\Lambda_0 = \{ D_j \}_{j \in J}$ of closed n-cells such that X is contained in $\bigcup_j \text{Int}\, D_j$ and $\text{Int}\, D_j \cap C \neq \emptyset$ for all $j \in J$.

Therefore, by an application of lemma 4.2, say with $\gamma = 1/2$ and $\epsilon = \infty$, we get the desired expansion.

4.4 Proof of the main theorem.

When the dimension of the manifold M is less or equal two, the theorem follows from the classification of second countable manifolds of dimensions one and two (see Ahlfors and Sario [1]).

Assume now that the dimension of M is greater or equal to three. Let d be a complete metric on M. Let Λ_1 , Λ_2 , ... be a sequence of locally finite covers of M such that each element of Λ_i

is a closed n-cell of diameter less than $1/2^{i+1}$ and $\mathrm{Int}\, M = \cup \{ \mathrm{Int}\, D \mid D \in \Lambda_i \}$. For each i, let X_i be a locally finite set of points such that $X_i \subset \mathrm{Int}\, M$ and $\mathrm{Int}\, D \cap X_i \neq \emptyset$ if $D \in \Lambda_i$.

Let C_1 be the image under $\psi_\star$ where $\psi_\star$ is the embedding given by the above lemma, and assume that $X_1 \subset \mathrm{Int}\, C_1$. Applying Lemma 4.2 with $X = X_2$, $\Lambda = \Lambda_1$ and γ small, we get a 1/2-homeomorphism f_1 of M onto itself such that

$$M \supset C_2 = f_1(C_1) \supset f_1(\overset{o}{C_1}) \supset C_1 \cup X_2 \text{ and } f_1|_{(1-\gamma)C_1} = \mathrm{Id},$$

where $(1-\gamma)C_1 = C_1 \setminus \partial C_1 \times \langle -\gamma, 0]$.

Repeated applications of 4.2 give a sequence $f_1, f_2, \ldots$ of homeomorphisms of M such that for each $m \in \mathbb{N}\setminus\{0\}$,

f_m is a $(1/2)^m$-homeomorphism;

$$M \supset f_m \circ \ldots \circ f_1(C_1) \supset f_m \circ \ldots \circ f_1(\overset{o}{C_1}) \supset C_1 \cup \cup_i \{ X_i \mid 1 \leq i \leq m+1 \};$$

f_{m+1} restricted to $f_m \circ \ldots \circ f_1((1-\gamma/2^m)C_1)$ is the identity.

Clearly $f_m \circ \ldots \circ f_1$ converges to a map ψ such that

$$\psi(C_1) = \lim_{m\to\infty} f_m \circ \ldots \circ f_1 (C_1) = M;$$

ψ is a homeomorphism on $\overset{o}{C}_1$;

$$\psi^{-1}(\psi(\partial C_1)) = M \setminus \overset{o}{C}_1;$$

so that when ψ is restricted to C_1 we get the required map.

4.5 Remark.

Let $\psi : I^n \setminus E \longrightarrow M$ be a mapping given by the main theorem above. Then, measures (having the boundary of the unit n-cube as a null set) and homeomorphisms of the unit n-cube fixing ∂I^n pointwise can be thrown, respectively, into measures and homeomorphisms of M via ψ. This provides us with a tool for the topological and algebraic study of various groups of (measure preserving) homeomorphisms of M (see [3]).

Appendix 1.

A1.1 Definitions.

A subset X of an n-manifold M is cellular if for every neighbourhood U of X there is an n-cell Q such that $X \subset \operatorname{Int} Q \subset U$.

If B is an (n-1)-dimensional, bicollared submanifold of M and $\delta_i : B \to \langle 0, 1 \rangle$ $(i = 1, 2)$ continuous are given, define

$$B \times \langle\langle -\delta_1, \delta_2 \rangle\rangle = \{ (b,t) \mid -\delta_1(b) < t < \delta_2(b) \} .$$

$$B \times \{\{ (-1)^i \delta_i \}\} = \{ (b,t) \mid (-1)^i \delta_i(b) = t \} .$$

We divide the proof of lemma 4.2 into two.

A1.2 Lemma.

Let M^n be a manifold with $n \geq 3$ and let d be a metric on M. Let C^n be a closed n-dimensional manifold with bicollared boundary ∂C in M.

Let $\epsilon > 0$ and a continuous function $\delta : \partial C \longrightarrow \langle 0,1 \rangle$ be given. Suppose $\Lambda = \{ D_j \}_{j \in J}$ is an (at most countable) locally finite family of sets in M such that each D_j is a closed n-cell of diameter less than $\epsilon/2$ whose interior intersects C. Let $X = \{ x_i \}_{i \in L}$ be a locally finite set of points in $\bigcup_{j \in J} \operatorname{Int} D_j \setminus C$. Then there is a locally finite set of points $X' = \{ x'_i \}_{i \in L}$ in $\partial C \times \langle\langle 0 , \delta \rangle\rangle$ and an $\epsilon/2$-homeomorphism $h : M \longrightarrow M$ such that $\operatorname{supp} h \subset \bigcup_{j \in J} \operatorname{Int} D_j \setminus C$ and $h(x'_i) = x_i$ for each $i \in L$.

Proof. We may assume, without loss of generality, that $x_{i_1} = x_{i_2}$ for $i_1 \neq i_2$ (hence L is at most countable).

Associate with each x_i some element, say $D_{j(i)}$, of Λ which contains x_i in its interior. Associate with each D_j a point y_j in $C \cap \operatorname{Int} D_j$. For $i \in L$ let α_i be a poligonal arc (relative to some combinatorial structure on $D_{j(i)}$) in $\operatorname{Int} D_{j(i)}$ from x_i to $y_{j(i)}$. Since an n-dimensional connected manifold cannot be disconnected by a subset of dimension less or equal n-2 (see Hurewicz and Wallman [5]), this can be done in such a manner that α_{i_1} and α_{i_2} are disjoint or intersect only in the common end point $y_{j(i_1)} = y_{j(i_2)}$.

Let x'_i be a point of $\alpha_i \cap \partial C \times \langle\langle 0 , \delta \rangle\rangle$ such that the segment $[x_i , x'_i]$ of α_i does not intersect C. Since α_i is poligonal in $D_{j(i)}$, so is $[x_i , x'_i]$. Hence $[x_i , x'_i]$ is cellular in $D_{j(i)}$ and therefore cellular in M. Hence there exists a (locally finite) family $\{ Q_i \}_{i \in L}$ of n-cells such that

$$Q_i \cap Q_j = \emptyset \qquad \text{if } i = j ;$$

$$[x_i , x'_i] \subset \overset{o}{Q}_i ;$$

$$Q_i \cap C = \emptyset ;$$

$$Q_i \subset \text{Int } D_{j(i)} .$$

Let h be a homeomorphism of M onto M such that

h restricted to $M \setminus \bigcup_i Q_i$ equals the identity ;

$h(Q_i) = Q_i$;

$h(x'_i) = x_i$.

Then h is the required homeomorphism.

A1.3 Lemma.

Suppose that $0 < \gamma < 1$ and that the hypotheses of the above lemma are satisfied. Then there is an ϵ-homeomorphism f of M onto M such that $f(C) \supset f(\overset{o}{C}) \supset C \cup X$ and $\operatorname{supp} f \subset (\bigcup_j \operatorname{Int} D_j \setminus C) \cup \partial C \times \langle -\gamma, \gamma \rangle$. In particular, f fixes pointwise the "inner" n-manifold bounded by $\partial C \times \{-\gamma\}$.

Proof. Choose $\delta : \partial C \longrightarrow \langle 0, \gamma/2 \rangle$ continuous and such that for each $c \in \partial C$ the diameter (with respect to the induced metric) of $\{c\} \times [-2\delta(c), 2\delta(c)]$ - in the collar $C \times [-1, 1]$ - is less than $\epsilon/2$.

Let $\alpha : \partial C \longrightarrow H([-1, 1])$ be defined by the formula

$$\alpha_c(t) = \begin{cases} t & -1 \le t \le -2\delta(c) \\ (3/2)\,t + \delta(c) & -2\delta(c) \le t \le 0 \\ (1/2)\,t + \delta(c) & 0 \le t \le 2\delta(c) \\ t & 2\delta(c) \le t \le 1 \end{cases}$$

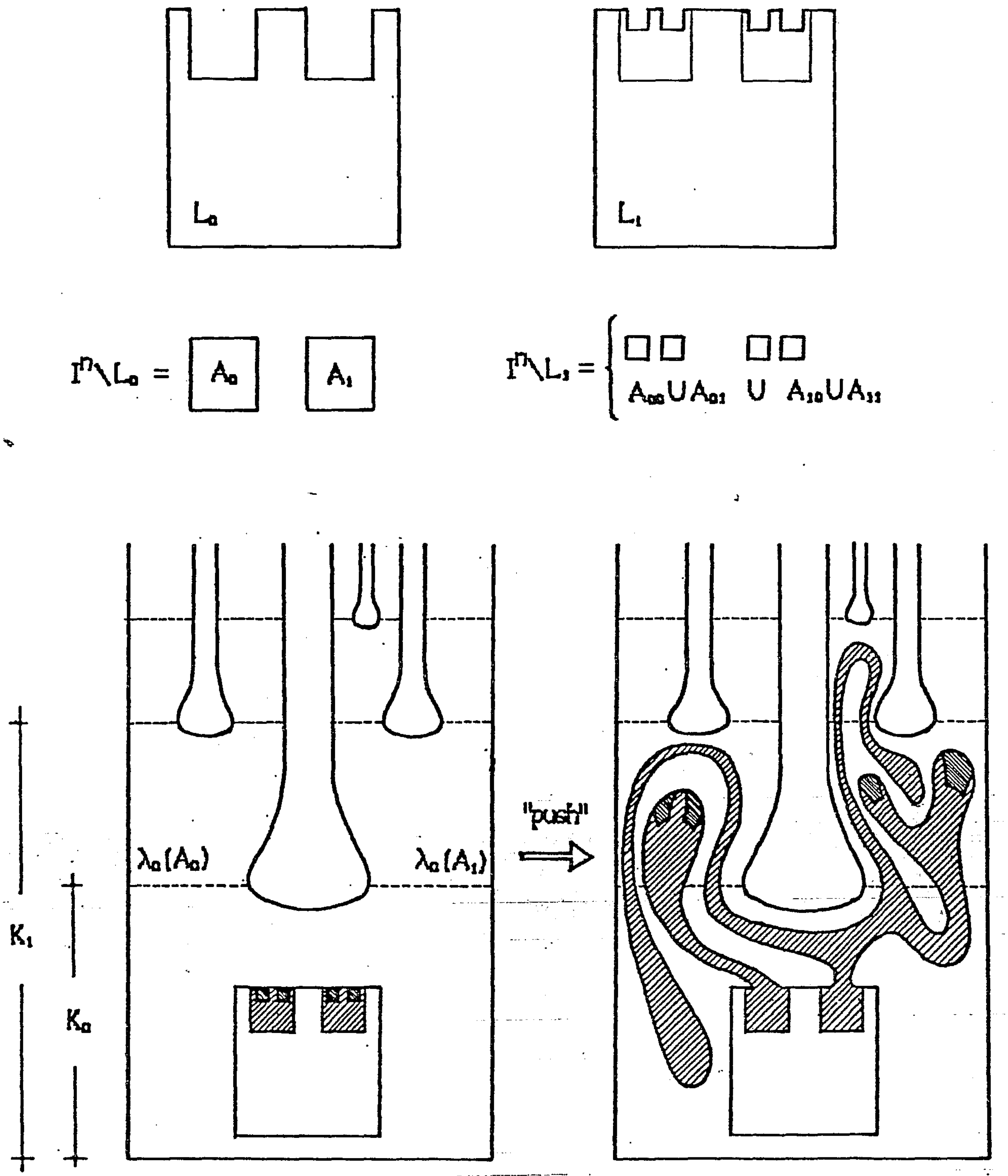
Since $\partial C \times [-1, 1]$ is closed in M, we can define a homeomorphism $g \in H(M)$ such that g is the identity outside $\partial C \times \langle -1, 1 \rangle$ and is given by $g(c,t) = (c, \alpha_c(t))$ for each $(c,t) \in \partial C \times [-1, 1]$.

Therefore, g is fixed on the manifold bounded by $\partial C \times \{\{-2\delta\}\}$, stretches $\partial C \times \{0\}$ parametrically onto $\partial C \times \{\{\delta\}\}$ and is fixed outside $\partial C \times \{\{2\delta\}\}$. Furthermore, g is an $\epsilon/2$-homeomorphism and if h is the homeomorphism obtained in the conclusion of the above lemma, then $f = h \circ g$ is the required ϵ-homeomorphism.

Appendix 2.

We would like to embed a copy of I^n in K_0 and start an inductive process with the aid of the combinatorial scheme constructed in Step 1. Suppose for a moment that I^n is actually contained in $\mathring{K}_0$ and that A_0 is a component of $I^n \setminus L_0$. Then, we want to "push" A_0 (or some part of A_0) to where it corresponds. That is, into $\lambda_0(A_0)$. Now let A_{00} be a component of $I^n \setminus L_1$ contained in A_0. A further push should take A_{00} (or some part of A_{00}) into $\lambda_1(A_{00})$. And so on.

Many things can go wrong in the process. The following diagram intends to show some of the difficulties.



There is nothing wrong with the push we gave to A_0 , but A_1 is so badly deformed that we cannot push, say A_{00} , any further (and achieve convergence at the end of the story).

The situation for A_{10} is bad as well. Certainly we can push it once more as to move some part of it into $\lambda_1(A_{10})$. But then a third push most probably will be impossible.

Although the example is two-dimensional (we are assuming dimension no less than three in this lemma), it is easy to extend it to dimension three (the "tentacles" will now look like "domes").

There can be problems with the compact sets $\{K_i\}_i$ as well, so we will need to assume some "connectivity" properties.

Proceed now into Step 2 ....

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