Deformations of branched Lefschetz's pencils

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ABSTRACT. Let M be a projective manifolds of dimension ≥ 3, with $H^1(M,\mathbb{C}) = 0$. We will show that a deformation of a codimension one singular foliation $\mathcal{F}$ arising from the fibers of a generic meromorphic map of the form f^p/g^q, $p,q > 0$ has a meromorphic first integral of the same type.

0 INTRODUCTION

Recently, Goméz-Mont and Lins [GM-L] have shown the following result which is an extension to codimension one holomorphic foliations with singularities of the Thurston-Reeb stability theorem:

THEOREM. [GM-L] *Let M be a projective manifold:*
(a) If $H^1(M,\mathbb{C}) = 0$, and $\dim_{\mathbb{C}} M \geq 3$, then Lefschetz Pencils are C^0-structurally stable foliations.
(b) If $\pi_1(M) = 0$, and $\dim_{\mathbb{C}} \geq 4$, then branched Lefschetz Pencils are C^0-structurally stable foliations.

Let L_1 and L_2 be positive holomorphic line bundles on M with holomorphic sections f_i such that $L_1^{\otimes p} = L_2^{\otimes q}$ with p, q relatively prime positive integers. The fibers of the meromorphic map $\phi = f_1^p/f_2^q$ define a codimension one holomorphic foliation with singularities represented by the twisted one-form

$$\omega = pf_2\, df_1 - qf_1\, df_2 \in Fol(M, L_1 \otimes L_2)$$

In what follows, we shall say that ϕ is a *meromorphic first integral* of the foliation ω. By a *generic* meromorphic map we mean the following:
(1) The sets $\{f_i = 0\}_{i=1,2}$ are smooth and meet transversally on a codimension two submanifold K.
(2) The subvarieties defined by $\lambda f_1^p + \mu f_2^q = 0$ with $(\lambda : \mu) \in \mathbb{P}^1$ are smooth on $M - K$ except for a finite set of points $\{(\lambda_i : \mu_i)\}_{\{i=1,\dots,k\}}$.
A meromorphic map satisfying conditions (1) and (2) is called a *Lefschetz pencil* if $p = q = 1$ and a *branched Lefschetz pencil* otherwise.
Our main result is the following:

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THEOREM A. *Let M be a projective manifold with $H^1(M,\mathbb{C})=0$ and dimension ≥ 3. Let $\omega = pf_2\,df_1 - qf_1\,df_2$ with $p \neq q$ be as above and satisfying condition (1). Then any deformation ω' of the foliation defined by ω has a meromorphic first integral $\phi' = f_1'^p/f_2'^q$ where $f_i' \in H^0(M,\mathcal{O}(L_i))$.*

As consequence of this result, we may state the Goméz-Mont-Lins Theorem as follows:

THEOREM B. *Let M be a projective manifold whose complex dimension is at least 3, with $H^1(M,\mathbb{C}) = 0$. If $\mathcal{F}$ has a generic meromorphic first integral, then $\mathcal{F}$ is a C^0-structurally stable foliation*

In this theorem, generic means a Lefschetz or a branched Lefschetz pencil.

THEOREM C. *Let M be a projective manifold whose complex dimension is at least 3, and with $H^1(M,\mathbb{C}) = 0$. Let $\mathcal{B}(c)$ be an irreducible component of $Fol(M,c)$ that contains a foliation which has a generic meromorphic first integral. Then there exists a Zariski dense open subset of $\mathcal{B}(c)$ parametrizing the C^0-structurally stable foliations; all of them are topologically equivalent, and have a generic rational first integral.*

In [M], Muciño analyses the tangent space of the space of foliations on a Lefschetz pencil. From these infinitesimal methods, he gives an independent proof of part (a) of the Goméz-Mont-Lins theorem.

1 CODIMENSION ONE FOLIATIONS

A codimension one holomorphic foliation (with singularities) on a complex manifold M may be given by a family of integrable 1-forms ω_α defined on an open cover $\mathcal{U} = \{U_\alpha\}$ of M, $\omega_\alpha \wedge d\omega_\alpha = 0$, satisfying $\omega_\alpha = \lambda_{\alpha\beta}\omega_\beta$ in $U_\alpha \cap U_\beta$, where $\lambda_{\alpha\beta}$ are never vanishing holomorphic functions. If L denotes the holomorphic line bundle on M obtained with the cocycles $\{\lambda_{\alpha\beta}\}$, then the 1-forms glue to give a holomorphic section of the bundle $T^*M \otimes L$.

1.1 DEFINITION. *A codimension one holomorphic foliation $\mathcal{F}$ (with singularities) in the complex manifold M is an equivalence class of sections $\omega \in H^0(M,\Omega^1(L))$ where L is a holomorphic line bundle such that ω does not vanish on any connected component of M and satisfies the integrability condition $\omega \wedge d\omega = 0$. The **singular set** of the foliation $\mathcal{F}$ is the set of points $S(\mathcal{F}) = \{p \in M | \omega(p) = 0\}$. The **leaves of the foliation** are the leaves of the non-singular foliation in $M - S(\mathcal{F})$.*

When a leaf $\mathcal{L}$ of $\mathcal{F}$ is such that its closure $\overline{\mathcal{L}}$ is a closed analytic subspace of M of codimension 1, we will also call $\overline{\mathcal{L}}$ a leaf of $\mathcal{F}$.

A ***holomorphic family*** $\{\mathcal{F}_t\}_{t\in T}$ *of* ***codimension one holomorphic foliations*** with singularities parametrized by a complex analytic space T consists of:

(1) A holomorphic family of complex manifolds $\{M_t\}$, given as a smooth map $\pi : \mathcal{M} \to T$ between complex spaces with $\pi^{-1}(t) = M_t$

(2) A holomorphic foliation with singularities $\tilde{\mathcal{F}}$ on $\mathcal{M}$ such that its leaves are contained in the t-fibers and the restriction $\tilde{\mathcal{F}}|_{M_t} = \mathcal{F}_t$ is a codimension one holomorphic foliation with singularities on M_t.

Given a family of foliations $\{\mathcal{F}_t\}$, the *perturbed holonomy* of a leaf $\mathcal{L}$ of the foliation $\mathcal{F}_0$ is the holonomy of $\mathcal{L}$ as a leaf of the foliation $\tilde{\mathcal{F}}$. It is clear that the perturbed holonomy has the form:

$$H_\alpha(t,z) = (t, h_\alpha(t,z))$$

where h_α is a holomorphic function such that $h_\alpha(0,z)$ is the holonomy map associated to $\alpha \in \pi_1(\mathcal{L})$ as a leaf of $\mathcal{F}_0$.

We will assume that M is compact and has complex dimension ≥ 3, in this case, the set $Fol(M,c)$ of those foliations defined by an equivalence class of sections $\omega \in H^0(M,\Omega^1(L))$ where L is a line bundle with Chern class $c(L) = c$, is an algebraic variety [GM-M p. 133].

1.2 DEFINITION. *Consider $\omega \in H^0(M,\Omega^1(L))$. A section $\varphi \in H^0(M,\mathcal{O}(L))$ is an integrating factor of ω if and only if the meromorphic one form*

$$\Omega := \frac{\omega}{\varphi}$$

is closed.

If a section $\omega \in H^0(M,\Omega^1(L))$ has an integrating factor then it is integrable.

1.3 THEOREM. *Let M be a projective manifold. If $\varphi = \varphi_1^{r_1}\cdots\varphi_k^{r_k} \in H^0(M,\mathcal{O}(L))$ is an integrating factor of ω with $r_i > 0$ $i = 1,\dots,k$ then:*

$$\frac{\omega}{\varphi} = \sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i} + d(\Psi) + \eta$$

where $\lambda_i \in \mathbf{C}$, Ψ is a meromorphic function with poles at the divisor $\sum_{i=1}^{k} l_i\{\varphi_i = 0\}$ with $l_i < r_i$ and η is a holomorphic closed 1-form

PROOF: Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M such that U_α and $U_\alpha \cap U_\beta$ are simply connected for all α,β.

In [C.M. p. 37] is showed that

$$\Omega_\alpha := \frac{\omega_\alpha}{\varphi_\alpha} = \sum_{i=1}^{k} \lambda_i \frac{d\varphi_{\alpha,i}}{\varphi_{\alpha,i}} + d(\Psi_\alpha) + dh_\alpha$$

here Ψ_α is a meromorphic function on U_α with poles at the divisor $\sum_{i=1}^{k} l_i\{\varphi_{\alpha i} = 0\}$ for some integers $l_i < r_i$ for all $i = 1\cdots,k$ and $h_\alpha \in \mathcal{O}(U_\alpha)$, the complex numbers λ_i are given by

$$\lambda_i = \frac{1}{2\pi i}\int_{\gamma_i} \Omega$$

here $\gamma_i \in \pi_1(M - \{\varphi_i = 0\})$ is the generator of the kernel of the map $i_* : \pi_1(M - \{\varphi_i = 0\}) \to \pi_1(M)$ where $i : M - \{\varphi_i = 0\} \to M$ denotes the inclusion. Since $\Omega_\alpha = \Omega_\beta$ in $U_\alpha \cap U_\beta$, we get:

$$\sum_{i=1}^{k}\lambda_i\frac{d\varphi_{\alpha,i}}{\varphi_{\alpha,i}}+d(\Psi_\alpha)+dh_\alpha=\sum_{i=1}^{k}\lambda_i\frac{d\varphi_{\beta,i}}{\varphi_{\beta,i}}+d(\Psi_\beta)+dh_\beta$$

In $U_\alpha\cap U_\beta$ one has the never-vanishing holomorphic functions $\mu_{\alpha\beta,i}:=\varphi_{\alpha,i}/\varphi_{\beta,i}$. By choosing some branch of the logarithm we get:

$$\sum_{i=1}^{k}\lambda_i\log\mu_{\alpha\beta,i}+h_\alpha-h_\beta+c_{\alpha,\beta}=\Psi_\beta-\Psi_\alpha$$

for some $c_{\alpha\beta}\in\mathbf{C}$. The left side of the equation is holomorphic, thus the rigth side vanishes, this implies that Ψ_α defines a meromorphic function Ψ on M.

2 KUPKA TYPE SINGULARITIES

2.1 DEFINITION. *Let $\mathcal{F}$ be a codimension-one holomorphic foliation with singularities represented by $\omega\in H^0(M,\Omega^1(L))$. The Kupka singular set denoted by $K(\mathcal{F})\subset S(\mathcal{F})$ is defined by:*

$$K(\mathcal{F}):=\{p\in M|\omega(p)=0\quad d\omega(p)\neq 0\}$$

The proof of the following theorem may be found in [Me].

2.2 THEOREM. *Let ω and $K(\mathcal{F})$ as above, then:*

(1) $K(\mathcal{F})$ is a codimension two locally closed submanifold of M.

(2) For every connected component $K\subset K(\mathcal{F})$ there exist a holomorphic 1-form

$$\eta=A(x,y)dx+B(x,y)dy$$

defined in a neighborhood V of $0\in\mathbb{C}^2$ and vanishing only at 0, a covering $\{U_\alpha\}$ of a neighborhood of K in M and a family of submersions $\varphi_\alpha:U_\alpha\to\mathbb{C}^2$ such that $\varphi_\alpha^{-1}(0)=K\cap U_\alpha$ and $\omega_\alpha=\varphi_\alpha^*\eta$ defines $\mathcal{F}$ in U_α.

(3) $K(\mathcal{F})$ is persistent under variation of $\mathcal{F}$; namely, for $p\in K(\mathcal{F})$ with defining 1-form $\varphi^*\eta$ as above, and for any foliation $\mathcal{F}'$ sufficiently close to $\mathcal{F}$, there is a holomorphic 1-form η' defined on a neighborhood of $0\in\mathbb{C}^2$ and a submersion φ' close to φ such that $\mathcal{F}'$ is defined by $(\varphi')^*\eta'$ on a neighborhood of p.

Remarks: The germ at $0\in\mathbb{C}^2$ of η is well defined up to biholomorphism and multiplication by non-vanishing holomorphic functions. We will call it the ***transversal type*** of $\mathcal{F}$ at K. If X is the dual vector field of η, since $d\omega\neq 0$ we have that $\mathrm{Div}X(0)\neq 0$, thus the linear part $D=DX(0)$ is well defined up to linear conjugation and multiplication by scalars. We will say that D is the linear type of K. Since $\mathrm{tr}D\neq 0$, it has at least one non-zero eigenvalue. Normalizing, we may assume that the eigenvalues are 1 and μ. We will distiguish three possible types of Kupka type singularities:

(a) Degenerate: If $\mu=0$

(b) Semisimple: If $\mu\neq 0$ and D is semisimple.

(c) Non-semisimple, $\mu=1$ and D is not semisimple.

Moreover in [GM-L p. 320-324] is showed the following theorem:

2.3 THEOREM. *Let K be a compact connected componet of $K(\mathcal{F})$ such that the first Chern class of the normal bundle of K in M is non-zero, then the transversal type is non-degenerate, linearizable with eigenvalues 1 and $\mu \in \mathbb{Q}$ and for any deformation $\{\mathcal{F}_t\}$ of $\mathcal{F} = \mathcal{F}_0$ the transversal type is constant through the deformation.*

Let f_1, f_2 be holomorphic sections of the positive line bundles L_1, L_2 such that $L_1^{\otimes p} = L_2^{\otimes q}$ with p, q relatively positive integers. Supoose that the hypersurfaces $\{f_i = 0\}_{i=1,2}$ are smooth and meet transverselly. Consider the holomorphic section of the bundle $T^*M \otimes L_1 \otimes L_2$ given by

$$\omega = pf_2\, df_1 - qf_1\, df_2$$

The associated foliation has the meromorphic first integral $\phi = f_1^p/f_2^q$. Observe that if $x \in \{f_1 = 0\} \cap \{f_2 = 0\}$ then $\omega(x) = 0$ and $d\omega(x) = -(p+q)(df_1 \wedge df_2)(x) \neq 0$. In this case we have that $\{f_1 = 0\} \cap \{f_2 = 0\} = K(\omega)$ and the transversal type is given by the one-form $\eta = px\, dy - qy\, dx$. The normal bundle $\nu_K = (L_1 \oplus L_2)|_K$ thus has non-vanishing first Chern class and by theorem (2.3) the transversal type is constant under deformations.

3 PROOF OF THEOREM A

In this section we will prove the following result:

THEOREM A. *Let M be a projective manifold with $H^1(M, \mathbb{C}) = 0$ and dimension ≥ 3. Let $\omega = pf_2\, df_1 - qf_1\, df_2$ with $p \neq q$ where $f_i \in H^0(M, \mathcal{O}(L_i)$ and $L_1^{\otimes p} = L_2^{\otimes q}$. If $\{f_i = 0\}_{i=1,2}$ are smooth and meet transverselly, then any deformation ω' of the foliation defined by ω has a meromorphic first integral $\phi' = f_1'^p/f_2'^q$ (where $f_i' \in H^0(M, \mathcal{O}(L_i))$).*

Let $\omega = pf_2\, df_1 - qf_1\, df_2$ be as in theorem A. Observe that $f_1 \cdot f_2$ is an integrating factor for ω, converselly, if $q/p, p/q \notin \mathbb{N}$ and ω' is an integrable section close to ω, we will show that the leaves $\{f_i = 0\}_{i=1,2}$ have non-trivial holonomy and are stable under deformations, namely, there are sections $f_i' \in H^0(M, \mathcal{O}(L_i))\; i = 1, 2$ such that $\{f_i' = 0\}_{i=1,2}$ are compact leaves of the foliation ω'. We will show that $f_1' \cdot f_2'$ is an integrating factor for ω' and the conclusion follows from (1.3)

3.1 THEOREM. *Let M be a smooth projective manifold of dimension ≥ 3. If $\omega = pf_2\, df_1 - qf_1\, df_2 \in \mathcal{F}ol(M, L_1 \otimes L_2)$ is a section as in theorem A, then at least one of the leaves $\{f_i = 0\}$ is stable under deformations*

PROOF: Let ω_t be a family of foliations such that $\omega_0 = \omega$. The idea is to find a fixed point of the perturbed holonomy. In order to do this, we will find a central element which has non-trivial linear holonomy.

Recall that if V is a smooth algebraic manifold of complex dimension ≥ 2, $W \subset V$ is a smooth, positive divisor on V, $i : V - W \hookrightarrow V$ is the inclusion. The generator γ_W of the kernel $i_* : \pi_1(V - W) \to \pi_1(V)$ is central in $\pi_1(V - W)$ [No 315-316].

Since $K \subset \{f_i = 0\}\; i = 1, 2$ is a positive divisor, the loop $\gamma(i) := \gamma_K^i \in \pi_1(\{f_i = 0\} - K)$ is central and the holonomy is given by

$$h_{\gamma(1)}(y) = e^{2\pi i \frac{p}{q}} \cdot y$$
$$h_{\gamma(2)}(y) = e^{2\pi i \frac{q}{p}} \cdot y$$

hypothesis that $p/q \neq 1$ and that they are relatively prime implies that the holonomy $h_{\gamma(i)}$ has non-trivial linear part for some $i = 1, 2$. The perturbed holonomy is given by

$$H_{\gamma(i)}(t, z) = (t, h_{\gamma(i)}(t, z))$$

by the implicit function theorem, there exits an analytic function $t \mapsto z_t$ such that $h_{\gamma(i)}(t, z_t) = z_t$, and since $\gamma(i)$ is central, for any $\beta \in \pi_1(\{f_i = 0\} - K)$

$$H_\beta(t, z_t) = (t, z_t)$$

Let $\mathcal{L}_t$ be the leaf of ω_t through the point (t, z_t). The closure of this leaf contains the local separatrix of the Kupka set hence $\overline{\mathcal{L}_t}$ is a compact leaf of ω_t.

Remark: Note that if $p/q \notin \{2, 3, \dots, 1/2, 1/3, \dots\}$, both leaves $\{f_i = 0\}_{i=1,2}$ have non-trivial holonomy and are stable.

3.2 LEMMA. *Let L_i $i = 1, 2$ and $\{f_i\}$ be as in (3.1). If $H^1(M, \mathbb{C}) = 0$ then any deformation of ω has an integrating factor*

PROOF: Let ω_t be an analytic family of foliations with $\omega_0 = (pf_2\, df_1 - qf_1\, df_2) \in Fol(M, L_1 \otimes L_2)$.

We will consider two cases:

(1) If $p/q \notin \mathbb{N}$ and $q/p \notin \mathbb{N}$.

In this case we have seen that the leaves $\{f_i = 0\}_{i=1,2}$ are stable, thus there exists an analytic family of sections $f_{it} \in H^0(M, \mathcal{O}(L_i))$ $i = 1, 2$ such that $\{f_{it} = 0\}_{i=1,2}$ are compact leaves of the foliation represented by ω_t.

We claim that the product $f_{1t} \cdot f_{2t} \in H^0(M, \mathcal{O}(L_1 \otimes L_2))$ is an integrating factor of the section ω_t.

By (2.3), we have that the transversal type of the Kupka set is constant through the deformation, thus, on a neighborhood of the Kupka set, $\omega_t/(f_{1,t}f_{2,t})$ has the local expresion:

$$\frac{\omega_t}{(f_{1,t}f_{2,t})}|_{U_\alpha} = p\frac{dx_{\alpha,t}}{x_{\alpha,t}} - q\frac{dy_{\alpha,t}}{y_{\alpha,t}}$$

moreover $\{x_{\alpha,t} = 0\} = \{f_{1,t} = 0\} \cap U_\alpha$ and $\{y_{\alpha,t} = 0\} = \{f_{2,t} = 0\} \cap U_\alpha$ and hence the meromorphic 1-form $\omega_t/(f_{1,t}f_{2,t})$ is closed.

(2) Assume that $p = 1 < q$. In this case only the compact leaf $\{f_2 = 0\}$ is stable, thus there exists an analytic family $f_{2t} \in H^0(M, \mathcal{O}(L_2))$ such that $\{f_{2t} = 0\}$ is a leaf of the foliation defined by ω_t

We claim that f_{2t}^{q+1} is an integrating factor for ω_t

Since $H^1(M, \mathbb{C}) = 0$, the Hodge decomposition theorem [G-H p. 116] implies that a holomorphic line bundle L is classified by its Chern class. This implies that f_{2t}^{q+1} is a holomorphic section of the bundle $L = L_1 \otimes L_2$

On a neighborhood of the Kupka set K_t of ω_t we have

$$\frac{\omega_t}{f_{2,t}^{q+1}}|_{U_\alpha} = \frac{1}{y_{\alpha t}^{q+1}}(y_{\alpha t}\, dx_{\alpha t} - qx_{\alpha t}\, dy_{\alpha t}) = d(\frac{x_{\alpha t}}{y_{\alpha t}^q})$$

and $\{y_{\alpha,t} = 0\} = \{f_{2t} = 0\} \cap U_\alpha$, thus the meromorphic 1-form $\omega_t / f_{2,t}^{q+1}$ is closed and $f_{2,t}^{q+1}$ is an integrating factor.

Now we will finish the proof of theorem A.

(1) Since $f_{1t} f_{2t}$ is an integrating factor for ω_t we have seen that:

$$\frac{\omega_t}{f_{1t} f_{2t}} = p\frac{df_{1t}}{f_{1t}} - q\frac{df_{2t}}{f_{2t}} + \eta_t$$

If $H^1(M, \mathbb{C}) = 0$ the holomorphic closed 1-form η_t is zero and we get a meromorphic first integral for ω_t

(2) Since f_{2t}^{q+1} is an integrating factor for ω_t again by (1.3) we have:

$$\frac{\omega_t}{f_{2t}^{q+1}} = d\left(\frac{f_{1t}}{f_{2t}^{q}}\right) + \eta_t$$

where $f_{1t} \in H^0(M, \mathcal{O}(L_2^q)) = H^0(M, \mathcal{O}(L_1))$ and η_t is zero because $H^1(M, \mathbb{C}) = 0$, thus we get:

$$\omega_t = f_{2t}^{q+1} d\left(\frac{f_{1t}}{f_{2t}^{q}}\right) = f_{2t}\, df_{1t} - q f_{1t}\, df_{2t}$$

This finish the proof.

REMARKS: (1) If we begin with a unbranched rational function (That is, $L_1 = L_2$) and we consider deformations keeping one leaf stable, then it is possible to find an integrating factor.

4 UNIVERSAL FAMILIES

We will describe irreducible components of the universal families of foliations of codimension 1.

Let M be a projective manifold with $H^1(M, \mathbb{C}) = 0$, we have seen that every holomorphic line bundle on M is determined by its Chern class $c_1(L) \in H^2(M, \mathbb{Z})$. It may be shown that $\cup_c Fol(M, c)$ parametrizes the universal family of foliations of codimension 1 in M ([GM]).

4.1- DEFINITION: The fibers $\{\varphi^{-1}(c)\}$ of a rational map $\varphi : M \to \mathbb{P}^1$ defined on a connected projective manifold M form a *Branched Lefschetz Pencil* If there are global sections f_i of positive line bundles L_i, $i = 1, 2$ with $L_1^p = L_2^q$, $p, q > 0$ such that:

(1) The subvarities $\{f_i = 0\}_{i=1,2}$ are smooth and meets transversely in a smooth manifold K called the *center of the Pencil.*

(2) The subvarities defined by $\lambda f_1^p + \mu f_2^q = 0$ with $\lambda\mu \neq 0$ are smooth on $M - K$ except for a finite set of points $\{(\lambda_i : \mu_i)\}_{i=1,\ldots,k}$ where it has just a Morse type singularity over the critical value in $M - K$.

Note that the meromorphic maps satisfying only condition (1), are dense in the set of branched Lefschetz Pencils, thus as a consequence of theorem A we have:

THEOREM B. *Let M be a projective manifold whose complex dimension is at least 3, with $H^1(M, \mathbb{C}) = 0$. If $\mathcal{F}$ has a generic meromorphic first integral, then $\mathcal{F}$ is a C^0-structurally stable foliation*

Proof. We have two cases:

(1) If The meromorphic first integral is a Lefschetz Pencil, it is part (a) of the Goméz-Mont Lins theorem [GM-L].

(2) If the meromorphic first integral is a branched Lefchetz Pencil, by Theorem A, we have that any deformation of a branched Lefschetz pencil has a meromorphic first integral, this implies that the Kupka set is locally structurally stable, we can repeat the proof of part (b) of the Gomez-Mont Lins theorem given in [GM-L].

THEOREM C. *Let M be a projective manifold of complex dimension at least 3 and with $H^1(M, \mathbb{C}) = 0$. Let $\mathcal{B}(c)$ be an irreducible component of $Fol(M, c)$ that contains a branched Lefschetz Pencil; then there exists a Zariski dense open subset of $\mathcal{B}(c)$ parametrizing C^0-estructurally stable foliations, all of them topologically equivalent and branched Lefschetz pencils*

Proof. Let L_i be positive line bundles with chern classes $c(L_i) = c_i$ such that $L_1 \otimes L_2 = L$ where $c(L) = c = c_1 + c_2$.

Consider the map

$$\begin{aligned} \Phi : \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} &\to Fol(M, c_1 + c_2) \\ ([f_1], [f_2]) &\mapsto p f_1 \, df_2 - q f_2 \, df_1 \end{aligned}$$

where $n_i = \dim_{\mathbb{C}} H^0(M, \mathcal{O}(L_i)) - 1, \quad i = 1, 2.$

This a well defined algebraic map. Let $\mathcal{W}$ be the Zariski's closure of the image of Φ. We claim that $\mathcal{W}$ is an irreducible component of $\mathcal{F}ol(M, c)$, where $c = c_1 + c_2$. We know that any deformation of a branched Lefschetz pencil is again a branched pencil and then is in the image of Φ. This show that $\mathcal{W}$ and $\mathcal{F}ol(M, c)$ coincide in a neighborhood of a foliation $\mathcal{F}_0$. Since $\mathcal{W}$ is irreducible, then it is an irreducible component of $Fol(M, c)$. Hence $\mathcal{W} = \mathcal{B}(c)$. This proves the theorem.

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