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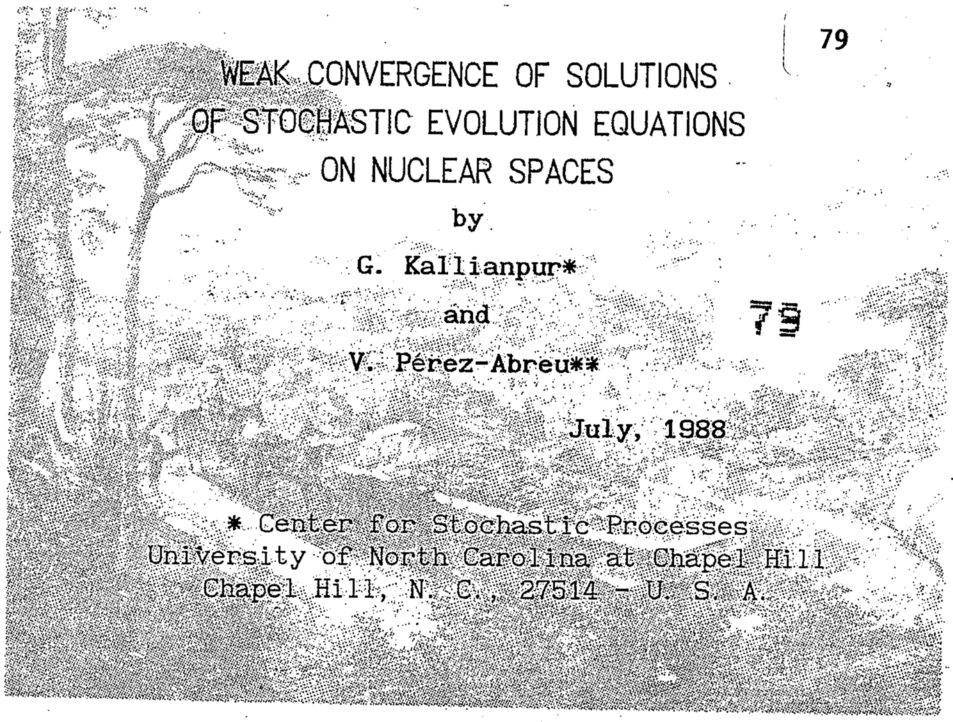

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WEAK CONVERGENCE OF SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS ON NUCLEAR SPACES

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WEAK CONVERGENCE OF SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS ON NUCLEAR SPACES

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INTRODUCTION

In recent years there has been interest in the study of fluctuation limits of infinite particle systems of independent Brownian motions and different types of interacting diffusion systems. Such problems have been studied, among others, by Hitsuda and Mitoma [2], Itô [3], Tanaka and Hitsuda [8] and Mitoma [11]. In all cases the limit process is given by a generalized Langevin equation or a stochastic evolution equation driven by a Gaussian martingale on a nuclear space of distributions Φ' which is the dual of a countably Hilbertian nuclear space Φ. Solutions of Φ'-valued stochastic evolution equations have been investigated by Dawson and Gorostiza [12], Bojdecki and Gorostiza [1], Kallianpur and Pérez-Abreu [5] and Mitoma [7,11]. In the present work we study the weak convergence of solutions of stochastic evolution equations driven by Φ'-valued martingales $M^n = (M^n_t)t \geq 0$ as n goes to infinity. It is worth noting that in this work it is not assumed that all the martingales M^n live in one of the Hilbert or Banach spaces whose norms define the topology of Φ.

In a later publication, we hope to apply Theorem 1.2 of this paper to investigate the fluctuation limit of "weakly" interacting systems.

In all this work the techniques and results from $(C_0,1)$-reversed evolution systems on Φ developed in [5] play an important role.

1. NOTATION AND MAIN RESULT

Let Φ be a countably Hilbertian nuclear space whose topology τ is defined by an increasing sequence of Hilbertian norms $|\cdot|_1 \leq |\cdot|_2 \leq \dots \leq |\cdot|_q \dots$. Let Φ_q be the completion of Φ by $|\cdot|_q$, Φ'_q the topological dual of Φ_q, $|\cdot|_{-q}$ the dual norm of Φ'_q and Φ' the strong topological dual of Φ. Denote by $\mathcal{L}(\Phi,\Phi)$ (respectively $\mathcal{L}(\Phi',\Phi')$)

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the space of continuous linear operators from Φ to Φ (resp. Φ' to Φ'). Let $\{\|\cdot\|_q: q\geq 0\}$ be any sequence of increasing norms on Φ also defining the τ-topology of Φ. Such a sequence of norms will henceforth be called τ-compatible and we will denote by $\Phi_{|q|}$ the $\|\cdot\|_q$-completion of Φ.

A C_0-semigroup $\{S(s): s\geq 0\}$ on Φ is said to be a *$(C_0,1)$-semigroup* if for each $q\geq 0$ there exist numbers M_q, σ_q and $p\geq q$ such that

$$|S(s)\phi|_q \leq M_q e^{\sigma_q s}|\phi|_p \quad \text{for all } \phi\in\Phi,\ s\geq 0. \tag{1.1}$$

A family $\{A(t)\}_{t\geq 0}$ of infinitesimal generators of $(C_0,1)$-semigroups $\{S_t(s): s\geq 0\}_{t\geq 0}$ on Φ is called *stable* if there exists a sequence of τ-compatible norms $\{\|\cdot\|_q: q\geq 0\}$ on Φ such that for each $T>0$ there exists $q_0\geq 0$ and for $q\geq q_0$ there are constants $M_q= M_q(T)$ and $\sigma_q = \sigma_q(T)$ satisfying the following condition:

$$\Big\|\prod_{j=1}^{k} S_{t_j}(s_j)\phi\Big\|_q \leq M_q e^{\sigma_q \sum_{j=1}^{k} s_j}\|\phi\|_q \quad \text{for all } \phi\in\Phi,\ s_j\geq 0 \tag{1.2}$$

whenever $0\leq t_1\leq t_2\leq\ldots\leq t_k\leq T$, $k\geq 0$. Here and in the sequel the time ordered product $\Pi_{j=1}^{k} S_{t_j}(s_j)\phi$ is $S_{t_1}(s_1)S_{t_2}(s_2)\ldots S_{t_k}(s_k)\phi$.

A two parameter family of operators $\{T(s,t): 0\leq s<t<\infty\}$ in $\mathcal{L}(\Phi,\Phi)$ is said to be a *reversed evolution system* on Φ if the following two conditions are satisfied:

$$T(s,t)\phi = T(s,r)T(r,t)\phi \quad \text{for all } \phi\in\Phi,\ 0\leq s\leq r\leq t,\ T(t,t)=I, \tag{1.3}$$

$$\text{for each } \phi\in\Phi \text{ the map } (s,t)\rightarrow T(s,t)\phi \text{ is } \Phi \text{ continuous.} \tag{1.4}$$

The following result has been proved in [5]. It gives sufficient conditions for the existence of a reversed evolution system on Φ generated by a family of operators $\{A(t)\}_{t\geq 0}$ in $\mathcal{L}(\phi,\phi)$.

Theorem 1.1: Let $\{A(t)\}_{t\geq 0}$ be a family of continuous linear operators on Φ such that for each $t\geq 0$ $A(t)$ is the infinitesimal generator of a $(C_0,1)$-semigroup on Φ. Let $\{\|\cdot\|_q: q\geq 0\}$ be a sequence of τ-compatible norms on Φ such that the following two conditions hold:

a) $\{A(t)\}_{t\geq 0}$ is a stable family on Φ with respect to $\{\|\cdot\|_q: q\geq 0\}$.

b) For each $q\geq 0$ there exists $p\geq q$ such that for $t\geq 0$ $A(t)$ has a continuous linear extension from $\Phi_{|p|}$ to $\Phi_{|q|}$ (also denoted by $A(t)$) and $t\rightarrow A(t)$ is $\mathcal{L}(\Phi_{|p|},\Phi_{|q|})$-continuous.

Then there exists a unique reversed evolution system $\{T(s,t): 0\leq s\leq t<\infty\}$ on Φ such that for each $T>0$ the following two conditions are satisfied:

(1) For some $q_0 \geq 0$ and all $q \geq q_0$

$$\|T(s,t)\phi\|_q \leq M_q e^{\sigma_q(t-s)} \|\phi\|_q \quad \text{for all } \phi\in\Phi,\ 0\leq s\leq t\leq T \tag{1.5}$$

where $M_q = M_q(T)$ and $\sigma_q = \sigma_q(T)$ are the stability constants.

(2) The forward and backward equations hold, i.e.

$$\frac{d}{dt}T(s,t)\phi = T(s,t)A(t)\phi \text{ for all } \phi\in\Phi,\ 0\leq s\leq t\leq T, \tag{1.6}$$

$$\frac{d}{dt}T(s,t)\phi = -A(s)T(s,t)\phi \text{ for all } \phi\in\Phi,\ 0\leq s\leq t\leq T. \tag{1.7}$$

A reversed evolution system satisfying (1.5) is called a *$(C_0,1)$-reversed evolution system.*

Let $(\Omega,\mathcal{F},P)$ be a complete probability space with a right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$, $\mathcal{F}_0$ containing all the P-null sets of $\mathcal{F}$. A Φ'-valued stochastic process $M = (M_t)_{t\geq 0}$ is said to be a Φ'-valued martingale if for each $\phi\in\Phi$ $(M_t[\phi])_{t\geq 0}$ is a real valued martingale. In this work we further assume $M \in C([0,\infty);\Phi')$ where $C([0,\infty);\Phi')$ is the space of all continuous mappings of $[0,\infty)$ to Φ' with the strong topology and that M is an L^2-martingale, i.e., $EM_t[\phi]^2 < \infty$ for all $\phi\in\Phi$. The topology of $C([0,\infty);\Phi')$ is described in R.2.1 of Mitoma [6], who shows that $C([0,\infty);\Phi')$ is a completely regular topological space whose compact sets are all metrizable.

Let $\{A(t)\}_{t\geq 0}$ be a family in $\mathcal{L}(\Phi,\Phi)$ and $A'(t)$ denote the adjoint of $A(t)$. A Φ'-valued stochastic process $\xi = (\xi_t)_{t\geq 0}$ is said to be a solution of the stochastic evolution equation

$$d\xi_t = A'(t)\xi_t dt + dM_t \quad t>0,\ \xi_0 = \gamma \tag{1.8}$$

if for each $\phi\in\Phi$

$$\xi_t[\phi] = \xi_0[\phi] + \int_0^t \xi_s[A(s)\phi]ds + M_t[\phi] \quad t\geq 0 \text{ a.s.} \tag{1.9}$$

It has been shown in [5] that if $\{A(t)\}_{t\geq 0}$ satisfies the assumptions of Theorem 1.1 and if $E|\gamma|_{-r}^2 < \infty$ for some $r>0$, then the unique solution of (1.8) is given by the evolution solution

$$\xi_t = T'(t,0)\gamma + \int_0^t T'(t,s)dM_s \tag{1.10}$$

where for each $s < t$ the operator $T'(t,s)\colon \Phi' \to \Phi'$ is defined by the relation

$$(T'(t,s)\psi)[\phi] = \psi[T(s,t)\phi] \text{ for all } \phi\in\Phi,\ \psi\in\Phi' \tag{1.11}$$

and $\{T(s,t): 0 \leq s \leq t \leq \infty\}$ is the $(C_0,1)$-reversed evolution system on Φ generated by the family $\{A(t)\}_{t \geq 0}$. The stochastic integral in (1.10) has the following property

$$\int_0^t T'(t,s)dM_s = M_t + \int_0^t T'(t,s)A'(s)M_s ds \quad \text{for all } t \geq 0 \text{ a.s. ,} \tag{1.12}$$

i.e.,

$$(\int_0^t T'(t,s)dM_s)[\phi] = M_t[\phi] + \int_0^t M_s[A(s)T(s,t)\phi]ds \tag{1.13}$$

$$\text{for all } \phi \in \Phi,\ t \geq 0 \text{ a.s.}$$

The main result of this paper is the following theorem. We denote by "$\Rightarrow$" weak convergence of measures on the indicated spaces.

Theorem 1.2: Assume the following five conditions:

(1) $\{A_n(t)\}_{t \geq 0}$ $n \geq 1$ is a sequence of families of continuous linear operators on Φ such that for each $n \geq 1$ $\{A_n(t)\}_{t \geq 0}$ satisfies the assumptions of Theorem 1.1, the indices appearing in the latter, the sequence of τ-compatible norms $\{\|\cdot\|_q: q \geq 0\}$ and the stability constants M_q, σ_q being the same for all n.

(2) The corresponding evolution systems $\{T_n(s,t): 0 \leq s \leq t < \infty\}$ generated by $\{A_n(t)\}_{t \geq 0}$ $n \geq 1$ satisfy the condition: For each $T>0$ and $q>0$ there exist M_q and σ_q such that for $0 \leq s \leq t \leq T$

$$\|T_n(s,t)\phi\|_q \leq M_q e^{\sigma_q(t-s)} \|\phi\|_q \quad \text{for all } \phi \in \Phi \text{ and } n \geq 1. \tag{1.14}$$

(3) $\{A(t)\}_{t \geq 0}$ is a family of continuous linear operators on Φ satisfying the assumptions of Theorem 1.1 and such that for each $T>0$ and $q>0$ there exists $p>q$ satisfying

$$\sup_{0 \leq t \leq T} \|A_n(t)-A(t)\|_{\mathcal{L}(\Phi_{|p|},\Phi_{|q|})} \to 0 \quad \text{as } n \to \infty. \tag{1.15}$$

(4) $M^n = (M^n_t)_{t \geq 0}$, $n \geq 0$ and $M = (M_t)_{t \geq 0}$ are Φ'-valued L^2-martingales vanishing at the origin and such that $M^n \Rightarrow M$ in $C([0,\infty): \Phi')$.

(5) γ^n, $n \geq 1$ and γ are Φ'-valued random variables such that $\gamma^n \Rightarrow \gamma$ on Φ' and for each $n \geq 1$ γ^n and M^n are independent.

For each $n \geq 1$ suppose that the stochastic evolution equation

$$d\eta_t = A_n'(t)\eta_t dt + dM^n_t \quad t>0,\ \eta_0=\gamma^n \tag{1.16}$$

has the unique solution $\xi^n_\cdot = (\xi^n_t)$ and that.

$$d\eta_t = A'(t)\eta_t dt + dM_t \quad t>0,\ \eta_0=\gamma. \tag{1.17}$$

has the unique solution $\xi_\cdot = (\xi_t)$. Then $\xi^n_\cdot \Rightarrow \xi_\cdot$ in $C([0,\infty);\Phi')$.

The proof of this theorem is given in the next section.

2. PROOF OF THE MAIN THEOREM

In order to prove Theorem 1.2 we need the following lemmas. We will denote the space $C([0,\infty);\Phi')$ by $C_{\Phi'}$.

Lemma 2.1: For $n\geq 1$ let $G_n: C_{\Phi'}\to C_{\Phi'}$ and $G: C_{\Phi'}\to C_{\Phi'}$ be such that $G_n(x)\to G(x)$ as $n\to\infty$ uniformly over compact sets of $C_{\Phi'}$. Let P_n $n\geq 1$ and P be probability measures on $C_{\Phi'}$, and $Q_n = P_n G_n^{-1}$ $n\geq 1$ and $Q = PG^{-1}$. If $P_n \Rightarrow P$ in $C_{\Phi'}$ and G is continuous then $Q_n \Rightarrow Q$ in $C_{\Phi'}$.

Proof: Let $C = C([0,\infty);\mathbb{R})$ be the space of continuous function of $[0,\infty)$ to $\mathbb{R}$ with the topology given in Whitt [13]. For $\phi\in\Phi$ denote by Π_ϕ the mapping of $C_{\Phi'}$ to C defined by

$$(\Pi_\phi x)_. = x_.[\phi]. \tag{2.1}$$

Since $P_n \Rightarrow P$ in $C_{\Phi'}$ then $P_n\Pi_\phi^{-1} \Rightarrow P\Pi_\phi^{-1}$ in C for all $\phi\in\Phi$ and therefore $\{P_n\Pi_\phi^{-1}\}_{n\geq 1}$ is tight in C since C is a Polish space. Then by (R.2.1) and Theorem 3.1 of Mitoma [6] $\{P_n\}_{n\geq 1}$ is itself tight in $C_{\Phi'}$. The remainder of the proof goes as in the case of complete separable metric spaces as we now show: Let α be a bounded real valued continuous function on $C_{\Phi'}$ and let $\epsilon>0$. Then there exists a compact set A in $C_{\Phi'}$ such that

$$\left|\int_{A^c} \alpha(G_n(a))dP_n(a) - \int_{A^c} \alpha(G(a))dP_n(a)\right| \tag{2.2}$$

$$\leq 2\|\alpha\|_\infty (P_n(A^c) < \epsilon\|\alpha\|_\infty \quad \text{for each } n\geq 1$$

where $\|\alpha\|_\infty = \sup_{x\in C_{\Phi'}} |\alpha(x)|$.

Next, by assumption, on the compact set A, $G_n(a) \to G(a)$ uniformly, i.e., for each $\epsilon>0$ there exist $N_\epsilon>0$ and a neighborhood V_ϵ of zero in $C_{\Phi'}$ such that

$$G_n(a) - G(a) \in V_\epsilon \quad \text{for all } n\geq N_\epsilon,\ a\in A.$$

Therefore since α is a bounded continuous function on $C_{\Phi'}$ to $\mathbb{R}$

$$\sup_{a\in A} |\alpha(G_n(a))-\alpha(G(a))| < \epsilon/2 \text{ for all } n\geq N_\epsilon. \tag{2.3}$$

Then for $n\geq N_\epsilon$

$$\left|\int_A \alpha(G_n(a))dP_n(a) - \int_A \alpha(G(a))dP_n(a)\right| \leq \epsilon/2,$$

so that from (2.2),

$$\left|\int \alpha(G_n(a))dP_n(a) - \int \alpha(G(a))dP_n(a)\right| \leq \epsilon \quad \text{for } n\geq N_\epsilon. \tag{2.4}$$

Since $P_n \Rightarrow P$, we have

$$\int \alpha(G(a))dP_n(a) \rightarrow \int \alpha(G(a))dP(a) \tag{2.5}$$

since G is continuous. The assertion of the lemma follows from (2.4) and (2.5) which together imply

$$\int \alpha(x)dQ_n(x) \rightarrow \int \alpha(x)dQ(x). \qquad \square$$

The proof of the above result holds without change if, in its statement, $C_{\Phi'}$ is replaced by $C^T_{\Phi'} := C([0,T];\Phi')$ where $T<\infty$. We will need to use the lemma only in this form.

<u>Lemma 2.2</u>: Let $\{A_n(t)\}_{t\geq 0}$ $n\geq 1$ and $\{A(t)\}_{t\geq 0}$ be continuous linear operators in Φ as in (1) and (3) of Theorem 1.2 and let $\{T_n(s,t): 0\leq s\leq t<\infty\}$ $n\geq 1$ and $\{T(s,t): 0\leq s\leq t<\infty\}$ be the corresponding $(C_0,1)$-reversed evolution systems generated by them such that $T_n(s,t)$ satisfy (2) in Theorem 1.2. Restrict $X\in C_{\Phi'}$ to $C^T_{\Phi'}$, and define

$$(G_n(X))_t[\phi] = X_0[\phi] + \int_0^t X_s[A_n(s)T_n(s,t)\phi]ds, \ \phi\in\Phi, \ 0\leq t\leq T, \ n\geq 1. \tag{2.6}$$

and

$$(G(X))_t[\phi] = X_0[\phi] + \int_0^t X_s[A(s)T(s,t)\phi]ds, \ \phi\in\Phi, \ 0\leq t\leq T. \tag{2.7}$$

Then $G_n(X) \rightarrow G(X)$ as $n\rightarrow\infty$ uniformly on compact sets of $C^T_{\Phi'}$ and G is a continuous map of $C^T_{\Phi'}$ to $C^T_{\Phi'}$.

<u>Proof</u>: We first show that if $X \in C^T_{\Phi'}$ the map, for $\phi\in\Phi$,

$$\phi \rightarrow \int_0^t X_s[A(s)T(s,t)\phi]ds \tag{2.8}$$

defines an element in $C^T_{\Phi'}$. Since $X \in C^T_{\Phi'}$, X sends the compact set [0,T] into a compact set of Φ'. Thus $\{X_s: 0\leq s\leq T\}$ is a compact set in Φ' and therefore bounded in the strong topology of Φ', i.e., for any $\epsilon>0$ and any bounded set B in Φ there exists N>0 such that

$$\left\{X_s: 0\leq s\leq T\right\} \subset N\left\{F \in \Phi': \sup_{\phi\in B} |F[\phi]| < \epsilon\right\}.$$

Taking $\epsilon = 1$ and $B = \{\phi\}$, ϕ in Φ, we have that for each ϕ in Φ there exists $N_\phi>0$ such that

$$\sup_{0\leq s\leq T} |X_s[\phi]| \leq \frac{1}{N_\phi} < \infty. \tag{2.9}$$

Define

$$V_T(\phi) = \sup_{0\le s\le T} |X_s[\phi]|. \tag{2.10}$$

From a Baire category argument (see Lemma I.2.3 in [10] or Lemma 2.2 in [4]) it follows that $V_T(\phi)$ is a continuous function in Φ and hence there exist $\theta_T>0$ and $q_T>0$ such that

$$V_T(\phi) \le \theta_T \|\phi\|_{q_T} \quad \text{for all } \phi \text{ in } \Phi. \tag{2.11}$$

Hence from (2.10) and (2.11) we have that $X_s \in \Phi'_{q_T}$, $0\le s\le T$ and

$$C_T := \sup_{0\le s\le T} \|X_s\|_{-q_T} < \infty. \tag{2.12}$$

Next if a family of linear operators $\{A(t)\}_{t\ge 0}$ on Φ satisfies the conditions of Theorem 1.1, there exists $r_T>q_T$ such that

$$\|A(s)T(s,t)\phi\|_{q_T} \le \|A(s)\|_{\mathcal{L}(\Phi_{|r_T|},\Phi_{|q_T|})} \|T(s,t)\phi\|_{r_T} \tag{2.13}$$

$$\le K_T M_{r_T} e^{T\sigma_{r_T}} \|\phi\|_{r_T} \quad \text{for all } \phi \text{ in } \Phi,\ 0\le s\le t\le T,$$

where M_{r_T} and σ_{r_T} are stability constants and

$$K_T = \sup \|A(s)\|_{\mathcal{L}(\Phi_{|r_T|},\Phi_{|q_T|})} < \infty. \tag{2.14}$$

Then using (2.12) and (2.13) we have that for $0\le t\le T$

$$Y_t[\phi]: \int_0^t X_s[A(s)T(s,t)\phi]ds$$

defines a continuous linear map on Φ, i.e., $Y_t \in \Phi'$ for all $t\ge 0$. Also, if $0\le t\le T$,

$$|Y_t[\phi]| \le TC_T K_T M_{r_T} e^{T\sigma_{r_T}} \|\phi\|_{r_T} \quad \text{for all } \phi \text{ in } \Phi. \tag{2.15}$$

It has been shown in Step 2 of Theorem 2.1 in [5] that there exists $p_T>r_T$ such that $Y^T_\cdot := (Y_t\colon 0\le t\le T) \in C([0,T];\Phi'_{p_T})$. Then $Y^T_\cdot \in C([0,T];\Phi')$ for all $T>0$. Hence the map (2.8) sends $C^T_{\Phi'}$ into $C^T_{\Phi'}$.

Let K be a compact set in $C^T_{\Phi'}$. By R.2.1 and Proposition 2.1 of Mitoma [6], there exists $q_T>0$ such that K is compact in $C([0,T];\Phi'_{q_T})$. Then if $X \in B$,

$$D_T(X) := \sup_{0\le s\le T} \|X_s\|_{-q_T} < \infty$$

and

$$|(G_n(X)-G(X))_t[\phi]| \le \int_0^t \|X_s\|_{-q_T} \|A_n(s)T_n(s,t)\phi - A(s)T(s,t)\phi\|_{q_T} ds \quad (2.16)$$

$$\le D_T(X) \int_0^t \|A_n(s)T_n(s,t)\phi - A(s)T(s,t)\phi\|_{q_T} ds.$$

Writing

$$A_n(s)T_n(s,t)\phi - A(s)T(s,t)\phi$$

$$= A_n(s)(T_n(s,t)\phi - T(s,t)\phi) + (A_n(s)-A(s))T(s,t)\phi$$

for all ϕ in Φ and $0\le s\le t\le T$ we have

$$\|A_n(s)T_n(s,t)\phi - A(s)T(s,t)\phi\|_{q_T} \quad (2.17)$$

$$\le \|A_n(s)(T_n(s,t)\phi - T(s,t))\phi\|_{q_T} + \|(A_n(s)-A(s))T(s,t)\phi\|_{q_T}.$$

Now by (3) in Theorem 1.2 there exists $p_T>q_T$ such that

$$\|A_n(s)(T_n(s,t)\phi - T(s,t)\phi\|_{q_T} \le E_T\|T_n(s,t)\phi - T(s,t)\phi\|_{p_T} \quad (2.18)$$

where using (1.15), for $n\ge n_0$, some $n_0>0$

$$E_T := \sup \|A_n(s)\|_{\mathcal{L}(\Phi_{|p_T|},\Phi_{|q_T|})} < \infty. \quad (2.19)$$

Next since $A_n(s)$ and $A(s)$ generate the $(C_0,1)$-reversed evolution systems $T_n(s,t)$ and $T(s,t)$ respectively, using the corresponding forward and backward equations we obtain

$$T_n(s,t)\phi - T(s,t)\phi = \int_s^t T_n(s,r)(A_n(r)-A(r))T(r,t)\phi dr \quad (2.20)$$

and therefore for each $m\ge 0$

$$\|T_n(s,t)\phi - T(s,t)\phi\|_m \le \int_s^t \|T_n(s,r)(A_n(r)-A(r))T(r,t)\phi\|_m dr. \quad (2.21)$$

Next, using the equi-$(C_0,1)$-evolution property (1.14) we obtain that for each $m\ge 0$

$$\|T_n(s,t)\phi - T(s,t)\phi\|_m \le M_n \int_s^t e^{\sigma_m(r-s)} \|A_n(r)-A(r))T(r,t)\phi\|_m dr \quad (2.22)$$

where M_m and σ_m are stability constants. Taking $m = q_T$ and using again (3) in Theorem 1.2 there exists $p_T>q_T$ such that

$$\|(A_n(r)-A(r))T(r,t)\phi\|_{q_T} \le \|A_n(r)-A(r)\|_{\mathcal{L}(\Phi_{|p_T|},\Phi_{|q_T|})} \|T(r,t)\phi\|_{p_T}.$$

Using the $(C_0,1)$-evolution property of $T(r,t)$ we obtain from (1.5) of Theorem 1 (taking p_T large enough)

$$\|(A_n(r)-A(t))T(r,t)\phi\|_{q_T} \tag{2.23}$$

$$\leq M_{p_T} e^{\sigma_{p_T}(t-r)} \|\phi\|_{p_T} \|A_n(r)-A(r)\|_{\mathcal{L}(\Phi_{|p_T|},\Phi_{|q_T|})}.$$

Thus by (2.23) and (2.22) we have that if $\sigma = \max(\sigma_{q_T},\sigma_{p_T})$ and $M = \max(M_{q_T},M_{p_T})$ by (1.15)

$$\|T_n(s,t)\phi - T(s,t)\phi\|_{q_T} \tag{2.24}$$

$$\leq TM^2\|\phi\|_{p_T}\, e^{\sigma(t-s)} \sup_{0\leq r\leq t} \|A_n(r)-A(r)\|_{\mathcal{L}(\Phi_{|p_T|},\Phi_{|q_T|})}$$

$$\rightarrow 0 \quad \text{as } n\rightarrow\infty \text{ for all } \phi \text{ in } \Phi,\ 0\leq s\leq t\leq T.$$

Also using (1.15) from (2.23) we have that

$$\sup_{0\leq r\leq t\leq T} \|(A_n(r)-A(r))T(r,t)\phi\|_{q_T} \rightarrow 0 \quad \text{as } n\rightarrow\infty \quad \text{for all } \phi \text{ in } \Phi \tag{2.25}$$

and using (2.24) and (2.19) in (2.18) we obtain

$$\sup_{0\leq s\leq t\leq T} \|A_n(s)(T_n(s,t)\phi - T(s,t))\phi\|_{q_T} \rightarrow 0 \quad \text{as } n\rightarrow\infty \quad \text{for all } \phi \text{ in } \Phi. \tag{2.26}$$

Then applying (2.25) and (2.26) in (2.17), from (2.16) it follows that if $X \in B$,

$$(G_n(X)-G(X))_t[\phi] \rightarrow 0 \quad \text{as } n\rightarrow\infty \quad \text{for all } \phi \text{ in } \Phi,\ (0\leq t\leq T). \tag{2.27}$$

Moreover,

$$F^n_T := \left(TE_T M^2 e^{\sigma T} + M_{p_T} e^{\sigma_p T}\right) \sup_{0\leq r\leq T} \|A_n(r)-A(r)\|_{\mathcal{L}(\Phi_{|p_T|},\Phi_{|q_T|})} \tag{2.28}$$

$$\rightarrow 0 \quad \text{as } n\rightarrow\infty.$$

Thus there exist $p_T > q_T$ and F^n_T such that for any $X \in B$ and $0\leq t\leq T$

$$|(G_n(X)-G(X))_t[\phi]| \leq D_T(X) F^n_T \|\phi\|_{p_T} \quad \text{for all } \phi \text{ in } \Phi. \tag{2.29}$$

Moreover since B is a compact set in $C([0,T];\Phi'_{q_T})$ and the latter is a metric space with norm $\sup_{0\leq t\leq T} \|X_t\|_{-q_T}$,

$$\sup_{X\in B}\ \sup_{0\leq t\leq T} \|X_t\|_{-q_T} =: H < \infty,$$

i.e.,

$$|(G_n(X)-G(X))_t[\phi]| \leq HF^n_T \|\phi\|_{p_T} \quad \text{for all } \phi \text{ in } \Phi \text{ and } X \text{ in } B. \tag{2.30}$$

Finally let V be the collection of neighborhoods of zero defining the topology of Φ', i.e., if $v \in V$

$$v := v(\epsilon,\bar{v}) := \{F \in \Phi' : \sup_{\phi \in \bar{v}} |F[\phi]| < \epsilon\}$$

where $\bar{v}$ is a bounded set in Φ and $\epsilon>0$. Let

$$|||X|||_v = \sup_{0\le t\le T} \sup_{\phi\in\bar{v}} |X_t[\phi]|, \quad v \text{ in } V.$$

From Mitoma [6], we have that $C([0,T];\Phi')$ has the projective limit topology of $\{|||X|||_v : v\in V\}$. Then if $v \in V$ and $X \in B$

$$|||G_n(X)-G(X)|||_v = \sup_{0\le t\le T} \sup_{\phi\in v} |(G_n(X)-G(X))_t[\phi]|$$

and by (2.30)

$$|||G_n(X)-G(X)|||_v \le HF_T^n \sup_{\phi\in\bar{v}} ||\phi||_{p_T} \to 0 \quad \text{as } n\to\infty.$$

Hence $G_n(X)$ converges to $G(X)$ uniformly over compact sets of $C^T_{\Phi'}$.

The fact that $G(X)$ is a continuous map from $C^T_{\Phi'}$ to itself is easily shown. The proof of the lemma is complete. □

From the proof of the above lemma (see (2.24)) we obtain the following corollary.

Corollary 2.1: Let $\{A_n(t)\}_{t\ge0}$, $\{T_n(s,t): 0\le s\le t<\infty\}$ $n\ge1$ and $\{A(t): t\ge0\}$, $\{T(s,t): 0\le s\le t\le T\}$ be as in Lemma 2.2. Then for each ϕ in Φ

$$T_n(s,t)\phi \to T(s,t)\phi \quad \text{as } n\to\infty$$

uniformly in $0\le s\le t\le T$ for each $T>0$.

Proof of Theorem 1.2:

Let P_n, $n\ge1$, and P be the probability measures on $C([0,\infty);\Phi')$ induced by M^n, $n\ge1$, and M respectively. By assumption $P_n \Rightarrow P$ in $C([0,\infty);\Phi')$.

From (1.10) and (1.12) we have that for each $n\ge1$ the solution $\xi^n = (\xi^n_t)$ of (1.16) can be written as

$$\xi^n_t = T'_n(t,0)\gamma^n + \int_0^t T'_n(t,s)A'_n(s)M^n_s ds \tag{2.31}$$

and using (2.6)

$$\xi^n_t = T'_n(t,s)\gamma^n + G_n(M^n)_t. \tag{2.32}$$

Similarly,

$$\xi_t = T'(t,s)\gamma + G(M)_t. \tag{2.33}$$

We will first prove that $\xi_\cdot^{n,T} \Rightarrow \xi_\cdot^T$ in $C_{\Phi'}^T$ for each $T>0$. Here $\xi_\cdot^{n,T}(\xi_\cdot^T)$ is the restriction of $\xi_\cdot^n(\xi_\cdot)$ to $[0,T]$.

Define $f_n(X)$: $C_{\Phi'}^T \to C_{\Phi'}^T$ by

$$(f_n(X))_t[\phi] = X_0[T_n(0,t)\phi], \quad (0\leq t\leq T). \tag{2.34}$$

Then for $X \in C_{\Phi'}^T$ using Corollary 2.1 we have

$$(f_n(X))_t[\phi] \underset{n\to\infty}{\rightarrow} (f(X))_t := X_0[T(0,t)\phi] \quad \text{for each } \phi \text{ in } \Phi.$$

Let $B \subset C_{\Phi'}^T$ be a compact set. Then using the notation of the proof of Lemma 2.2 for v in V,

$$|||f_n(X)-f(X)|||_V := \sup_{0\leq t\leq T} \sup_{\phi\in\bar{V}} \left|X_0\left[T_n(0,t)]\phi - T(0,t)\phi\right]\right| \tag{2.35}$$

$$\leq H \sup_{0\leq t\leq T} \sup_{\phi\in\bar{V}} ||T_n(0,t)\phi - T(0,t)\phi||_{q_T}$$

$$\to 0 \quad \text{as } n\to\infty \text{ for all } X \text{ in } B.$$

Finally, for $X \in C_{\Phi'}^T$ define

$$h_n(X) = f_n(X) + G_n(X) \quad \text{and} \quad h(X) = f(X) + G(X).$$

Then $h_n(X) \to h(X)$ uniformly over the compacts of $C_{\Phi'}^T$. The assumption $P_n \Rightarrow P$ implies $P_n^T \Rightarrow P^T$ where P_n^T and P^T are the probability measures induced on $C_{\Phi'}^T$ by $M^{n,T}$ and M^T respectively where $M_t^{n,T} = M_t^n$ $(0\leq t\leq T)$ (similarly, $M_t^T = M_t$, $0\leq t\leq T$). Since $\gamma^n \Rightarrow \gamma$ and γ^n and M^n are independent, it follows from (2.31), (2.32) and Lemmas 2.1 and 2.2 that $Q^{n,T} \Rightarrow Q^T$ in $C_{\Phi'}^T$. For each $\phi\in\Phi$ let Π_ϕ be the mapping introduced in Lemma 2.1, suitably modified. Then the relation $Q_n^j \Rightarrow Q^j$ $\forall j$ implies that $\mu_n^{\phi j} \Rightarrow \mu^{\phi j}$ where $\mu_n^{\phi j}$ (and similarly $\mu^{\phi j}$) is defined by $\mu_n^{\phi j} = Q_n^j \Pi_\phi^{-1}$. Thus, for each j, $\{Q_n^j \Pi_\phi^{-1}\}$ is tight for every $\phi\in\Phi$. Now, from the fact that C has the projective limit topology of $\{C^j\}_{j\geq 1}$ it can be shown that the sequence of measures $\mu_n^\phi = Q_n \Pi_\phi^{-1}$ is tight for each $\phi\in\Phi$. By Theorem 3.1 of Mitoma [6] it follows that $\{Q_n\}$ is tight. On the other hand, the weak convergence of Q_n^j to Q^j

for every j clearly implies finite dimensional convergence under Q_n of $(X_{t_1}[\phi_1],\dots, X_{t_k}[\phi_k])$ to its law under Q. Proposition 5.1 of [6] then implies $Q_n \Rightarrow Q$. □

Remark: For (1.16) and (1.17) to have unique solutions, it is sufficient, according to Theorem 2.1 of [5] that for some $r_n>0$, $r>0$ we have

$$E|\gamma^n|^2_{-r_n} < \infty \quad \text{and} \quad E|\gamma|^2_{-r} < \infty.$$

With this assumption and the other conditions of Theorem 1.2 in force we have the following result as an immediate corollary.

Corollary to Theorem 1.2: Let $\xi^n_\cdot$ and $\xi_\cdot$ be respectively the unique solutions of the equations

$$d\eta_t = A'(t)\eta_t dt + dM^n_t, \quad \eta_0 = \gamma^n,$$

and

$$d\eta_t = A'(t)\eta_t dt + dM_t, \quad \eta_0 = \gamma$$

where $\{A(t)\}_{t\geq 0}$ satisfies the conditions of Theorem 1.1. Then $\xi^n_\cdot \Rightarrow \xi_\cdot$.

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