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Abstract

We prove that every semi-Fredholm operator on an arbitrary Banach space can be approximated by injective or surjective operators. In the case of a complex separable Hilbert space, we show that the set of semi-Fredholm operators having a fixed index is connected. Last, we present a simple approach for calculating some known distances from a bounded operator to certain sets related to semi-Fredholm operators.

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1 INTRODUCTION

There are several semigroups which appear naturally in connection with the Banach algebra $B(X)$ of all (bounded linear) operators acting from a Banach space X into itself. We have the injective operators

$$Inj(X) = \{T \in B(X) : T \text{ is one-to-one and } R(T) \text{ is closed}\},$$

the surjective operators

$$Sur(X) = \{T \in B(X) : R(T) = X\},$$

the upper semi-Fredholm operators

$$\Phi_+(X) = \{T \in B(X) : \dim N(T) < \infty \text{ and } R(T) \text{ is closed}\},$$

and the lower semi-Fredholm operators

$$\Phi_-(X) = \{T \in B(X) : \dim X/R(T) < \infty\};$$

here $N(T)$ denotes the null space of T and $R(T)$ is its range. Clearly we have $Inj(X) \subset \Phi_+(X)$ and $Sur(X) \subset \Phi_-(X)$. Although this contentions are generally proper, Theorem 2.1 shows that $Inj(X) \cup Sur(X)$ is dense in $\Phi_+(X) \cup \Phi_-(X)$.

Recall that the index of a (upper or lower) semi-Fredholm operator T is ind $T = n(T) - d(t)$, where $n(T) = \dim N(T)$ and $d(T) = \dim X/R(T)$. Let Z^* consist of all the integers together with $-\infty$ and ∞. For $m \in Z^*$, we define $\Phi_m(X)$ as the set of semi-Fredholm operators $T \in B(X)$ such that ind $T = m$. When H is a complex separable Hilbert, we prove in Theorem 2.2 that $\Phi_m(H)$ is connected.

If H is a complex separable Hilbert space , R. Bouldin [2-4], J. Zemánek [9] C. Apostol-L. A. Fialkow-D. A. Herrero [1], S. Izumino-Y. Kato [6], and P. Y. Wu [8] have calculated the distance of $T \in B(H)$ to several subsets related to semi-Fredholm operators on H. In Section 3 we develop a somewhat simpler approach for obtaining some of their results.

2 TWO PROPERTIES

Let us consider

$$F_m(X) = \Phi_m(X) \cap Sur(X),\ m \geq 0, \quad F_m(X) = \Phi_m(X) \cap Inj(X),\ m \leq 0.$$

Theorem 2.1 *$F_m(X)$ is dense in $\Phi_m(X)$, $m \in Z^*$.*

Proof. First we will assume that $m \leq 0$. Take $T \in \Phi_m(X)$. Since $m \leq 0$, note that $n = n(T) < \infty$. Thus we can express $X = N(T) \oplus V$, for some closed subspace $V \subset X$. Next, by using that $d(T) \geq n$, we can find a vector space $W \subset X$ such that dim W = n and $W \cap R(T) = 0$. Take S to be any linear isomorphism from $N(T)$ onto W. Let us define $T_k : X \to X$ by

$$T_k(u+v) = 1/k\ Su + Tv,\ u \in N(T),\ v \in V.$$

Since the projections onto N(T) and V are continuous, as well as S, it follows that $T_k \in B(X)$. It is clear that $T_k \in Inj(X) \cap \Phi_m(X)$. Noticing that $T_k \to T$, we obtain the desired conclusion.

Let us now consider $m \geq 0$. Then we have $d = d(T) < \infty$ and so we can express $X = N \oplus V$ where, $N \subset N(T)$ and dim $N = d$. The rest of the proof follows along similar lines as the above and we will omit it.

Remark 2.1. The proof of Theorem 2.1 shows that if $T \in \Phi_m(X)$, then there is a compact operator $K \in B(X)$ such that $T + K \in F_m(X)$.

It will now be convenient to introduce some notation. Hereafter H is always a fixed complex separable Hilbert space. If $A \subset H$, then $A^c \equiv H \setminus A$, and $A^\perp$ is the orthogonal subspace of A. For $T \in B(H)$, $|T|$ denotes the square root of T^*T; here T^* is the adjoint operator of T. Note that, in particular, we have the polar form $T = U|T|$, for some partial isometry $U \in B(H)$ [5, Ch. 4]. The group of invertible operators on H will be indicated by $G(H)$, moreover $\Phi_+ \equiv \Phi_+(H)$, $\Phi_- \equiv \Phi_-(H)$, $\Phi_m \equiv \Phi_m(H)$.

Theorem 2.2 *Φ_m is connected, $m \in Z^*$.*

Proof. First we will assume that $m \leq 0$. Take $\tilde{S}, \tilde{T} \in \Phi_m$. Next, choose compact operators $C, K \in B(X)$ such that $S = \tilde{S} + C, T = \tilde{T} + K \in F_m(H)$. Since the curves $\tilde{\alpha}(t) = \tilde{S} + tC$ and $\tilde{\beta}(t) = \tilde{T} + tK$, $0 \leq t \leq 1$, lie in Φ_m and,respectively, connect $\tilde{S}$ with S and $\tilde{T}$ with T, all we need is to show that S and T can be connected by some curve in Φ_m.

Let us express S and T in its polar form: $S = U|S|, T = V|T|$. Since U takes $R(|S|)$ isometrically onto $R(S)$, it follows that $R(|S|)$ is closed. Moreover, $|S|$ is one-to-one because $N(S) = N(|S|)$. Since $|S|$ is self-adjoint, this implies that $|S|$ is invertible. Analogously, we have that $|T|$ is invertible.

Note now that $VU^{-1} : R(S) \to R(T)$ is an isometry. Since $\dim R(S)^\perp = -m = \dim R(T)^\perp$ and H is separable (this is required when $-m = \infty$), VU^{-1} can be extended to obtain a unitary operator $J \in B(H)$.

Take $\alpha(t)$ to be a curve in $G(H)$ connecting J with I, and $\beta(t)$ to be a curve in $G(H)$ connecting $|T|$ with $|S|$; this is possible because $G(H)$ is open and connected [7, p. 317]. It is now easily verified that $\alpha(t)U\beta(t)$ is a curve in $\Phi_m(H)$ which joins T and S.

Finally, let us consider the case $m \geq 0$. Since ind $T^* = -$ ind T, we can verify that $\Phi_m(H)$ is the image of Φ_{-m} under the map $T \to T^*$. The conclusion follows now from the continuity of this map and the connectedness of $\Phi_{-m}(H)$.

3 APPROXIMATION BY SEMI-FREDHOLM OPERATORS

Let us denote the semigroup of Fredholm operators on H by Φ; notice that $\Phi = \Phi_+ \cap \Phi_-$. The essential spectrum of $T \in B(H)$ is then given by $\sigma_e(T) = \{\lambda : T - \lambda I \notin \Phi\}$, and the essential minimum modulus is

$$m_e(T) = \inf \{\lambda : \lambda \in \sigma_e(|T|)\}.$$

We will also consider the set

$$N(H) = \{T \in B(H) : m_e(T) = m_e(T^*) = 0\}.$$

Our approach for calculating the distance $\text{dist}(T, A)$ of $T \in B(H)$ to certains subsets A of $B(H)$ is based upon the following result; part(i) is due to R. Bouldin [2, Thm.3], and part (ii) is due to J. Zemánek [9]

Theorem 3.1
(i) If $T \in N(H)$, *then* $\text{dist}(T, G(H)) = 0$.
(ii) $m_e(T) = \text{dist}(T, \Phi_+^c)$, $m_e(T^*) = \text{dist}(T, \Phi_-^c)$.

Remark 3.1. $N(H)$ is a closed subgroup of $B(H)$.
Proof. Let $T \in B(H)$. The following properties are well known [2]:

$$m_e(T) > 0 \quad \text{if and only if} \quad T \in \Phi_+.$$
$$T \in \Phi_- \quad \text{if and only if} \quad T^* \in \Phi_+.$$

This implies that $N(H) = (\Phi_+ \cup \Phi_+)^c$ and so $N(H)$ is clearly closed.

Assume $S, T \in N(H)$. Since

$$TS \in \Phi_- \text{ implies } T \in \Phi_-, \tag{1}$$

and $TS \in \Phi_+$ implies $S \in \Phi_+$ [5, p. 35], we have $TS \in N(H)$.

In our next result bdy$A$ denotes the boundary of $A \subset H$.

Theorem 3.2 $N(H) = \text{bdy}\Phi_m$, $m \in Z^*$.

Proof. By the continuity of the index on $\Phi_+ \cup \Phi_-$, it follows that $\text{bdy}\Phi_m \subset N(H)$.

Next, let us assume $m \in Z \cup \{-\infty\}$. Fix $R \in \Phi_m \subset \Phi_+$. Notice now that we can choose $S \in B(H)$ such that

$$SR = I + K, \tag{2}$$

where $K \in B(H)$ is a compact operator. Take $T \in N(H)$ and note that $T \notin \Phi m$. By (1), $TS \notin \Phi_-$. Suppose that $TS \in \Phi_+$. Then, using (2), we would have $T \in \Phi_+$, which is contradictory.

The above shows that $TS \in N(H)$. Thus, by Theorem 3.1, there is a sequence $\{L_k\} \subset G(H)$ such that $L_k \to TS$. Hence, applying (2), we have $L_k R \to T + TK$. Since $\text{ind}(L_k R - TK) = m$, it follows that $T \in \text{bdy}\Phi_m$. This shows that $N(H) = \text{bdy}\Phi_m$.

Finally let us treat the case case $m = \infty$. Take $T \in N(H)$. Then, $T^* \in H$ and so, applying what we have just proved, there is a sequence $\{S_k\} \subset \Phi_{-m}$ such that $S_k \to T^*$. Hence $S_k^* \to T$ and the conclusion follows.

Corollary 3.1 *Let $m \in Z^*$. If $T \notin \Phi_m$, then*

$$\text{dist}(T, \Phi_m) = \max\{m_e(T), m_e(T^*)\}.$$

Proof. First, notice that

$$\begin{aligned} \max\{m_e(T), m_e(T^*)\} &= m_e(T) \text{ if } T \in \Phi_+, \\ \max\{m_e(T), m_e(T^*)\} &= m_e(T^*) \text{ if } T \in \Phi_-. \end{aligned}$$

Take $d \equiv \text{dist}(T, \Phi_m)$. We will consider several cases. If $T \notin \Phi_+ \cup \Phi_-$, the conclusion follows readily from Theorem 3.2.

Assume now $T \in \Phi_+$. Since $T \notin \Phi_m$, from Theorem 3.1 and using the continuity of the index on Φ_+, we find

$$m_e(T) \leq d. \tag{3}$$

Let $\epsilon > 0$. Applying again Theorem 3.1, we can obtain $S \notin \Phi_+$ satisfying $\|T - S\| \leq m_e(T) + \epsilon$. Note now that, because of the continuity of the index on $\Phi_+ \cup \Phi_-$, the curve $\alpha(t) = (1-t)T + tS$, $0 \leq t \leq 1$, cannot be contained in $\Phi_+ \cup \Phi_-$. Thus we can find $0 \leq t \leq 1$ such that $\alpha(T) \in N(H)$. Since $\|T - \alpha(T)\| \leq m_e(T) + \epsilon$, we have $d \leq m_e(T) + \epsilon$. Letting $\epsilon \to 0$ and using (3) the conclusion follows.

Finally, the case $T \in \Phi_-$ can be proved analogously.

Remark 3.2. Corollary 3.1 was established independently by Apostol-Fialkow- Herrero [1, Thm. 12.2], and Izumino-Kato [6, Thm. 4.1]; it is also discussed by Bouldin [3]. However, our method of proof is different from the ones they employed.

Corollary 3.2 *If $n \leq 0$, then*

$$\text{dist}(T, F_n) = \begin{cases} \max\{m_e(T), m_e(T^*)\} & \text{if ind} T \neq n \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It follows from Theorem 2.1 that $F_n(H)$ is dense in Φ_n. Thus, $\text{dist}(T, F_n(H)) = \text{dist}(T, \Phi_n)$. The conclusion is now obtained by using Corollary 3.1.

Remark 3.3. Corollary 3.3 was established by Wu in [8].

Let us denote by ν the function assigning $n(T)$ to each $T \in B(H)$.

Corollary 3.3 *The function ν is continuous at $T \in B(H)$ if and only if $T \in \Phi_+ \cup \Phi_-$ and $n(T) = 0$ or $d(T) = 0$.*

Proof. Let $A \equiv \{T \in \Phi_+ \cup \Phi_- : n(T) = 0 \text{ or } d(T) = 0\}$. If $T \notin A$, it follows from theorems 2.1 3.2 that ν is discontinuous at T. Assume now $T \in A$. Take $\rho > 0$ such that $\|T - S\| \leq \rho$ implies $S \in (\Phi_+ \cup \Phi_-)$, $n(S) \leq n(T)$, and $d(T) \leq d(S)$ [5, p. 36]. If $n(T) = 0$ and $\|T - S\| \leq \rho$, we clearly have $n(T) = n(S)$. If $d(T) = 0$ and $\|T - S\| \leq \rho$, then $n(T) = n(S)$ because $\text{ind}(T) = \text{ind}(S)$. This proves our assertion.

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