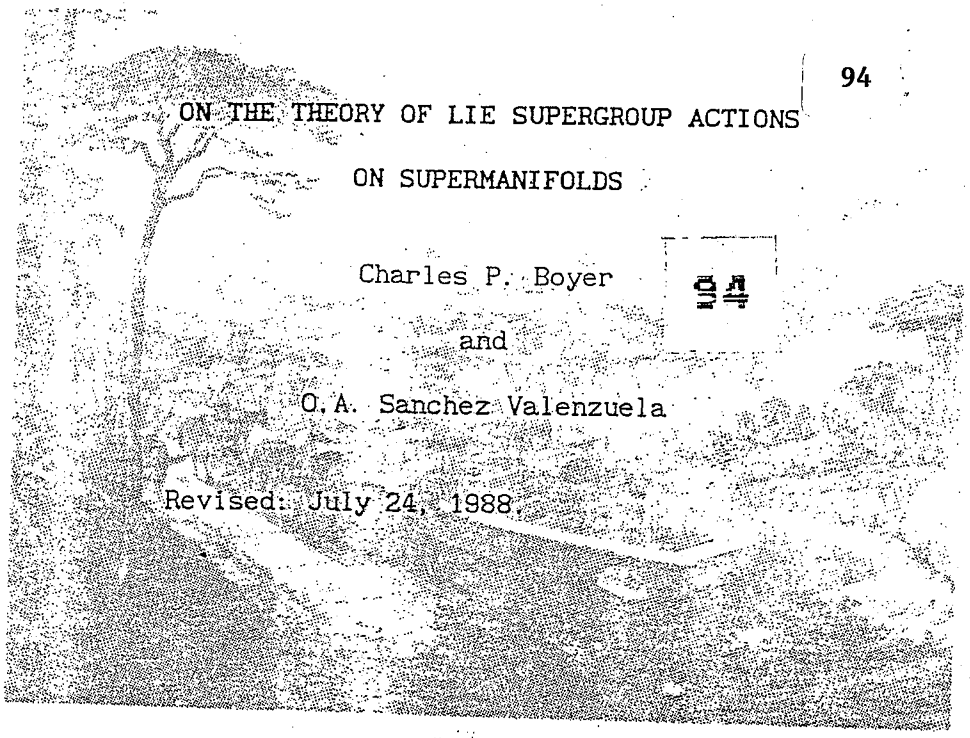
COMUNICACIONES DEL CIMAT



94

ON THE THEORY OF LIE SUPERGROUP ACTIONS ON SUPERMANIFOLDS

Charles P. Boyer

and

O.A. Sanchez Valenzuela

94

Revised: July 24, 1988.

CENTRO DE INVESTIGACION EN MATEMATICAS

Apartado Postal 402
Guanajuato, Gto.
México
Tels. (473) 2-25-50
2-02-58

On the theory of

Lie supergroup actions on supermanifolds

Charles P. Boyer(1)

and

O. A. Sánchez Valenzuela(2)

abstract

Lie supergroups are here understood as group objects in the category of supermanifolds (as in [10]). Actions of Lie supergroups in supermanifolds are defined in general by means of diagrams of supermanifold morphisms. Examples of such actions are given. Among them emerge the linear actions discussed in [7], and the natural actions on the Grassmannian supermanifolds studied in [4] and [8]. The nature of the *isotropy subsupergroup* associated to an action is fully elucidated; it is exhibited as an *embedded subsupergroup* within the spirit of the theory of smooth manifolds and Lie groups, and with no need of the Lie-Hopf algebraic approach of Kostant in [2] The notion of *orbit* is also discussed. Explicit calculations of isotropy subsupergroups are included. Also, an alternative description of the space of global sections of the structural sheaf of a Lie supergroup is given, based on the triviality of its supertangent bundle.

(1) Department of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico, 87131
U.S.A.

(2) Centro de Investigación en Matemáticas, A.C.
Apartado postal 402; C.P. 36000
Guanajuato, Gto.
México.

Introduction

The first treatise on Lie supergroups and superhomogeneous spaces was given by Kostant in his pioneering work [2]. His definition of supermanifold, however, is very general. For the examples that have appeared elsewhere (cf., [4], [7]) and those we are about to present in this paper, supermanifolds in the sense of [3] and [4] suffice. (We shall refer the reader to section A.4 in the appendix, for a brief discussion on how Kostant's approach cannot be immediately compared to that of other authors). Nevertheless, Kostant's Lie supergroups are actually supermanifolds in the more restricted sense. Indeed, they are characterized by means of abstract group-like properties and one can prove quite generally that the supertangent bundle (in the sense of [6]) of a Lie supergroup is trivial. The trivialization is accomplished by the existence of a basis of left invariant superderivations which can also be used to pick up function and exterior factors globally (cf., [2]); hence the structural sheaf of a Lie supergroup is exhibited from the outset in the Batchelor-trivial form (cf., [1]). The odd generators are the elements of a basis for the dual of the odd subspace of the Lie superalgebra of the supergroup. In sheaf-theoretical terms, the Lie superalgebra is a constant sheaf (cf., 2.7) and this makes the pressumed complications in Kostant's definition to be abscent from the theory. (Lie supergroups are defined in § 1. Left invariance is discussed in § 2 where the Lie superalgebra of a Lie supergroup is defined.)

Our discussion of Lie supergroup actions on supermanifolds is differential-geometric oriented, too. Our original motivation was to obtain the isotropy subsupergroup of an action, say $\psi : (G, A_G) \times (M, A_M) \longrightarrow (M, A_M)$, in the same way one does in the smooth theory. Thus, for each $p \in M$, we make sense of the *partial map* $\psi_p : (G, A_G) \to (M, A_M)$, and the *constant map* $\varepsilon_p : (G, A_G) \longrightarrow (M, A_M)$, whose *image* is the $(0, 0)$-dimensional point (p, R) of (M, A_M). The *isotropy subsupergroup at p,* (G_p, A_{G_p}), is obtained by making sense of the *locus in* (G, A_G) *where* $\psi_p = \varepsilon_p$. It turns out to be an embedded subsupermanifold of (G, A_G) and it inherits naturally a Lie supergroup structure. The precise result is stated in theorem 4.6. Concrete examples are given in § 5.

Then, following the ideas in [2], it is immediate to define a supermanifold sheaf, A_{G/G_p}, on the space of cosets G/G_p, giving thus rise to a *superhomogeneous space*. There is also a morphism

$$(G/G_p, A_{G/G_p}) \longrightarrow (M, A_M).$$

naturally induced by ψ_p. The *orbit through p,* (O_p, A_{O_p}), is the image of ψ_p in the category of supermanifolds (cf., 4.9); it is a subsupermanifold of (M, A_M) and its supermanifold structure is the one that makes $(G/G_p, A_{G/G_p}) \longrightarrow (O_p, A_{O_p})$ into a superdiffeomorphism.

Acquintance with some of the references will be assumed (particularly, [2], [3], and [5]). Our inclusion of an appendix is to provide an appropriate setting for the comparison of the definitions in [2] and [3]. It is by no means complete, even though it may be useful as a quick reference for basic definitions and notation.

Acknowledgements

One of us (O.A.S.V.) would like to thank professor R. Berlanga for the helpful discussions maintained during the preparation of this work and for his most valuable criticism.

1. Abstract characterization of a Lie supergroup

1.1 Definition: A Lie supergroup is a finite dimensional supermanifold, (G, A_G), equipped with the following additional structure:

(*i*) a supermanifold morphism

$$\mu : (G, A_G) \times (G, A_G) \longrightarrow (G, A_G)$$

satisfying the *associativity property,*

$$\mu \circ (\pi_1 \times \mu \circ (\pi_2 \times \pi_3)) = \mu \circ (\mu \circ (\pi_1 \times \pi_2) \times \pi_3)$$

[Both sides are morphisms $(G, A_G) \times (G, A_G) \times (G, A_G) \longrightarrow (G, A_G)$; π_i denotes the projection of $(G, A_G) \times (G, A_G) \times (G, A_G)$ onto the $i^{\underline{th}}$ factor $(i = 1, 2, 3)$].

(*ii*) a distinguished point in the underlying manifold, $e \in G$, and hence, a distinguished supermanifold morphism

$$\mathcal{E}_e := \delta_e \circ C_{(G, A_G)} : (G, A_G) \longrightarrow (G, A_G)$$

that satisfies the *identity property,*

$$\mu \circ (id \times \mathcal{E}_e) = id = \mu \circ (\mathcal{E}_e \times id).$$

(iii) an involutive superdiffeomorphism

$$\sigma : (G, A_G) \longrightarrow (G, A_G)$$

that satisfies the *inverse property*,

$$\mu \circ (id \times \sigma) = \mathcal{E}_e = \mu \circ (\sigma \times id).$$

1.2 Example : $\mathbf{R}^{m|n} = (\mathbf{R}^m, \mathcal{R}^{m|n})$.

Recall from [2] and [3] that the sheaf $\mathcal{R}^{m|n}$ of the superafine space $\mathbf{R}^{m|n}$ is

$$C^\infty{}_{\mathbf{R}^m} \otimes \wedge[\theta_1, \ldots, \theta_n]$$

where $\{\theta_1, \ldots, \theta_n\}$ is a set of generators for an n-dimensional vector space over $\mathbf{R}$. Thus, for any non-empty open subset $U \subset \mathbf{R}^m$, $\mathcal{R}^{m|n}(U) = C^\infty{}_{\mathbf{R}^m}(U) \otimes \wedge[\theta_1, \ldots, \theta_n]$.

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ be a basis of $\mathbf{R}^m$ and let $\{x^1, \ldots, x^m\}$ be the dual basis. Then, $\{x^1, \ldots, x^m; \theta_1, \ldots, \theta_n\}$ is a global coordinate system for the supermanifold $\mathbf{R}^{m|n}$. We shall keep this coordinate system fixed.

Associated to the vector spaces $\mathbf{R}^m$ and $\wedge[\theta_1, \ldots, \theta_n]$ there is a natural morphism

$$\mu : (\mathbf{R}^m, \mathcal{R}^{m|n}) \times (\mathbf{R}^m, \mathcal{R}^{m|n}) \longrightarrow (\mathbf{R}^m, \mathcal{R}^{m|n})$$

defined over each non-empty open subset $U \subset R^m$ as the one that corresponds to the superalgebra morphism

$$\mu^{\#} : R^{m|n}(U) \longrightarrow R^{m|n} \times R^{m|n}(U \times U)$$

given in terms of the specified coordinates by,

$$\mu^{\#} x^i = p_1{}^{\#} x^i + p_2{}^{\#} x^i; \qquad i = 1, \ldots, m,$$
$$\mu^{\#} \theta_\nu = p_1{}^{\#} \theta_\nu + p_2{}^{\#} \theta_\nu; \qquad \nu = 1, \ldots, n;$$

$p_i : (R^m, R^{m|n}) \times (R^m, R^{m|n}) \to (R^m, R^{m|n})$ being the projection morphism onto the $i^{\underline{th}}$ factor $(i = 1, 2)$.

The origin $e = (0, \ldots, 0) \in R^m$ is obviously a distinguished point for which the morphism ε_e has the identity property. Furthermore, there is an involutive isomorphism $\sigma : (R^m, R^{m|n}) \longrightarrow (R^m, R^{m|n})$ having the inverse property; namely, the one defined over each non-empty open subset $U \subset R^m$, via the superalgebra morphism,

$$\sigma^{\#} : R^{m|n}(U) \longrightarrow R^{m|n}(U)$$

whose effect on the fixed set of coordinates is,

$$\sigma^{\#} x^i = -x^i; \quad i = 1, \ldots, m, \qquad \text{and} \qquad \sigma^{\#} \theta_\nu = -\theta_\nu; \quad \nu = 1, \ldots, n.$$

These considerations work for $m = 0$. Indeed, $R^0 \simeq \{*\}$. The sheaf $R^{0|n}$ becomes the constant sheaf, $\Lambda[\theta_1, \ldots, \theta_n]$, over $\{*\}$. The morphism ε_e has no choice. The morphisms μ and σ are those given by the same expressions above on the odd coordinates θ_ν; $\nu = 1, \ldots, n$. What results is then a $(0,n)$-dimensional Lie supergroup.

1.3 **Example**: $(\mathbf{R}^*, \mathbf{R}^{1|1}|_{\mathbf{R}^*})$, where $\mathbf{R}^* = \mathbf{R} - \{0\}$.

In this case, the morphism μ is the supermultiplication morphism of $\mathbf{R}^{1|1} = (\mathbf{R}, \mathbf{R}^{1|1})$, as defined in [6], restricted to the open subsupermanifold $(\mathbf{R}^*, \mathbf{R}^{1|1}|_{\mathbf{R}^*})$. Thus, in terms of the standard coordinate system $\{x, \theta\}$ of $\mathbf{R}^{1|1}$,

$$\mu^* x = p_1^* x \; p_2^* x + p_1^* \theta \; p_2^* \theta$$
$$\mu^* \theta = p_1^* x \; p_2^* \theta + p_1^* \theta \; p_2^* x$$

The distinguished point in $\mathbf{R}^*$ is the unit $1 \in \mathbf{R}^*$, and the superdiffeomorphism $\sigma : (\mathbf{R}^*, \mathbf{R}^{1|1}|_{\mathbf{R}^*}) \longrightarrow (\mathbf{R}^*, \mathbf{R}^{1|1}|_{\mathbf{R}^*})$ is the one that corresponds to the superalgebra morphism $\sigma^* : \mathbf{R}^{1|1}(\mathbf{R}^*) \longrightarrow \mathbf{R}^{1|1}(\mathbf{R}^*)$ given on generators by,

$$\sigma^* x = 1/x \qquad \text{and,} \qquad \sigma^* \theta = -(1/x)^2 \theta.$$

Remark: one may simply assume that $\sigma^* x = f$, and $\sigma^* \theta = g\theta$, with f and g some C^∞ functions on $\mathbf{R}^*$. Then, by requiring $\mu \circ (id \times \sigma)$ to be identical to the morphism ε_1, one concludes that $f(x) = 1/x$, and $g(x) = -(1/x)^2$.

1.4 Example: The supergroup $GL_S(m|n)=GL_S(V_0|V_1)=(GL(V_0 \oplus V_1), GL_S(m+n|m+n))$, associated to an (m,n)-dimensional supervector space $V=V_0 \oplus V_1$.

Recall that $Hom(V,V)$ is a supervector space of dimension $(m^2+n^2, 2mn)$ whose supermanifoldification, $Hom(V,V)_S$, as defined in [6], is an afine supermanifold of dimension $((m+n)^2, (m+n)^2)$; its underlying smooth manifold is $Hom(V,V)$ itself. Following [7], we can introduce even and odd linear coordinates $(\{A^{bj}, \pi\Gamma^{bJ}, \pi\Theta^{Bj}, D^{BJ}\}$ and $\{\pi A^{bj}, \Gamma^{bJ}, \Theta^{Bj}, \pi D^{BJ}\}$, respectively) on it, arrange them in matrix form,

$$\begin{pmatrix} A^{bj}+\pi A^{bj} & \Gamma^{bJ}+\pi\Gamma^{bJ} \\ \Theta^{Bj}+\pi\Theta^{Bj} & D^{BJ}+\pi D^{BJ} \end{pmatrix}, \tag{1}$$

and define a composition map $\mu : Hom(V,V)_S \times Hom(V,V)_S \to Hom(V,V)_S$, according to the rules of linear superalgebra for left supermodule morphisms (cf., [9]); that is,

$$\begin{pmatrix} \mu^*(A^{bj}+\pi A^{bj}) & \mu^*(\Gamma^{bJ}+\pi\Gamma^{bJ}) \\ \mu^*(\Theta^{Bj}+\pi\Theta^{Bj}) & \mu^*(D^{BJ}+\pi D^{BJ}) \end{pmatrix} =$$

$$= \begin{pmatrix} p_1^*(A^{bj}+\pi A^{bj}) & p_1^*(\Gamma^{bJ}+\pi\Gamma^{bJ}) \\ p_1^*(\Theta^{Bj}+\pi\Theta^{Bj}) & p_1^*(D^{BJ}+\pi D^{BJ}) \end{pmatrix} \begin{pmatrix} p_2^*(A^{bj}+\pi A^{bj}) & p_2^*(\Gamma^{bJ}+\pi\Gamma^{bJ}) \\ p_2^*(\Theta^{Bj}+\pi\Theta^{Bj}) & p_2^*(D^{BJ}+\pi D^{BJ}) \end{pmatrix}$$

where $p_i : Hom(V,V)_S \times Hom(V,V)_S \longrightarrow Hom(V,V)_S$ denotes the projection morphism onto the $i^{\underline{th}}$ factor $(i=1,2)$. Thus,

$$\mu^*(A^{bj}+\pi A^{bj}) = \sum_k p_1^*(A^{bk}+\pi A^{bk})\, p_2^*(A^{kj}+\pi A^{kj})$$
$$+\sum_J p_1^*(\Gamma^{bJ}+\pi\Gamma^{bJ})\, p_2^*(-\Theta^{Jj}+\pi\Theta^{Jj})$$
$$\mu^*(\Gamma^{bJ}+\pi\Gamma^{bJ}) = \sum_k p_1^*(A^{bk}+\pi A^{bk})\, p_2^*(\Gamma^{kJ}+\pi\Gamma^{kJ})$$
$$+\sum_B p_1^*(\Gamma^{bB}+\pi\Gamma^{bB})\, p_2^*(D^{BJ}-\pi D^{BJ})$$
$$\mu^*(\Theta^{Bj}+\pi\Theta^{Bj}) = \sum_J p_1^*(D^{BJ}+\pi D^{BJ})\, p_2^*(\Theta^{Jj}+\pi\Theta^{Jj}) \qquad (2)$$
$$+\sum_k p_1^*(\Theta^{Bk}+\pi\Theta^{Bk})\, p_2^*(A^{kj}-\pi A^{kj})$$
$$\mu^*(D^{BJ}+\pi D^{BJ}) = \sum_K p_1^*(D^{BK}+\pi D^{BK})\, p_2^*(D^{KJ}+\pi D^{KJ})$$
$$+\sum_k p_1^*(\Theta^{Bk}+\pi\Theta^{Bk})\, p_2^*(-\Gamma^{kJ}+\pi\Gamma^{kJ})$$

Now, any square matrix with entries in a superalgebra like $A_M(U)$, A_M being the structural sheaf of a given supermanifold, is invertible, if and only if the matrix obtained by projecting the entries onto the commutative algebra $(A_M/J_M)(U)=C^\infty(U)$, is (cf., [2], [3]). Since the condition,

$$\det\begin{pmatrix} A^{bj} & \pi\Gamma^{bJ} \\ \pi\Theta^{Bj} & D^{BJ} \end{pmatrix}^{\sim} \neq 0 \qquad (3)$$

defines the open subset $GL(V)=GL(V_0\oplus V_1)$ of $\mathrm{Hom}(V,V)$, the same condition picks up an open subsupermanifold of $\mathrm{Hom}(V,V)_S$; such subsupermanifold is, by definition, $GL_S(V_0|V_1)$. Thus, for its structural sheaf, $GL_S(m+n|m+n)$, we have,

$$GL_S(m+n|m+n) \simeq R^{(m+n)^2|(m+n)^2}\big|_{GL(m+n)} \qquad (4)$$

Then, by restricting the morphism μ above to such an open subsupermanifold and defining the inverse morphism σ in terms of the given coordinates so as to obtain the inverse matrix of (1), $GL_S(V_0|V_1)$ becomes a Lie supergroup.

1.5 **Remark**: Recall how inverse matrices are obtained. Let B be any given supercommutative and associative superalgebra. If the supermatrix associated (as in [9]) to a left B-supermodule morphism has the block decomposition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

then, such a morphism is invertible if and only if the blocks A and B are both invertible, in which case the left inverse corresponds to the supermatrix,

$$\begin{pmatrix} (A - B(D^{-1})^*C^*)^{-1} & -(A - B(D^{-1})^*C^*)^{-1}B(D^{-1})^* \\ -(D - C(A^{-1})^*B^*)^{-1}C(A^{-1})^* & (D - C(A^{-1})^*B^*)^{-1} \end{pmatrix},$$

where, for any (non-graded) matrix F, F^* denotes the matrix obtained from F by changing the signs of the odd components of its entries (cf., [9]). On the other hand, the right inverse is given by

$$\begin{pmatrix} (A - B(D^{-1})^*C^*)^{-1} & -A^{-1}B(D^* - C^*A^{-1}B)^{-1} \\ -D^{-1}C(A^* - B^*D^{-1}C)^{-1} & (D - C(A^{-1})^*B^*)^{-1} \end{pmatrix}.$$

It is easy to check that both, left and right inverses, are actually the same.

1.6 **Example**: The supergroup $GL(V_0|V_1) = (GL(V_0)\times GL(V_1), GL(m|n))$; $V = V_0 \oplus V_1$ being an (m,n)-dimensional supervector space.

For the supergroup $GL_S(V_0|V_1)$ of the example above, the conditions

$$\pi\Gamma^{bJ}=0, \qquad \pi\Theta^{Bj}=0, \qquad \pi A^{bj}=0, \qquad \pi D^{BJ}=0, \qquad \det(A^{bj})\det(D^{BJ})\neq 0,$$

define an embedded $(m^2+n^2, 2mn)$-dimensional subsupermanifold of $GL(V)$. The restriction of the same morphisms μ and σ and the same identity element make this subsupermanifold into a Lie supergroup. This is the Lie supergroup $\{SHom(V,V)\}^*$, also introduced in [6] and studied from an algebraic point of view in [7].

1.7 **Definition**: Let (G, A_G) be a Lie supergroup and let $\mu_G, \varepsilon_G, \sigma_G$, be its multiplication, identity and inversion morphisms, respectively. Let (H, A_H) be an immersed (resp., embedded) subsupermanifold of (G, A_G) and let $\iota:(H, A_H)\longrightarrow(G, A_G)$ be the corresponding immersion (resp., embedding). Then, (H, A_H) is a Lie subsupergroup of (G, A_G) if (H, A_H) is a Lie supergroup itself and ι is a *homomorphism* (cf., [10]); that is, if $\mu_H, \varepsilon_H, \sigma_H$, are the multiplication, identity and inversion morphisms of (H, A_H), then,

$$\mu_G\circ(\iota\circ\pi_1\times\iota\circ\pi_2)=\iota\circ\mu_H\,, \tag{5}$$

regarded as morphisms from $(H, A_H)\times(H, A_H)$ into (G, A_G). As usual, π_i denotes the projection of the product $(H, A_H)\times(H, A_H)$ into the i^{th} factor.

1.8 **Remark**: Just as in the theory of Lie groups, from the single condition above, one obtains,

$$\varepsilon_G\circ\iota=\iota\circ\varepsilon_H \qquad \text{and} \qquad \sigma_G\circ\iota=\iota\circ\sigma_H. \tag{6}$$

The proof of these properties follows easily from the following two lemmas which we shall have occasion to use again in this work.

1.9 **Lemma**: Let (M, A_M) be a supermanifold and let $p \in M$ be an arbitrary point. Let $\mathcal{E}_p : (M, A_M) \longrightarrow (M, A_M)$ be the composition $\delta_p \circ C_{(M, A_M)}$. Then, for any morphism $\alpha : (M, A_M) \longrightarrow (M, A_M)$,

$$\alpha \circ \mathcal{E}_p = \mathcal{E}_{\tilde{\alpha}(p)} \qquad \text{and} \qquad \mathcal{E}_p \circ \alpha = \mathcal{E}_p.$$

Proof: Note, on the one hand, that for any $f \in A_M(M)$, and for any $q \in M$, we have, $\mathcal{E}_q{}^* f = \tilde{f}(q) 1_{A(M)}$. On the other hand, $(\alpha^* f)^{\tilde{}}(p) = \tilde{f}(\tilde{\alpha}(p))$ (cf., [3]). Hence,

$$(\mathcal{E}_p \circ \alpha)^* f = \mathcal{E}_p{}^* (\alpha^* f) = \tilde{f}(\tilde{\alpha}(p)) 1_{A(M)} = \mathcal{E}_{\tilde{\alpha}(p)}{}^* f.$$

Similarly, since $\alpha^* : A_M(M) \longrightarrow A_M(M)$ is a morphism of $\mathbf{R}$-superalgebras, we have,

$$(\mathcal{E}_p \circ \alpha)^* f = \alpha^* (\mathcal{E}_p{}^* f) = \alpha^* (\tilde{f}(p) 1_{A(M)}) = \tilde{f}(p) 1_{A(M)} = \mathcal{E}_p{}^* f.$$

This verifies that the effect of $(\alpha \circ \mathcal{E}_p)^*$ (resp., $(\mathcal{E}_p \circ \alpha)^*$) on the superalgebra $A_M(M)$ of global sections is the same as that of $\mathcal{E}_{\tilde{\alpha}(p)}{}^*$ (resp., $\mathcal{E}_p{}^*$). Therefore, the morphisms are the same $\square$

1.10 **Lemma**: Let (G, A_G) be a Lie supergroup and let μ be its composition morphism. Then, μ is an epimorphism. More generally, the following cancellation laws holds true: for any morphisms $\alpha, \beta, \gamma : (G, A_G) \longrightarrow (G, A_G)$,

$$\mu\circ(\alpha\times\beta)=\mu\circ(\gamma\times\beta) \implies \alpha=\gamma \qquad \text{and} \qquad \mu\circ(\alpha\times\beta)=\mu\circ(\alpha\times\gamma) \implies \beta=\gamma$$

Proof: Let σ and $\mathcal{E}_e$ be the inversion and identity morphims of the supergroup. Assume that $\mu\circ(\alpha\times\beta)=\mu\circ(\gamma\times\beta)$, and consider the composite

$$(G,A_G)\xrightarrow{\lambda}(G,A_G)\times(G,A_G)\xrightarrow{\mu}(G,A_G)$$

where $\lambda=\mu\circ(\alpha\times\beta)\times\sigma\circ\beta$. By hypothesis, $\lambda=\mu\circ(\gamma\times\beta)\times\sigma\circ\beta$. If we denote by π_i the projection morphism of the product of three copies of (G,A_G) onto the $i^{\underline{th}}$ factor, we have,

$$\begin{aligned}\mu\circ(\mu\circ(\alpha\times\beta)\times\sigma\circ\beta)&=\mu\circ(\mu\circ(\pi_1\times\pi_2)\times\pi_3)\circ(\alpha\times\beta\times\sigma\circ\beta)\\&=\mu\circ(\pi_1\times\mu\circ(\pi_2\times\pi_3))\circ(\alpha\times\beta\times\sigma\circ\beta)\\&=\mu\circ(\pi_1\times\mu\circ(id\times\sigma))\circ(\alpha\times\beta\times\beta)\\&=\mu\circ(\pi_1\times\mathcal{E}_e)\circ(\alpha\times\beta\times\beta)=\pi_1\circ(\alpha\times\beta\times\beta)=\alpha\end{aligned}$$

But, the same string of equalities, with α replaced by γ, shows that

$$\mu\circ(\mu\circ(\gamma\times\beta)\times\sigma\circ\beta)=\gamma$$

Therefore, $\mu\circ(\alpha\times\beta)=\mu\circ(\gamma\times\beta)\Rightarrow\alpha=\gamma$. The other cancelation law is proved similarly $\square$

1.11 Remark: There is another point which is worth observing from 1.7. The definition, as it stands, allows the irrational flow on the torus ($H\simeq\mathbb{R}$ and $G=S^1\times S^1$) to be the underlying manifold of a Lie subsupergroup. This degree of generality, on the other hand, keeps the morphism $\iota : (H, A_H) \longrightarrow (G, A_G)$ from being a monomorphism (i.e., a left cancelable morphism). It follows, however, in a straightforward manner, that ι will be a monomorphism if and only if (H, A_H) is an embedded subsupermanifold of (G, A_G), which is true, if and only if H is a closed Lie subgroup of G and $\iota^* : A_G \longrightarrow \tilde{\iota}_* A_H$ is an epimorphism; $\tilde{\iota}$ being the embedding of H into G.

As it was pointed out to us by R. Berlanga, this raises the question of classifying the different subsupergroup structures $\iota : (H, A_H) \longrightarrow (G, A_G)$ that a given closed subgroup H of G can support. We shall deal with the classification problem elsewhere. However, we would only like to observe here that there can be several different such structures. In fact, this is evident from the example 1.2 above with $(G, A_G) = (R^m, R^{m|n})$, and $H = \{e\} \simeq \{*\}$; there are n different subsupergroup structures in this case.

2. Left and right invariance on Lie supergroups

2.1 Let (G, A_G) be a Lie supergroup. For each point $g \in G$, we define left and right translations by g, as the supermanifold morphisms,

$$L_g := \mu \circ (\mathcal{E}_g \times id) \quad \text{and} \quad R_g := \mu \circ (id \times \mathcal{E}_g), \tag{1}$$

respectively. Here, $\mathcal{E}_g : (G, A_G) \longrightarrow (G, A_G)$ denotes the morphism $\delta_g \circ C_{(G, A_G)}$, whose corresponding superalgebra morphism $\mathcal{E}_g^* : A_G(U) \longrightarrow A_G(\tilde{\mathcal{E}}_g^{-1}(U))$ is,

$$\mathcal{E}_g^* f = \begin{cases} \tilde{f}(g)\, 1_{A_G(\tilde{\mathcal{E}}_g^{-1}(U))} & \text{if} \quad g \in U \\ 0 & \text{if} \quad g \notin U \end{cases} \tag{2}$$

We claim that *both,* L_g *and* R_g *are superdiffeomorphisms whose inverses are respectively given by,*

$$(L_g)^{-1} = L_{g^{-1}} \quad \text{and} \quad (R_g)^{-1} = R_{g^{-1}} \tag{3}.$$

2.2 **Proof**: We shall only prove here that $(R_g)^{-1} = R_{g^{-1}}$; that is,

$$\mu \circ (id \times \mathcal{E}_{g^{-1}}) \circ \mu \circ (id \times \mathcal{E}_g) = id \quad \text{and} \quad \mu \circ (id \times \mathcal{E}_g) \circ \mu \circ (id \times \mathcal{E}_{g^{-1}}) = id \tag{4}$$

as morphisms from (G, A_G) into itself. We start by making use of the definition of the product; thus,

$$\mu\circ(id\times\mathcal{E}_{g^{-1}})\circ\mu\circ(id\times\mathcal{E}_g)=\mu\circ(id\circ[\mu\circ(id\times\mathcal{E}_g)]\times\mathcal{E}_{g^{-1}}\circ[\mu\circ(id\times\mathcal{E}_g)])$$
$$=\mu\circ([\mu\circ(id\times\mathcal{E}_g)]\times\mathcal{E}_{g^{-1}}\circ\mu\circ(id\times\mathcal{E}_g))$$

But now, it follows from 1.9 above that,

$$\mathcal{E}_{g^{-1}}\circ\mu\circ(id\times\mathcal{E}_g)=\mathcal{E}_{g^{-1}} \tag{5}$$

Therefore,

$$\mu\circ(id\times\mathcal{E}_{g^{-1}})\circ\mu\circ(id\times\mathcal{E}_g)=\mu\circ([\mu\circ(id\times\mathcal{E}_g)]\times\mathcal{E}_{g^{-1}})$$
$$=\mu\circ(\mu\circ(\pi_1\times\pi_2)\times\pi_3)\circ(id\times\mathcal{E}_g\times\mathcal{E}_{g^{-1}})$$

where in the last step we have used again the definition of the product. Now, in view of the associativity property of μ, we get,

$$\mu\circ(id\times\mathcal{E}_{g^{-1}})\circ\mu\circ(id\times\mathcal{E}_g)=\mu\circ(\pi_1\times\mu\circ(\pi_2\times\pi_3))\circ(id\times\mathcal{E}_g\times\mathcal{E}_{g^{-1}})$$
$$=\mu\circ(id\times\mu\circ(\mathcal{E}_g\times\mathcal{E}_{g^{-1}}))$$

But since, $\tilde{\sigma}(g)=g^{-1}$, it follows from 1.9 again that

$$\mathcal{E}_{g^{-1}}=\sigma\circ\mathcal{E}_g \tag{6}$$

Hence,

$$\mu\circ(id\times\mathcal{E}_{g^{-1}})\circ\mu\circ(id\times\mathcal{E}_g)=\mu\circ(id\times\mu\circ(\mathcal{E}_g\times\mathcal{E}_{g^{-1}}))=\mu\circ(id\times\mu\circ(id\times\sigma)\circ\mathcal{E}_g)$$
$$=\mu\circ(id\times\mathcal{E}_e\circ\mathcal{E}_g)$$

where in the last step use has been made of the inverse property for σ. Finally, using 1.9 one more time to conclude that $\varepsilon_e \circ \varepsilon_g = \varepsilon_e$, we end up with,

$$\mu \circ (id \times \varepsilon_{g^{-1}}) \circ \mu \circ (id \times \varepsilon_g) = \mu \circ (id \times \varepsilon_e) = id$$

In a similar manner one proves that $R_{g^{-1}}$ is a right inverse, too. Thus, $(R_g)^{-1} = R_{g^{-1}}$ □

2.3 **Proposition**: Let (G, A_G) be a Lie supergroup and let L_g and R_g be the left and right translations by $g \in G$, as in 2.1(1). Then, for any other $h \in G$,

$$L_g \circ L_h = L_{gh} \qquad \text{and} \qquad R_g \circ R_h = R_{hg}$$

Proof: This is a straightforward consequence of the definitions and lemma 1.9. In fact,

$$\begin{aligned} R_g \circ R_h &= \mu \circ (id \times \varepsilon_g) \circ \mu \circ (id \times \varepsilon_h) = \mu \circ (\mu \circ (id \times \varepsilon_h) \times \varepsilon_g \circ \mu \circ (id \times \varepsilon_h)) \\ &= \mu \circ (\mu \circ (id \times \varepsilon_h) \times \varepsilon_g) = \mu \circ (id \times \mu \circ (\varepsilon_h \times \varepsilon_g)) = \mu \circ (id \times \varepsilon_{\tilde{\mu}(h,g)}) \\ &= \mu \circ (id \times \varepsilon_{hg}) = R_{hg} \end{aligned}$$

The corresponding property for left translations is similarly verified □

2.4 In what follows we shall be concerned with the left invariant supervector fields on (G, A_G). We shall show that their characterization is exactly the same as in the smooth theory. Let $\mathrm{Der}\, A_G$ be the sheaf (over G) of superderivations of the structural sheaf A_G. Let $\mathrm{Der}\, A_G(G)$ be its corresponding superspace of global sections. Recall that $\mathrm{Der}\, A_G(G)$ is the real subsupervector space of $\mathrm{End}\, A_G(G)$,

$$\mathrm{Der}\,A_G(G) = (\mathrm{Der}\,A_G(G))_0 \oplus (\mathrm{Der}\,A_G(G))_1$$

where,

$$(\mathrm{Der}\,A_G(G))_\mu = \{X \in \mathrm{End}\,A_G(G) \mid (\forall f, g \in A_G(G); f \text{ homogeneous})\, X(fg) = X(f)g + (-1)^{|f|\mu} f X(g)\}$$

Also recall that the supervector space $\mathrm{Der}\,A_G(G)$ inherits from $\mathrm{End}\,A_G(G)$ a Lie superalgebra structure, the Lie superbracket of which is given on homogeneous elements X and Y by,

$$[X, Y] = X \circ Y - (-1)^{|X||Y|} Y \circ X \tag{7}$$

2.5 **Definition**: We shall say that a superderivation, $X \in \mathrm{Der}\,A_G(G)$, is *left-invariant* if for each $g \in G$, $L_{g*}X = X$ (compare with [2]), where,

$$L_{g*}X = (L_{g^{-1}})^{\#} \circ X \circ (L_g)^{\#} \tag{8}$$

2.6 **Remark**: The map $g \mapsto L_{g*}$ defines a representation of the Lie group G on $\mathrm{Der}\,A_G(G)$ acting via automorphisms of the Lie superalgebra structure. In particular, the subsuperspace consisting of left invariant superderivations is itself a Lie superalalgebra. It will be denoted by $\mathfrak{g}\ (= \mathfrak{g}_0 \oplus \mathfrak{g}_1)$.

2.7 **Proposition**: There exists a supervector space isomorphism

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \longrightarrow (ST)_e(G, A_G) = \{(ST)_e(G, A_G)\}_0 \oplus \{(ST)_e(G, A_G)\}_1$$

where, $\{(ST)_e(G, A_G)\}_\mu$ is the space of germs at e of homogeneous superderivations of degree μ of the superalgebra $A_G(U)$, with $U \ni e$.

Proof : We claim that the isomorphism is given by assigning, to each left invariant superderivation, its germ at the identity. Let us verify first that this yields a surjection.

Let $\xi \in (ST)_e(G, A_G)$ be arbitrary and consider, for each $g \in G$, the following morphism induced on the stalk of A_G over g:

$$(A_G)_g \xrightarrow{(L_g)^*_g} (A_G)_e \xrightarrow{\xi} (A_G)_e \xrightarrow{(L_{g^{-1}})^*_e} (A_G)_g$$

Then, define the section $g \mapsto \hat{\xi}_g = (L_{g^{-1}})^*_e \circ \xi \circ (L_g)^*_g$ of the sheaf space, $\mathbf{L}\mathrm{Der}\,A_G$, associated to the presheaf $\mathrm{Der}\,A_G$ (cf., [11]). Since $\mathrm{Der}\,A_G$ is in fact a sheaf, we have,

$$\mathrm{Der}\,A_G(\cdot) \simeq \Gamma(\cdot, \mathbf{L}\mathrm{Der}\,A_G) \tag{9}$$

and therefore, $g \mapsto \hat{\xi}_g$ defines a global section, $\hat{\xi} \in \mathrm{Der}\,A_G(G)$. We shall now show that this section is left-invariant. In fact, for each $h \in G$,

$$\begin{aligned}(L_{h*}\hat{\xi})_g &= \{(L_{h^{-1}})^* \circ \hat{\xi} \circ (L_h)^*\}_g = (L_{h^{-1}})^*_{h^{-1}g} \circ \hat{\xi}_{h^{-1}g} \circ (L_h)^*_g \\ &= (L_{h^{-1}})^*_{h^{-1}g} \circ (L_{(h^{-1}g)^{-1}})^*_e \circ \xi \circ (L_{h^{-1}g})^*_{h^{-1}g} \circ (L_h)^*_g \\ &= (L_{(h^{-1}g)^{-1}} \circ L_{h^{-1}})^*_e \circ \xi \circ (L_h \circ L_{h^{-1}g})^*_g \\ &= (L_{g^{-1}})^*_e \circ \xi \circ (L_g)^*_g = \hat{\xi}_g\end{aligned}$$

where use has been made of the contravariance of $*$, and the multiplicativity property of left translations given in 2.3.

Finally, to prove that the morphism $X \mapsto X_e$ is injective, note that if $X_e = Y_e$, then X and Y coincide in some open neighborhood of the identity. By left invariance, they coincide everywhere (this actually means that we cover G by open subsets which are left translates of an open neighborhood of e, over each of which the restrictions of X and Y coincide; since $\mathrm{Der}\, A_G$ is a sheaf, $X = Y$) □

2.8 **Remark**: It has been proved in [2] and [3], that for any (m,n)-dimensional supermanifold (M, A_M), the sheaf $\mathrm{Der}\, A_M$ is a locally free sheaf of A_M-modules over M, with m even generators and n odd. Hence, for any point $p \in M$, the supervector space of superderivations at p is an (m,n)-dimensional supervector space. In particular, $(ST)_e(G, A_G)$, and hence $\mathfrak{g}$, is a finite dimensional supervector space, whose dimension is precisely that of the supergroup.

More importantly, via the isomorphism given in 2.7, one can define a global frame on the supermanifold (G, A_G) consisting entirely of left invariant supervector fields; namely, by choosing any homogeneous basis $\{\xi_1, \ldots, \xi_r; \zeta_1, \ldots, \zeta_s\}$ of $(ST)_e(G, A_G)$ and looking at their corresponding left invariant superderivations. In this way, we get,

$$\mathrm{Der}\, A_G(G) \simeq A_G(G) \otimes \mathfrak{g} \tag{10}$$

and therefore, the supertangent bundle of (G, A_G), is trivial; that is,

$$(STG, STA_G) \simeq (G, A_G) \times (\mathfrak{g}_0 \oplus \mathfrak{g}_1)_S, \tag{11}$$

where $(\mathfrak{g}_0 \oplus \mathfrak{g}_1)_S$ is the supermanifoldification of the supervector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$; it is a $(\dim \mathfrak{g}_0 + \dim \mathfrak{g}_1, \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1)$-dimensional supermanifold (cf., [6]).

Let us include here a brief argument, based on the approach of [6], that proves that a frame over some open subset $U \subset G$ yields a trivialization of the supertangent bundle over the same open subset (and hence, provide a proof for (11)). That is, we shall see how, to give such a frame, is the same as to give an isomorphism φ_U which makes the following diagram to commute:

$$\begin{tikzcd} (\tilde{\tau}^{-1}(U), STA_G|_{\tilde{\tau}^{-1}(U)}) \arrow[rr, "\varphi_U"] \arrow[dr, "\tilde{\tau}"'] & & (U, A_G|_U) \times (\mathfrak{g}_0 \oplus \mathfrak{g}_1)_S \arrow[dl, "p_1"] \\ & (U, A_G|_U) & \end{tikzcd} \tag{12}$$

where, $\tilde{\tau}$ denotes the submersion morphism of the supertangent bundle into the base supermanifold, and p_1 the projection morphism of the product onto the first factor.

We racall that vector bundles were approached in [6] in such a way that supermanifold morphisms $\sigma : (U, A_G|_U) \longrightarrow (\tilde{\tau}^{-1}(U), STA_G|_{\tilde{\tau}^{-1}(U)})$, satisfying $\tilde{\tau} \circ \sigma = id : (U, A_G|_U) \longrightarrow (U, A_G|_U)$, correspond in a one-to-one fashion with sections $\sigma \in \mathrm{Der}\, A_G(U)$. If $\{\xi_1, \ldots, \xi_r; \zeta_1, \ldots, \zeta_s\}$ is a graded frame over U, with $|\xi_i| = 0$ and $|\zeta_\mu| = 1$, each $\sigma \in \mathrm{Der}\, A_G(U)$ can be written uniquely in the form $\sigma = \sum f^i \xi_i + \sum \varphi^\mu \zeta_\mu$ where $f^i, \varphi^\mu \in A_G(U)$. That is, σ is uniquely characterized by the ordered $(r+s)$-tuple $\{f^1, \ldots, f^r; \varphi^1, \ldots, \varphi^s\} \in A_G(U)^r \oplus A_G(U)^s$. Now, the key point observed in [6] is that

$A_G(U)^r \oplus A_G(U)^s$ is isomorphic to the set of all supermanifold morphisms from $(U, A_G|_U)$ into the supermanifoldification of any (r,s)-dimensional supervector space. That is,

$$A_G(U)^r \oplus A_G(U)^s \simeq \mathfrak{Mor}\,((U, A_G|_U), (\mathfrak{g}_0 \oplus \mathfrak{g}_1)_S) \tag{13}$$

Thus, if we denote by Ψ^σ the morphism $(U, A_G|_U) \longrightarrow (\mathfrak{g}_0 \oplus \mathfrak{g}_1)_S$ that corresponds to the (r+s)-tuple $\{f^1, \ldots, f^r; \varphi^1, \ldots, \varphi^s\} \in A_G(U)^r \oplus A_G(U)^s$ that σ gives rise to, then the trivialization φ_U is uniquely determined by the pair of conditions:

$$(\forall \sigma \in \mathrm{Der}\, A_G(U)) \qquad id = p_1 \circ \varphi_U \circ \sigma \qquad \text{and} \qquad \Psi^\sigma = p_1 \circ \varphi_U \circ \sigma \tag{14} \;\square$$

2.9 **Remark**: Once the triviality of the supertangent bundle of a Lie supergroup (G, A_G) is settled, one may argue as in [2] to conclude that for the superalgebra of global sections of its structural sheaf one has,

$$A_G(G) \simeq C^\infty(G) \otimes \wedge(\mathfrak{g}_1{}^*) \tag{15}$$

where, $\mathfrak{g}_1{}^*$ is the dual of the odd part of the Lie superalgebra of the supergroup.

More precisely, one may regard $\{\xi_1, \ldots, \xi_r; \zeta_1, \ldots, \zeta_s\}$ as the basis for $\mathfrak{g}$ obtained via 2.7 from the germs at $e \in G$ of the set $\{\partial_{z^1}, \ldots, \partial_{z^r}; \partial_{\eta^1}, \ldots, \partial_{\eta^s}\}$ of superderivations of the superalgebra $A_G(U)$, with $(U, \{z^1, \ldots, z^r; \eta^1, \ldots, \eta^s\})$, a coordinate neighborhood around e. Then,

$$C^\infty(G) \simeq \{ f \in A_G(U) \,|\, (\forall \mu = 1, \ldots, s) \;\; \zeta_\mu f = 0 \}$$

and (16)

$$\Lambda(\mathfrak{g}_1{}^*) \simeq \{ f \in A_G(U) \,|\, (\forall i = 1, \ldots, r) \;\; \xi_i f = 0 \}.$$

In fact, the idea is simply to propagate the supermanifold structure over U, which is already given locally by definition, to the entire group via left translations. G is thus covered by the family of open subsets $\{L_g(U) : g \in G\}$, and an easy sheaf-theoretical argument then proves (16) □

3. Lie supergroup actions on supermanifolds

3.1 Definition: Let (G, A_G) be a Lie supergroup and let (M, A_M) be a supermanifold. (G, A_G) is said to act on (M, A_M) from the left if there is a supermanifold morphism

$$\psi : (G, A_G) \times (M, A_M) \longrightarrow (M, A_M)$$

satisfying the following two identities:

$$(i) \qquad \psi \circ (\pi_1 \times \psi \circ (\pi_2 \times \pi_3)) = \psi \circ (\mu \circ (\pi_1 \times \pi_2) \times \pi_3)$$

[Both sides are morphisms $(G, A_G) \times (G, A_G) \times (M, A_M) \longrightarrow (M, A_M)$ and this time, the projections π_i are defined on $(G, A_G) \times (G, A_G) \times (M, A_M)$]

and

$$(ii) \qquad \psi \circ (\varepsilon_e \circ \pi_1 \times \pi_2) = \pi_2$$

[as morphisms $(G, A_G) \times (M, A_M) \rightarrow (M, A_M)$]

3.2 Example: $GL_S(V_0 | V_1)$ acting on V_S; $V = V_0 \oplus V_1$ being an (m,n)-dimensional supervector space and V_S its supermanifoldification (cf., [6] and [7]).

Let $\{A^{bJ}, \pi\Gamma^{bJ}, \pi\Theta^{BJ}, D^{BJ}\}$ and $\{\pi A^{bJ}, \Gamma^{bJ}, \Theta^{BJ}, \pi D^{BJ}\}$ be the (linear) even and odd coordinates in $\mathrm{Hom}(V,V)_S$, respectively, introduced in 1.4 above. Also, let $\{x^J, \pi\xi^J\}$ and $\{\pi x^J, \xi^J\}$ respectively be even and odd (linear) coordinates in V_S (as explained in [9]). Define the morphism

$$\psi : GL_S(V_0|V_1) \times V_S \longrightarrow V_S,$$

so as to have a left action in the sense of [9]; that is,

$$\psi^*(x^b + \pi x^b) = \sum_k p_1^*(A^{bk} + \pi A^{bk})\, p_2^*(x^k + \pi x^k) + \sum_J p_1^*(\Gamma^{bJ} + \pi\Gamma^{bJ})\, p_2^*(-\xi^J + \pi\xi^J)$$

$$\psi^*(\xi^B + \pi\xi^B) = \sum_J p_1^*(D^{BJ} + \pi D^{BJ})\, p_2^*(\xi^J + \pi\xi^J) + \sum_k p_1^*(\Theta^{Bk} + \pi\Theta^{Bk})\, p_2^*(x^k - \pi x^k),$$

where $p_1 : \mathrm{Hom}(V,V)_S \times V_S \longrightarrow \mathrm{Hom}(V,V)_S$ and $p_2 : \mathrm{Hom}(V,V)_S \times V_S \longrightarrow V_S$ are the projection morphisms onto the first and second factors, respectively. It is now easy to see that ψ yields an action in the sense of 3.1.

3.3 **Example** : $GL(V_0|V_1)$ acting on the Grassmannian supermanifold $G_{k|h}(V^{m|n})$; $V = V_0 \oplus V_1 = V^{m|n}$ being an (m,n)-dimensional supervector space.

Let $G_{k|h}(V^{m|n})$ be the Grassmannian supermanifold of (k,h)-dimensional supervector subspaces in $V^{m|n}$ (cf., [4] and [8]). There is obviously a natural action of the Lie group

$GL(m)\times GL(n)$ on $G_{k|h}(V^{m|n})$. But more generally, it was emphasized in [8] that there is also a natural action of the Lie supergroup $GL(V_0|V_1)$ (cf., 1.6 above). In fact, one may introduce local coordinates in $G_{k|h}(V^{m|n})$ and arrange them in matrix form as follows (cf., [4] and [8]):

$$\begin{pmatrix} x & \xi \\ \zeta & y \end{pmatrix}$$

where, $x=(x^{ij})$ and $y=(y^{ab})$ are the even coordinates, and $\xi=(\xi^{ib})$ and $\zeta=(\zeta^{aj})$ are the odd ones $(1\leqq i\leqq m-k;\ 1\leqq j\leqq k;\ 1\leqq a\leqq n-h;\ 1\leqq b\leqq h)$. According to [8], this set of local coordinates can be used to define the following *element* of $GL(V_0|V_1)$, (*supercoset representative*),

$$\begin{pmatrix} 1 & x & 0 & \xi \\ 0 & 1 & 0 & 0 \\ 0 & \zeta & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, in terms of local coordinates, the action of the supergroup can be explicitly obtained (under the assumption that the transformation takes place within the same coordinate patch), by solving for the coordinates $\tilde{x}$, $\tilde{y}$, $\tilde{\xi}$, and $\tilde{\zeta}$ of the new supercoset representative in the equation (cf., [8]),

$$\begin{pmatrix} a & b & \alpha & \beta \\ c & d & \gamma & \delta \\ \pi & \rho & p & r \\ \sigma & \tau & s & t \end{pmatrix}\begin{pmatrix} 1 & x & 0 & \xi \\ 0 & 1 & 0 & 0 \\ 0 & \zeta & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & \tilde{x} & 0 & \tilde{\xi} \\ 0 & 1 & 0 & 0 \\ 0 & \tilde{\zeta} & 1 & \tilde{y} \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} J & 0 & K & 0 \\ c & M & \gamma & T \\ U & 0 & H & 0 \\ \sigma & W & s & N \end{pmatrix}$$

There is, however, a more succint way of writing the action in terms of matrices; the idea (suggested by J. A. Wolf) is to conjugate both sides of this equation by the matrix that interchanges the second and third rows. The inverse of such a matrix is evidently itself

Since (cf., [9] for an explanation of how matrix computations are performed),

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & \alpha & \beta \\ c & d & \gamma & \delta \\ \pi & \rho & p & r \\ \sigma & \tau & s & t \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & \alpha & b & \beta \\ -\pi & p & -\rho & r \\ c & -\gamma & d & -\delta \\ \sigma & s & \tau & t \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & 0 & \xi \\ 0 & 1 & 0 & 0 \\ 0 & \zeta & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x & \xi \\ 0 & 1 & -\zeta & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

the redifinition of the original blocks as

$$\left(\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right) = \left(\begin{array}{cc|cc} a & \alpha & b & \beta \\ -\pi & p & -\rho & r \\ \hline c & -\gamma & d & -\delta \\ \sigma & s & \tau & t \end{array}\right) \qquad \left(\begin{array}{c|c} 1 & Z' \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & x & \xi \\ 0 & 1 & -\zeta & y \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

and

$$\left(\begin{array}{c|c} 1 & \tilde{Z}' \\ \hline 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & \tilde{x} & \tilde{\xi} \\ 0 & 1 & -\tilde{\zeta} & \tilde{y} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

yields,

$$\tilde{Z}' = (A'Z' + B')(C'^{*}Z' + D'^{*})^{-1},$$

where the multiplication of the matrix blocks is, in this equation, the usual one. It is easy to verify, however, that the same result can be rewritten in the form,

$$\tilde{Z} = (AZ + B)(CZ + D)^{-1};$$

this time the multiplication of matrices is to be performed according to the rules of linear superalgebra (as in [9]), and,

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|cc} a & \alpha & b & \beta \\ \pi & p & \rho & r \\ \hline c & \gamma & d & \delta \\ \sigma & s & \tau & t \end{array}\right) \qquad Z = \left(\begin{array}{c|c} x & \xi \\ \hline \zeta & y \end{array}\right) \qquad \tilde{Z} = \left(\begin{array}{c|c} \tilde{x} & \tilde{\xi} \\ \hline \tilde{\zeta} & \tilde{y} \end{array}\right)$$

Therefore, the action morphism for the action of $GL(V_0|V_1)$ in the grassmannian supermanifold $G_{k|h}(V^{m|n})$ is given in local coordinates by,

$$\psi^{*}Z = \{(p_1^{*}A)(p_2^{*}Z) + (p_1^{*}B)\}\{(p_1^{*}C)(p_2^{*}Z) + (p_1^{*}D)\}^{-1};$$

where, the p_i's are the projection morphisms of the product supermanifold $GL(V_0|V_1)\times G_{k|h}(V^{m|n})$ and the blocks A, B, C, and D are to be understood as in the decomposition above, but with the entries given in terms of the corresponding local coordinates of $GL(V_0|V_1)$ introduced in 1.4 and 1.6 (see also 5.2 below).

4. The isotropy subsupergroup of an action

4.1 Let (G, A_G) be a Lie supergroup acting on the supermanifold (M, A_M) via the morphism

$$\psi : (G, A_G) \times (M, A_M) \longrightarrow (M, A_M) \tag{1}$$

and fix some point $p \in M$. Then, ψ induces a morphism

$$\psi_p : (G, A_G) \longrightarrow (M, A_M) \tag{2}$$

by letting,

$$\psi_p := \psi \circ (id \times \delta_p \circ C_{(G, A_G)}) = \psi \circ (id \times \varepsilon_p) \tag{3}$$

where, $\varepsilon_p := \delta_p \circ C_{(G, A_G)} : (G, A_G) \longrightarrow (M, A_M)$ is the supermanifold morphism whose corresponding map of presheaves is given by

$$(\forall V \subset M, \text{open})(\forall f \in A_M(V)) \qquad \varepsilon_p{}^* f = \begin{cases} \tilde{f}(x)\, 1_{A_G(G)} & \text{if } x \in V \\ 0 & \text{if } x \notin V \end{cases} \tag{4}$$

The underlying continuous maps $\tilde{\psi}_p$ and $\tilde{\varepsilon}_p$ of ψ_p and ε_p, respectively, are

$$\tilde{\psi}_p : \mathrm{h} \mapsto \tilde{\psi}(\mathrm{h}, p) \qquad \text{and} \qquad \tilde{\varepsilon}_p : \mathrm{h} \mapsto p \quad \text{(constant map)}$$

In a similar way, each point $g \in G$ gives rise to a superdiffeomorphism

$$\psi_g : (M, A_M) \longrightarrow (M, A_M)$$

defined by,

$$\psi_g := \psi \circ (\delta_g \circ C_{(G, A_G)} \times id) = \psi \circ (\mathcal{E}_g \times id) \tag{5}$$

and whose underlying continuous map is $\tilde{\psi}_g : q \mapsto \tilde{\psi}(g, q)$. By the methods of § 2, one proves that,

$$(\psi_g)^{-1} = \psi_{g^{-1}} \quad , \quad \psi_g \circ \psi_h = \psi_{gh} \quad , \quad \psi_p \circ R_g = \psi_{\tilde{\psi}(g,p)} \quad , \quad \psi_g \circ \psi_p = \psi_p \circ L_g \tag{6}$$

4.2 **Definition** : Let (G, A_G) be a Lie supergroup acting on the supermanifold (M, A_M) via ψ as in (1). We shall say that the action is *transitive* if there exists a point $p \in M$, such that, $\psi_p : (G, A_G) \longrightarrow (M, A_M)$ is an epimorphism (that is, if for any supermanifold (N, A_N) and for any pair of supermanifold morphisms $\alpha, \beta : (M, A_M) \longrightarrow (N, A_N)$, $\alpha \circ \psi_p = \beta \circ \psi_p \Longrightarrow \alpha = \beta$).

Note that if ψ_p is an epimorphism for some $p \in M$, then ψ_q is an epimorphism for *any* $q \in M$. Indeed, ψ transitive implies $\tilde{\psi}$ transitive and hence, $\exists\, g \in G$ such that, $q = \tilde{\psi}(g, p)$; since R_g is a superdiffeomorphism, $\psi_q = \psi_p \circ R_g$ is an epimorphism □

4.3 **Observation** : Roughly speaking, we would like to define the *orbit through* p as the *image* of the morphism ψ_p and the *isotropy subsupergroup* at p as the *locus in* (G, A_G) on which the morphisms ψ_p and $\mathcal{E}_p$ *coincide*. On ordinary smooth or holomorphic manifolds these notions can be defined immediately because one can evaluate functions on points and the set of all possible values of a function completely determines the function itself. In supermanifold theory, however, the values of a given morphism on all the points of the underlying domain do not determine the morphism completely; a fact stressed in [2], [3], and [5]. What does determine it, is a knowledge of the superalgebra map from the global sections of the sheaf of the target into the global sections of the direct image sheaf of the source (cf., the appendix).

Thus, even though the notions of *image* and *locus over which two given morphisms coincide* in supermanifold theory have to be defined in a sheaf theoretical manner, this observation gives at least an easy way to state some defining conditions for the isotropy subsupergroup; namely, if we are given an action of (G, A_G) on (M, A_M) as in (1), we should be able to extract a supermanifold (and in fact, a supergroup) structure on the underlying isotropy subgroup $G_p = \{g \in G : \tilde{\psi}_p(g) = p\}$, from the condition,

$$\text{Image}(\psi_p{}^*) = \text{Image}(\varepsilon_p{}^*), \tag{7}$$

where both, $\psi_p{}^*$ and $\varepsilon_p{}^*$ are the superalgebra maps determined from their corresponding sheaf morphisms. (At this point, the reader might want to look first at the explicit examples given in §5 below, and come back for the general argument later).

Once the problem is put this way, the superalgebra of global sections of the isotropy subsupergroup is, if it exists at all, a coequalizer for the diagram

$$A_M(M) \underset{\varepsilon_p{}^*}{\overset{\psi_p{}^*}{\rightrightarrows}} A_G(G) \tag{8}$$

Hence, up to isomorphism, it must be the superalgebra

$$(\tilde{\psi}_{p*}A_G)(M)/\mathrm{Im}(\psi_p^* - \mathcal{E}_p^*)(M) \tag{9}$$

where,

$$\mathrm{Im}(\psi_p^* - \mathcal{E}_p^*)(M) = \{\psi_p^*(f) - \mathcal{E}_p^*(f) : f \in A_M(M)\} \tag{10}$$

and of course,

$$(\tilde{\psi}_{p*}A_G)(M) = A_G(\tilde{\psi}_p^{-1}(M)) = A_G(G) \tag{11}$$

4.4 **Observation**: We pause for a moment in order to further explain the ideas involved in 4.3. What we are implying is that the isotropy subsupergroup of the given action must be defined as an object, (G_p, A_{G_p}), together with a morphism

$$i_p : (G_p, A_{G_p}) \longrightarrow (G, A_G) \tag{12}$$

that makes the diagram

$$\begin{array}{ccc} (G_p, A_{G_p}) & \xrightarrow{\textit{unique}} & (\{*\}, \mathbf{R}) \\ i_p \downarrow & & \downarrow \delta_p \\ (G, A_G) & \xrightarrow[\psi_p]{} & (M, A_M) \end{array} \tag{13}$$

to commute, and with the following universal property: for any other object (H, A_H) and morphism $j : (H, A_H) \longrightarrow (G, A_G)$ making commutative an analogous square, the existence of a unique morphism

$$\mu : (H, A_H) \longrightarrow (G_p, A_{G_p}),$$

can be deduced, with $i_p \circ \mu = j$. In particular, it follows that the morphism i_p is necessarily monic .

Now, the problem of determining both, the superalgebra of global sections $A_{G_p}(G_p)$ and the morphism i_p, is that of constructing a pushout diagram for

$$\begin{array}{ccc} A_M(M) & \xrightarrow{\delta_p^*} & \mathbf{R} \\ {\scriptstyle \psi_p^*}\downarrow & & \\ A_G(G) & & \end{array}$$

Or, since δ_p^* is just the composition

$$A_M(M) \xrightarrow{\delta_p^*} \mathbf{R} \xrightarrow{unique} A_G(G)$$

the problem is that of constructing a coequalizer for the diagram (8) as we claimed.

Our immediate goal is to prove in the next few paragraphs that indeed, (9) carries all the information needed to define an embedded subsupergroup of (G, A_G). The first observation to be made is the following:

4.5 **Proposition**: There is a natural bijection

$$G_p = \{ g \in G : \tilde{\psi}_p(g) = p \} \longleftrightarrow \{\text{Superalgebra maps } (\tilde{\psi}_{p*} A_G)(M) / \mathrm{Im}(\psi_p^* - \delta_p^*)(M) \to \mathbf{R}\}$$

Hence, if the superalgebra of global sections of the sheaf of a supergroup is to be given by (9), its underlying smooth manifold is, up to diffeomorphism, G_p (cf., [2], [3]).

proof : Each superalgebra map $(\tilde{\psi}_{p*} A_G)(M)/\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*) \to R$ corresponds naturally to a superalgebra map, $(\tilde{\psi}_{p*} A_G)(M) \to R$, whose kernel contains $\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)$. Since, the points of G are in natural one-to-one correspondence with the superalgebra morphisms $A_G(G) \longrightarrow R$, via $g \longleftrightarrow \delta_g{}^*$, it follows that $\delta_g{}^*$ defines a superalgebra map $(\tilde{\psi}_{p*} A_G)(M)/\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*) \to R$, if and only if, $\mathrm{Ker}\,\delta_g{}^* \supset \mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)$; that is, if and only if,

$$(\forall f \in A_M(M)) \qquad \delta_g{}^*(\psi_p{}^* f - \varepsilon_p{}^* f) = \delta_g{}^*(\psi_p{}^* f) - \delta_g{}^*(\varepsilon_p{}^* f) = 0.$$

Hence, if and only if, $(\forall \tilde{f} \in C^\infty(M))\ \tilde{f}(\tilde{\psi}_p(g)) = \tilde{f}(\tilde{\varepsilon}_p(g))$, which obviously holds true, if and only if, $\tilde{\psi}_p(g) = p = \tilde{\varepsilon}_p(g)$; that is, if and only if, $g \in G_p$ □

4.6 **Theorem** : Let $\tilde{\imath}_p : G_p \longrightarrow G$ be the embedding of the closed Lie subgroup G_p into G. Let A_{G_p} be the pullback to G_p, via $\tilde{\imath}_p$, of the sheafification of the presheaf of superalgebras over G,

$$U \longmapsto A_G(U)/(\tilde{\psi}_p{}^* \mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*))(U),$$

where, $\tilde{\psi}_p{}^* \mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)$ denotes the pullback to G of the sheaf $\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)$ under $\tilde{\psi}_p : G \longrightarrow M$. Then, (G_p, A_{G_p}) is an embedded Lie subsupergroup of (G, A_G). The embedding $\imath_p : (G_p, A_{G_p}) \longrightarrow (G, A_G)$ is defined by the natural sheaf morphism obtained from the composition,

$$A_G \longrightarrow A_G/\tilde{\psi}_p{}^* \mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*) \longrightarrow \tilde{\imath}_{p*} \tilde{\imath}_p{}^* \{A_G/\tilde{\psi}_p{}^* \mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)\}.$$

The supergroup structure is inherited from (G_p, A_{G_p}) by defining the composition morphism $\nu:(G_p, A_{G_p})\times(G_p, A_{G_p})\longrightarrow(G_p, A_{G_p})$ by way of the diagram

$$\begin{array}{ccc} A_G & \xrightarrow{\mu^*} & \tilde{\mu}_*(A_G\times A_G) \\ {\scriptstyle \iota_p^*}\downarrow & & \downarrow{\scriptstyle (\iota_p\circ\pi_1\times\iota_p\circ\pi_2)^*} \\ A_{G_p} & \dashrightarrow & \tilde{\nu}_*(A_{G_p}\times A_{G_p}) \end{array}$$

Finally, the embedding $\iota_p:(G_p, A_{G_p})\to(G, A_G)$ has the universal property stated in 4.4 with respect to any othermorphism $j:(H, A_H)\to(G, A_G)$, such that

$$\psi_p\circ j=\delta_p\circ C_{(H, A_H)}.$$

Proof: We shall keep $p\in M$ fixed. The map $\tilde{\psi}_p:G\longrightarrow M$, can be thought of as a map into the orbit through p, say O_p. We shall endow O_p with the topology and differentiable structures that make the induced map $G/G_p\longrightarrow O_p$ a diffeomorphism. In particular, $\tilde{\psi}_p$ becomes an open map onto O_p. This condition is sufficient for the natural $\tilde{\psi}_p$-morphism, $A_G\longrightarrow\tilde{\psi}_{p*}A_G|O_p$ (cf., [11]), to induce isomorphisms on each stalk. In fact, given $g\in G$, and letting W run through all open neighborhoods of $\tilde{\psi}_p(g)$, the stalk $(A_G)_g$ becomes a target for the direct system defined by the sheaf $\tilde{\psi}_{p*}A_G$. Hence, we have the following commutative diagram:

$$
\begin{tikzcd}[column sep=small]
 & & & & \exists\, W \subset \tilde{\psi}_p^{-1}(W) \\
(\tilde{\psi}_{p*}A_G)(W) \arrow[rr, equal] \arrow[dd, "(\tilde{\psi}_p(g)\in V\subset W)"'] \arrow[dr] & & A_G(\tilde{\psi}_p^{-1}(W)) \arrow[rr, dashed] \arrow[dd] & & A_G(\tilde{W}) \arrow[dl] \arrow[dd, "(g\in\tilde{V}\subset\tilde{W})"] \\
 & (\tilde{\psi}_{p*}A_G)_{\tilde{\psi}_p(g)} \arrow[rr] & & (A_G)_g & \\
(\tilde{\psi}_{p*}A_G)(V) \arrow[rr, equal] \arrow[ur] & & A_G(\tilde{\psi}_p^{-1}(V)) \arrow[rr, dashed] & & A_G(\tilde{V}) \arrow[ul] \\
 & & & & \exists\, \tilde{V} \subset \tilde{\psi}_p^{-1}(V)
\end{tikzcd}
$$

where the dotted horizontal arrows can be drawn simply from the continuity of $\tilde{\psi}_p$. But, we can reverse the arrow $(\tilde{\psi}_{p*}A_G)_{\tilde{\psi}_p(g)} \longrightarrow (A_G)_g$. Indeed, since $\tilde{\psi}_p$ is open, for each open neighborhood $\tilde{V}$ of $g \in G$, there exists an open neighborhood V of $\tilde{\psi}_p(g) \in M$, with $\tilde{\psi}_p^{-1}(V) \subset \tilde{V}$; hence, the dotted arrows may be reversed by means of restriction maps. Therefore, $(\tilde{\psi}_{p*}A_G)_{\tilde{\psi}_p(g)}$ is a target for the directed system of A_G. By uniqueness of direct limits, we get

$$(\forall g \in G) \qquad (\tilde{\psi}_{p*}A_G)_{\tilde{\psi}_p(g)} \simeq (A_G)_g \tag{14}$$

Since, $\{\mathrm{Im}(\psi_p^* - \mathcal{E}_p^*)\}_{\tilde{\psi}_p(g)} \simeq \{\tilde{\psi}_p^*\mathrm{Im}(\psi_p^* - \mathcal{E}_p^*)\}_g$, we furthermore have,

$$(\tilde{\psi}_{p*}A_G)_{\tilde{\psi}_p(g)} / \{\mathrm{Im}(\psi_p^* - \mathcal{E}_p^*)\}_{\tilde{\psi}_p(g)} \simeq (A_G)_g / \{\tilde{\psi}_p^*\mathrm{Im}(\psi_p^* - \mathcal{E}_p^*)\}_g \tag{15}$$

for all $g \in G$. In particular, we obtain a sheaf epimorphism

$$A_G \longrightarrow A_G / \tilde{\psi}_p^*\mathrm{Im}(\psi_p^* - \mathcal{E}_p^*) \tag{16}$$

Note next that the natural morphism

$$A_G / \tilde{\psi}_p{}^*\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*) \longrightarrow \tilde{\imath}_{p*}\tilde{\imath}_p{}^*\{A_G / \tilde{\psi}_p{}^*\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)\} \qquad (17)$$

is an isomorphism. In fact, since G_p is a closed Lie subgroup of G, the sheaf on the right is *the* extension of $A_{G_p} = \tilde{\imath}_p{}^*\{A_G / \tilde{\psi}_p{}^*\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)\}$ by zero (cf., [11]). Therefore, for any open subset $U \subset G$, the map

$$\begin{aligned} \tilde{\imath}_{p*}\tilde{\imath}_p{}^*\{A_G / \tilde{\psi}_p{}^*\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)\}(U) &\longrightarrow \tilde{\imath}_p{}^*\{A_G / \tilde{\psi}_p{}^*\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)\}(U \cap G_p) \\ z &\longmapsto z|U \cap G_p \end{aligned}$$

is an isomorphism and the same conclusion holds true stalkwise. In summary, we get a sheaf epimorphism

$$\imath_{p*} : A_G \longrightarrow \tilde{\imath}_{p*}\tilde{\imath}_p{}^*\{A_G / \tilde{\psi}_p{}^*\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)\} \qquad (18)$$

Now, if this sheaf morphism is to be combined with $\tilde{\imath}_p$ so as to define a supermanifold morphism, the following compatibility condition must be established:

$$(\forall f \in A_G(U))(\forall g \in G_p \cap U) \qquad (\imath_p{}^* f)\tilde{}(g) = \tilde{f} \circ \tilde{\imath}_p(g) \qquad (19)$$

This follows easily: by definition of the pullback sheaf, and taking the isomorphism (17) into account, the sections of $\mathrm{Im}\, \imath_p{}^*$ over $\mathrm{Im}\, \tilde{\imath}_p \subset G$ are presicely of the form $\tilde{\imath}_p(g) \mapsto$ germ at $\tilde{\imath}_p(g)$ of some section of the sheaf $A_G / \tilde{\psi}_p{}^*\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)$. Since both, $A_G \to A_G / \tilde{\psi}_p{}^*\mathrm{Im}(\psi_p{}^* - \varepsilon_p{}^*)$ and $A_G \to C^\infty{}_G|G_p$, are morphisms of sheaves of superalgebras, the compatibility condition follows.

Now, to show that the supergroup structure $\{\mu, \sigma, \varepsilon_e\}$ of (G, A_G) restricts to (G_p, A_{G_p}) in the appropriate manner so as to yield a subsupergroup, we shall make use of the fact that μ^* is a monomorphism (cf., 1.10) in combination with the epimorphism $\iota_p{}^*$. The idea is to show that there is an unambiguous way of clossing the following diagram so as to make it commutative for each pair $(g, h) \in G_p \times G_p$:

$$\begin{tikzcd}[column sep=huge]
(A_G)_{\tilde{\mu}(\tilde{\iota}_p(g), \tilde{\iota}_p(h))} \arrow[r, "(\mu^*)_{\tilde{\mu}(\tilde{\iota}_p(g), \tilde{\iota}_p(h))}"] \arrow[d, "{\begin{array}{c}(\iota_p{}^*)_{\tilde{\mu}(\tilde{\iota}_p(g), \tilde{\iota}_p(h))} = \\ = (\iota_p{}^*)_{\tilde{\iota}_p(\tilde{\nu}(g,h))}\end{array}}"'] & (A_G \times A_G)_{(\tilde{\iota}_p(g), \tilde{\iota}_p(h))} \arrow[d, "(\iota_p \circ \pi_1 \times \iota_p \circ \pi_2)^*_{(\tilde{\iota}_p(g), \tilde{\iota}_p(h))}"] \\
(A_{G_p})_{\tilde{\nu}(g,h)} \arrow[r, dashed, "(\nu^*)_{\tilde{\nu}(g,h)}"'] & (A_{G_p} \times A_{G_p})_{(g,h)}
\end{tikzcd}$$

Note that, since G_p is already a Lie subgroup of G, its smooth composition map, $\tilde{\nu} : G_p \times G_p \to G_p$, satisfies $(\forall (g, h) \in G_p \times G_p)\ \tilde{\mu}(\tilde{\iota}_p(g), \tilde{\iota}_p(h)) = \tilde{\iota}_p(\tilde{\nu}(g, h))$.

Thus, we shall define $(\nu^*)_{\tilde{\nu}(g,h)}$ germwise by going first to $(A_G)_{\tilde{\mu}(\tilde{\iota}_p(g), \tilde{\iota}_p(h))}$ using the surjectivity of $(\iota_p{}^*)_{\tilde{\iota}_p(\tilde{\nu}(g,h))}$ and then to $(A_{G_p} \times A_{G_p})_{(g,h)}$ by just following the arrows in the diagram. The only point that has to be checked, is that this prescription is well defined; that is,

$$\xi \in \mathrm{Ker}\,(\iota_p{}^*)_{\tilde{\mu}(\tilde{\iota}_p(g), \tilde{\iota}_p(h))} \Longrightarrow (\iota_p \circ \pi_1 \times \iota_p \circ \pi_2)^*_{(\tilde{\iota}_p(g), \tilde{\iota}_p(h))} (\mu^*)_{\tilde{\mu}(\tilde{\iota}_p(g), \tilde{\iota}_p(h))}\, \xi = 0 \qquad (20)$$

which amounts to check that, for any germ, $f_p \in (A_M)_p$,

$$\{\psi_p \circ \mu \circ (\iota_p \circ \pi_1 \times \iota_p \circ \pi_2)\}^*{}_p f_p = \{\varepsilon_p \circ \mu \circ (\iota_p \circ \pi_1 \times \iota_p \circ \pi_2)\}^*{}_p f_p \qquad (21)$$

This is so because the morphism $\iota_p{}^*$ has been defined so as to have its kernel isomorphic to $\mathbf{Im}(\psi_p{}^* - \varepsilon_p{}^*)$. In fact, since (17) is an isomorphism, we have,

$$\begin{aligned} \mathrm{Ker}\,(\iota_p{}^*)_{\tilde\mu(\tilde\iota_p(g),\tilde\iota_p(h))} &\simeq (\tilde\psi_p{}^* \mathbf{Im}(\psi_p{}^* - \varepsilon_p{}^*))_{\tilde\mu(\tilde\iota_p(g),\tilde\iota_p(h))} \\ &\simeq (\mathbf{Im}(\psi_p{}^* - \varepsilon_p{}^*))_{\tilde\psi(\tilde\mu(\tilde\iota_p(g),\tilde\iota_p(h)))} \\ &\simeq (\mathbf{Im}(\psi_p{}^* - \varepsilon_p{}^*))_p \end{aligned} \qquad (22)$$

More generally, $\mathrm{Ker}\,(\iota_p{}^*)_g \simeq (\mathbf{Im}(\psi_p{}^* - \varepsilon_p{}^*))_{\tilde\psi(g)}$ holds true for any $g \in G$. In particular,

$$(\psi_p \circ \iota_p)^* f = \iota_p{}^*(\psi_p{}^* f) = \iota_p{}^*(\psi_p{}^* f - \varepsilon_p{}^* f + \varepsilon_p{}^* f) = \iota_p{}^*(\varepsilon_p{}^* f) = (\varepsilon_p \circ \iota_p)^* f$$

for each $f \in A_M(M)$, and therefore,

$$\psi_p \circ \iota_p = \varepsilon_p \circ \iota_p \qquad (23)$$

But now, (21) follows when this identity is used in conjunction with the results of the following:

4.7 **Lemma**: Let (G, A_G) be a Lie supergroup acting on the supermanifold (M, A_M) via the morphism ψ as in (1), and let $p \in M$ be a given point. Let $\{\mu, \sigma, \varepsilon_e\}$ be the multiplication, inversion and identity morphisms of (G, A_G). Let q_1 and q_2 be the projection morphisms of the product $(G, A_G) \times (G, A_G)$ onto the first and second factors, respectively. Then,

$$(i) \qquad \mathcal{E}_p \circ \mu = \mathcal{E}_p \circ q_1 = \mathcal{E}_p \circ q_2$$

$$(ii) \qquad \psi_p \circ \mu = \psi \circ (q_1 \times \psi_p \circ q_2)$$

$$(iii) \qquad \psi \circ (\sigma \times \psi_p) = \psi \circ (\mathcal{E}_e \times \mathcal{E}_p) = \mathcal{E}_p$$

Proof: The proof of (i) reduces to compute the effect of $(\mathcal{E}_p \circ \mu)^*$, $(\mathcal{E}_p \circ q_1)^*$ and $(\mathcal{E}_p \circ q_2)^*$ on global sections, but this is simple with the help of (4). Now, for (ii), let τ_i be the projection morphism of the product $(G, A_G) \times (G, A_G) \times (G, A_G)$ onto the i^{th} factor $(i = 1, 2, 3)$. Then,

$$\begin{aligned}
\psi_p \circ \mu &= \psi \circ (id \times \mathcal{E}_p) \circ \mu = \psi \circ (\mu \times \mathcal{E}_p \circ \mu) = \psi \circ (\mu \circ (q_1 \times q_2) \times \mathcal{E}_p \circ q_2) = \\
&= \psi \circ (\mu \circ (\tau_1 \times \tau_2) \times \tau_3) \circ ((q_1 \times q_2) \times \mathcal{E}_p \circ q_2) = \\
&= \psi \circ (\tau_1 \times \psi \circ (\tau_2 \times \tau_3)) \circ ((q_1 \times q_2) \times \mathcal{E}_p \circ q_2) = \\
&= \psi \circ (q_1 \times \psi \circ (q_2 \times \mathcal{E}_p \circ q_2)) = \psi \circ (q_1 \times \psi \circ (id \times \mathcal{E}_p) \circ q_2) = \\
&= \psi \circ (q_1 \times \psi_p \circ q_2)
\end{aligned}$$

Finally, (iii) is proved similarly; one only has to note that $\mathcal{E}_p \circ \sigma = \mathcal{E}_p$ □

We shall now conclude the proof of 4.6 with a result of Leites (cf., [3]) that shows that a coordinate neighborhood of the identity can be found in (G, A_G), say $(U, \{z^i; \eta^\mu\})$, for which $(G_p \cap U, A_{G_p}|G_p \cap U)$ exhibits the subsupermanifold property. Note that just as in the smooth theory, it is enough to restrict ourselves to such a neighborhood, since we can propagate this local supermanifold structure to the entire embedded subgroup G_p via left translations. The result that we invoke is the following:

4.8 **Lemma** (Leites): Let $U \subset G$ be some open coordinate neighborhood of $e \in M$ and let $I_p|_U$ be the restriction to U of the sheaf of ideals $\tilde{\psi}_p{}^* \mathrm{Im}\,(\psi_p{}^* - \varepsilon_p{}^*)$ in A_G. Let

$$H_p = \{g \in U : \forall f \in I_p(U),\ \tilde{f}(g) = 0\}.$$

For each $g \in H_p$, there is an open neighborhood V ($U \supset V \ni g$) and there are homogeneous sections $\{f^1, \ldots, f^k; \eta^1, \ldots, \eta^l\}$, $f^j \in (I_p(V))_0$ and $\eta^\nu \in (I_p(V))_1$, that generate $I_p(V)$. Furthermore, $\{(df^1)_g, \ldots, (df^k)_g; (d\eta^1)_g, \ldots, (d\eta^l)_g\}$ is a linearly independent set and therefore, $\{f^1, \ldots, f^k; \eta^1, \ldots, \eta^l\}$ may be extended to a coordinate system $\{f^1, \ldots, f^k, f^{k+1}, \ldots, f^m; \eta^1, \ldots, \eta^l, \eta^{l+1}, \ldots, \eta^n\}$ on V.

Proof: (cf., [3]) □

Note that because of 4.5, H_p above is precisely $G_p \cap U$. Therefore, noting that the assertion regarding the universal character of (G_p, A_{G_p}) is automatic by the very definition of A_{G_p} (cf., 4.3 and 4.4), 4.6 follows □

4.9 **Observation**: Let us briefly discuss how the orbits of an action ψ are to be understood. Following [2], the idea is to show that a natural supermanifold sheaf can be defined on the space of cosets G/G_p and then, carry this structure over the orbit $O_p = \tilde{\psi}_p(G) \subset M$, via $\tilde{\psi}_p$, so as to have $(G/G_p, A_{G/G_p}) \simeq (O_p, A_{O_p})$.

In this way, it suffices to show that for any supergroup (G, A_G) and a given embedding of supergroups $i : (H, A_H) \longrightarrow (G, A_G)$ (cf., 1.11), there is a natural way of defining a supermanifold sheaf over the coset space G/H, say $A_{G/H}$, and a sheaf monomorphism $q^* : A_{G/H} \longrightarrow \tilde{q}_* A_G$, where, $\tilde{q} : G \longrightarrow G/H$, is the canonical projection.

This is obtained as follows (compare with [2]): For any open subset $U \subset G/H$, the assignment

$$U \mapsto \{ f \in A_G(\tilde{q}^{-1}(U)) \,|\, (\forall h \in H)\, R_h^* f = f \} = \bigcap_{h \in H} \mathrm{Ker}\,(R_h^* - id^*)(\tilde{q}^{-1}(U)) \quad (24)$$

defines a sheaf of superalgebras over G/H. It is, in fact, a subsheaf of $\tilde{q}_* A_G$. This is precisely the sheaf $A_{G/H}$ and q^* is simply the natural inclusion into $\tilde{q}_* A_G$. Note, for example, that if $i : (H, A_H) \to (G, A_G)$ is the embedding of the isotropy subsupergroup of a transitive action ψ as in (1) (cf., 4.2), then, the sheaf defined in (24) is isomorphic to $\mathrm{Im}\, \psi_p^*$, in view of the third relation in (6).

5. Examples

5.1 Example : $GL_S(1|1)$ acting on the supermanifold $R^{2|2} = (R^2, R^{2|2})$, (cf., [7]).

Let

$$\psi : (GL(2), GL_S(1|1)) \times (R^2, R^{2|2}) \longrightarrow (R^2, R^{2|2}) \tag{1}$$

be the action in 3.2. Let $\{A^{bJ}, \pi\Gamma^{bJ}, \pi\Theta^{BJ}, D^{BJ}\}$ and $\{\pi A^{bJ}, \Gamma^{bJ}, \Theta^{BJ}, \pi D^{BJ}\}$ be the even and odd linear coordinates introduced in 1.4. In this example the indices can be omitted altogether. To simplify the writing even further, we shall redefine the coordinates as

$$\begin{array}{cccc} A = a & \pi\Gamma = b & \pi A = \alpha & \Gamma = \beta \\ \pi\Theta = c & D = d & \Theta = \gamma & \pi D = \delta \end{array} \tag{2}$$

Now, let $\{x, \pi\xi\}$ and $\{\pi x, \xi\}$ respectively be the even and odd (linear) coordinates in V_S in $R^{2|2}$ as introduced in 3.2. This time we redefine $\pi\xi$ and πx as

$$\pi\xi = y \qquad \pi x = \zeta \tag{3}$$

Therefore, the action morphism of 3.2 becomes,

$$\begin{aligned} \psi^* x &= ax - \beta\xi + \alpha\zeta + by \\ \psi^* y &= -\gamma\zeta + dy + cx + \delta\xi \\ \psi^* \zeta &= a\zeta + \beta y + \alpha x - b\xi \\ \psi^* \xi &= \gamma x + d\xi - c\zeta + \delta y \end{aligned} \tag{4}$$

where we have further omitted the explicit reference to the projection morphisms of the product supermanifold $GL_S(1|1)\times R^{2|2}$. We know that these equations can be written in matrix form as (cf., [9], [7]),

$$\psi^*\begin{pmatrix} x+\zeta \\ y+\xi \end{pmatrix} = \begin{pmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{pmatrix}\begin{pmatrix} x+\zeta \\ y+\xi \end{pmatrix} \tag{5}$$

If we now fix some $p\in R^2$ and consider

$$\psi_p = \psi\circ(id\times\delta_p\circ C) : GL_S(1|1) \longrightarrow R^{2|2}$$

we find that

$$\begin{aligned} \psi_p{}^* x &= a\tilde{x}(p) + b\tilde{y}(p) \quad , \quad \psi_p{}^*\zeta = \alpha\tilde{x}(p) + \beta\tilde{y}(p) \\ \psi_p{}^* y &= c\tilde{x}(p) + d\tilde{y}(p) \quad , \quad \psi_p{}^*\xi = \gamma\tilde{x}(p) + \delta\tilde{y}(p) \end{aligned} \tag{6}$$

In particular, let us note that the map $\tilde{\psi}_p : GL(4) \longrightarrow R^2$ is given by,

$$\tilde{\psi}_p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tilde{x}(p) \\ \tilde{y}(p) \end{pmatrix} \tag{7}$$

Observe that, since the coordinate expressions for $\psi_p{}^* x$ and $\psi_p{}^* y$ only involve C^∞ functions, we have,

$$(\forall f\in C^\infty(R^2)) \qquad \psi_p{}^* f = f\circ\tilde{\psi}_p$$

This is to be contrasted with the effect of ψ^* on $C^\infty(R^2)$: there, we have the underlying continuous map $\tilde{\psi}: GL(2)\times R^2 \longrightarrow R^2$ given by,

$$\tilde{\psi}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and for any $f \in C^{\infty}(\mathbf{R}^2)$, we have,

$$\psi^* f = f \circ \tilde{\psi} + (\alpha\zeta - \beta\xi) f_x \circ \tilde{\psi} + (\delta\xi - \gamma\zeta) f_y \circ \tilde{\psi} +$$
$$+ (^1/_2)\{(\alpha\zeta - \beta\xi)^2 f_{xx} \circ \tilde{\psi} + 2(\alpha\zeta - \beta\xi)(\delta\xi - \gamma\zeta) f_{xy} \circ \tilde{\psi} + (\delta\xi - \gamma\zeta)^2 f_{yy} \circ \tilde{\psi}\}$$

where, f_x stands for the partial derivative of f with respect to x, etc.

Let us now consider the morphism $\varepsilon_p : GL_S(1|1) \longrightarrow \mathbf{R}^{2|2}$. According to 4.1, its effect on the coordinate functions is this:

$$\varepsilon_p^* x = \tilde{x}(p) 1_{GL(2|4)} \quad ; \quad \varepsilon_p^* \zeta = 0 \qquad (8)$$
$$\varepsilon_p^* y = \tilde{y}(p) 1_{GL(2|4)} \quad ; \quad \varepsilon_p^* \xi = 0$$

Therefore, the condition $\mathrm{Im}(\psi_p^*) = \mathrm{Im}(\varepsilon_p^*)$ imposes the following relations among the coordinates of $GL_S(1|1)$:

$$\begin{aligned} \psi_p^* x &= a\tilde{x}(p) + b\tilde{y}(p) = \tilde{x}(p) 1_{GL(2|4)} = \varepsilon_p^* x \\ \psi_p^* y &= c\tilde{x}(p) + d\tilde{y}(p) = \tilde{y}(p) 1_{GL(2|4)} = \varepsilon_p^* y \qquad (9) \\ \psi_p^* \zeta &= \alpha\tilde{x}(p) + \beta\tilde{y}(p) = 0 = \varepsilon_p^* \zeta \\ \psi_p^* \xi &= \gamma\tilde{x}(p) + \delta\tilde{y}(p) = 0 = \varepsilon_p^* \xi \end{aligned}$$

For example, under the assumption that $\tilde{x}(p) = 1$ and $\tilde{y}(p) = 0$ (choice of p), these equations imply that

$$a = 1_{GL(1|1)}\ ,\ c = 0\ ,\ \alpha = 0\ ,\ \gamma = 0 \tag{10}$$

Note that these conditions define indeed an embedded subsupergroup of $GL_S(1|1)$, for the set of all matrices of the form

$$\begin{pmatrix} 1_{GL(1|1)} & b+\beta \\ 0 & d+\delta \end{pmatrix} \tag{11}$$

is closed under the composition morphism μ defined in 1.4. Note that the underlying Lie group of such subsupergroup is the semidirect product of the multiplicative group of the nonzero real numbers (d being a local coordinate) with the additive group of the reals (b being the corresponding coordinate). Also note that it has odd dimension equal to 2.

It is worth mentioning that the calculations in this example are only slightly simpler from those required in the more general case of the Lie supergroup $GL_S(m|n)$ acting on the supermanifoldification V_S of the (m,n)-dimensional supervector space $V = V_0 \oplus V_1$. Indeed, one only has to interprete the definitions in (2) and (3) as equations between matrices of the appropriate sizes and proceed accordingly. For example, the relations (9) look exactly the same in the general case, where $\tilde{x}(p)$ and $\tilde{y}(p)$ represent the column vectors with the coordinates in V_0 and V_1 of the point p. The relations that define the isotropy subsupergroup, and hence, isotropy subsupergroups themselves, vary depending on the chosen point, as expected.

5.2 **Example** : $GL(2|2) = GL(V_0|V_1)$ acting on $G_{1|1}(V^{2|2})$; $V = V_0 \oplus V_1 = V^{m|n}$ being a (2, 2)-dimensional supervector space.

Let $\{A^{bJ}, D^{BJ}\}$ and $\{\Gamma^{bJ}, \Theta^{BJ}\}$ respectively be the even and odd coordinates in $GL(2|2)$ as in 1.5 above. To avoid writing so many indices we shall relabel these coordinates as follows:

$$\begin{array}{llll} A^{11} = a & A^{12} = b & \Gamma^{11} = \alpha & \Gamma^{12} = \beta \\ A^{21} = c & A^{22} = d & \Gamma^{21} = \gamma & \Gamma^{22} = \delta \\ \Theta^{11} = \pi & \Theta^{12} = \rho & D^{11} = p & D^{12} = r \\ \Theta^{21} = \sigma & \Theta^{22} = \tau & D^{21} = s & D^{22} = t \end{array} \tag{12}$$

Let $\{x, y\}$ and $\{\xi, \zeta\}$ respectively be the even and odd local coordinates on $G_{1|1}(V^{2|2})$ defined on the open neighborhood, $\{x \neq 0\} \cap \{y \neq 0\}$. According to 3.3 (and omiting the projection morphisms of the product) we have,

$$\left(\begin{pmatrix} a & \alpha \\ \pi & p \end{pmatrix}\begin{pmatrix} x & \xi \\ \zeta & y \end{pmatrix} + \begin{pmatrix} b & \beta \\ \rho & r \end{pmatrix}\right)\left(\begin{pmatrix} c & \gamma \\ \sigma & s \end{pmatrix}\begin{pmatrix} x & \xi \\ \zeta & y \end{pmatrix} + \begin{pmatrix} d & \delta \\ \tau & t \end{pmatrix}\right)^{-1}$$

$$= \begin{pmatrix} ax + b - \alpha\zeta & \alpha y + \beta + a\xi \\ \pi x + \rho + p\zeta & py + r - \pi\xi \end{pmatrix}\begin{pmatrix} cx + d - \gamma\zeta & \gamma y + \delta + c\xi \\ \sigma x + \tau + s\zeta & sy + t - \sigma\xi \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} ax + b - \alpha\zeta & \alpha y + \beta + a\xi \\ \pi x + \rho + p\zeta & py + r - \pi\xi \end{pmatrix}\begin{pmatrix} A^{-1}(1 + BD^{-1}CA^{-1})^{-1} & -A^{-1}BD^{-1}(1 + CA^{-1}BD^{-1})^{-1} \\ -D^{-1}CA^{-1}(1 + BD^{-1}CA^{-1})^{-1} & D^{-1}(1 + CA^{-1}BD^{-1})^{-1} \end{pmatrix}$$

where,

$$\begin{array}{ll} A = cx + d - \gamma\zeta & B = \gamma y + \delta + c\xi \\ C = \sigma x + \tau + s\zeta & D = sy + t - \sigma\xi \end{array} \tag{13}$$

and the assumption to be made in making sense of these calculations is that we do not leave the same coordinate patch after the transformation. That is, we are assuming that the coordinates are constrained in such a way that,

$$A = cx + d - \gamma\zeta \quad \text{and} \quad D = sy + t - \sigma\xi \quad \text{are both invertible.} \tag{14}$$

In particular, $cx + d$ and $sy + t$ are both invertible, and,

$$A^{-1} = (cx+d)^{-1}\{1 + (cx+d)^{-1}\gamma\zeta\}$$

and

$$D^{-1} = (sy+t)^{-1}\{1 + (sy+t)^{-1}\sigma\xi\}.$$

Therefore,

$$\begin{aligned}
\psi^* x &= \{(ax+b-\alpha\zeta)(cx+d-\gamma\zeta)^{-1} + (\alpha y+\beta+a\xi)D^{-1}CA^{-1}\}(1+BD^{-1}CA^{-1})^{-1} \\
\psi^* y &= \{(py+r-\pi\xi)(sy+t-\sigma\xi)^{-1} + (\pi x+\rho+p\zeta)A^{-1}BD^{-1}\}(1+CA^{-1}BD^{-1})^{-1} \\
\psi^* \xi &= \{(\alpha y+\beta+a\xi)(sy+t-\sigma\xi)^{-1} - (ax+b-\alpha\zeta)A^{-1}BD^{-1}\}(1+CA^{-1}BD^{-1})^{-1} \\
\psi^* \zeta &= \{(\pi x+\rho+p\zeta)(cx+d-\gamma\zeta)^{-1} - (py+r-\pi\xi)D^{-1}CA^{-1}\}(1+BD^{-1}CA^{-1})^{-1}
\end{aligned} \tag{15}$$

Since we are interested in comparing the image of $\psi_p{}^*$ with that of $\varepsilon_p{}^*$, we apply the morphism $(id \times \varepsilon_p)^*$ to both sides of each of these equations. After a little simplification, we find that,

$$
\begin{aligned}
\psi_p{}^* x &= \left(\frac{(a\tilde{x}(p)+b)}{(c\tilde{x}(p)+d)} + \frac{(\sigma\tilde{x}(p)+\tau)(\alpha\tilde{y}(p)+\beta)}{(c\tilde{x}(p)+d)(s\tilde{y}(p)+t)} \right)\left(1 - \frac{(\sigma\tilde{x}(p)+\tau)(\gamma\tilde{y}(p)+\delta)}{(c\tilde{x}(p)+d)(s\tilde{y}(p)+t)} \right) \\
\psi_p{}^* y &= \left(\frac{(p\tilde{y}(p)+r)}{(s\tilde{y}(p)+t)} + \frac{(\pi\tilde{x}(p)+\rho)(\gamma\tilde{y}(p)+\delta)}{(c\tilde{x}(p)+d)(s\tilde{y}(p)+t)} \right)\left(1 - \frac{(\sigma\tilde{x}(p)+\tau)(\gamma\tilde{y}(p)+\delta)}{(c\tilde{x}(p)+d)(s\tilde{y}(p)+t)} \right) \\
\psi_p{}^* \xi &= \left(\frac{(\alpha\tilde{y}(p)+\beta)}{(s\tilde{y}(p)+t)} - \frac{(a\tilde{x}(p)+b)(\gamma\tilde{y}(p)+\delta)}{(c\tilde{x}(p)+d)(s\tilde{y}(p)+t)} \right)\left(1 - \frac{(\sigma\tilde{x}(p)+\tau)(\gamma\tilde{y}(p)+\delta)}{(c\tilde{x}(p)+d)(s\tilde{y}(p)+t)} \right) \\
\psi_p{}^* \zeta &= \left(\frac{(\pi\tilde{x}(p)+\rho)}{(c\tilde{x}(p)+d)} - \frac{(\sigma\tilde{x}(p)+\tau)(p\tilde{y}(p)+r)}{(c\tilde{x}(p)+d)(s\tilde{y}(p)+t)} \right)\left(1 - \frac{(\sigma\tilde{x}(p)+\tau)(\gamma\tilde{y}(p)+\delta)}{(c\tilde{x}(p)+d)(s\tilde{y}(p)+t)} \right)
\end{aligned}
\tag{16}
$$

Let us now choose the point p as that whose coordinates are

$$\tilde{x}(p) = 0 \qquad \text{and} \qquad \tilde{y}(p) = 0 \tag{17}$$

so that

$$
\begin{aligned}
\varepsilon_p{}^* x &= \tilde{x}(p) = 0 & \varepsilon_p{}^* y &= \tilde{y}(p) = 0 \\
\varepsilon_p{}^* \xi &= 0 & \varepsilon_p{}^* \zeta &= 0
\end{aligned}
\tag{18}
$$

It is then easy to verify that the condition $\mathrm{Im}(\psi_p{}^*) = \mathrm{Im}(\varepsilon_p{}^*)$ yields the coordinate relations,

$$
\begin{aligned}
A^{12} &= b = 0 & \Gamma^{12} &= \beta = 0 \\
\Theta^{12} &= \rho = 0 & D^{12} &= r = 0
\end{aligned}
\tag{18}
$$

But, these conditions define an embedded (6,6)-dimensional subsupergroup of the (8,8)-dimensional supergroup GL (2| 2).

We remark again that most of the computations above remain valid (and the final results take exactly the same form) for the more general case of the supergroup GL (m|n) acting on the supergrassmannian $G_{k|h}(V^{m|n})$. All that is required is to interprete A^{11}, $A^{12}, \ldots, D^{21}, D^{22}$ in (12) above as block matrices of the appropriate sizes. What comes out of the same analysis is that for the point p whose coordinates are given by (17) (understood as equations between matrices), the condition $\mathrm{Im}(\psi_p{}^*) = \mathrm{Im}(\mathcal{E}_p{}^*)$ yields the same coordinate relations (19), to be interpreted as conditions on the corresponding blocks of coordinates in the supergroup. The only expressions that look different in the general case are those in (16), where we have used quotients and have permuted some of the factors, but the reader will have no trouble in finding what the general expressions should be. In fact, he/she will note that the common factor on the right of (16) is invertible. Taking this into account, it is not difficult to see then that (18) define an embedded subsupergroup of $GL(m|n)$ of dimension

$$(m^2 + n^2 - k(m-k) - h(n-h),\ 2mn - h(m-k) - k(n-h)).$$

Appendix

definitions and notation

A.1 Various definitions of supermanifolds can be found in the literature and not all of them are equivalent (cf., [5] and references therein). The approach we have adopted in previous works ([6], [7], and [8]), and the one followed here, is that of Leites and Manin (cf., [3], [4]). Thus, a *real smooth supermanifold* is a ringed space (M, A_M); M being a smooth m-dimensional manifold, and A_M a sheaf of supercommutative **R**-superalgebras over M. The conditions imposed on A_M (cf., A.3 below) require some preliminaries.

A.2 Let $J_M = ((A_M)_1)$ be the sheaf of ideals over M generated by the *odd subsheaf* $(A_M)_1$. Consider the J_M-adic filtration of A_M,

$$A_M = J_M{}^0 \supset J_M{}^1 \supset J_M{}^2 \supset \cdots J_M{}^k \supset \cdots$$

and form the corresponding sheaf of graded algebras associated with it:

$$\mathrm{Gr}\, A_M = \bigoplus_{k \geqq 0} \mathrm{Gr}^k A_M\,; \qquad \mathrm{Gr}^k A_M = J_M{}^k / J_M{}^{k+1} \tag{1}$$

Then, $\mathrm{Gr}^0 A_M = A_M / J_M$ is a sheaf of commutative algebras over M and there is a sheaf epimorphism

$$\delta : A_M \longrightarrow \mathrm{Gr}^0 A_M \tag{2}$$

naturally defined. Also, each $\mathrm{Gr}^k A_M$ is a sheaf of $\mathrm{Gr}^0 A_M$-modules and, when viewed as a sheaf of $Gr^0 A_M$-algebras, $Gr A_M$ is generated by $Gr^1 A_M$. In fact, $Gr A_M$ has the structure of a sheaf of augmented $Gr^0 A_M$-algebras over M, with augmentation given by the projection onto the direct summand $Gr^0 A_M$; i.e.,

$$\varepsilon : Gr A_M \longrightarrow Gr^0 A_M \tag{3}$$

Since A_M is supercommutative, $Gr A_M$ is a homomorphic image of the sheaf $\Lambda_{Gr^0 A} Gr^1 A_M$. If there is some k, such that $J_M{}^k = 0$, then $Gr A_M$ is actually isomorphic to the latter.

A.3 (M, A_M) is a real (m,n)-dimensional smooth supermanifold, if:

(i) For each $x \in M$, the stalk $A_{M,x}$ is a local super-ring.

(ii) The sheaf $Gr^0 A_M$ is isomorphic to the sheaf $C^\infty{}_M$ of real smooth functions over M.

(iii) $Gr^1 A_M$ is a locally free sheaf of $Gr^0 A_M$-modules of finite rank, n, over M. (The rank is called the *odd dimension* of the supermanifold; it is the largest integer, such that, $J_M{}^n \neq 0$).

(iv) For each point $x \in M$, there is an open neighborhood U of x, and an isomorphism of sheaves of supercommutative superalgebras over U,

$$\varphi_U : A_M|_U \longrightarrow Gr A_M|_U$$

such that, $\varepsilon \circ \varphi_U = \varepsilon$.

A.4 Observation: For the sake of comparison, we include here the definition of a supermanifold as originally given by Kostant in [2]: an (m,n)-dimensional supermanifold is a pair (M, A_M) consisting of an ordinary m-dimensional C^∞ manifold M, and a sheaf A_M of supercommutative superalgebras, such that,

(i) for each non-empty open subset $U \subset M$, there is defined a superalgebra homomorphism $A_M(U) \ni f \mapsto \tilde{f} \in C^\infty{}_M(U)$ that commutes with restrictions,

and,

(ii) each open subset U of M can be covered by open neighborhoods U_i $(i \in I)$, such that,

(ii.1) $\exists$ a subalgebra $C(U_i) \subset (A_M(U_i))_0$ (called a *function factor* of $A_M(U_i)$), such that the map $C(U_i) \ni f \mapsto \tilde{f} \in C^\infty{}_M(U_i)$, is an isomorphism,

and,

(ii.2) $\exists$ odd elements $s_1(i), s_2(i), \ldots, s_n(i) \in (A_M(U_i))_1$, such that,

$$s_1(i)\, s_2(i) \cdots s_n(i) \neq 0,$$

and if $D(U_i)$ denotes the subsuperalgebra of $A_M(U_i)$ generated by them (called an *exterior factor* of $A_M(U_i)$), the map $C(U_i) \otimes D(U_i) \ni f \otimes w \mapsto f w \in A_M(U_i)$, is an isomorphism of superalgebras.

The U_i's are called A_M-*splitting neighborhoods* of odd dimension n, and $C(U_i)$ and $D(U_i)$ are said to be a pair of *splitting factors for* A_M over U_i.

Now, Kostant asserts in his proposition 2.4.2 [2] that if U is an A_M-splitting neighborhood with $(C(U), D(U))$ a given pair of splitting factors, and if V is an open subset contained in U, there exists a unique function factor $C(V)$ of $A_M(V)$, such that, $\rho^U{}_V(C(U)) \subset C(V)$; furthermore, after setting $D(V) = \rho^U{}_V(D(U))$ a commutative diagram of superalgebra morphisms is obtained; namely,

$$\begin{tikzcd}
C(U) \otimes D(U) \arrow[r] \arrow[d, "\rho^U{}_V \otimes \rho^U{}_V"'] & A_M(U) \arrow[d, "\rho^U{}_V"] \\
C(V) \otimes D(V) \arrow[r] & A_M(V)
\end{tikzcd}$$

However, it does not follow from Kostant's definitions alone that this commutative diagram *factors* so as to yield a commutative diagram of the form

$$\begin{tikzcd}
C(U) \otimes D(U) \arrow[rr] \arrow[dd] \arrow[dr] & & A_M(U) \arrow[dd] \\
& C^\infty(U) \otimes \Lambda[n] \arrow[ur] \arrow[dd, dashed] & \\
C(V) \otimes D(V) \arrow[rr] \arrow[dr] & & A_M(V) \\
& C^\infty(V) \otimes \Lambda[n] \arrow[ur] &
\end{tikzcd}$$

where the vertical dotted arrow is the restriction map of the sheaf $C^\infty \otimes \Lambda[n]$. Note that if it does, Kostant's definition is the same as the one above. As far as we know, however, no examples have been given yet of supermanifolds in this sense which are not supermanifolds in the sense of [3] and [4].

A.5 If (M, A_M) is an (m,n)-dimensional supermanifold and if $U \subset M$ is any open subset of the underlying manifold M, then $(U, A_M|_U)$ is an (m,n)-dimensional supermanifold. It is called an *open subsupermanifold* of (M, A_M).

A.6 A supermanifold *morphism* from (M, A_M) into (N, A_N), is a pair $\varphi = (\tilde{\varphi}, \varphi^*)$ consisting of a continuous map

$$\tilde{\varphi} : M \longrightarrow N$$

and a sheaf homomorphism over N,

$$\varphi^* : A_N \longrightarrow \tilde{\varphi}_* A_M$$

which is local on each stalk (cf., [11]).

It is a well known fact (c.f., [2], and [3]) that a supermanifold morphism is completely determined by the superalgebra morphism on global sections that the sheaf homomorphism gives rise to; that is, by

$$\varphi^* : A_N(N) \longrightarrow A_M(\tilde{\varphi}^{-1}(N))$$

In particular, every supermanifold comes equipped with the supermanifold morphism

$$\delta : (M, C^\infty{}_M) \longrightarrow (M, A_M) \tag{4}$$

determined by the canonical projection

$$\begin{array}{ccc} A_M(M) & \longrightarrow & (A_M / J_M)(M) \simeq C^\infty{}_M(M) \\ f & \longmapsto & \delta(f) = \tilde{f} \end{array} \tag{5}$$

Moreover, each point $p \in M$, defines a morphism

$$\delta_p : (\{*\}, \mathbf{R}) \longrightarrow (M, A_M), \tag{6}$$

by letting

$$(\forall f \in A_M(M)) \qquad \delta_p{}^* f = \tilde{f}(p) \tag{7}$$

and each superalgebra morphism $A_M(M) \longrightarrow \mathbf{R}$ is of this form. The object $(\{*\}, \mathbf{R})$ is the supermanifold consisting of a single point and the constant sheaf **R**, the reals, over it. It is a terminal object, for there is only one *constant morphism*

$$C_{(M,\, A_M)} : (M, A_M) \longrightarrow (\{*\}, \mathbf{R}) \tag{8}$$

from any supermanifold into it; namely, the one determined by the only **R**-superalgebra map $\mathbf{R} \longrightarrow A_M(M)$:

$$(\forall \lambda \in \mathbf{R}) \qquad C_{(M,\, A_M)}{}^* \lambda = \lambda\, 1_{A_M(M)}. \tag{9}$$

A.7 A *supercoordinate system* in (N, A_N) consists of an open neighborhood, $U \subset N$, together with a collection of homogeneous sections $\{x^1, \ldots, x^m; \zeta^1, \ldots, \zeta^n\}$, with $x^i \in (A_N(U))_0$ and $\zeta^\mu \in (A_N(U))_1$, such that,

(i) the set of C^∞ functions, $\{\tilde{x}^1, \ldots, \tilde{x}^m\}$, forms a coordinate system (in the usual sense) over the open set $U \subset N$, and,

(ii) the collection $\{\zeta^1, \zeta^2, \ldots, \zeta^n\}$ is maximal among all collections of odd sections on U with the property that $\zeta^1 \zeta^2 \cdots \zeta^n \neq 0$.

It is clear from the definitions that supercoordinate neighborhoods always exist.

A.8 Let $(\tilde{\varphi}, \varphi^*)$ be a morphism from (M, A_M) into (N, A_N). Let $U \subset N$ be a coordinate neighborhood with supercoordinates $\{x^1, \ldots, x^m; \zeta^1, \ldots, \zeta^n\}$. It is a result of Leites [3] that the restriction of φ to $(\tilde{\varphi}^{-1}(U), A_M|_{\tilde{\varphi}^{-1}(U)})$ is uniquely determined by the sections $\varphi^* x^i$ and $\varphi^* \zeta^\mu$. In fact, given m even sections, $y^i \in A_M(\tilde{\varphi}^{-1}(U))_0$, and n odd sections, $\xi^\mu \in A_M(\tilde{\varphi}^{-1}(U))_1$, there is a unique morphism $(\tilde{\varphi}^{-1}(U), A_M|_{\tilde{\varphi}^{-1}(U)}) \longrightarrow (U, A_N|_U)$, such that,

$$\varphi^* x^i = y^i \qquad \text{and} \qquad \varphi^* \zeta^\mu = \xi^\mu$$

A.9 Let (N, A_N) be an (m,n)-dimensional supermanifold. A supermanifold (M, A_M) is *immersable* into (N, A_N) if there is a morphism $\varphi : (M, A_M) \longrightarrow (N, A_N)$ such that, for each point $p \in M$, there exists an open neighborhood $V \ni p$, such that $\tilde{\varphi}(V)$ has the *subsupermanifold property*; that is, if there exist local coordinates, $\{y^1, \ldots, y^p; \xi^1, \ldots, \xi^q\}$ in V and $\{x^1, \ldots, x^m; \zeta^1, \ldots, \zeta^m\}$ in some open neighborhood $U \supset \tilde{\varphi}(V)$, such that,

$$\rho^{\tilde{\varphi}^{-1}(U)}{}_V\, \varphi^* x^j = y^j \quad \text{if} \quad 1 \leqq j \leqq p \qquad ; \qquad \rho^{\tilde{\varphi}^{-1}(U)}{}_V\, \varphi^* x^j = 0 \quad \text{if} \quad p+1 \leqq j \leqq m$$
$$\rho^{\tilde{\varphi}^{-1}(U)}{}_V\, \varphi^* \zeta^\nu = \xi^\nu \quad \text{if} \quad 1 \leqq \nu \leqq q \qquad ; \qquad \rho^{\tilde{\varphi}^{-1}(U)}{}_V\, \varphi^* \zeta^\nu = 0 \quad \text{if} \quad q+1 \leqq \nu \leqq n.$$

The supermanifold (M, A_M) is *regularly immersable* into (N, A_N) if it is immersable and $\tilde{\varphi}$ is a homeomorphism onto its image. Also, (M, A_M) is *embeddable* into (N, A_N) if it is regularly immersable and $\tilde{\varphi}(M) \subset N$ is a closed subset. We shall say, by abuse of language, that (M, A_M) is an *immersed* (resp., *regular*; *embedded*) subsupermanifold of (N, A_N) whenever it is *immersable* (resp., *regularly immersable*; *embeddable*).

References

[1] **M. Batchelor**, The structure of supermanifolds; Transactions of the American Mathematical Society, **253**, (1980), 329-338.

[2] **B. Kostant**, Graded manifolds, graded Lie theory and prequantization; Springer-Verlag, Lecture Notes in Mathematics, **570**, (1977), 177-306.

[3] **D. A. Leites**, Introduction to the theory of supermanifolds; Russian Mathematical Surveys, **35**, (1980), 1-64.

[4] **Y. I. Manin**, Holomorphic supergeometry and Yang-Mills superfields (preprint).

[5] **M. Rothstein**, The axioms of supermanifolds and a new structure arising from them; Transactions of the American Mathematical Society, **297** (1986) 159-180.

[6] **O. A. Sanchez Valenzuela**, On supergeometric structures; Harvard thesis (may 1986), and, On supervector bundles; IIMAS-UNAM preprint * 457, México, (1986).

[7] **O. A. Sanchez Valenzuela**, Linear supergroup actions I (on the defining properties); Transactions of the American Mathematical Society, **307** (1988) 569-595.

[8] **O. A. Sanchez Valenzuela**, On Grassmannian supermanifolds; to appear in Transactions of the American Mathematical Society, **307** (1988) 597-614.

[9] **O. A. Sanchez Valenzuela**, Matrix computations in linear superalgebra; to appear in Linear Algebra and its Applications.

[10] **T. Schmitt**, Super differential geometry; Akademie der Wissenschaften der DDR, Institut für Mathematik, report R-MATH-05/84, Berlin, (1984).

[11] **O. Tennison**, Sheaf theory; London Mathematical Society, Lecture Notes Series, **20** (1975).