On the extension of smooth functions by means of orthogonal polynomials

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ON THE EXTENSION OF SMOOTH FUNCTIONS BY MEANS OF ORTHOGONAL POLYNOMIALS.

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1. Introduction.

As is well known, the theory of orthogonal polynomials was developed mainly for constructing solutions of differential equations. Another aspect in which this theory plays an important role is in approximation theory, e.g. the role played by the Tchebycheff polynomials. The purpose of this note is to show how the theory of orthogonal polynomials can also be applied to the problem of extending a function f of class C^m over a finite interval, say $[0,\alpha]$, $\alpha > 0$, to a function $\tilde{f}$ defined on a larger interval, say $[-\beta,\alpha]$, $\beta > 0$, and preserving the degree of smoothness.

Of course, this problem is not new, and has been studied in a more general setting for domains in euclidean n-space, among others by, Babič [1], Calderón [2], Nikolsky [4], and Stein [6]. Compared with their methods and scopes, our approach can be considered as elementary. As a matter of fact, here, except for a more precise estimation of the bound for the norm of the extension operator, we do not prove anything new, we only give new proofs of old results using the techniques of orthogonal polynomials.

Our starting point is the following result due essentially to Babič.

THEOREM. (extension by C^m reflection). Let α and β be real numbers with $0 < \alpha \le \beta$. Given $f \in C^m([0,\alpha])$, $m = 0,1,2, \ldots$, define

$$E_m f(x) = \begin{cases} f(x), & 0 \le x \le \alpha \\ g(x), & -\beta \le x \le 0, \end{cases} \tag{1.1}$$

where

$$g(x) = \sum_{j=0}^{m} c_j f(-\lambda_j x), \tag{1.2}$$

$$\lambda_j = \frac{j+1}{m+1} \frac{\alpha}{\beta} \qquad (j = 0,1, \ldots, m), \tag{1.3}$$

and $(c_0, c_1, \ldots, c_m)$ is the unique solution of the system

$$\sum_{j=0}^{m} c_j \lambda_j^k = (-1)^k \qquad (k = 0,1,\ldots,m). \tag{1.4}$$

Then, $E_m f \in C^m([-\beta,\alpha])$, and for $1 \le p < \infty$,

$$\|g^{(k)}\|_{L^p([-\beta,0])} \le K_{m,p} \left(\frac{\alpha}{\beta}\right)^{-m-1/p} \|f^{(k)}\|_{L^p([0,\alpha])}, \tag{1.5}$$

$k = 0, 1, \ldots, m$, where $K_{m,p}$ is a constant depending only on m and p.

As Fraenkel [3] has pointed out, in (1.5) the dependence on α/β, although easily calculated, is usually not stated.

From (1.5) it follows immediately that

$$\|(E_m f)^{(k)}\|_{L^p([-\beta,\alpha])} \le K_{m,p} \left[\left(\frac{\alpha}{\beta}\right)^{-mp-1} + 1\right]^{1/p} \|f^{(k)}\|_{L^p([0,\alpha])}, \tag{1.6}$$

$k = 0,1,\ldots,m$.

What we are going to show next, is that the method of extension by C^m reflection is essentially a discretized version of an extension method involving orthogonal polynomials.

2. An integral operator suggested by C^m reflection.

We start by noting that (1.2) can be written in the form

$$g(x) = \int_0^{m+1} c(t)f(-\lambda(t)x)\,dt, \qquad -\beta \le x \le 0, \tag{2.1}$$

subject to (e.g. (1.4))

$$\int_0^{m+1} c(t)\lambda^k(t)\,dt = (-1)^k \qquad (k = 0,1,\dots,m), \tag{2.2}$$

where $c(t)$ and $\lambda(t)$ are the step functions given by

$$c(t) = c_j \text{ and } \lambda(t) = \lambda_j, \text{ if } t \in [j,j+1] \quad (j = 0,1,..,m).$$

In (2.1) and (2.2) let us take λ in place of t as integration variable, replacing $\lambda(t)$ by a strictly increasing function of t, which we denote the same $\lambda = \lambda(t)$, and such that

$$\lambda(0) = \frac{1}{m+1}\frac{\alpha}{\beta}, \qquad \lambda(m+1) = \frac{\alpha}{\beta}.$$

If we let $H(\lambda) = c(t(\lambda))$ and $w(\lambda) = dt/d\lambda$, where $t = t(\lambda)$ is the inverse function of $\lambda = \lambda(t)$, then from (2.1) we obtain

$$g(x) = \int_{\gamma_m}^{\gamma} H(\lambda)f(-\lambda x)w(\lambda)\,d\lambda, \qquad -\beta \le x \le 0, \tag{2.3}$$

where

$$\gamma = \frac{\alpha}{\beta}, \qquad \gamma_m = \frac{\gamma}{m+1}. \tag{2.4}$$

Also, from (2.2) we see that for the continuity of the derivatives up to the order m at $x = 0$ of the extension $E_m f$ given by (1.1), where g is now defined by (2.3), $H(\lambda)$ must satisfy

$$\int_{\gamma_m}^{\gamma} H(\lambda)\lambda^k w(\lambda)\,d\lambda = (-1)^k \qquad (k = 0,1,\dots,m). \tag{2.5}$$

We see then, that the problem we are now facing is the following: For a given weight w, find a function H satisfying condition (2.5); so that, if $f \in C^m([0,\alpha])$ and if $E_m f$ is as in (1.1) where g is given by (2.3), then $E_m f \in C^m([-\beta,\alpha])$.

In the following section we will show that given any weight w, such an H can be found, and it is a polynomial of degree m which can be expressed as a sum of orthogonal polynomials with respect to the weight function w.

3. A polynomial kernel H(λ).

From now on we shall assume that $w(\lambda)$ is a weight over $]\gamma_m,\gamma[$, i.e., $w(\lambda)$ is a measurable function such that $w(\lambda) > 0$ a.e. on $]\gamma_m,\gamma[$ and $w \in L^1(]\gamma_m,\gamma[)$.

We transform the interval $[\gamma_m,\gamma]$ into the interval $[-1,1]$ by means of

$$\lambda = \tilde{\alpha} + \tilde{\beta}s, \quad \tilde{\alpha} = \frac{\gamma+\gamma_m}{2} = \frac{2+m}{2(m+1)}\,\gamma, \quad \tilde{\beta} = \frac{\gamma-\gamma_m}{2} = \frac{m}{2(m+1)}\,\gamma. \tag{3.1}$$

Then, (2.3) is now written as

$$g(x) = \int_{-1}^{1} \tilde{H}(s)f(-(\tilde{\alpha}+\tilde{\beta}s)x)\tilde{w}(s)\, ds, \tag{3.2}$$

where

$$\tilde{H}(s) = \tilde{\beta}H(\tilde{\alpha}+\tilde{\beta}s), \text{ and } \tilde{w}(s) = w(\tilde{\alpha}+\tilde{\beta}s). \tag{3.3}$$

Also, (2.5) becomes

$$\int_{-1}^{1} \tilde{H}(s)(\tilde{\alpha}+\tilde{\beta}s)^k\tilde{w}(s)\, ds = (-1)^k \qquad (k = 0,1,\dots,m). \tag{3.4}$$

LEMMA. Condition (3.4) is equivalent to the condition

$$\int_{-1}^{1} \tilde{H}(s)s^k\tilde{w}(s)\, ds = (-1)^k\left(\frac{1+\tilde{\alpha}}{\tilde{\beta}}\right)^k \qquad (k = 0,1,\dots,m). \tag{3.5}$$

PROOF. Suppose that condition (3.4) holds. Then

$$\int_{-1}^{1} \tilde{H}(s)(\tilde{\beta}s)^k\tilde{w}(s)\, ds = \int_{-1}^{1} \tilde{H}(s)\tilde{w}(s)\left\{ \sum_{j=0}^{k} \binom{k}{j}(\tilde{\alpha}+\tilde{\beta}s)^{k-j}(-\tilde{\alpha})^j \right\} ds$$
$$= \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(-1)^j\tilde{\alpha}^j = (-1)^k(1+\tilde{\alpha})^k.$$

Suppose now that condition (3.5) holds. Then

$$\int_{-1}^{1} \tilde{H}(s)(\tilde{\alpha}+\tilde{\beta}s)^k\tilde{w}(s)\ ds = \int_{-1}^{1} \tilde{H}(s)\tilde{w}(s)\left\{ \sum_{j=0}^{k} \binom{k}{j}\tilde{\alpha}^{k-j}(\tilde{\beta}s)^j \right\} ds$$
$$= \sum_{j=0}^{k} \binom{k}{j}\tilde{\alpha}^{k-j}(-1)^j(1+\tilde{\alpha})^j = (\tilde{\alpha}-1-\tilde{\alpha})^k$$
$$= (-1)^k.$$

If we let

$$\sigma = \frac{1+\tilde{\alpha}}{\tilde{\beta}} = \frac{2(m+1)+(m+2)\gamma}{m\gamma}, \tag{3.6}$$

then condition (3.5) is now written as

$$\int_{-1}^{1} \tilde{H}(s)s^k\tilde{w}(s)\ ds = (-\sigma)^k \qquad (k = 0,1,\ldots,m). \tag{3.5'}$$

Let $\{\tilde{\phi}_n\}$ be a complete orthonormal sequence of polynomials with respect to the weight function $\tilde{w}$ over the interval $[-1,1]$, where the degree of $\tilde{\phi}_n$ equals n for $n = 0,1,2,\ldots$.

We define

$$\tilde{H}(s) = \sum_{k=0}^{m} C_k\tilde{\phi}_k(s), \qquad -1 \le s \le 1, \tag{3.7}$$

where the constants C_0, C_1, ..., C_m are going to be determined in such a way that condition (3.5') holds. We have

$$s^k = A_{k,0}\tilde{\phi}_0(s) + \ldots + A_{k,k}\tilde{\phi}_k(s) \qquad (k = 0,1,\ldots,m), \tag{3.8}$$

where $A_{k,k} \neq 0$ $(k = 0, 1, \ldots, m)$. So, for (3.5') to hold we must have

$$(-\sigma)^k = \int_{-1}^{1} \left[C_0\tilde{\phi}_0 + \ldots + C_m\tilde{\phi}_m\right]\left[A_{k,0}\tilde{\phi}_0 + \ldots + A_{k,k}\tilde{\phi}_k\right]\tilde{w}(s)\ ds$$
$$= A_{k,0}C_0 + \ldots + A_{k,k}C_k \qquad (k = 0,1,\ldots,m). \tag{3.9}$$

Since $A_{k,k} \neq 0$ $(k = 0,1,\ldots,m)$, we can solve successively (3.9) for the C_k $(k = 0,1,\ldots,m)$, and from (3.6) we obtain the following growth estimate

$$|C_k| \le K\ \gamma^{-k} \qquad (k = 0,1,\ldots,m), \tag{3.10}$$

where K is a constant depending on the coefficients $A_{k,\ell}$ and of m

(Recall that we are assuming that α and β satisfy $0 < \alpha \le \beta$, and hence that $0 < \gamma \le 1$).

Now, we reverse the transformation (3.1)

$$s = \frac{\lambda-\tilde{\alpha}}{\tilde{\beta}}, \qquad H(\lambda) = \frac{1}{\tilde{\beta}}\,\tilde{H}\left(\frac{\lambda-\tilde{\alpha}}{\tilde{\beta}}\right).$$

Then

$$H(\lambda) = \frac{1}{\tilde{\beta}} \sum_{k=0}^{m} C_k \phi_k(\lambda), \tag{3.11}$$

where

$$\phi_n(\lambda) = \tilde{\phi}_n\left(\frac{\lambda-\tilde{\alpha}}{\tilde{\beta}}\right) \qquad (n = 0, 1, \dots\ ), \tag{3.12}$$

and $\{\phi_n\}$ is a complete orthonormal sequence of polynomials with respect to the weight function w over the interval $[\gamma_m, \gamma]$, and the degree of ϕ_n equals n for $n = 0, 1, \dots$.

We have shown that if $w \in L^1(]\gamma_m, \gamma[)$ is any weight, then there exists a polynomial kernel $H(\lambda)$ of degree m, which can be expressed as a sum of orthogonal polynomials with respect to the weight w, such that $H(\lambda)$ satifies condition (2.5). Hence, if $f \in C^m([0,\alpha])$ and if we define $E_m f$ by (1.1) where g is given by (2.3), then $E_m f \in C^m([-\beta,\alpha])$.

In the following section we are going to show that for our extension operator E_m, formula (1.5) holds.

4. Bounding the derivatives in L^p, $1 \le p < \infty$.

Since our extension procedure works well as long we are dealing with orthogonal polynomials with respect to an arbitrary weight function $\tilde{w}$ over $]-1,1[$, we choose here the simplest one, namely, the weight function $\tilde{w}(s) \equiv 1$ on $[-1,1]$. Obtaining in this particular case the orthonormal polynomials

$$\tilde{\phi}_n(s) = \left[\frac{2n+1}{2}\right]^{1/2} P_n(s) \qquad (n = 0,1,\ldots),$$

where $P_n(s)$ is the Legendre polynomial of degree n over $[-1,1]$. As is well known, [5;p.181], the following estimate holds:

$$|P_n(s)| \le 1, \quad -1 \le s \le 1, \quad (n = 0,1,\ldots).$$

Hence, the orthonormal polynomials $\{\phi_n\}$ on $[\gamma_m,\gamma]$ given by (3.12) satisfy

$$|\phi_n(\lambda)| \le \left[\frac{2n+1}{2}\right]^{1/2}, \quad \gamma_m \le \lambda \le \gamma, \quad (n = 0,1,\ldots). \tag{4.1}$$

Also, in this case, the coefficients $A_{k,\ell}$ appearing in (3.8) satisfy the relations [5;p.193]

$A_{k,\ell} = 0$, when $k - \ell$ is odd or negative

$A_{k,\ell} = \dfrac{(2\ell+1)2^{\ell}((k+\ell)/2)!}{((k-\ell)/2)!(k+\ell+1)!}$, when $k - \ell$ is even and positive.

From this last fact and (3.1), (3.10) ,(3.11) ,(4.1) we obtain

$$|H(\lambda)| \le K_m\, \gamma^{-m-1}, \quad \gamma_m \le \lambda \le \gamma, \tag{4.2}$$

where K_m is a constant depending on m.

Now, since $w(\lambda) \equiv 1$ on $[\gamma_m,\gamma]$, from (2.3) we see that

$$g^{(k)}(x) = \int_{\gamma_m}^{\gamma} H(\lambda) f^{(k)}(-\lambda x)(-\lambda)^k \, d\lambda \tag{4.3}$$

for $k = 0,1,\ldots,m$. Let $1 \le p < \infty$. Applying Minkowsky inequality for integrals to (4.3) we get

$$\begin{aligned}
\|g^{(k)}\|_{L^p([-\beta,0])} &= \left\{ \int_{-\beta}^{0} |g^{(k)}(x)|^p \, dx \right\}^{1/p} \\
&\le \left\{ \int_{-\beta}^{0} \left[\int_{\gamma_m}^{\gamma} |H(\lambda)|\,|f^{(k)}(-\lambda x)|\lambda^k \, d\lambda \right]^p dx \right\}^{1/p} \\
&\le \int_{\gamma_m}^{\gamma} \left\{ \int_{-\beta}^{0} |H(\lambda)|^p |f^{(k)}(-\lambda x)|^p \lambda^{kp} \, dx \right\}^{1/p} d\lambda \\
&= \int_{\gamma_m}^{\gamma} |H(\lambda)|\lambda^k \left\{ \int_{-\beta}^{0} |f^{(k)}(-\lambda x)|^p \, dx \right\}^{1/p} d\lambda
\end{aligned}$$

$$= \int_{\gamma_m}^{\gamma} |H(\lambda)|\lambda^k \left\{ \int_0^{\lambda\beta} |f^{(k)}(t)|^p \frac{dt}{\lambda} \right\}^{1/p} d\lambda$$

$$\leq \left\{ \int_{\gamma_m}^{\gamma} |H(\lambda)|\lambda^{k-1/p} \, d\lambda \right\} \|f^{(k)}\|_{L^p([0,\alpha])} . \tag{4.4}$$

Also, from (4.2) we have

$$\int_{\gamma_m}^{\gamma} |H(\lambda)|\lambda^{k-1/p} \, d\lambda \leq K_m \, \gamma^{-m-1} \int_{\gamma_m}^{\gamma} \lambda^{k-1/p} \, d\lambda, \tag{4.5}$$

and if we analyze separately the cases $p > 1$ and $p = 1$, from (4.4) and (4.5) we easily obtain (1.5).

5. Final remarks.

Stein, in [6], has constructed a remarkable extension operator E, which extends to $\mathbb{R}^n$ functions defined on a domain D in $\mathbb{R}^n$ with a "minimally smooth boundary". This extension operator E is universal, in the sense that simultaneously extends all orders of differentiablity, and is such that $E : W^{m,p}(D) \longrightarrow W^{m,p}(\mathbb{R}^n)$ is a bounded linear operator for $m = 0,1,\dots$, $1 \leq p < \infty$, where $W^{m,p}(D)$ is the usual Sobolev space. This is to be contrasted, for example, with the hierarchy of extension operators E_m of the Babič type.

Here, in our more modest setting we ask ourselves the following question: Does there exists a kernel $H(\lambda)$ that serves for all extension operators E_m as defined above ?. The answer is no, as one should expect. More precisely we have the following

PROPOSITION. Let $0 \leq t_0 < t_1 < \infty$, and let w be a weight function over $]t_0,t_1[$. Then, there exists no function $\phi \in L^1(]t_0,t_1[,wdt)$ such that

$$\int_{t_0}^{t_1} \phi(t)t^m w(t) \, dt = (-1)^m, \qquad (m = 0,1,\dots). \tag{5.1}$$

PROOF. The function ϕ corresponds to the function H in (2.5). We transform the integral in (5.1) to an integral over the interval $[-1,1]$, by means of the change of variable

$$t = a+bs, \quad a = (t_1+t_0)/2, \quad b = (t_1-t_0)/2.$$

Then (5.1) becomes

$$\int_{-1}^{1} \psi(s)(a+bs)^m v(s)\, ds = (-1)^m \qquad (m = 0,1,\dots), \tag{5.2}$$

where $\psi(s) = b\phi(a+bs)$, and $v(s) = w(a+bs)$, or equivalently (see the proof of the Lemma in Section 3),

$$\int_{-1}^{1} \psi(s)s^m v(s)\, ds = (-1)^m \left[\frac{1+a}{b}\right]^m \qquad (m = 0,1,\dots). \tag{5.3}$$

Now,

$$\frac{1+a}{b} = \frac{2+t_1+t_0}{t_1-t_0} \geq \frac{2+t_1}{t_1} = 1+\delta, \qquad \delta > 0,$$

and if (5.3) holds

$$(1+\delta)^m \leq \left[\frac{1+a}{b}\right]^m \leq \int_{-1}^{1} |\psi(s)|v(s)\, ds \qquad (m = 0,1,\dots),$$

i.e.

$$\int_{-1}^{1} |\psi(s)|v(s)\, ds = \infty,$$

in contradiction with our hypothesis.

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