An explicit classification of (1,1)-dimensional Lie supergroup structures

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AN EXPLICIT CLASSIFICATION OF (1,1)-DIMENSIONAL LIE SUPERGROUP STRUCTURES.

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INTRODUCTION.

In recent works on the geometric theory of supermanifolds, the fundamental importance of the Lie supergroup structure of $\mathbb{R}^{1|1}$ and its actions has been stressed in connection with the problem of integration of supervector fields [4-5]. The most striking fact in this direction is the following: even though $\mathbb{R}^{1|1}$ plays in supermanifold theory the role that $\mathbb{R}$ does in the C^∞ category, $\mathbb{R}^{1|1}$ does not possess an essentially unique Lie supergroup structure, but it has **three**, non-isomorphic, equally acceptable ones. This fact was pointed out in [4] as a consequence of a new theory of integration of supervector fields developed after the methods of [3] and [5].

The three supergroup structures of $\mathbb{R}^{1|1}$ were obtained in [4] by representing each $(1,1)$-dimensional Lie superalgebra as graded derivations of the structure sheaf of some $(1,1)$-dimensional supermanifold, and then solving the differential equations they give rise to. This process yields the local coordinate expressions for the multiplication laws on the supergroups corresponding to the three different $(1,1)$-dimensional Lie superalgebras. It was then verified *a posteriori* that the graded derivations were actually left invariant for their corresponding supergroup structures.

In this note we give a direct analytic derivation of the three supergroup structures on $\mathbb{R}^{1|1}$. We thus provide an alternative proof of the results in [4]. Our proof, however, is independent of the integration theory of supervector fields and independent of the Lie theoretic relationship between Lie groups and Lie algebras via the exponential map. Moreover, using only the notion of Lie supergroup homomorphism as in [1], we are able to describe directly the isomorphism classes of Lie supergroup structures on $\mathbb{R}^{1|1}$. Finally, we also provide analogous results for the $(1,1)$-dimensional "supercircle" $S^{1|1}$, together with the corresponding "exponential covering maps".

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1. Lie supergroup structures on $\mathbb{R}^{1|1}$

A Lie supergroup consists of a supermanifold G, together with supermanifold morphisms, $\mu: G \times G \to G$, $\varepsilon: G \to G$, and $\alpha: G \to G$, satisfying the usual group conditions (cf. [1]): μ is associative, which means that $\mu \circ (\mu \circ (\pi_1 \times \pi_2) \times \pi_3) = \mu \circ (\pi_1 \times \mu \circ (\pi_2 \times \pi_2))$, where π_i denotes the projection of $G \times G \times G$ onto the i^{th} factor; $\mu \circ (\varepsilon \times id) = id = \mu \circ (id \times \varepsilon)$, and $\mu \circ (\alpha \times id) = \varepsilon = \mu \circ (id \times \alpha)$. In this note we will only need the associativity of the morphism μ, and for its explicit description we shall make use of the fact that supermanifold morphisms are determined by their effect on global sections of the sheaf of superfunctions (cf. [2]).

Let G be the supermanifold $\mathbb{R}^{1|1} = (\mathbb{R}, C^\infty_{\mathbb{R}} \otimes \bigwedge(\tau))$, and let (t, τ), and $(t_1, t_2, \tau_1, \tau_2)$ be sets of (linear global) supercoordinates for $\mathbb{R}^{1|1}$ and $\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}$, respectively. Then μ is determined by the equations,

$$\mu^* t = \widetilde{\mu}(t_1, t_2) + c(t_1, t_2)\,\tau_1 \tau_2 \qquad\qquad \mu^* \tau = a(t_1, t_2)\,\tau_1 + b(t_1, t_2)\,\tau_2$$

where $\widetilde{\mu}$, a, b, and c are smooth functions of t_1, and t_2.

Now the associativity of μ is equivalent to the equations

$$\begin{aligned}\{\mu \circ (\mu \circ (\pi_1 \times \pi_2) \times \pi_3)\}^* t &= \{\mu \circ (\pi_1 \times \mu \circ (\pi_2 \times \pi_2))\}^* t \\ \{\mu \circ (\mu \circ (\pi_1 \times \pi_2) \times \pi_3)\}^* \tau &= \{\mu \circ (\pi_1 \times \mu \circ (\pi_2 \times \pi_2))\}^* \tau,\end{aligned}$$

and the explicit computation of these equations gives a set of functional relations among the coefficients $\widetilde{\mu}$, a, b, and c. The easiest to solve is the one for $\widetilde{\mu}$: up to a transformation of coordinates, $\widetilde{\mu}(t_1, t_2) = t_1 + t_2$. Using this fact we can state the following proposition, which describes all the possible Lie supergroup structures on $\mathbb{R}^{1|1}$:

Proposition 1. *The functions a, b, and c define a Lie supergroup structure on $\mathbb{R}^{1|1}$ if and only if, there exists a smooth function $g: \mathbb{R} \to \mathbb{R}$, and real constants K_1, K_2 ($K_2 \neq 0$), such that, either*

$$\begin{aligned} a(t_1, t_2) &= e^{\left(g(t_1+t_2) - g(t_1)\right)} \\ b(t_1, t_2) &= a(t_2, t_1) \\ c(t_1, t_2) &= K_1\, e^{-\left(g(t_1) + g(t_2)\right)} \end{aligned}$$

or,

$$\begin{aligned} a(t_1, t_2) &= e^{\left(g(t_1+t_2) - g(t_1)\right)} \\ b(t_1, t_2) &= a(t_2, t_1)\, e^{K_2 t_1} \\ c(t_1, t_2) &\equiv 0. \end{aligned}$$

2. THE ISOMORPHISM CLASSES

An isomorphism between two supergroups (G_1, μ_1), and, (G_2, μ_2), is a superdiffeomorphism $\Psi : G_1 \to G_2$ such that (cf. [1]) $\mu_2 \circ (\Psi \times \Psi) = \Psi \circ \mu_1$. Applied to the case of $\mathbb{R}^{1|1}$ this translates into the existence of a non-zero constant λ and a smooth, non-vanishing function $\psi : \mathbb{R} \to \mathbb{R}$, such that the following set of equations is satisfied:

$$\psi(t_1 + t_2) a_1(t_1, t_2) = a_2(\lambda t_1, \lambda t_2)\psi(t_1)$$
$$\psi(t_1 + t_2) b_1(t_1, t_2) = b_2(\lambda t_1, \lambda t_2)\psi(t_2)$$
$$\lambda c_1(t_1, t_2) = c_2(\lambda t_1, \lambda t_2)\psi(t_1)\psi(t_2).$$

Here, a_i, b_i, and c_i correspond to the product μ_i.

We then obtain the following classification theorem:

Theorem 1. *There are three equivalence classes of Lie supergroup structures on the $(1,1)$-dimensional superline. Each class is determined by the data:*

1. *Type I: $a(t_1,t_2)$, $b(t_1,t_2) = a(t_2,t_1)$, $c \equiv 0$*
2. *Type II: $a(t_1,t_2)$, $b(t_1,t_2) = a(t_2,t_1)$, $c = K_1 e^{-(g(t_1)+g(t_2))}$, for some $K_1 \neq 0$.*
3. *Type III: $a(t_1,t_2)$, $b(t_1,t_2) = a(t_2,t_1)e^{K_2 t_1}$, for some $K_2 \neq 0$, $c \equiv 0$*

where a and g are as in proposition 1.

Remark. In particular, one has the following normal forms for the Lie supergroup structures on the $(1,1)$-dimensional superline (cf. [4]):

Type I	$a = b \equiv 1$, $c \equiv 0$	$\binom{t_1}{\tau_1} * \binom{t_2}{\tau_2} = \binom{t_1+t_2}{\tau_1+\tau_2}$
Type II	$a = b = c \equiv 1$	$\binom{t_1}{\tau_1} * \binom{t_2}{\tau_2} = \binom{t_1+t_2+\tau_1\tau_2}{\tau_1+\tau_2}$
Type III	$a \equiv 1$, $b(t_1,t_2) = e^{t_1}$, $c \equiv 0$	$\binom{t_1}{\tau_1} * \binom{t_2}{\tau_2} = \binom{t_1+t_2}{\tau_1+e^{t_1}\tau_2}$

3. SUPERGROUP STRUCTURES ON $S^{1|1}$

We now discuss the case of the supercircle $S^{1|1} = (S^1, C^\infty(S^1) \otimes \bigwedge(\sigma))$. Let $(e^{i\theta}, \sigma)$, be a set of supercoordinates on $S^{1|1}$. We may write the smooth functions on the circle as products of the form $e^{i\theta} f(e^{i\theta})$, with f a smooth real-valued function.

Now, as in the case of the superline $\mathbb{R}^{1|1}$, Lie supergroup structures on $S^{1|1}$ are specified by triples of smooth functions, a, b, and c, defined on $S^1 \times S^1$. Thus,

$$\mu^* e^{i\theta} = e^{i\theta_1} e^{i\theta_2} + c(e^{i\theta_1}, e^{i\theta_2})\sigma_1\sigma_2$$
$$\mu^* \sigma = a(e^{i\theta_1}, e^{i\theta_2})\sigma_1 + b(e^{i\theta_1}, e^{i\theta_2})\sigma_2$$

Similar considerations to those used for $\mathbb{R}^{1|1}$ yield the following proposition that completely characterizes the Lie supergroup structures on the supercircle:

Proposition 2. *The functions a, b, and c define a Lie supergroup structure on the supercircle $S^{1|1}$ if and only if there exists a smooth function $g : S^1 \to \mathbb{R}$, with $g(1) = 0$, and real constants λ, μ, and κ ($\mu \neq 0$), such that, either*

$$\begin{aligned} a(e^{i\theta_1}, e^{i\theta_2}) &= e^{i\lambda\theta_2} e^{g(e^{i\theta_1}e^{i\theta_2}) - g(e^{i\theta_1})} \\ b(e^{i\theta_1}, e^{i\theta_2}) &= a(e^{i\theta_2}, e^{i\theta_1}) \\ c(e^{i\theta_1}, e^{i\theta_2}) &= \kappa e^{i(1-\lambda)(\theta_1+\theta_2) - (g(e^{i\theta_1}) + g(e^{i\theta_2}))} \end{aligned}$$

or

$$\begin{aligned} a(e^{i\theta_1}, e^{i\theta_2}) &= e^{i\lambda\theta_2} e^{g(e^{i\theta_1}e^{i\theta_2}) - g(e^{i\theta_1})} \\ b(e^{i\theta_1}, e^{i\theta_2}) &= e^{i\mu\theta_1} a(e^{i\theta_2}, e^{i\theta_1}) \\ c(e^{i\theta_1}, e^{i\theta_2}) &\equiv 0 \end{aligned}$$

Similarly to theorem 1, one has the following characterization of the equivalence classes of Lie supergroup structures on $S^{1|1}$:

Theorem 2. *There are three equivalence classes of Lie supergroup structures on the $(1,1)$-dimensional supercircle. Each class is determined by the data:*

(1) *Type I: $a(e^{i\theta_1}, e^{i\theta_2})$, $b(e^{i\theta_1}, e^{i\theta_2}) = a(e^{i\theta_2}, e^{i\theta_1})$, $c \equiv 0$,*
(2) *Type II: $a(e^{i\theta_1}, e^{i\theta_2})$, $b(e^{i\theta_1}, e^{i\theta_2}) = a(e^{i\theta_2}, e^{i\theta_1})$, $c(e^{i\theta_1}, e^{i\theta_2}) = K_1 e^{i(1-\lambda)(\theta_1+\theta_2) - (g(e^{i\theta_1}) + g(e^{i\theta_2}))}$, for some $K_1 \neq 0$,*
(3) *Type III: $a(e^{i\theta_1}, e^{i\theta_2})$, $b(e^{i\theta_1}, e^{i\theta_2}) = a(e^{i\theta_2}, e^{i\theta_1}) e^{iK_2\theta_1}$, for some $K_2 \neq 0 \bmod 2\pi$, $c \equiv 0$,*

where a, g and λ are as in proposition 2. The normal forms for these supergroup structures are:

Type I	$a = b \equiv 1,\ c \equiv 0$	$\binom{e^{i\theta_1}}{\sigma_1} * \binom{e^{i\theta_2}}{\sigma_2} = \binom{e^{i\theta_1} e^{i\theta_2}}{\sigma_1 + \sigma_2}$
Type II	$a = b \equiv 1,\ c = e^{i(\theta_1+\theta_2)}$	$\binom{e^{i\theta_1}}{\sigma_1} * \binom{e^{i\theta_2}}{\sigma_2} = \binom{e^{i\theta_1} e^{i\theta_2}(1+\sigma_1\sigma_2)}{\sigma_1 + \sigma_2}$
Type III	$a \equiv 1,\ b(e^{i\theta_1}, e^{i\theta_2}) = e^{i\theta_1},\ c \equiv 0$	$\binom{e^{i\theta_1}}{\sigma_1} * \binom{e^{i\theta_2}}{\sigma_2} = \binom{e^{i\theta_1} e^{i\theta_2}}{\sigma_1 + e^{i\theta_1}\sigma_2}$

From the comparison of theorems 1 and 2, the existence of the corresponding "exponential covering maps" from $\mathbb{R}^{1|1}$ to $S^{1|1}$ now follows:

Corollary. *Up to conjugacy by isomorphisms, there are unique Lie supergroup covering homomorphisms from each of the three types of Lie supergroup structures on $\mathbb{R}^{1|1}$ onto their corresponding types on $S^{1|1}$.*

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