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Measures on Hyperbolic Surface Laminations

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INTRODUCTION

A lamination or foliated space is a topological space that can be seen as a disjoint union of manifolds, whose local aspect is that of a product while its global geometry is usually quite complicated. The first non-trivial example of a lamination is the phase space of a continuous dynamical system -written as the disjoint union of orbits-, where the problem of understanding the *asymptotic behaviour* of orbits is most naturally posed.

In this thesis, however, we will be concerned exclusively with laminations whose leaves are hyperbolic Riemann surfaces. Examples of these objects arise naturally in the field of ordinary differential equations with complex time, but may come from other constructions such as Sullivan's definition of universal solenoids (see [S] or the Appendix) or Hilbert modular foliations (see Chapter III).

In an attempt to give an interpretation to the problem of the asymptotic behaviour of the leaves, and to understand it from the probabilistic (or ergodic-theoretical) point of view, many different measures have been associated to laminations. The aim of this work is to shed some light on the way some of these measures relate.

The first measures we will consider are holonomy-invariant measures. These are not measures on the foliated space, but on its transversals. Nevertheless. there is a natural sense in which they generalize the concept of invariant measures for flows. While invariant measures always exist for flows on compact metric spaces, holonomy-invariant measures not always exist on laminations, and an example of this phenomenon can be found in [Ga]. The main theorem concerning the existence of holonomy-invariant measures. due to Plante ([P]), guarantees their existence if there are leaves whose volume has subexponential growth-which is not usually the case for laminations by hyperbolic surfaces.

The second class of measures we will consider are defined and studied by Lucy Garnett in [Ga]. These are measures on the foliated space invariant under the heat flow on leaves. called *harmonic* or *stationary*, of which holonomy-invariant measures are essentially a special case.

The fact that leaves are hyperbolic Riemann surfaces has led to the study of three flows which are particulary interesting on hyperbolic surfaces: the geodesic, the stable horocycle and the unstable horocycle laminated flows. The third class of measures we will be interested in is that of measures invariant under one or more of these flows. The hope is that they will help us to understand the geometric complexity of the lamination; that is, the way leaves are "wrapping" inside the space–although most of the work in this direction is still to be done. In [B-GM], for example, for certain Riccati foliations unique ergodicity of the horocycle flow is proved, and the unique SRB (for Sinai-Ruelle-Bowen) or physical measure of the geodesic foliated flow is found and described.

In this thesis $\mathcal{L}$ will always denote a compact, nonsingular lamination whose leaves are hyperbolic Riemann surfaces, and $T^1\mathcal{L}$ its unit tangent bundle. The laminated geodesic flow on $T^1\mathcal{L}$ is called g, and the stable and unstable horocycle flows on $T^1\mathcal{L}$ are called h^+ and h^-, respectivley.

In chapter two, certain relationships between harmonic measures and measures invariant under the laminated flows g, h^+ and h^- are discussed. The following two theorems are proved:

THEOREM II.2.5. *Any harmonic measure on $\mathcal{L}$ is the projection of a measure invariant under the horocycle flow h^+ on $T^1\mathcal{L}$.*

THEOREM II.3.2. *Any measure on $T^1\mathcal{L}$ invariant under both g and h^+ projects onto a harmonic measure on $\mathcal{L}$.*

Of course these theorems remain true if we change h^+ for h^-.

The converse of Theorem II.2.5 is not known, but does not hold if the lamination is not compact. And the proof of this theorem does not indicate if the correspondence "harmonic measure $\mapsto$ measure invariant under h^+" is one-to-one. The converse of Theorem II.3.2 is not known either, nor is it known if the correspondence "measure invariant under g and $h^+ \mapsto$ harmonic measure" is surjective.

A theorem that characterizes harmonic measures in terms of measures invariant under one or more laminated flows is likely to hold, but is still to be proved. In fact, the converse of Theorem II.3.2 should be true. Such a result would allow us to express problems related to harmonic measures, which are usually very difficult, in terms of laminated flows. For example, the question 'Does a minimal lamination by hyperbolic surfaces have a unique harmonic measure?' would be generalized to 'Is there a unique measure ergodic for the joint action of g and h^+?' This problem seems, at first sight, to be easier to handle.

The study of measures simultaneously invariant under the three laminated flows g, h^+ and h^- does not lead to new measures. Showing this is the aim of the following proposition, which can also be found in Chapter two.

PROPOSITION II.1.1. *There is a canonical bijection between measures invariant under g, h^+ and h^- and holonomy-invariant measures.*

In Chapters three and four, two families of examples are studied: Hilbert modular foliations and a Riccati foliations.

For Hilbert modular foliations, the following result is proved:

PROPOSITION III.2.3. *The volume is the unique harmonic measure of the Hilbert modular foliation.*

The volume in this foliated space is the projection of the volume in the unit tangent bundle to the foliation, which is an ergodic measure for the three flows g, h^+ and h^-.

Also in Chapter III, Proposition III.3.1 describes the geometry of the leaves of Hilbert modular foliations, and easily implies that the horocycle flow has no closed orbits. Observe that in the unit tangent bundle of a single hyperbolic Riemann surface S of finite volume, there are relatively few measures invariant under the horocycle flow. Namely, all ergodic probability measures of the horocycle flow h^+ are the Liouville measure and measures on closed horocycles. This does not generalize to foliated spaces: In Example III.3.2 we see that the Hilbert Modular foliation has measures invariant under h^+ which are not invariant under g, and are therefore not at all similar to the Liouvlle measure on T^1S. These measures do not come from closed horocycles, since there are none.

Chapter four is actually the chronological beggining of this thesis. In this chapter, a measure that was found and studied by Bonatti and Gómez-Mont in [B-GM] is identified as the only harmonic measure of the generic Riccati foliation. It is obtained in [B-GM] as the projection of two measures μ_+ and μ_- on the unit tangent bundle to the foliation, each of which is invariant under two foliated flows. ($\mu_\pm$ is invariant under g and $h^\pm$.) The fact that this measure is harmonic, although easily derived from the results and techniques in [B-GM], suggested the ideas for the whole of this thesis.

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CHAPTER I

PRELIMINARIES

1 Hyperbolic Surface Laminations

DEFINITION: $\mathcal{L}$ is a *lamination* if it is a separable, locally compact metrizable space that has an open covering $\{E_i\}$ and an atlas $\{(E_i, \varphi_i)\}$ satisfying:

1) $\varphi_i : E_i \to D_i \times T_i$, for some open disk D_i in $\mathbb{R}^d$ and topological space T_i, and
2) the coordinate changes $\varphi_j \circ \varphi_i^{-1}$ are of the form $(z,t) \mapsto (\zeta(z,t), \tau(t))$ where each ζ is smooth in the z variable.

This last condition says that the sets of the form $\varphi_i^{-1}(D_i \times \{t\})$, called *plaques*, glue together to form d-dimensional manifolds that we call *leaves*. The open sets E_i are called *flow boxes*.

When the topological spaces T_i are disks in $\mathbb{R}^k$ and the changes of coordinates are smooth, the total space $\mathcal{L}$ is a manifold and the lamination is called *foliation*. In this case, we use the notation M for the manifold and $\mathcal{F}$ for the foliated structure.

DEFINITION: $\mathcal{L}$ is a *Riemann surface lamination* if the disks D_i are open subsets of the complex plane and the maps ζ are holomorphic in the z variable. We say that $\mathcal{L}$ is a *hyperbolic surface lamination* if its leaves are hyperbolic Riemann surfaces.

DEFINITION: $\mathcal{L}$ is *oriented* if the atlas $\{(E_i, \varphi_i)\}$ induces an oriented atlas on each leaf.

Remark that a Riemann surface lamination is always oriented.

Each leaf on a hyperbolic surface lamination $\mathcal{L}$ has a Poincaré metric, which is the only Riemannian metric of constant curvature -1 compatible with the conformal structure. According to a theorem due to Candel [C1], these metrics on the leaves, as well as all their derivatives, have a continuous variation in the transverse direction.

In this thesis, $\mathcal{L}$ will always denote a hyperbolic surface lamination, and, unless otherwise stated, it will be compact.

For examples of these objects, see the Appendix.

If μ is any measure on the Borel σ-algebra of $\mathcal{L}$, the integral of any function f with respect to μ will be written $\int f \, d\mu$ or $\mu(f)$.

2 Laminated Geodesic and Horocycle Flows

2.1 Geodesic and Horocycle Flows on Hyperbolic Surfaces

We will work with the upper-half plane model of the hyperbolic plane, given by

$$\mathfrak{H} = \{z = x + iy \in \mathbb{C} : y > 0\},$$

where the Riemannian metric

$$\langle \cdot, \cdot \rangle_{x+iy}$$

is $1/y^2$ times the Euclidean scalar product.

The isometries of $\mathfrak{H}$ are the Möbius transformations that preserve the horizontal axis $\partial\mathfrak{H} = \mathbb{R} \cup \{\infty\}$ sending $\mathfrak{H}$ onto $\mathfrak{H}$; so the group of isometries can be identified with $PSL(2, \mathbb{R})$, whose left action on $\mathfrak{H}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

DEFINITION: A *geodesic* on $\mathfrak{H}$ is an isometric immersion of the real line $\mathbb{R}$ into $\mathfrak{H}$.

Geodesics are circles orthogonal to the real axis $\mathbb{R}$ or vertical lines. In fact, it is not difficult to verify that the curve

$$\gamma : \mathbb{R} \to \mathfrak{H}; \ \gamma(t) = ie^t$$

is a geodesic, and Möbius transformations preserving $\mathbb{R} \cup \{\infty\}$ send it to other vertical lines or circles orthogonal to the real axis. Remark that since the action of the group of isometries is transitive in the unit tangent bundle $T^1\mathfrak{H}$ of $\mathfrak{H}$, all geodesics must be of this form. The geodesic flow in $T^1\mathfrak{H}$, that for any $x \in \mathfrak{H}$ and any vector $v \in T^1_x\mathfrak{H}$ gives the geodesic starting at x with velocity v, will be called g_t.

The action of $PSL(2, \mathbb{R})$ on $T^1\mathfrak{H}$, being transitive and free, allows us to identify $T^1\mathfrak{H}$ with $PSL(2, \mathbb{R})$. We will choose one of the possible identifications: Let v be the vector $(1, 0)$ based at the point $i \in \mathfrak{H}$, i.e. the unit tangent vector to $\mathfrak{H}$ at i pointing upwards. We will associate to $A \in PSL(2, \mathbb{R})$ the element $A \cdot v$ in $T^1\mathfrak{H}$.

We will call D_t the diagonal matrix

$$\begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.$$

Under the identification given above, D_t corresponds to the point $g_t(v) \in T^1\mathfrak{H}$, and therefore the geodesic starting at v, when viewed in $PSL(2, \mathbb{R})$, is the one-parameter subgroup $\{D_t\}$. Consider the geodesic starting at some other point $B \cdot v$ in $T^1\mathfrak{H}$–that is, at the element B of $PSL(2, \mathbb{R})$. It can be obtained by applying the isometry B to the geodesic $\{D_t\}$; namely, it is the curve

$$\mathbb{R} \to T^1\mathfrak{H}$$
$$t \mapsto BD_t.$$

This tells us that the geodesic flow on $T^1\mathfrak{H}$, when seen in $PSL(2, \mathbb{R})$, is the action by right translations of the one-parameter subgroup $D = \{D_t\}$.

A circle in $\mathfrak{H} \cup \partial\mathfrak{H}$ tangent to $\partial\mathfrak{H} = \mathbb{R} \cup \{\infty\}$ is called a *horosphere*.

Consider the horizontal line l_+ in $\mathfrak{H}$ passing through the point i, and in each point $t+i$ of l_+ take the unit tangent vector to $\mathfrak{H}$ at $t+i$ pointing upwards, which we will call v_t. The curve

$$\mathbb{R} \to T^1\mathfrak{H}$$
$$t \mapsto v_t$$

(which is parametrized by unit length) is a *horocycle*.

Remark that the geodesic starting at $t+i$ with velocity v_t is the curve $s \mapsto t+e^s i$, which approaches exponentially, as s increases, the geodesic starting at i with velocity $v_0 = v$.

DEFINITION: We say that the curve h is the *stable horocycle* at the point $v \in T^1\mathfrak{H}$.

Viewing geodesics as curves in $T^1\mathfrak{H}$, we see that all geodesics passing through points in h are headed towards $\infty \in \partial\mathfrak{H}$; conversely, all geodesics heading towards ∞ intersect h at some point.

Considering the horosphere l_- through i that is tangent to $0 \in \partial\mathfrak{H}$, and the unit vectors tangent to $\mathfrak{H}$ and orthogonal to l_- pointing "outwards" (like v on the point i), we get vectors directing geodesics that come from 0, and that approach exponentially, in the past, the geodesic with initial condition v.

DEFINITION: We call the curve in $T^1\mathfrak{H}$ whose elements are these points of $T^1\mathfrak{H}$ and which is parametrized by unit length, the *unstable horocycle* at v.

DEFINITION: For any other point w in $T^1\mathfrak{H}$ and the isometry B taking v to w, we define the stable (unstable) horocycle at w as the image under B of the stable (unstable) horocycle at v.

REMARK. *Any two points on this curve direct geodesics that approach exponentially in the future (past), and that head to (come from) the same point in $\partial\mathfrak{H}$.*

DEFINITION: The *stable (unstable) horocycle flow* is the flow $h^+ : T^1\mathfrak{H} \to T^1\mathfrak{H}$ (h^-) that, for any initial condition $w \in T^1\mathfrak{H}$, gives the stable (unstable) horocycle at w. When viewed in $PSL(2, \mathbb{R})$, it is given by the action by right translations of the one-parameter group

$$H_+ = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}$$

$$\left(H_- = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\} \right).$$

REMARK. *The geodesic and horocycle flows satisfy the following important relationship, which can be easily verified by means of matrix multiplication:*

$$g_s \circ h_t^+ = h^+_{te^{-s}} \circ g_s \quad \text{and} \quad g_s \circ h_t^- = h^-_{te^s} \circ g_s. \qquad (*)$$

From this we can see that the horocycle flows are preserved by the geodesic flow, one of them being exponentially contracted and the other one being exponentially expanded.

Namely, the geodesic flow is Anosov and its stable and unstable manifolds are precisely given by the orbits of the horocycle flows. (For the definition of Anosov flow see, for example, [K-H], Ch.17, sect. 4, pg.545.)

If Γ is a discrete subgroup of $PSL(2,\mathbb{R})$, it acts on the left by isometries on $\mathfrak{H}$ and the quotient space

$$S = \Gamma\backslash\mathfrak{H}$$

is a hyperbolic surface or orbifold, depending on whether Γ contains elliptic elements –matrices in $PSL(2,\mathbb{R})$ whose canonical form is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, the only ones whose action on $\mathfrak{H}$ has a fixed point. S inherits from $\mathfrak{H}$ a Riemannian metric of constant curvature -1, that can be of finite or infinite area.

The unit tangent bundle T^1S to S is $\Gamma\backslash T^1\mathfrak{H} \cong \Gamma\backslash PSL(2,\mathbb{R})$, and the flows g_t, h_t^+ and h_t^- on $T^1\mathfrak{H}$ project onto flows in T^1S which are also called geodesic, stable horocycle and unstable horocycle flows. The three of them preserve the *Liouville measure*, which is locally given by the hyperbolic area times the angular measure on the unit tangent space to each point (or simply the projection of the Haar measure on $PSL(2,\mathbb{R})$). The geodesic flow $g_t : T^1S \to T^1S$ is Anosov, and its stable and unstable manifolds are the orbits of h^+ and h^-. The *weakly stable* (*weakly unstable*) foliation of the geodesic flow on T^1S is the two-dimensional foliation on T^1S tangent to the vector field directing g and h^+ (h^-).

The following two theorems, due to Hopf and Hedlund (respectively), are not used in the sequel, strictly speaking. Nevertheless, they suggest the idea of using the laminated geodesic and horocycle flows as a means to see "where leaves are going", and constitute beautiful examples of the interplay between g, h^+ and h^-. The proofs given are taken from [Gh].

THEOREM. (Hopf) *If S has finite area, the geodesic flow is ergodic with respect to the Liouville measure in T^1S.*

Proof:

Remark that any matrix in $PSL(2,\mathbb{R})$ can be written as the product of matrices in the subgroups D, H_+ and H_-; this means that any function invariant under g, h^+ and h^- must be constant on T^1S.

Let λ be the Liouville measure, and f a square-integrable function on T^1S which is invariant under the geodesic flow. We wish to prove that f is constant λ-almost everywhere.

Using the formula labeled (*) and the fact that f is invariant under g_s, we have:

$$\int f\cdot(f\circ h_t^+)\,d\lambda = \int (f\circ g_s)\cdot(f\circ h_t^+\circ g_s)\,d\lambda$$
$$= \int f\cdot f\circ h^+_{e^{-s}t}\,d\lambda \xrightarrow[s\to\infty]{} \int f^2\,d\lambda.$$

This means (Cauchy-Schwarz) that $f\circ h_t^+ = f$ in $L^2(\lambda)$, and therefore almost everywhere. In the same way we can prove that f is h^--invariant, and therefore constant almost everywhere. □

THEOREM. (Hedlund) *If S has finite area, the horocycle flows are ergodic with respect to the Liouville measure in T^1S.*

Proof:

If we take the linear (usual) action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$, the stabilizer of the point $(1,0)$ is $H_+ = \left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$. Therefore, the homogeneous space $SL(2,\mathbb{R})/H_+$ can be identified with $\mathbb{R}^2\backslash\{(0,0)\}$, and the left H_+-action on $SL(2,\mathbb{R})/H_+$ is by horizontal translations. A continuous function on $\mathbb{R}^2\backslash\{(0,0)\}$ invariant under such translations is a function of the variable y; in particular, it is constant on the x-axis, which is the orbit of $(1,0)$ under the right action of the diagonal group D.

The preceding discussion tells us the following: if Φ is a continuous function on $SL(2,\mathbb{R})$ invariant on double H_+-classes, then it is constant on the one-parameter subgroup $D = \left\{\begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}\right\}$.

Consider now an h^+-invariant function $f \in L^2(\lambda)$, where λ is again the Liouville measure on T^1S. The function Φ on $SL(2,\mathbb{R})$ defined by

$$\Phi(g) = \int f(x)\cdot f(xg)\,d\lambda(x)$$

is continuous and satisfies

$$\Phi\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g\right) = \Phi(g) = \Phi\left(g\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right),$$

and therefore it is invariant under the right action of D. This means that

$$\int f^2\,d\lambda = \int f\cdot(f\circ g_s)\,d\lambda,$$

and f is invariant under the geodesic flow. We know, using Hopf's theorem, that f must be constant λ-almost everywhere. □

When the surface $S = \Gamma\backslash\mathfrak{H}$ has finite volume, Dani and Smillie give in [D–S] the complete description of the ergodic measures for the horocycle flow h^+. If the surface S is compact, the horocycle flow has no closed orbits. If S is not compact, it has a finite number of ends, each corresponding to a parabolic generator of its fundamental group Γ. Each end is foliated by a one-parameter family of closed horocycles, and these families include all closed horocycles of S. Dani and Smillie prove the following:

THEOREM. (Dani-Smillie) *The ergodic measures for the horocycle flow on T^1S are the Liouville measure and measures having support in closed horocycles.*

REMARK. *This means that if S is compact, the horocycle flow on T^1S is uniquely ergodic. Such a result does not hold for the geodesic flow g, which has many ergodic invariant measures.*

2.2 Laminated flows

DEFINITION: If $\mathcal{L}$ is a hyperbolic surface lamination, we call $T^1\mathcal{L}$ the lamination whose three dimensional leaves are the unit tangent bundles of the leaves of $\mathcal{L}$ and that has "the same" charts as $\mathcal{L}$.

DEFINITION: The *laminated geodesic flow* is a flow (also called g_t, which will not lead to confusion since this will be the only g_t in the sequel), that, restricted to the unit tangent bundle of a leaf L of $\mathcal{L}$, coincides with the geodesic flow in T^1L. The *laminated horocycle flows* are defined in an analogous way.

In general we will only work with the stable laminated horocycle flow, which will be called h_t. All these flows are continuous on $T^1\mathcal{L}$, as a consequence of Candel's theorem (see [C1]).

It is an important fact that in the lamination $T^1\mathcal{L}$ there is a right $PSL(2,\mathbb{R})$-action whose orbits are the leaves, which is as *good* (meaning continuous, smooth or analytic) as the lamination itself. Exactly as in the case of surfaces, the geodesic and the horocycle flows correspond to the action on $T^1\mathcal{L}$ of the diagonal and upper-triangular matrices respectively.

Measures associated to flows are inspired by the analogy between a lamination and a surface: a leaf of $\mathcal{L}$ that is very recurrent should behave in a way similar to that of a single compact Riemann surface S. For such a surface, the Loiuville measure of its unit tangent bundle may be obtained from any single orbit of the horocycle flow. Namely, for any point $v \in T^1S$, if λ is the Liouville measure,

$$\int f\, d\lambda = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(h_t^+(v))\, dt,$$

for any continuous function f on T^1S. The same is true for the geodesic flow, if we restrict ourselves to considering almost all initial conditions v (according to λ).

In a sense, if we knew h^+ but not S, we would be able to "recover" the manifold T^1S, and therefore the surface S. In an analogous way, using laminated flows, we would like to get a better understanding of the behaviour of a leaf L from the study of the orbits that lie on L of the laminated flows.

The following remark, which is a well-known fact, will not be used in the sequel. Nevertheless, it states a similarity between compact hyperbolic Riemann surfaces and compact hyperbolic surface laminations.

REMARK. *The laminated horocycle flow h has no closed orbits on $T^1\mathcal{L}$.*

Proof:

Any closed orbit of h would be contracted exponentially by the geodesic flow, and would therefore converge to a fixed point of h in the compact space $T^1\mathcal{L}$. But h has no fixed points. □

3 Holonomy-invariant measures

If $\mathcal{L}$ is a compact lamination and $\varphi : E \to D \times T$ is a coordinate chart, we say that a subset of $\mathcal{L}$ of the form $\varphi^{-1}(\{x\} \times T)$ is a *transversal* of $\mathcal{L}$.

Consider a finite set $\tau_1, \ldots, \tau_n$ of transversals such that any leaf of $\mathcal{L}$ intersects at least one of the τ_i.

DEFINITION: A finite measure μ on $\tau_1 \cup \ldots \cup \tau_n$ is *holonomy-invariant* if it is invariant under all the elements f of the holonomy pseudogroup of $\mathcal{L}$ that have domain and range contained in $\tau_1 \cup \ldots \cup \tau_n$. (For the definition of the holonomy pseudogroup of $\mathcal{L}$, see [C-L], Ch.4, secc.1 or [C-C], vol.I, Ch.11, secc.2.)

REMARK:. *A holonomy-invariant measure on $\tau_1 \cup \ldots \cup \tau_n$ induces in a canonical way a holonomy-invariant measure on any other finite union of transversals that intersects all leaves of $\mathcal{L}$. This allows us to speak of holonomy-invariant measures on $\mathcal{L}$.*

We say that a Riemannian manifold L has subexponential growth if

$$\lim_{r \to \infty} \frac{1}{r} \log \operatorname{vol}(B(x,r)) = 0,$$

where $\operatorname{vol}(B(x,r))$ is the volume of the ball of radius r centered at a point $x \in L$.

Suppose there is a Riemannian metric on the leaves of $\mathcal{L}$ that is continuous in $\mathcal{L}$. (If $\mathcal{L} = (M, \mathcal{F})$ is a foliation, one may simply take a Riemannian metric on M.) Plante proved in [P] a sufficient condition for the existence of holonomy-invariant measures, which is the following:

THEOREM. (Plante) *If $\mathcal{L}$ has a leaf whose volume has sub-exponential growth, then there is a holonomy-invariant measure.*

However, leaves of subexponential growth need not exist in hyperbolic surface laminations.

An example of a hyperbolic surface lamination that has no holonomy-invariant measures is the foliation $(T^1S, \mathcal{F})$, where S is a compact hyperbolic Riemann surface and $\mathcal{F}$ is the weakly stable foliation of the geodesic flow on T^1S. This example can be found in [Ga].

DEFINITION: A *measure class* on a measurable space is a family of measurable sets that is closed under countable unions.

We say that two measures μ and ν on a given measurable space are *equivalent* if each of them is absolutely continuous with respect to the other one. An equivalence class $\bar{\mu}$ of measures determines a measure class: that of the measurable sets of μ-measure zero.

DEFINITION: Let $\tau_1, \ldots, \tau_n$ be transversals to $\mathcal{L}$ such that $\tau_1 \cup \ldots \cup \tau_n$ intersects all leaves. A measure class $\mathcal{Q}$ in $\tau_1 \cup \ldots \cup \tau_n$ is *holonomy-invariant* if,

$$Q \in \mathcal{Q} \Leftrightarrow f(Q) \in \mathcal{Q},$$

for every holonomy transformation f whose domain and range are contained in $\tau_1 \cup \ldots \cup \tau_n$.

4 Harmonic Measures

Introduction

In this section, a brief introduction to harmonic measures will be given. These measures were defined by Lucy Garnett in [Ga], where everything which is said here is stated and proved in much more detail. Nevertheless, certain background on heat diffusion on Riemannian manifolds is assumed in Garnett's paper, so we make here a brief review of the necessary definitions. The following books can be referred to for further study of this topic: [Ch] gives in Chapters VI to VIII an exhaustive treatment of the heat kernel in Riemannian manifolds. [D] explains the interplay between the classical and probabilistic diffusion theories–in particular Brownian motion, although in $\mathbb{R}^n$. [Y] develops in Chapter XIII, sections 1 to 5, the theory of Markov processes in locally compact spaces–which include Browninan motion on Riemannian manifolds and foliated Brownian motion–, and in Chapter IX, the theory of semigroups and their infinitesimal generators.

4.1 Preliminaries on Heat Diffusion.

Let M be a complete Riemannian manifold of bounded Ricci curvature. Associated to M is its Laplace-Beltrami operator, called Δ_M, which is the infinitesimal generator of the *heat semigroup* $\{D_t\}$ on M, whose transition probabilities are given by the *heat kernel*. We will briefly recall what all these terms refer to.

DEFINITION: Given a differentiable function f on M, the *gradient* of f is the vector field grad f on M for which

$$\langle \operatorname{grad} f, v\rangle = vf \quad \forall v \in TM,$$

where vf is the directional derivative of f at the basepoint of v in the direction of v.

DEFINITION: If ∇ denotes the Levi-Civita connection associated to the metric, the *divergence* of a vector field X is the real-valued function on M defined by

$$(\operatorname{div} X)(p) = \operatorname{trace}(v \mapsto \nabla_v X),$$

where the variable v ranges over T_pM.

DEFINITION: The *Laplacian* (or *metric Laplacian* or *Laplace-Beltrami operator*) is defined on functions f which are at least twice-differentiable, by the expression

$$\Delta_M f = \operatorname{div} \operatorname{grad} f.$$

DEFINITION: A function u on $M \times (0, +\infty)$ satisfies the *heat equation* on M if

$$\frac{\partial u}{\partial t} = \Delta_M u.$$

DEFINITION: A heat kernel on M is a continuous function which is a fundamental solution for the heat equation on M, i.e. it is a real-valued function p on $M \times M \times (0, +\infty)$ for which

$$u(x,t) = \int_M p(x,y,t) f(y)\, dy$$

always gives a solution to the initial value problem for the heat equation on M, with initial values f, where f is bounded and continuous on M.

Furthermore (see [Ch]), the heat kernel

$$p : M \times M \times (0, +\infty) \to \mathbb{R}$$

is of class C^∞ and unique, and satisfies the following:
1) It is symmetric in the space variables, i.e. $p(x,y,t) = p(y,x,t)$,
2) it is harmonic and has mass 1 as a function of x for fixed y and t and
3) for all $t, s > 0$, $\int_M p(x,z,t)p(z,y,s)\, dz = p(x,y,t+s)$.

The last equation is clearly identifiable as Chapman-Kolmogorov's equation (see, for example, [Y], chapter XIII), and tells us that p defines transition probabilities for a Markov process on M, which is the *Weiner process*, usually called *Brownian motion*. This means that if x is a point in M and E is a measurable subset of M, the probability that a Brownian path starting at x is in E at time t is given by

$$\int_E p(x,y,t)\, dy.$$

These transition probabilities determine the *Weiner measure* on the space of continuous paths Ω_x, as follows:

DEFINITION:

$$\Omega_x = \{\omega : [0, +\infty) \to M, \text{ continuous, and such that } \omega(0) = x\}.$$

The Weiner measure is determined by its value on *cylinders*; that is, sets of the form

$$C(t_1, \ldots, t_n; E_1, \ldots, E_n) = \{\omega \in \Omega_x : \omega(t_1) \in E_1, \ldots, \omega(t_n) \in E_n\},$$

for positive times $t_1 < \ldots < t_n$ and measurable subsets $E_1, \ldots, E_n$ of M.

DEFINITION: The Weiner measure of $C(t_1, \ldots, t_n; E_1, \ldots, E_n)$ is

$$\int_{E_n} \cdots \int_{E_2}\int_{E_1} p(x, y_1, t_1)p(y_1, y_2, t_2 - t_1) \ldots p(y_{n-1}, y_n, t_n - t_{n-1})\, dy_1\, dy_2 \ldots dy_n.$$

DEFINITION: The heat semigroup on M is the semigroup $\{D_t\}$, that acts on real-valued continuous bounded functions on M solving the initial-value problem for the heat equation. Namely,

$$D_t f(x) = \int_M p(x,y,t) f(y)\, dy.$$

This quantity should be thought of as "the temperature at x at time t if the initial temperature distribution is f"–assuming M is a homogeneous medium.

There is another expression for $D_t f$, in terms of the Brownian motion, which is

$$D_t f(x) = \int_{\Omega_x} f(\omega(t))\, d\omega,$$

where the integration is performed with respect to the Weiner measure.

4.2 Definition of Harmonic Measures.

Let $\mathcal{L}$ be a lamination, such that every leaf has a Riemannian metric with bounded Ricci curvature, that varies continously in $\mathcal{L}$ (in the topology of uniform convergence in compact subsets of the leaves). Under these hypotheses, associated to each leaf L there is a semigroup $\{D_L(t)\}_{t\geq 0}$ that corresponds to the heat diffusion in L, whose infinitesimal generator is the Laplace-Beltrami operator of L, Δ_L.

We define an operator Δ on $\mathcal{L}$ that is the *collage* of the Laplace-Beltrami operators on leaves:

DEFINITION: $\Delta f(x)$ is $\Delta_L f|_L(x)$, if L is the leaf of $\mathcal{L}$ through x and $f|_L$ is the restriction of f to L. We call Δ *laminated* (or *foliated) Laplacian.*

To this operator there corresponds a semigroup $\{D_t\}_{t\geq 0}$, the *laminated heat semigroup*, that does heat diffusion along the leaves. It can also be expressed in terms of the *laminated heat kernel*, $p : \mathcal{L} \times \mathcal{L} \times (0, +\infty) \to \mathbb{R}$, which restricted to a leaf coincides with its heat kernel, and such that $p(x, y, t) = 0$ if x and y lie on different leaves. In this notation, we make the following definition:

DEFINITION:

$$D_t f(x) = \int_{\mathcal{L}} p(x, y, t) f(y)\, dy.$$

DEFINITION: A probability measure m on $\mathcal{L}$ is *harmonic* if

$$\int_{\mathcal{L}} \Delta f\, dm = 0$$

for every measurable, bounded function f, which is twice differentiable in the leaf direction.

It is true and intuitively clear, although not easy to prove, that a measure is harmonic if and only if it is fixed by the laminated heat semigroup; that is, if

$$\int_{\mathcal{L}} D_t f\, dm = \int_{\mathcal{L}} f\, dm$$

for every measurable bounded function f on $\mathcal{L}$. (And indeed the apparently weaker condition $D_1 m = m$ implies that $D_t m = m$ for all t.) In fact, the existence of harmonic measures on compact laminations can be shown by applying a fixed-point theorem to the action of the laminated heat semigroup on the compact convex set of probability measures on $\mathcal{L}$ (see [Ga]), although Candel gives in [C2] an elementary proof of the existence of harmonic measures, using Hahn-Banach's theorem.

Two properties of harmonic measures, one of which is its local characterization, can be found in Lucy Garnett's paper. Here we will only state them:

THEOREM. (Garnett)

(1) Measurable functions on $\mathcal{L}$ having null laminated Laplacian are constant on almost all leaves with respect to any harmonic measure.

(2) As a consequence of Rokhlin's disintegration theorem, any harmonic measure m on $\mathcal{L}$ decomposes in any flow box E as

$$\int_E f\, dm = \int_I \int_E f(x)\, d\sigma_s(x)\, d\gamma(s),$$

the set I being the set of plaques of E (that is, the transversal of E), $\pi : E \to I$ the projection of E onto I, $\gamma = m\pi^{-1}$ the projection of m and each σ_s a probability measure on E that has its support in the plaque $\pi^{-1}(s)$. A probability measure m is harmonic if and only if the conditional measure σ_s is the Riemannian measure of the plaque $\pi^{-1}(s)$ times a nonnegative harmonic function on the plaque, for almost every $s \in I$.

An important fact about harmonic measures is that they determine holonomy-invariant measure classes–which implies that these classes exist, although holonomy-invariant measures may not. Conversely, any holonomy-invariant measure, combined with the volume measure on the leaf direction, determines a harmonic measure. These harmonic measures are called *totally invariant* harmonic measures.

DEFINITIONS: We can also define the *laminated Brownian motion*, as the Markov process on $\mathcal{L}$ whose transition probabilities are given by the *laminated heat kernel*, in a way analogous to that of the previous section. Each of its paths lies on a single leaf, and the Weiner measure on

$$\Omega_x = \{\omega : [0, +\infty) \to \mathcal{L},\ \text{continuous, such that } \omega(0) = x \text{ and } \omega(t) \text{ lies on the leaf through } x\}$$

is defined exactly as it was before.

Of course, the laminated heat semigroup can be expressed as

$$D_t f(x) = \int_{\Omega_x} f(\omega(t))\, d\omega.$$

4.3 Ergodic Theorems and Ergodic Decomposition

Two of the main theorems in Lucy Garnett's paper are *ergodic theorems for harmonic measures*, that state the following:

LAMINATION ERGODIC THEOREM. (Garnett) *Let m be a finite harmonic measure. For any m-integrable function f there exists an m-integrable function $\bar{f}$ which is constant along leaves and has the following two properties:*

1) $\bar{f}(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1} D_t f(x)$ for m-almost all x,

2) $\int_{\mathcal{L}} \bar{f}\, dm = \int_{\mathcal{L}} f\, dm$.

LEAF PATH ERGODIC THEOREM. (Garnett) *Let m be a finite harmonic measure. For any real-valued integrable function f on $\mathcal{L}$ the $\lim_{n\to\infty} \frac{1}{n}\sum_{t=0}^{n-1} f(\omega(t))$ exists for m-almost every point x and almost any path ω (in the sense of Weiner measure) starting at x and lying on the leaf of x. This limit is independent of ω, constant on leaves and equals the leaf diffused time average $\bar{f}$ of f.*

DEFINITION: A harmonic measure m is *ergodic* if $\mathcal{L}$ can not be partitioned into two measurable leaf-saturated subsets having positive m measure.

In the last section of her paper, Lucy Garnett gives an ergodic decomposition of harmonic measures on compact foliated manifolds, which holds for compact laminations.

The ergodic harmonic measures can be briefly described as follows: we take a point x in $\mathcal{L}$ and consider the Dirac delta at x, which we call δ_x. We can diffuse this measure, obtaining for each positive time t a probability measure $D_t\delta_x$, whose integral on any continuous function f on $\mathcal{L}$ is $D_t f(x)$. For almost all x according to any harmonic measure, the sequence of Birkhoff means

$$\frac{1}{n}\sum_{t=0}^{n-1} D_t\delta_x$$

has a limit which we call $\bar{\delta}_x$. Again for almost all x, this limit does not depend on x but only on the leaf where it lies–and, of course, the integral of a continuous function f with respect to $\bar{\delta}_x$ is nothing but $\bar{f}(x)$.

These measures are ergodic, and the Ergodic Decomposition Theorem reads as follows:

ERGODIC DECOMPOSITION THEOREM. (Garnett) *If m is any harmonic probabillity measure and f is any bounded measurable function on $\mathcal{L}$, then*

$$\int_{\mathcal{L}} f\, dm = \int_{x\in R}\left(\int_{\mathcal{L}} f\, d\bar{\delta}_x\right) dm(x),$$

where R is a leaf-saturated subset of $\mathcal{L}$ such that

1) at every point $x \in R$, $\bar{\delta}_x$ exists, and x is contained in its support,

2) for any two point $x, y \in R$ lying on the same leaf, $\bar{\delta}_x = \bar{\delta}_y$, and

3) any harmonic probability measure gives full measure to R.

CHAPTER II

RELATIONSHIPS BETWEEN MEASURES ON HYPERBOLIC SURFACE LAMINATIONS

Introduction

As before, $\mathcal{L}$ is a compact lamination whose leaves are hyperbolic Riemann surfaces, each endowed with its Poincaré metric. We can consider harmonic measures on $\mathcal{L}$, as well as measures invariant under the horocycle and geodesic laminated flows, which are both continuous in the unit tangent bundle $T^1\mathcal{L}$ to $\mathcal{L}$. The aim of this chapter is to relate these measures, proving the following:

PROPOSITION 1.1. *There is a one to one correspondence between any two of the following:*

1) measures on $T^1\mathcal{L}$ invariant under the three laminated flows g, h^+ and h^-,

2) measures on $T^1\mathcal{L}$ invariant under the $PSL(2,\mathbb{R})$-action and

3) holonomy-invariant measures on $\mathcal{L}$.

THEOREM 2.5. *Any harmonic measure on $\mathcal{L}$ is the projection of a measure invariant under the horocycle flow in $T^1\mathcal{L}$.*

THEOREM 3.2. *Any measure on $T^1\mathcal{L}$ invariant under both the (stable) horocycle and the geodesic flows, projects onto a harmonic measure on $\mathcal{L}$.*

Of course, the measures mentioned in Theorem 3.2 do exist, as we shall briefly see at the end of section 3.

In section 1 we will prove Proposition 1.1, in section, 2 Theorem 2.5 and in section 3, Theorem 3.2.

1 Laminated flows and holonomy-invariant measures

As we saw in the previous chapter, there is a right action of $PSL(2,\mathbb{R})$ on $T^1\mathcal{L}$, whose orbits are the three-dimensional leaves of the lamination $T^1\mathcal{L}$. It is therefore a natural question if measures invariant under this action exist, and what they look like.

Since $PSL(2,\mathbb{R})$ is generated by the one parameter subgroups D, H_+ and H_- (defined in section I.2), measures on $T^1\mathcal{L}$ invariant under the right $PSL(2,\mathbb{R})$-action are measures invariant under the three laminated flows.

REMARK. *If $S = \Gamma\backslash\mathfrak{H}$ is a hyperbolic surface and m is a measure on an open subset U of $T^1S = \Gamma\backslash PSL(2,\mathbb{R})$, invariant under right translations by small elements of $PSL(2,\mathbb{R})$, it must be a scalar multiple of the Liouville measure of U. This is true because the Liouville measure is the Haar measure of the unimodular group $PSL(2,\mathbb{R})$.*

Proof of Proposition 1.1.

Suppose there is a measure μ on $T^1\mathcal{L}$ invariant under the three flows g, h^+ and h^-. The disintegration of μ in a flow box $E \cong U \times T$ of $T^1\mathcal{L}$ gives a measure ν in the transversal T and conditional measures μ_t on the plaques of E such that, for every continuous function f whose support is contained in E,

$$\int f\,d\mu = \int_T \left(\int_{U\times\{t\}} f\,d\mu_t \right) d\nu(t).$$

The invariance of μ and the uniqueness of the decomposition imply that each conditional measure μ_t is a scalar multiple of the Liouville measure. This means (see [C2]) that μ is a totally invariant harmonic measure for the lamination $T^1\mathcal{L}$; that is, ν is a holonomy-invariant measure.

On the other hand, any holonomy-invariant measure on $T^1\mathcal{L}$ induces, by multiplication with the Liouville measure on the leaves, a measure on $T^1\mathcal{L}$ which must be invariant under the flows g, h^+ and h^-.

This proves that measures invariant under the three laminated flows correspond to holonomy-invariant measures of the lamination $T^1\mathcal{L}$. More precisely, we know that measures invariant under the three laminated flows are the same as totally-invariant harmonic measures on $T^1\mathcal{L}$.

It is easy to see that $T^1\mathcal{L}$ has the same holonomy-invariant measures as $\mathcal{L}$. Namely, the natural projection $\pi : T^1\mathcal{L} \to \mathcal{L}$ embeds transversals onto transversals, and projects any holonomy-invariant measure ν onto a holonomy-invariant measure on $\mathcal{L}$. Now, suppose ν is holonomy-invariant in $\mathcal{L}$. To prove that it is invariant in $T^1\mathcal{L}$ it is enough to verify that any path of the form $\gamma_x = \pi^{-1}(\{x\})$ has trivial holonomy. This is immediately noticed from the fact that, if E is a flow box containing x, $\pi^{-1}(E) \sim T \times U \times S^1$, and the path γ_x is simply one of the S^1 factors. □

2 Horocycle flow and harmonic measures

This section is devoted to the proof of Theorem 2.5. We will first prove Theorem 2.5 for ergodic harmonic measures, and we will then use the Ergodic Decomposition Theorem to handle the general case.

The basic idea underlying the proof of this theorem is the following: Birkhoff means converging to ergodic harmonic measures have radial symmetry, and are therefore convex combinations of the Lebesgue measures of circles. These circles can be embedded in $T^1\mathcal{L}$, and the embedded *big* circles are *very close to horocycles.* One proves that, in the convex combination mentioned above, *only big circles matter.* This will imply that, in the limit, a measure invariant under the horocycle flow is obtained.

Step 1: Ergodic measure generated by a simply connected leaf.

Let us pick a point x in $\mathcal{L}$ for which the ergodic harmonic measure

$$\bar{\delta}_x = \lim \frac{1}{n} \sum_{t=0}^{n-1} D_t\delta_x$$

exists; and for the moment let us assume that the leaf L_x of $\mathcal{L}$ through x is simply connected. The Birkhoff sums $\frac{1}{n}\sum_{t=0}^{n-1} D_t\delta_x$ shall be called $\delta_x^{(n)}$.

LEMMA 2.1. *Let p be the heat kernel on the hyperbolic plane. There exists a function $h : [0,+\infty) \to [0,+\infty)$ such that, for all z,*
1) $\lim_{t\to\infty} h(t) = \infty$ and
2) $\int_{D(z,h(t))} p_t(z,y)\,dy \xrightarrow[t\to\infty]{} 0$.

Proof:

An example of such a function is the radius of the disk whose area is $1/\sqrt{p_t(0,0)}$. We use an h such that $h(r) < r$.

DEFINITION: For each n, let r_n be $h(\sqrt{n})$, and we consider the measure η_n whose integral on continuous functions on $\mathcal{L}$ is given by

$$f \mapsto \frac{1}{n} \sum_{t=0}^{n-1} \int_{D(x,r_n)^c} f(y)p_t(x,y)\,dy.$$

LEMMA 2.2. *The mass of the measure η_n approaches 1 as n goes to infinity.*

Proof:

$$1-\eta_n(1)=\frac{1}{n}\sum_{t=0}^{n-1}\int_{D(x,r_n)} p_t(x,y)\,dy$$
$$=\frac{1}{n}\sum_{t=0}^{[\sqrt{n}]}\int_{D(x,r_n)} p_t(x,y)\,dy+\frac{1}{n}\sum_{t=[\sqrt{n}]+1}^{n-1}\int_{D(x,r_n)} p_t(x,y)\,dy$$
$$\leq\frac{[\sqrt{n}]+1}{n}+\int_{D(x,r_n)} p_{\sqrt{n}}(x,y)\,dy\longrightarrow_{n\to\infty}0.\quad\square$$

DEFINITION. Let $R: L_x\backslash\{x\}\to T^1L_x\subset T^1\mathcal{L}$ be the inward pointing radial unit vector field. Namely, $R(y)$ is the velocity at y of the only geodesic starting at y and passing through x. We consider the measures $\mu_n=R_*\delta_x^{(n)}$ and $\nu_n=R_*\eta_n$.

The measures μ_n are probabilities and the ν_n are not, but, according to Lemma 2.2, these sequences have the same limit points.

As the η_n are convex combinations of the normalized lengths of circles whose radii are larger than r_n, the ν_n are convex combinations of the push-forwards of these measures by R, which we shall call $\lambda_r(x)$, r being the radius.

LEMMA 2.3. *The limit points of the sequence μ_n –equivalently ν_n– are probability measures on $T^1\mathcal{L}$ invariant under the laminated horocycle flow.*

Proof:

Since the lamination $T^1\mathcal{L}$ is oriented, it is possible to define for each $r>0$ a flow φ^r in $T^1\mathcal{L}$ whose orbits (which we shall parametrize by unit lenght) are the circles of radius r with the unit normal vector pointing inwards. Remark that the measures $\lambda_r(x)$ are invariant under the flow φ^r. When $r\to\infty$, the angle between φ^r and the horocycle flow tends to zero uniformly. This means that for any positive δ, if r is large enough, $d(\varphi_t^r(y),h_t(y))<\delta$, for any $y\in T^1\mathcal{L}$ and any time t between -1 and 1. (See for example Theorem 3.8 in [B-N], which is easily proved using Gronwall's inequality.)

Let μ be a limit point of the sequence ν_n, and therefore of the sequence μ_n. We will see that μ is invariant under h_t for all times t between -1 and 1.

Let f be a continuous function on $T^1\mathcal{L}$, $\varepsilon>0$, and $\delta>0$ as in the definition of uniform continuity.

For a fixed t,

$$|\mu(f)-\mu(f\circ h_t)|\leq|\mu_n(f)-\mu_n(f\circ h_t)|+\varepsilon,$$

if n is large enough and such that μ_n belongs to the subsequence converging to μ.

Now, taking a bigger n if necessary,

$$|\mu_n(f)-\mu_n(f\circ h_t)|=$$
$$=\left|\frac{1}{n}\sum_{t=0}^{n-1}\int_{r_n}^{\infty}p_t(r)\mathrm{long}(S_r)\left[\int f(y)\,d(\lambda_r(x))(y)-\int f\circ h_t(y)\,d(\lambda_r(x))(y)\right]\right|$$
$$\leq\frac{1}{n}\sum_{t=0}^{n-1}\int_{r_n}^{\infty}p_t(r)\mathrm{long}(S_r)\int|f\circ\varphi_t^r(y)-f\circ h_t(y)|\,d(\lambda_r(x))(y)$$
$$<\varepsilon\frac{1}{n}\sum_{t=0}^{n-1}\int_{r_n}^{\infty}p_t(r)\mathrm{long}(S_r)<\varepsilon,$$

which achieves the proof of Lemma 2.3. $\square$

Step 2: Ergodic measure generated by a leaf which is not necessarily simply connected.

Now we can drop our assumption that the leaf L_x is simply connected. Of course the definition of the vector field R on L_x no longer makes sense, but we can define R and the measures $\delta_x^{(n)}$ on the universal cover of the leaf and λ_r, ν_n and μ_n on its unit tangent bundle, and then project all the measures on $T^1\mathfrak{H}$ onto $T^1\mathcal{L}$. Remark that the Birkhoff sums for the δ_x and the heat diffusion in the universal cover of L_x project onto those corresponding to L_x, with the heat kernel of the leaf. And as for the measures on $T^1\mathfrak{H}$, their projections are still convex combinations of those of λ_r, and the whole argument holds.

Since the μ_n project onto the Birkhoff sums $\delta_x^{(n)}$, any of its limit points (which in fact is only one) projects onto $\tilde{\delta}_x$, and a consequence of Lemma 2.3 is the following:

LEMMA 2.4. *Any ergodic harmonic measure is the projection of a measure on $T^1\mathcal{L}$ invariant under the laminated horocycle flow.*

Step 3: Harmonic measure not necessarily ergodic.

Now let μ be any harmonic measure on $\mathcal{L}$. The measures $\tilde{\delta}_x$ exist for x in a full-measure subset of $\mathcal{L}$, and the ergodic decomposition theorem for harmonic measures states that for any continuous function f on $\mathcal{L}$

$$\int_{\mathcal{L}} f\,d\mu=\int_{\mathcal{L}}\left(\int_{\mathcal{L}} f\,d\tilde{\delta}_x\right)\,d\mu(x).$$

For each $\tilde{\delta}_x$, let α_x be the measure invariant under the horocycle flow that projects onto $\tilde{\delta}_x$, constructed as above. Remark that the map $x\mapsto\alpha_x$ is measurable, since it is the limit of continuous functions.

DEFINITION. We define the measure β on $T^1\mathcal{L}$ by the expression

$$\int_{T^1\mathcal{L}} f\,d\beta=\int_{\mathcal{L}}\left(\int_{T^1\mathcal{L}} f\,d\alpha_x\right)\,d\mu(x).$$

β is clearly a measure invariant under the horocycle flow that projects onto μ.

This achieves the proof of

THEOREM 2.5. *Every harmonic measure on $\mathcal{L}$ is the projection of a measure invariant under the horocycle laminated flow on $T^1\mathcal{L}$.*

3 The flows g and h and harmonic measures

This section is devoted to the proof of Theorem 3.2, which will be done in two steps. The first consists of looking at a single leaf of the weakly stable foliation of the geodesic flow on the unit tangent bundle $T^1\mathfrak{H}$ to the hyperbolic plane, and studying measures invariant under both the geodesic and the horocycle flows there. The second is to disintegrate a measure invariant under the laminated flows in a chart of the lamination $T^1\mathcal{L}$, obtaining conditional measures which are of the type studied in step one, and to use our understanding of these to see how the whole measure projects onto $\mathcal{L}$.

Step 1: A leaf of the weakly stable foliation of the geodesic flow in $T^1\mathfrak{H}$.

We will work on the upper-half plane model of the hyperbolic plane $\mathfrak{H}$. In this model, the leaf of the weakly stable foliation of the geodesic flow consisting of all geodesics that have ∞ as an endpoint can be identified to the affine group

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}^+, b \in \mathbb{R} \right\},$$

since the left action of B on this leaf sending the vertical unit vector based on (x, y) to the one based on $(a^2x + ab, a^2y)$ is free and transitive.

This action of B, that corresponds under the identification to the left translation in B, is an action by isometries, and this tells us that the hyperbolic area is nothing but a left Haar measure–which we will call μ. Saying that a measure ν on B is invariant under both the geodesic and the horocyle flows is simply stating that it is invariant under right translations in B; namely, that it is a multiple of the right Haar measure. We wish to compare μ and ν, which amounts to computing their Radon-Nikodym derivative, as they are in the same measure class.

REMARK. *The Radon-Nikodym derivative $f = \frac{d\nu}{d\mu}$–which is, up to a scalar multiple, the right modular function of B–is harmonic.*

Proof:

An elementary calculation shows that this function is in fact, up to a scalar multiple,

$$f\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^2.$$

In terms of the upper-half plane (with the vertical vector at each point) this means that $f(x, y) = y$, which is harmonic in the euclidean sense. It is not difficult to see, from the definition of the Laplacian and the hyperbolic metric, that this is the same as being harmonic in the hyperbolic sense. □

Thanks to the preceding remark, we have

LEMMA 3.1. *Let L be a leaf of the weakly stable foliation of the geodesic flow on $T^1\mathfrak{H}$. If ν is a measure on L invariant under the geodesic and stable horocycle flows, it is a harmonic function times the area.*

Proof:

Let p be the point at infinity at which all geodesics on L end, and L_∞ the leaf of all geodesics ending at ∞. If φ is the isometry that takes L to L_∞, μ_L is the area and ν_L is invariant under both flows, the derivative $\frac{d\nu_L}{d\mu_L}$ is $f \circ \varphi$, where f is the harmonic function spoken of above. □

According to [N], measures that have the form "harmonic function×area" are harmonic; namely the Laplacian of a smooth function with compact suport with respect to these measures is zero.

Step 2: Measures invariant under g and h on $T^1\mathcal{L}$.

Let $\pi : T^1\mathcal{L} \to \mathcal{L}$ be the projection, and ν a measure on $T^1\mathcal{L}$ invariant under the geodesic and the horocycle flows. Let $E' = T \times U$ be a flow box in $\mathcal{L}$, whose preimage by π is of the form $E = T \times V = T \times (S \times U)$, where S is a circle and U an open subset of the hyperbolic plane. The variables t, s and u refer to points in T, S and U respectively.

To prove that $\mu = \pi_*\nu$ is a harmonic measure on $\mathcal{L}$, we will see that the integral with respect to μ of the Laplacian of a function that has compact support in E' is zero. To carry out this computation, we will disintegrate the measures ν and μ in E and E' respectively, and use lemma 3.1.

$$\int_{E'} f\, d\mu = \int_E f \circ \pi\, d\nu$$
$$= \int_T \int_{\{t\}\times V = V_t} f \circ \pi\, d\nu_t\, d\hat{\nu}(t),$$

where the ν_t are the conditional measures corresponding to the disintegration of ν on the subsets V_t of E, and $\hat{\nu}$ is the push-forward of ν under the projection of E onto T. On the other hand,

$$\int_{E'} f\, d\mu = \int_T \int_{\{t\}\times U = U_t} f\, d\mu_t\, d\hat{\mu}(t)$$
$$= \int_T \int_{\{t\}\times U = U_t} f\, d\mu_t\, d\hat{\nu}(t),$$

where the μ_t are the conditional measures corresponding to the disintegration of μ on the plaques of E', and $\hat{\mu}$ is the push-forward of μ under the projection of E' onto T. Remark that $\hat{\nu}$ coincides with $\hat{\mu}$.

The uniqueness of the decomposition implies that, for $\hat{\nu}$-almost all t, $\mu_t = \pi_* \nu_t$.

We will disintegrate each ν_t on $V_t = (S \times U)_t = S_t \times U_t$:

$$\int_{V_t} f \, d\nu_t = \int_{S_t} \int_{(U_t)_s} f \circ \pi \, d(\nu_t)_s \, d\eta_t(s),$$

where η_t is a probability measure on the circle $S_t = \{t\} \times S$.

We are now ready to compute $\int_{E'} \Delta f \, d\mu$, for a continuous function f which is twice differentiable along the leaves and that has compact support contained in E'.

$$\begin{aligned}\int_{E'} \Delta f \, d\mu &= \int_{E'} \Delta f \circ \pi \, d\nu \\ &= \int_T \int_{S_t} \int_{(U_t)_s} \Delta f \circ \pi \, d(\nu_t)_s \, d\eta_t(s) \, d\hat{\nu} \\ &= \int_T \int_{S_t} \int_{\pi((U_t)_s) = U_t} \Delta f \, d\pi_*(\nu_t)_s \, d\eta_t(s) \, d\hat{\nu}.\end{aligned}$$

Since the measures $(\nu_t)_s$ are harmonic on $(U_t)_s$, and the restriction of π to $(U_t)_s$ is an isometry onto U_t, we have, for all $(t, s) \in T \times S$,

$$\int_{U_t} \Delta f \, d\pi_*(\nu_t)_s = 0,$$

which achieves the proof of

THEOREM 3.2. *Any measure invariant under both the horocycle and the geodesic flows, projects onto a harmonic measure.*

REMARK. *There are measures invariant under both the geodesic and the horocycle flows.*

Proof:

A simple way to see this is the following: Take a probability measure μ invariant under the horocycle flow, and take any limit point of the set of measures defined by

$$f \mapsto \frac{1}{2t} \int_{-t}^{t} \int f \circ g_s \, d\mu \, ds. \quad \square$$

CHAPTER III

HILBERT MODULAR FOLIATIONS

If, as before, $\mathfrak{H}$ is the hyperbolic plane, the group of isometries of $\mathfrak{H} \times \mathfrak{H}$ is isomorphic to

$$PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R}) \times \mathbb{Z}_2.$$

In fact, the two copies of $PSL(2, \mathbb{R})$ correspond to the isometries of each factor in $\mathfrak{H} \times \mathfrak{H}$ and the remaining $\mathbb{Z}_2$ represents the isometry that interchanges the two factors.

Recall that a discrete subgroup Γ of a Lie Group G is *irreducible* if its projection onto all normal proper subgroups of G is dense.

In this chapter we will consider a family of quotients of $\mathfrak{H} \times \mathfrak{H}$ by irreducible lattices

$$\Gamma \subset PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$$

having finite volume, which are not compact. These spaces have two natural transverse (singular) foliations, that come from the "horizontal" and "vertical" foliations in $\mathfrak{H} \times \mathfrak{H}$. One of them will be called $\mathcal{F}$.

In section 1 we will define these foliated spaces: Hilbert modular surfaces. (See [H], Ch.5.) $\mathcal{F}$ is called *Hilbert modular foliation.* In section 2 we will see that $(M, \mathcal{F})$ has only one harmonic measure: the volume. Much of what is said in this section is true for other quotients of $\mathfrak{H} \times \mathfrak{H}$ by irreducible lattices, although we will restrict ourselves to working with Hilbert modular foliations. Section 3 includes some considerations on measures invariant under the horocycle foliated flow.

1 Definition of Hilbert modular foliations

Let us consider a real algebraic extension of degree two of the field of rational numbers, of the form $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer. The ring of algebraic integers of K, which we will denote by $\mathfrak{O}_d$, is of the form

$$\mathfrak{O}_d = \mathbb{Z} + \mathbb{Z}\alpha,$$

α being $\sqrt{d}$ if $d \equiv 2, 3 \bmod 4$, and $\frac{1+\sqrt{d}}{2}$ otherwise. If $x = a + b\alpha$ is an element of $\mathfrak{O}_d$, its Galois conjugate is $\bar{x} = a - b\alpha$.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Let us consider the following group homomorphism:

$$PSL(2, \mathfrak{O}_d) \to PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$$
$$A \mapsto (A, \bar{A}),$$

where $\bar{A}$ is the matrix obtained by taking the Galois conjugates of the elements of A. The image of this morphism, which we will call Γ, is an irreducible lattice in $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$, and the space of orbits of the action of Γ on $\mathfrak{H} \times \mathfrak{H}$ is a normal complex space, that we will call $M(d)$, or simply M.

This space has a finite number of singularities, is non-compact, and can be compactified by a finite number of points (the "cusps") in order to obtain an analytic space, which by means of dissingularization gives a smooth complex surface known as the *Hilbert modular surface* (see [H]). Nevertheless, we will restrict ourselves to working with the space M.

The "horizontal" foliation in $\mathfrak{H} \times \mathfrak{H}$, whose leaves are of the form $\mathfrak{H} \times \{z\}$, induces on the quotient M a foliation by hyperbolic Riemann surfaces (or orbifolds), that will be called $\mathcal{F}$. Remark that the irreducibility of Γ implies that $\mathcal{F}$ is minimal–that is, all leaves are dense. Therefore, all leaves are non-compact, and in fact all but a finite number of leaves are hyperbolic planes. It will be seen in section 3 what leaves which are not simply connected look like.

2 The volume measure

As we stated before, the purpose of this section is to prove that $\mathcal{F}$ has only one harmonic measure.

REMARK. *Recall that M has isolated singularities. For any point $x \in M$, the set of continuous paths starting at x and lying on the leaf of $\mathcal{F}$ through x that reach a singular point has Weiner measure zero. Consequently, although M is not a manifold, its singular points are irrelevant when studying the foliated heat flow, and therefore we can forget them in our study of the harmonic measures.*

On $(M, \mathcal{F})$ the ergodic decomposition theorem for harmonic measures still holds. Ergodic harmonic measures are the limits of Birkhoff means of the diffusion of Dirac deltas; that is, measures of the form

$$\bar{\delta}_x = \lim_n \sum_{t+0}^{n-1} D_t \delta_x.$$

This limit exists and is a probability measure which depends only on the leaf passing through x, for almost all x according to any harmonic probability measure–if there are any. This can be read in [Y] (chapter XIII), observing that the foliated heat semigroup acts on the space

$$\mathcal{S} = \{f : M \to \mathbb{R} \ : \ f \text{ is continuous and } \lim_{x\to\infty} f(x) = 0\}.$$

As the lattice Γ is acting by isometries on $\mathfrak{H} \times \mathfrak{H}$, the Riemannian metric on $\mathfrak{H} \times \mathfrak{H}$ induces a Riemannian metric on M that makes M locally isometric to $\mathfrak{H} \times \mathfrak{H}$. We will take V to be the volume measure associated to this metric, suitably normalized so that $V(M) = 1$.

It can be easily seen that V is a harmonic measure on M, and in this section we will prove that all the measures $\bar{\delta}_x$ coincide. This of course implies that V is the only harmonic probability measure on M.

LEMMA 2.1. *The volume V is a harmonic measure on M.*

This lemma holds because the volume induces on transversals a holonomy invariant measure. Namely, it is locally of the form "(holonomy invariant measure) $\times$ (area of the leaves)". □

LEMMA 2.2. *There is a subset of M that has full probability according to any harmonic measure, and where the ergodic harmonic measure $\bar{\delta}_x$ is independent of the point x.*

This has the following immediate consequence:

PROPOSITION 2.3. *$(M, \mathcal{F})$ has a unique harmonic measure, which is V.*

REMARK 2.4. *If L is a hyperbolic Riemann surface and $\mathfrak{H}$ is the hyperbolic plane, let p and $\hat{p}$ be the heat kernels on L and on $\mathfrak{H}$ respectively. If $\hat{x}$ is a point in $\mathfrak{H}$ that projects onto $x \in L$, then*

$$p(t, x, \cdot) = \sum_{\alpha \in \pi_1(L)} \hat{p}(t, \alpha\hat{x}, \cdot).$$

This is, of course, a very general fact, not only concerning hyperbolic Riemann surfaces. It can be found, for instance, in [Ch], but it is easy to conclude from the fact that L and $\mathfrak{H}$ are locally isometric, and therefore posess the same Laplace operator.

Proof of Lemma 2.2:

Let π be the canonical projection $\mathfrak{H} \times \mathfrak{H} \to M$ and G the dense subgroup $PSL(2,\mathbb{R}) \times \mathfrak{O}_d$ of $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$. We take two points x and y in M for which the measures $\bar{\delta}_x$ and $\bar{\delta}_y$ exist and are probabilities only depending on the leaves through x and y, respectivley, and two points $\hat{x}, \hat{y} = (y_1, y_2) \in \mathfrak{H} \times \mathfrak{H}$ such that $\pi(\hat{x}) = x$ and $\pi(\hat{y}) = y$. There is a sequence of isometries $\{\alpha_n\} \subset G$ such that $\alpha_n(\hat{x}) = \hat{x}_n = (y_1, z_n)$ with $\lim_n z_n = y_2$ in $\mathfrak{H}$. We call $\bar{\delta}_z^{(m)}$ the m-th Birkhoff mean $\frac{1}{m}\sum_{t=0}^{m-1} D_t\delta_z$ at a point $z \in M$, and $\hat{\delta}_z^{(m)}$ the m-th mean at a point z in $\mathfrak{H} \times \mathfrak{H}$.

Let f be a continuous function on M such that $\lim_{x\to\infty} f(x) = 0$–that is, an element of the space $\mathcal{S}$.

Since $\pi(\hat{x}_n)$ lies in the same leaf as x for all n,

$$\lim_{m\to\infty} \delta^{(m)}_{\pi(\hat{x}_n)} = \bar{\delta}_x.$$

On the other hand, the function $\hat{f} = f \circ \pi$ is uniformly continuous, since f is and π is a local isometry, so if n is large enough,

$$|\hat{f}(z, z_n) - \hat{f}(z, y_2)| < \varepsilon,$$

for any x in $\mathfrak{H}$. If $\hat{p}$ is the heat kernel on the hyperbolic plane and $\{\hat{D}_t\}$ the foliated heat semigroup in $\mathfrak{H} \times \mathfrak{H}$,

$$|\hat{D}_t\hat{f}(\hat{x}_n) - \hat{D}_t\hat{f}(\hat{y})| = \left| \int_{\mathfrak{H}} \hat{p}_t(y_1, z)[f(z, z_n) - f(z, y_2)]\, dz \right| < \varepsilon.$$

Therefore, for big values of n and any m, $\hat{\delta}^{(m)}_{\hat{x}_n}(\hat{f})$ and $\hat{\delta}^{(m)}_{\hat{y}}(\hat{f})$ are ε-close. Observing that $\hat{\delta}^{(m)}_{\hat{y}}(\hat{f}) = \delta^{(m)}_{y}(f)$ completes the proof. □

3 Measures invariant under foliated flows

In this section, we will consider the geodesic and horocycle foliated flows in $T^1\mathcal{F} = \Gamma\backslash(PSL(2,\mathbb{R}) \times \mathfrak{H})$, and see how these relate to the volume measure V on M. $PSL(2,\mathbb{R})$ acts on $T^1\mathcal{F}$ by right translations on the first factor, and the geodesic and horocycle flows come from the actions of the subgroups

$$D = \left\{ \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \right\} \text{ and } H_+ = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\},$$

respectively.

Theorem II.2.5 leads us to expect the existence of a measure invariant under the horocycle flow that projects onto V. Remark that, although we are not in the hypotheses of this theorem –since M is noncompact and $\mathcal{F}$ is singular–, its proof still holds in this case. Such a measure is easily identifiable:

REMARK. *The measure on $T^1\mathcal{F}$ coming from Liouville×hyperbolic area (which we will call μ) is invariant under the right action of $PSL(2,\mathbb{R})$, and in particular under the foliated geodesic and horocycle flows. So it is a probability that satisfies the thesis of theorem II.1.5., and it is also an example of theorem II.3.2.*

The measure μ is ergodic for both flows, and we know this from Moore's ergodicity theorem, which we shall state below. For its proof, see [Z], Chapter II.

DEFINITION: We say that the action of a group G in a measure space $(S, \mathcal{A}, \mu)$ is *irreducible* if all normal subgroups of G are ergodic.

MOORE'S ERGODICITY THEOREM. *Let G be a semisimple Lie group acting irreducibly on a probability space $(S, \mathcal{A}, \mu)$. If H is a closed, noncompact subgroup of G, its action on S is ergodic.*

COROLLARY. *The volume on $T^1\mathcal{F}$ is ergodic for the three foliated flows g, h^+ and h^-.*

Proof:

Take $G = PSL(2,\mathbb{R})$, which is simple. The irreducibility of the lattice Γ implies the ergodicity of the action (by right translations on the first factor) of the group G on $(\Gamma\backslash(G \times G)$,projection of the Haar measure) and therefore of the action of G on $(T^1\mathcal{F}, \mu)$. □

Next, we will see that the horocycle flow is not uniquely ergodic on $T^1\mathcal{F}$, by exhibiting another ergodic invariant measure. We will show that this measure is nontrivial, in the sense that it is not a measure supported on a closed orbit–because there are no closed orbits– and that it is not invariant under the geodesic flow.

PROPOSITION 3.1. *The leaves of $\mathcal{F}$ are either simply connected or hyperbolic cylinders.*

COROLLARY. *The horocycle foliated flow has no closed orbits.*

Proof of Proposition 3.1:

Take a leaf L of $\mathcal{F}$, and a point $w \in \mathfrak{H}$ such that $\mathfrak{H} \times \{w\}$ projects onto L via $\mathfrak{H} \times \mathfrak{H} \to \Gamma\backslash(\mathfrak{H} \times \mathfrak{H})$. Two points (x, w) and (y, w) in $\mathfrak{H} \times \mathfrak{H}$ project onto the same point in M if and only if there is a matrix A in $PSL(2, \mathfrak{O}_d)$ that sends x to y and whose Galois conjugate $\bar{A}$ has w as a fixed point. This means that the fundamental group of L is a quotient of the abelian group $\{A \in PSL(2, \mathfrak{O}_d) : \bar{A}(w) = w\}$, and is therefore abelian. On the other hand, $\pi_1 L$ is free since L is noncompact, so it must be either isomorphic to $\mathbb{Z}$ or trivial. This implies that the leaf L is, topologically, a disk or a cylinder.

As a hyperbolic surface, a leaf L which is not simply connected may be of two types: a hyperbolic cylinder or a parabolic one. (Only in the latter case do we have closed horocycles.) L is parabolic when its fundamental group is generated by a parabolic element in $PSL(2, \mathfrak{O}_d) \subset PSL(2,\mathbb{R})$. But if A is parabolic, its conjugate $\bar{A}$ is also parabolic, and therefore has no fixed point in $\mathfrak{H}$ –which means that A does not belong to the fundamental group of any leaf, completing the proof. □

EXAMPLE 3.2

Consider the Lie subgroup

$$P = \left\{ \left(\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) ;\ s, t \in \mathbb{R} \right\}$$

of $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$, which is isomorphic to the plane $\mathbb{R}^2$. To see that P intersects Γ in a lattice, it is enough to give two linearly independent elements of $P \cap \Gamma$; for example,

$$\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) \text{ and } \left(\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & -\alpha\\ 0 & 1\end{pmatrix}\right),$$

where α is the irrational number spoken of in section 1. This means that $T = \Gamma\backslash P$ is a torus in $\Gamma\backslash(PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}))$ that is invariant under the action of H_+ by right translations in the first factor. The probability ν on T which comes from the Haar measure of P is invariant under this action, and Moore's theorem says that it is ergodic. It is easy to see that ν projects onto a nontrivial ergodic measure for the horocycle flow on $T^1\mathcal{F}$. Of course, the torus T is not invariant under the action of the diagonal group D, so the measure ν is not invariant under the geodesic flow.

CHAPTER IV

RICATTI FOLIATIONS

Let S be a hyperbolic Riemann surface of finite area and ρ a representation of the fundamental group of S in $\mathbb{C}P^n$; that is a group homomorphism

$$\rho : \pi_1 S \to PSL(n+1, \mathbb{C}).$$

Throughout the chapter, n will be 1 or 2.

The suspension of this representation gives a foliated manifold $(M, \mathcal{F})$, and the following commutative diagram, as in the Appendix:

$$\begin{array}{ccc} \mathfrak{H} \times \mathbb{C}P^n & \xrightarrow{\ p\ } & M \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q} \\ \mathfrak{H} & \xrightarrow[P]{} & S \end{array}$$

The restriction of q to each leaf of $\mathcal{F}$ is a covering map, and $q : M \to S$ is a fibration over S whose fiber $\mathbb{C}P^n$ is transverse to the foliation.

As before, on each leaf of $\mathcal{F}$ we consider the Poincaré metric.

We will study the harmonic measures of this foliation (and this metric), relating them to measures invariant under the geodesic and horocycle flows, that are studied in [B-GM] and [B-GM-V].

The unit tangent bundle $T^1\mathcal{F}$ to the foliation $\mathcal{F}$, where these flows are defined, can be obtained as a quotient E of $T^1\mathfrak{H} \times \mathbb{C}P^n$ exactly as before, which gives a commutative diagram

$$\begin{array}{ccc} T^1\mathfrak{H} \times \mathbb{C}P^n & \xrightarrow{\ p'\ } & E \\ {\scriptstyle p_1'}\downarrow & & \downarrow{\scriptstyle q'} \\ T^1\mathfrak{H} & \xrightarrow[P']{} & T^1 S \end{array}$$

Via q', the foliated geodesic and horocycle flows project onto the geodesic and horocycle flows in T^1S, which are both ergodic with respect to the Liouville measure. Let us call π the projection from E to M.

Let x be a point in M. L the leaf of $\mathcal{F}$ through x, whose universal cover is the hyperbolic plane. Fix x_0 in $\mathfrak{H}$ that projects onto x.

DEFINITIONS: The circles containing points at distance r from x_0 in $\mathfrak{H}$ have a length measure that, after normalizing, we call $\lambda_r(x_0)$. The projection of this measure is a probability on M whose support is contained in L, that we call $\lambda_r(x)$.

In [B-GM-V], the foliated horocycle and geodesic flows on E are studied. Under certain conditions on the representation ρ, there exist measures called μ_+ and μ_- on $E = T^1\mathcal{F}$, that satisfy the following:

1) μ_+ is invariant under the foliated geodesic and stable horocycle flows, and for almost all v in E (according to the Lebesgue measure class) and any continuous function f, the measure μ_+ gives the statistics for the orbit of v under g_t in the future; i.e.

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T f\circ g_t(v)\,dt = \int_E f\,d\mu_+.$$

2) μ_- is invariant under the foliated geodesic and unstable horocycle flows, and for almost all v in E (according to the Lebesgue measure class) and any continuous function f, the measure μ_- gives the statistics for the orbit of v under g_t in the past; i.e.

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T f\circ g_{-t}(v)\,dt = \int_E f\,d\mu_-.$$

3) The measure μ_+ (μ_-) gives the statistics of Lebesgue-almost every orbit of the foliated stable (unstable) horocycle flow, both for the past and the future.

We will consider a representation ρ whose image $\rho(\pi_1 S)$ does not leave any invariant measure on $\mathbb{C}P^n$. This condition is generic on ρ (see [J]), and it is enough to guarantee the existence of the measures μ_+ and μ_- described above. The following theorem can be found in [B-GM] in a more general version:

THEOREM. (Bonatti, Gómez-Mont) *If $\rho(\pi_1 S)$ leaves no invariant measure on $\mathbb{C}P^n$ ($n = 1$ or 2), then there is a probability measure ν on M such that for any x and any sequence $\{r_n\}$ of positive real numbers going to infinity, the sequence $\{\lambda_{r_n}(x)\}$ converges weakly to ν. Furthermore, ν is the projection to M of both μ_+ and μ_-.*

Let x be any point in M, and consider the Birkhoff means

$$\delta_x^{(n)} = \frac{1}{n}\sum_{t=0}^{n-1} D_t\delta_x.$$

We know from [Ga] that these means have a limit which is a probability for almost every x with respect to any harmonic measure; nevertheless, in the non-compact case we do not yet know if there are any harmonic measures.

PROPOSITION 1. *The limit*

$$\bar{\delta}_x = \lim \delta_x^{(n)}$$

exists and is equal to ν for every point x.

As any harmonic measure can be decomposed into measures of this type, we have the following corollary:

COROLLARY 2. *ν is the unique harmonic measure of the (generic) Riccati foliation.*

The proof of this fact is very similar to and easier than the proof of Theorem II.2.5, and its main ingredient is Lemma II.2.1.

Proof of Proposition 1:

If $\pi_L : \mathfrak{H} \to L$ is the universal cover of the leaf L and $\{\hat{D}_t\}$ is the heat semigroup on $\mathfrak{H}$, we will call $\hat{\delta}_{x_0}^{(n)}$ the probablility on $\mathfrak{H}$ given by the expression $\frac{1}{n}\sum_{t=0}^{n-1}\hat{D}_t\delta_{x_0}$. It is clear that $\hat{\delta}_{x_0}^{(n)}$ projects onto $\delta_x^{(n)}$.

We can approximate the measures $\hat{\delta}_{x_0}^{(n)}$ by

$$\hat{\eta}_{x_0}^{(n)} = \frac{1}{n}\sum_{t=[h(\sqrt{n})]}^{n-1} \hat{D}_t\delta_{x_0},$$

where h is the function given by Lemma II.2.1 and $[x]$ stands for the largest integer smaller than or equal to the real number x. Exactly as in Chapter II, the $\hat{\eta}_{x_0}^{(n)}$ are not probabilities but their mass tends to one. Their projections, $\eta_x^{(n)} = (\pi_L)_*\hat{\eta}_{x_0}^{(n)}$ are measures on M whose mass approaches 1 and that can be written as convex combinations of the measures $\lambda_r(x)$ for large values of r. They must therefore have ν as their limit. This implies, again as in Chapter II, that ν must be the weak limit of the sequence $\delta_x^{(n)}$. □

APPENDIX

EXAMPLES OF COMPACT HYPERBOLIC SURFACE LAMINATIONS

Example 1: Surfaces and product laminations

A compact hyperbolic Riemann surface S is obviously a compact hyperbolic surface lamination. So is a product of the form $S \times T$, where T is a compact metrizable space.

Example 2: Suspensions

Let S be a compact hyperbolic Riemann surface and F a compact metrizable space. An automorphism of F is simply a homeomorphism from F to itself if we think of F as an object in the category τ of topological spaces. If F belonged to a subcategory of τ (e.g. that of differentiable manifolds, complex manifolds), we could consider automorphisms of F to be maps in this subcategory. We call $\mathrm{Aut}(F)$ the group of automorphisms of F.

Consider a group homomorphism

$$\rho : \pi_1(S) \to \mathrm{Aut}(F).$$

On the product $\mathfrak{H} \times F$, consider the equivalence relation given by

$$(x, f) \sim (\gamma \cdot x, \rho(\gamma) \cdot f),$$

where the elements γ of $\pi_1(S)$ act by deck transformations on the first factor of $\mathfrak{H} \times F$ and via the representation ρ on the second. This action preserves the "horizontal" foliation, whose leaves are of the form $\mathfrak{H} \times \{f\}$, and therefore the quotient

$$\mathcal{L} = \frac{\mathfrak{H} \times F}{\sim}$$

is a compact lamination by hyperbolic Riemann surfaces, which is called *suspension of* ρ. If we call p the projection $\mathfrak{H} \times F \to \mathcal{L}$, $P : \mathfrak{H} \to S$ the universal covering of S and $p_1 : \mathfrak{H} \times F \to \mathfrak{H}$ the projection onto the first factor, there is a map q from $\mathcal{L}$ onto S that makes the following diagram commute:

$$\begin{array}{ccc} \mathfrak{H} \times F & \xrightarrow{p} & \mathcal{L} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q} \\ \mathfrak{H} & \xrightarrow[P]{} & S \end{array}$$

The restriction of q to each leaf of $\mathcal{L}$ is a covering map, and $q : \mathcal{L} \to S$ is a fibration over S whose fiber is isomorphic to F. If F is a differentiable manifold, $\mathcal{L}$ is itself a foliated differentiable manifold and the fibration q is transverse to the foliation.

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Example 3: Weakly stable and unstable foliations for the geodesic flow on the unit tangent bundle of a compact hyperbolic surface

Consider a point p in the circle at infinity $\mathbb{R} \cup \{\infty\}$ of the hyperbolic plane. For any point $x \in \mathfrak{H}$, there is a unique vector $v(x,p) \in T^1_x\mathfrak{H}$ that directs a geodesic starting at x and heading towards p. The set

$$L_p^+ = \{v(x,p) \; : \; x \in \mathfrak{H}\}$$

is a hyperbolic plane embedded in $T^1\mathfrak{H}$. As p varies, the manifolds L_p^+ determine a foliation $\tilde{\mathcal{F}}$ of $T^1\mathfrak{H}$ by hyperbolic planes. It is called 'weakly stable' foliation for the geodesic flow in $T^1\mathfrak{H}$.

If $\Gamma \subset PSL(2,\mathbb{R})$ is a discrete subgroup such that $S = \Gamma\backslash\mathfrak{H}$ is smooth and compact, $\tilde{\mathcal{F}}$ projects onto a foliation $\mathcal{F}$ by hyperbolic surfaces in $T^1S = \Gamma\backslash T^1\mathfrak{H}$. This is the 'weakly stable' foliation for the geodesic flow on T^1S.

If $p \in \mathbb{R} \cup \{\infty\}$ and $x \in \mathfrak{H}$, we may also consider the vector $u(x,p) \in T^1_x\mathfrak{H}$ that directs the unique geodesic starting at x and coming from p (namely, heading towards p in the past).

The sets of the form

$$L_p^- = \{u(x,p) \; : \; x \in \mathfrak{H}\}$$

determine a foliation whose projection onto T^1S is called the 'weakly unstable' foliation for the geodesic flow on T^1S.

Example 4: Sullivan's lamination (Notes by Manuel Cruz)

A. The dyadic solenoid

I. *The dyadic completion of* $\mathbb{Z}$: If $n < m$, then $2^m\mathbb{Z}$ is a subgroup of $2^n\mathbb{Z}$ and there is a well defined quotient application

$$\rho_{nm} : \mathbb{Z}/2^m\mathbb{Z} \to \mathbb{Z}/2^n\mathbb{Z}, \quad x \bmod 2^m\mathbb{Z} \mapsto x \bmod 2^n\mathbb{Z}.$$

This determines an inverse directed system $\{\mathbb{Z}/2^n\mathbb{Z}, \rho_{nm}\}$ whose inverse limit is the *dyadic completion of* $\mathbb{Z}$ which is denoted by $\mathbb{Z}_2$. $\mathbb{Z}_2$, with the profinite topology, is an abelian, compact, perfect and totally disconnected topological group which is homeomorphic to the dyadic Cantor set.

II. *The dyadic solenoid:* From the theory of covering spaces, we know that, for every $n \in \mathbb{N}$, there is a non-ramified covering space $p_n : X_n \to S^1$ of degree 2^n. Here, we canonically identify X_n with $\mathbb{R}/2^n\mathbb{Z}$. Then, for any n, m with $n \leq m$, we have the corresponding covering maps $p_n : X_n \to S^1$ and $p_m : X_m \to S^1$. Therefore, there is a unique covering map $p_{nm} : X_m \to X_n$ such that $p_n \circ p_{nm} = p_m$. This determines an inverse directed system $\{X_n, p_n\}_{n \in \mathbb{N}}$, whose inverse limit is the *dyadic solenoid*

$$\mathbb{S}_2 = \varprojlim X_n,$$

with canonical projection $\pi : \mathbb{S} \to S^1$, determined by the projection of coordinates, that determines a structure of $\mathbb{Z}_2$-principal bundle.

$\mathbb{S}_2$ is an abelian, compact, conected topological group, and each leaf of this unidimensional lamination is a simply connected manifold of dimension 1, homeomorphic to the universal cover $\mathbb{R}$ of S^1. A typical fiber of this projection is homeomorphic to $\mathbb{Z}_2$. Furthermore, every leaf of this lamination is dense.

B.SULLIVAN'S LAMINATION

I. *Natural extension:* If $f : S^1 \to S^1$ is the two-to-one covering map given by $z \mapsto z^2$, then f induces a function $\tilde{f} : \mathbb{S}_2 \to \mathbb{S}_2$, called *natural extension* of f, given by

$$\tilde{f}(\ldots, z_n, \ldots, z_1, z_0) := (\ldots, z_n, \ldots, z_1, z_0, f(z_0)).$$

Clearly $\tilde{f}$ is a homeomorphism since its inverse is the shift map "forgetting the first coordinate". Furthermore, $\tilde{f}$ covers f: $\pi \circ \tilde{f} = f \circ \pi$.

II. *Hyperbolization:* Let us denote by $L_{\tilde{z}}$ the leaf passing through $\tilde{z} \in \mathbb{S}_2$, and by $T_{\tilde{z}}L_{\tilde{z}}$ the line tangent to $L_{\tilde{z}}$ at $\tilde{z}$. We define the lamination $\tilde{\mathcal{L}}$ associated to f by

$$\tilde{\mathcal{L}} := \{(\tilde{z}, v) \in \mathbb{S}_2 \times T_{\tilde{z}}L_{\tilde{z}} \, : \, d\pi(\tilde{z}) \cdot v > 0\}.$$

Then, $\tilde{\mathcal{L}}$ is a Riemann surface lamination with canonical projection π. Each leaf of this bidimensional lamination is a simply connected Riemann surface which is conformally equivalent to $\mathfrak{H}$. A typical fiber of this projection is homeomorphic to the dyadic Cantor set.

II. *Extension:* Finally, let $\tilde{F} : \tilde{\mathcal{L}} \to \tilde{\mathcal{L}}$ be the application given by

$$F(\tilde{z}, v) := (\tilde{f}(\tilde{z}), d\tilde{f}(\tilde{z}) \cdot v),$$

where $d\tilde{f}(\tilde{z})$ is the derivative of the restriction of $\tilde{f}$ to the leaf passing through $\tilde{z}$. Clearly, F is an extension of $\tilde{f}$.

In $\tilde{\mathcal{L}}$ we define the following equivalence relation: $(\tilde{z}, v) \sim (\tilde{z}', v')$ if and only if $(\tilde{z}, v)$ and $(\tilde{z}', v')$ are in the same orbit of the F-action.

Let $\mathcal{L} := \tilde{\mathcal{L}}/\sim$ be the quotient space of this relation. Then, $\mathcal{L}$ is a compact Riemann surface lamination. Each leaf of this lamination is the image under the quotient projection of a leaf in $\tilde{\mathcal{L}}$. Every leaf induced by a 'periodic' leaf of F is an annulus. Every leaf of this lamination is dense.

EXAMPLE 5: THE UNIVERSAL HYPERBOLIC SOLENOID (Notes by Manuel Cruz)

Let us suppose that $X := (X_g, x)$ is a compact Riemann surface of genus $g > 1$ with a base point, and let $G := \pi_1(X, x)$ be the fundamental group of X. Let us fix a covering map with base point $(\mathfrak{H}, *) \to X$ and let us recall that we can identify G with the group of deck transformations of this universal covering.

I. *The profinite completion of* G: Let us consider the set $\mathcal{N} = \mathcal{N}(G)$ consisting of all normal subgroups of G having finite index. We can define a partial ordering in this set by inclusion: if $H, K \in \mathcal{N}$, then $H \leq K$ if and only if $K \subset H$. Let us suppose that G is a base filter, i.e., if $H, K \in \mathcal{N}$, then there exists $L \in \mathcal{N}$ such that $L \subset H \cap K$.

If $H, K \in \mathcal{N}$ with $H \leq K$, then we have a well-defined canonical projection

$$\rho_{HK} : G/K \to G/H, \quad gK \mapsto gH.$$

This determines an inverse directed system whose inverse limit is called the *profinite completion of* G and is denoted by $\hat{G}$. Namely,

$$\hat{G} = \varprojlim G/H.$$

If G is also residually finite (i.e. $\bigcap_{H \in \mathcal{N}} H = \{1\}$), then we have an injective homomorphism $G \hookrightarrow \hat{G}$ induced by the projections in the factors G/H whose image is dense. $\hat{G}$, endowed with the profinite topology, is an abelian, compact and totally disconnected topological group, homeomorphic to a Cantor set.

II. *The universal solenoid:* From the theory of covering spaces, we know that to each $H \in \mathcal{N}$ we can associate a non-ramified covering space of degree n with base point, $p_H : X_H \to X$. Here, we canonically identify X_H to $\mathfrak{H}/H$. Then, for any $H, K \in \mathcal{N}$ with $H \leq K$, we have the corresponding covering applications $p_H : X_H \to X$ and $p_K : X_K \to X$. Then, there is a unique covering map $p_{HK} : X_K \to X_H$ such that $p_H \circ p_{HK} = p_K$. This determines an inverse directed system $\{X_H, p_H\}_{H \in \mathcal{N}}$, whose inverse limit is the *Universal Hyperbolic Solenoid*

$$\mathbb{S} = \varprojlim X_H,$$

with canonical projection $\pi : \mathbb{S} \to X$, determined by a coordinate projection that determines a $\hat{G}$-principal bundle structure.

$\mathbb{S}$ is a compact, connected topological space, and each leaf of this bidimensional lamination is a simply connected two-dimensional manifold, homeomorphic to the covering space $\mathfrak{H}$ of X. A typical fiber of this projection is isomorphic to the profinite completion of G.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

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