SOME REMARKS ON A CLASS OF UNITARY EXTENSIONS OF THE SYMPLECTIC GROUPS.

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ABSTRACT. A special type of central extension of the real symplectic groups by the special unitary groups is described in an explicit fashion by means of the Bargmann parameters of the symplectic groups

1. GROUP EXTENSIONS.

Recall that given the groups G, N and H, G is said to be a central extension of H by N if we have the following exact sequence of groups

$$0 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 0 \tag{1}$$

In this situation G, as a set, is the cartesian product $H \times N$ and the product in G may be specified by giving two functions

$$\begin{gathered} \psi : H \to \mathrm{Aut}N : h \mapsto \psi_h \\ \chi : H \times H \to N \end{gathered} \tag{2}$$

by the rule

$$(h_1, n_1)(h_2, n_2) = \big(h_1 h_2, \chi(h_1, h_2)\psi_{h_2}(n_1)n_2\big) \tag{3}$$

where ψ and χ satisfy certain relations (cf. [4]). Also recall that G in (1) is called a central extension if $\chi(H \times H) \subset Z$, where Z is the center of N; ψ is then a right action and χ a Z-valued 2-cocycle.

Assume now we are given a central extension G of H by N, defined by the maps ψ and χ. Then, since Z is invariant by all the automorphisms of N, formula (2) defines as well an extension $\hat{G}$ of H by the abelian group Z, and the canonical inclusion $Z \to N$ extends to give the following commutative diagram:

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$$
\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] \arrow[d] & \hat{G} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & N \arrow[r] & G \arrow[r] & H \arrow[r] & 0
\end{tikzcd}
\tag{4}
$$

where id denotes the identity homomorphism. Conversely, given the action ψ , extensions of a group H by the center Z of a group N yield extensions of H by the whole group N.

Now, in general $\hat{G}$ is not a normal subgroup of G, so G is not an extension of $\hat{G}$, but under certain circumstances this is so, for instance we have the following lemma:

Lemma: *Let G be a central extension of H by N, associated to the right action ψ and the Z-valued cocycle χ, and let $\hat{G}$ be the corresponding extension of H by Z as in diagram (4). Assume that for every $h \in H$ and $n \in N$ we have*

$$\psi_h(n)n^{-1} \in Z.$$

Then $\hat{G}$ is a normal subgroup of G and there exists a homomorphism $\beta : G \to N/Z$ such that the following diagram is commutative and has exact rows and columns

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z \arrow[r] \arrow[d] & \hat{G} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "\beta"] & H \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N/Z \arrow[r, "id"] \arrow[d] & N/Z \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
\tag{5}
$$

The proof is a straightforward calculation, but observe that the lemma applies to the special case of a trivial action of H in N.

2. Bargmann parameters and covering groups of the symplectic groups.

Let $Sp = Sp(2n, \mathbb{R})$ and let $\tilde{Sp}$ be its universal covering group (cf. [1]). We shall denote the connected k-fold coverings of Sp by $\tilde{Sp}_k$ so that we have the following commutative diagram, with exact rows and columns:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & k\mathbb{Z} \arrow[r] \arrow[d, "id"'] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}_k \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & k\mathbb{Z} \arrow[r] \arrow[d] & \tilde{Sp} \arrow[r] \arrow[d, "\beta"] & \tilde{Sp}_k \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & Sp \arrow[r, "id"] \arrow[d] & Sp \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
\tag{6}
$$

The special case $k = 2$ is usually called the metaplectic group and denoted $Mp(2n, \mathbb{R})$ (cf. [2]).

The groups $\tilde{Sp}_k$ are in fact special cases of central extensions of Sp and we can give an explicit description of them, in the language of the preceding section, by means of the so-called Bargmann parameters of Sp. To simplify the description we shall write up the formulae for the case $n = 1$ and $k = 2$:

A matrix $A \in Sp(2, \mathbb{R})$ may be written as

$$
\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{i\omega} \end{pmatrix} \begin{pmatrix} \lambda & \mu \\ \bar{\mu} & \lambda \end{pmatrix}
\tag{7}
$$

where $\omega = \arg\alpha$, $\lambda = |\alpha|$ and $\mu = e^{-i\omega}\beta$ are the **Bargmann parameters** of Sp (cf. [6]), and are subject to the following restrictions: $\omega \in \mathbb{R}\ (\mathrm{mod} 2\pi)$, $\lambda > 0$, $\mu \in \mathbb{C}$.

The product rule for $Sp(2)$ may be easily written down in terms of these parameters: with the obvious notations, if $A = A_1 A_2$ then

$$
\begin{aligned}
\omega &= \omega_1 + \omega_2 + \arg\nu \quad (\mathrm{mod} 2\pi) \\
\mu &+ e^{-i\arg\nu}|\lambda_1\mu_2 + e^{-2i\omega}\mu_1\lambda_2 \\
\lambda &= \lambda_1|\nu|\lambda_2
\end{aligned}
\tag{8}
$$

where $\nu = 1 + e^{-2i\omega}\lambda_1^{-1}\mu_1\bar{\mu}_2\lambda_2^{-1}$.

The notation is chosen for the sake of easy generalization to the higher dimensional case: for arbitrary n simply λ in (8) to be a positive definite matrix, μ becomes an arbitrary $n \times n$ matrix with complex entries and complex conjugates are replaced by the adjoints of the matrices. Also, these formulae allow the following explicit description of the covering groups (it is perhaps wortwhile to recall that these cannot be realized as matrix groups), one simply needs to modify the restriction upon ω, the new restriction being $\omega \in \mathbb{R}\ (\mathrm{mod} 2k\pi)$.

Thus, for instance, an element $A \in \tilde{Sp}_2(2)$ may be written as a 4-tuple $(\omega, \mu, \lambda; \sigma)$, where $\sigma \in \mathbb{Z}_2 = \{-1, 1\}$, and the product rule becomes:

$$(\omega_1, \mu_1, \lambda_1; \sigma_1)(\omega_2, \mu_2, \lambda_2; \sigma_2) = (\omega, \mu, \lambda; \sigma) \tag{9}$$

ω, λ and μ being as above, and

$$\sigma = \left\{ \begin{array}{lll} 1 & \text{if} \quad \omega_1 + \omega_2 + \arg\nu \in [0, 2\pi) & (\mathrm{mod} 4\pi) \\ -1 & \text{if} \quad \omega_1 + \omega_2 + \arg\nu \in [2\pi, 4\pi) & (\mathrm{mod} 4\pi) \end{array} \right\} \sigma_1\sigma_2 \tag{10}$$

and the expression between brackets is simply the cocycle χ of the general construction. The right action ψ is also seen from this formula to be the trivial one. Finally, to get the cocycles for the remaining extensions one needs only replace $\mathbb{Z}_2$ by the corresponding group of k-roots of the unity.

3. THE UNITARY EXTENSIONS OF THE SYMPLECTIC GROUPS.

In [5] A. Weil studied the following unitary extension of Sp: consider the exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow U(1) \xrightarrow[id^2]{} U(1) \longrightarrow 0 \tag{11}$$

then, the cocycle χ defining the group Mp, considered as a $U(1)$-valued map, defines a central extension, which will be denoted Tp, of Sp by $U(1)$; Weil called this extension the metaplectic group, but to avoid confusion we will use the term toroplectic, which was proposed in [3].

So, for the toroplectic group one has the following commutative diagram, with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & Mp \arrow[r] \arrow[d] & Sp \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & U(1) \arrow[r] \arrow[d, "id^2"'] & Tp \arrow[r] \arrow[d, "\eta"] & Sp \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & U(1) \arrow[r, "id"] \arrow[d] & U(1) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd} \tag{12}$$

where η is a non trivial character of Tp, and the extensions we shall now describe are higher dimensional analogues of this extension. (Observe however that the toroplectic group is not a special case of the lemma, since $U(1)$ is abelian.)

Now, since the center of $SU(k)$ is $\mathbb{Z}_k$, associated to each k-fold covering $\tilde{Sp}_k$ we get a central extension of Sp by $SU(k)$

$$0 \longrightarrow SU(k) \longrightarrow Sp_k \longrightarrow Sp \longrightarrow 0 \tag{13}$$

so that the following diagram is commutative and with exact rows and columns

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}_k \arrow[r] \arrow[d] & \tilde{Sp}_k \arrow[r] \arrow[d] & Sp \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & SU(k) \arrow[r] \arrow[d] & Sp_k \arrow[r] \arrow[d] & Sp \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & SU(k)/\mathbb{Z}_k \arrow[r, "id"] \arrow[d] & SU(k)/\mathbb{Z}_k \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd} \tag{14}$$

and the product rule for these extensions is given by formulae (9) and (10), where σ, σ_1 and σ_2 should now be interpreted as matrices in $SU(k)$.

Now, observe that some topological properties of these extensions may be inferred directly from these diagrams, for instance, (12) shows that the toroplectic group may be identified to the quotient of $U(1) \times Mp$ by the diagonal action of $\mathbb{Z}_2$, so in particular Tp is a connected Lie group. Similarly, (14) shows that the groups Sp_k may be identified to the quotients of $SU(k) \times \tilde{Sp}_k$ by the diagonal action of $\mathbb{Z}_k$, so Sp_k is a connected Lie group for each k. Also, since both $SU(k)$ and $\tilde{Sp}_k$ are normal subgroups of Sp_k, the Lie algebra of the latter decomposes as a direct sum of ideals

$$\mathfrak{sp}_k \simeq \mathfrak{su}(k) \oplus \mathfrak{sp}. \tag{15}$$

We may summarize the preceding discussion in the following proposition:

Proposition: *For each k there exists a central extension $Sp_k(2n, \mathbb{R})$, of the symplectic group $Sp(2n, \mathbb{R})$ by the special unitary group $SU(k)$, such that diagram (14) holds. $Sp_k(2n, \mathbb{R})$ is a connected semisimple Lie group with Lie algebra $\mathfrak{su}(k) \oplus \mathfrak{sp}(2n, \mathbb{R})$.*

As a concluding remark, for the special case $k = 2$, recall that $SU(2)$ is isomorphic to the spin group $Spin(3)$, so that $SU(2)/\mathbb{Z}_2 \simeq SO(3)$ and diagram (14) becomes

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}_2 \arrow[r] \arrow[d] & Mp \arrow[r] \arrow[d] & Sp \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & SU(2) \arrow[r] \arrow[d] & Sp_2 \arrow[r] \arrow[d] & Sp \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & SO(3) \arrow[r, "id"] \arrow[d] & SO(3) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
\tag{16}
$$

which shows how the metaplectic groups are included in Sp_2. Finally, by including $U(1)$ equatorially in $SU(2)$ we also get the toroplectic group as a subgroup of Sp_2, albeit not as a normal subgroup: rather it corresponds to the Hopf fibering of S^3.

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