An abstract theory of distributions and Sobolev spaces

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INTRODUCTION.

As is well known, both, Schwartz theory of distributions [13], and the theory of Sobolev spaces (Adams [1], Nečas [10], Sobolev [15]), are concerned with the family of differential operators

$$\mathcal{L} = \left\{\frac{\partial}{\partial x_j} : j = 1, \ \ldots, \ n \right\}.$$

In these theories, one of the fundamental problems is to extend such operators in a suitable way, being them originally densely defined in a Frechet space X, which is usually taken as $L^1_{\ell oc}(\Omega)$ or $L^p(\Omega)$, $1 \le p < \infty$, where Ω is an open subset of $\mathbb{R}^n$. The solution to this question is based on the equality

$$\int_\Omega \frac{\partial u}{\partial x_j} \psi \, dx = - \int_\Omega u \frac{\partial \psi}{\partial x_j} \, dx, \qquad u \in C^\infty(\Omega), \qquad \psi \in C_c^\infty(\Omega).$$

Using this as a starting point, Schwartz introduces a topology in the space of test functions $\Psi = C_c^\infty(\Omega)$ and "extends" the operators using the idea of duality, that is, as continuous linear functions on Ψ.

Following the duality approach, Gelfand and Shilov [6] have developed the theory of generalized functions, which are defined as the elements in the dual space of a countably normed space , called the *fundamental space.*

Under another perspective, L. Ehrenpreis [5], payed atention to the "structure" of the open set Ω, extending Schwartz theory to certain families of linear operators acting on functions defined on a countable at infinity locally compact space.

In this work, we consider a Frechet space X together with a non degenerate "compatible product", $[\cdot,\cdot]:X \times Y \longrightarrow \mathbb{C}$, and a finite family $\mathcal{L} = \{ L_1, \ldots, L_n \}$ of linear operators on X, satisfying

$$L_j(\Phi) \subseteq \Phi \text{ and } L_j^*(\Psi) \subseteq \Psi , \quad (j = 1, \ldots, n), \qquad (*)$$

where $\Phi \subseteq X$ and $\Psi \subseteq Y$ are "test spaces". We show then, that it is possible to develop an abstract theory, within which, many of the basic problems appearing in the theory of distributions and the theory of Sobolev spaces can be stated and solved.

This abstract theory of distributions is based on a notion of duality defined by means of the pairing $(X,Y,[\cdot,\cdot])$, and solves the extension problem posed for the family of operators $\mathcal{L}$. As in Gelfand's approach, distributions are defined as elements of the dual of the test space Ψ, which is endowed with a locally convex metrizable topology. Nevertheless, in our work the test space (corresponding to Gelfand's *fundamental space*) it is not arbitrary, but it is constructed by means of the family $\mathcal{L}$ and the given pairing.

On the other hand, within this general context, we establish many of the basic properties appearing in the theory of Sobolev spaces. When X is a Hilbert space, we show that it is possible to define the (abstract) gradient, divergence and Laplace operators. With these in hand we can formulate and solve the Dirichlet and Neumann (abstract) problems. As in the usual case, we do it by making use of the (abstract) Friedrichs' and Poincaré inequalities, respectively.

Our exposition is divided in 9 sections, grouped in 4 chapters, and an Appendix. In the first part of Section 1 we give the terminology and basic results utilized in this work. In the second part of this section we give a detailed description of what we call the "space induced by a family of linear operators". These spaces are quite usual, but we were unable to find an explicit reference where their properties were sistematically developed. In Section 2 we define the concept of P-space $(X,Y,[\cdot,\cdot])$ and the concept of test space. Given a linear operator $L : D(L) \subseteq X \longrightarrow X$, and if $D(L)$ is a test space, then its adjoint operator $L^{*} : D(L^{*}) \subseteq Y \longrightarrow Y$ is defined in the usual way:

$$[Lx,y] = [x,L^{*}y], \qquad x \in D(L),\ y \in D(L^{*}).$$

When $D(L^{*})$ is also a test space we define $\mathbf{L}$, the maximal closed extension of L. Finally, here we give some examples of P-spaces.

Given a P-space $(X,Y,[\cdot,\cdot])$, in Chapter II we define the space of distributions $\Psi(X;\mathcal{L})'$ for a family of operators $\mathcal{L}$ satisfying condition (*). For this purpose, in Section 3 we define of what we mean by the "weak extension" or in the "sense of distributions" of an operator. In Section 4 we study the space of distributions $\Psi(X;\mathcal{L})'$.

In Chapter III, in a similar vein than the previous one, we define the Sobolev spaces $W^{m}(X;\mathcal{L})$ and $W_{0}^{m}(X;\mathcal{L})$, $m = 0, 1, \ldots, +\infty$. The elementary properties of these spaces are established in Section 5. In Section 6 we consider the situation in which the base space X is a

Banach space. In this case, we define the spaces $W^{-m}(X;\mathcal{L})$, which allow us to obtain a family of spaces with continuous inclusions

$$\ldots \hookrightarrow W_0^m(X;\mathcal{L}) \hookrightarrow \ldots \hookrightarrow X \hookrightarrow \ldots \hookrightarrow W^{-m}(Y;\mathcal{L}^*) \hookrightarrow \ldots$$

When X is reflexive, Φ is dense in each of these spaces.

In Chapter IV, we study the Hilbert-Sobolev spaces, that is, the case when X is a Hilbert space. In Section 7 we define the gradient, divergence and Laplace operators, and obtain their basic properties. In Section 8 we study the corresponding Dirichlet problem, and its relation with Friedrichs' inequality. The same is done, in Section 9, for the Neumann problem and Poincaré inequality, following some ideas of Deny and Lions in [3].

Finally, in the Appendix, we give a detailed account of those properties of the spaces $L^p_{\ell oc}(\Omega)$ utilized along our exposition, mainly in the illustrative examples, which we believe some are well known, but we were unable to find them in the literature.

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CHAPTER IV : THE HILBERT-SOBOLEV SPACES.

CHAPTER I : PRELIMINARIES.

The purpose of this chapter is to review and study in a systematic way, some concepts and results that serve as a basis for the development of our theory, and have appeared dispersed in the mathematical literature in one way or another.

1. The Space Induced by a Family of Linear Operators.

1.1 Some Basic Notation and Terminology.

Throughout this work, $\mathbb{K}$ will allways denote the field of real numbers $\mathbb{R}$, or the field of complex numbers $\mathbb{C}$.

Our definition of locally convex topological vector space, which we refer simply as "locally convex space", it will assume, unless it is stated otherwise, that is Hausdorff. Given a locally convex metrizable space X, if a sequence $\{x_n\}$ in X converges to a point x in X, we will write

$$x_n \longrightarrow x \text{ in X}.$$

A locally convex metrizable and complete space will be called a *Frechet space*.

Let V and W be locally convex spaces, and $T : V \longrightarrow W$ a linear operator. If T is one-to-one, $T(V) = W$, and if T and T^{-1} are continuous, then T is said to be a *linear isomorphism.*

Given a locally convex space X, its dual space will be denoted by X′. If $\phi \in X'$, we shall often employ the notation

$$\langle x, \phi \rangle = \phi(x), \qquad x \in X.$$

The space X′ together with the corresponding strong topology, will be called the *strong dual* of X, and will be denoted by X'_s. The *strong bidual* of X is then defined as $(X'_s)'_s$. Every $x \in X$ determines a continuous linear functional $\hat{x} \in (X'_s)'_s$, given by the equation

$$\langle \phi, \hat{x} \rangle = \langle x, \phi \rangle, \qquad \phi \in X'_s.$$

The correspondence

$$X \longrightarrow (X'_s)'_s, \qquad x \longrightarrow \hat{x},$$

is linear one-to-one and continuous, and if X is a Frechet space, then it is a linear isomorphism onto its image $\hat{X}$. This mapping will be called the *canonical identification* of X with its strong bidual. If $X = (X'_s)'_s$, then we say that X is a *reflexive space*.

Let X be a locally convex space, and F be a subset of X'. We say that F is *total*, if $x \in X$, and if $\langle x, \phi \rangle = 0$ for all $\phi \in F$, then we must have $x = 0$. An immediate application of Hahn-Banach theorem gives the following

LEMMA 1.1.1. Let X be a locally convex space. If X is reflexive and if $F \subseteq X'$ is total, then F is dense in X'_s.

Given a linear operator T from X to Y, we denote its domain, range, and nullspace, by $D(T)$, $R(T)$, and $N(T)$ respectively. We write $T \subseteq S$, if S is a linear operator from X to Y which is an extension of T (i.e., $D(T) \subseteq D(S)$ and $Tx = Sx$ for all $x \in D(T)$).

Let X and Y be locally convex metrizable spaces. Given a linear operator $T : D \subseteq X \longrightarrow Y$, we say that:

a) T is *closed*, if for every sequence $\{x_k\}$ in D, $x \in X$ and $y \in Y$; if $x_k \longrightarrow x$ and $Tx_k \longrightarrow y$, then $x \in D$ and $Tx = y$.

b) T is *closable*, if there exists a closed linear operator S from X to

Y such that $T \subseteq S$.

As is well known, T is closable if and only if for any sequence $\{x_k\}$ in D: $x_k \longrightarrow 0$ and $Tx_k \longrightarrow y$ imply $y = 0$. In this case, it is possible to define the *minimal closed extension* $\tilde{T} : \tilde{D} \subseteq X \longrightarrow Y$ of T, where $\tilde{D}$ consists of those $x \in X$ such that there exists a sequence $\{x_k\}$ in D with $x_k \longrightarrow x$ and $Tx_k \longrightarrow y$. If this is the case, we define $\tilde{T}x = y$. The name given to $\tilde{T}$ results from the following properties:

$$\tilde{T} \text{ is a closed linear operator and } T \subseteq \tilde{T} \tag{1.1.1}$$

$$\text{If } S : D(S) \subseteq X \longrightarrow Y \text{ is a closed linear operator with } T \subseteq S, \text{ then } \tilde{T} \subseteq S \tag{1.1.2}$$

Let $T : D(T) \subseteq X \longrightarrow Y$ be a linear operator with D dense in X. Then, its *dual operator* $T' : D(T') \subseteq Y'_s \longrightarrow X'_s$ is defined by the condition

$$\langle Tx, \phi\rangle = \langle x, T'\phi\rangle, \qquad x \in D(T),\ \phi \in D(T').$$

As is well known, if $D(T) = X$ and if T is continuous, then so it is $T' : Y'_s \longrightarrow X'_s$.

To end this section, we give a last piece of notation. Let X and Y be locally convex spaces. If X is a subspace of Y, and if the inclusion from X to Y is continuous, then we will write

$$X \hookrightarrow Y.$$

1.2 The Spaces $V(X;\mathcal{A})$.

Let X be a locally convex metrizable space, and let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a finite family of linear operators on X:

$$A_j : D(A_j) \subseteq X \longrightarrow X, \qquad j = 1, \ldots, n. \tag{1.2.1}$$

We define

$$V(X;\mathcal{A}) = D(A_1) \cap \ldots \cap D(A_n). \tag{1.2.2}$$

Let $\mathcal{P} = \{p_\ell\}$ be an increasing family of seminorms generating the topology of X. In $V(X;\mathcal{A})$ let us consider the family of seminorms

$$Q = \{p_\ell \circ A_j : \ell = 1, 2, \ldots; j = 0, 1, \ldots, n\}, \tag{1.2.3}$$

where $A_0 : X \longrightarrow X$ is the identity operator. Since $\mathcal{P} \subseteq Q$, the family Q is a separating family of semi-norms on $V(X;\mathcal{A})$. Being Q countable, the topology generated by Q makes $V(X;\mathcal{A})$ a locally convex metrizable space. We call $V(X;\mathcal{A})$ the *space induced on X by the family $\mathcal{A}$*. The notion of convergence in this space is characterized by the following result

LEMMA 1.2.1. A sequence $\{x_n\}$ in $V(X;\mathcal{A})$ converges to a point x in $V(X;\mathcal{A})$ if and only if

$$x_k \longrightarrow x \text{ in X and } A_j x_k \longrightarrow A_j x \text{ in X}, \qquad j = 1, \ldots, n.$$

In particular we have

$$V(X;\mathcal{A}) \hookrightarrow X, \tag{1.2.4}$$

and

$$A_j : V(X;\mathcal{A}) \longrightarrow X \text{ is continuous}, \ j = 1, \ldots, n. \tag{1.2.5}$$

PROPOSITION 1.2.2. Let X be a Frechet space. If each A_j is a closed operator, $j = 1, \ldots, n$, then $V(X;\mathcal{A})$ is a Frechet space.

PROOF. Let $\{x_k\}$ be a Cauchy sequence in $V(X;\mathcal{A})$. From (1.2.4) and (1.2.5) we see that each $\{A_j x_k\}$ is a Cauchy sequence. Since X is complete, there exist $x \in X$ and $y_j \in X$, such that

$$x_k \longrightarrow x \text{ in X}, \ A_j x_k \longrightarrow y_j \text{ in X}, \qquad j = 1, \ldots, n.$$

Now, each A_j is closed, so we must have $x \in D(A_j)$ and $A_j x = y_j$. Thus $x \in V(X;\mathcal{A})$, and from Lemma 1.2.1 we conclude that $x_k \longrightarrow x$ in $V(X;\mathcal{A})$.

Given a set S and a positive integer n, from now on we will employ the notation

$$S^{(n)} = S \times \ldots \times S \quad (n \text{ times}).$$

Let X be locally convex metrizable space, and consider $X^{(n+1)}$ with its product topology. Then $X^{(n+1)}$ is a locally convex metrizable space, and when X is a Frechet space, then so it is $X^{(n+1)}$. Let us define the *natural embedding*

$$i : V(X;\mathcal{A}) \longrightarrow X^{(n+1)},$$

by

$$i(x) = (x, A_1x, \ldots , A_nx), \qquad x \in V(X;\mathcal{A}).$$

In view of Lemma 1.2.1, i is a linear isomorphism onto its image $R(i)$. This basic fact will allow us to show how certain properties of the space X are inherited to the space $V(X;\mathcal{A})$. We illustrate this with the following two results.

PROPOSITION 1.2.3. If X is separable, then so it is $V(X;\mathcal{A})$.

PROOF. It is enough to show that $R(i)$ is separable. But this follows immediately from the fact that $X^{(n+1)}$ is separable.

PROPOSITION 1.2.4. If X is a reflexive space, and the operators A_j, $j = 1, \ldots, n$, are closed, Then $V(X;\mathcal{A})$ is a reflexive space.

PROOF. Let us note first that it is enough to show that $R(i)$ is reflexive. Now, since X is reflexive, then so it is $X^{(n+1)}$; and from the fact that the A_j are closed it follows that $R(i)$ is a closed subspace of $X^{(n+1)}$. But, as is well known, a closed subspace of a reflexive Frechet space is reflexive. Therefore $R(i)$ is reflexive.

Let us consider next, the case of one closable linear operator $A : D(A) \subseteq X \longrightarrow X$. Then $\tilde{A}$ is a closed linear operator, and we can consider the locally convex metrizable space $V(X;\tilde{A})$. From Proposition 1.2.2 we know that if X is a Frechet space, then so it is $V(X;\tilde{A})$. Furthermore, we have the following facts:

$$D(A) \text{ is dense in } V(X;\tilde{A}) = D(\tilde{A}), \tag{1.2.6}$$

$$\tilde{A} : V(X;\tilde{A}) \longrightarrow X \text{ is continuous.} \tag{1.2.7}$$

REMARK 1.2.1. We want to point out that the previous construction is still valid in the more general situation when one has a finite family of linear operators $\mathcal{A} = \{A_1, \ldots, A_n\}$, such that

$$A_j : D(A_j) \subseteq X \longrightarrow Z, \quad j = 1, \ldots, n,$$

where Z is another locally convex metrizable space, with the same properties as X. In this case, the space induced on X by the family $\mathcal{A}$ will be denoted by $V(X,Z;\mathcal{A})$. This more general setting will be needed in Section 8.4.

To end this section, we want to describe the space $V(X;\mathcal{A})$ in the case when X is a normed space with norm $\|\cdot\|$. In this case, it is possible to norm $V(X;\mathcal{A})$ in several equivalent ways. To fix ideas, in $V(X;\mathcal{A})$ we consider the norm

$$\|x\|_{\mathcal{A}} = \max \{ \|x\|, \|A_1x\|, \ldots, \|A_nx\| \}. \tag{1.2.8}$$

If H is a space whose norm $\|\cdot\|$ comes from a scalar product $(\cdot,\cdot)$, then

$$(x,y)_{\mathcal{A}} = (x,y) + (A_1x, A_1y) + \ldots + (A_nx, A_ny), \tag{1.2.9}$$

defines a scalar product on $V(H;\mathcal{A})$. Its associated norm is

$$\|x\|_{\mathcal{A}} = (\|x\|^2 + \|A_1x\|^2 + \ldots + \|A_nx\|^2)^{1/2}. \tag{1.2.10}$$

Let us note that the norm (1.2.10) is equivalent to the norm (1.2.8). In this context, we see that if X is a normed space, then the natural embedding $i : V(X;\mathcal{A}) \longrightarrow X^{(n+1)}$, is an isometry if in the space $X^{(n+1)}$ we consider the norm

$$\|(x_0, x_1, \ldots, x_n)\| = \max\{\|x_0\|, \|x_1\|, \ldots, \|x_n\|\}.$$

Furthermore, if H is an space whose norm $\|\cdot\|$ comes from an scalar product $(\cdot,\cdot)$, then the embedding $i : V(H;\mathcal{A}) \longrightarrow H^{(n+1)}$ is a unitary operator, when we consider on $H^{(n+1)}$ the inner product

$$((x_0, \ldots, x_n),(x'_0, \ldots, x'_n)) = (x_0,x'_0) + \ldots + (x_n,x'_n).$$

From Proposition 1.2.2 we obtain the following

COROLLARY 1.2.5. Assume that $\mathcal{A}$ is a family of closed operators.

(i) If X is a Banach space, then so it is $V(X;\mathcal{A})$.

(ii) If H is a Hilbert space, then so it is $V(H;\mathcal{A})$.

1.3 The spaces $V^m(X;\mathcal{A})$.

Given a locally convex metrizable space X, and a family of linear operators $\mathcal{A}$ as in (1.2.1), we define

$$V^0(X;\mathcal{A}) = X, \quad V^1(X;\mathcal{A}) = V(X;\mathcal{A}).$$

Proceeding inductively, let us assume that we have define the locally convex spaces $V^1(X;\mathcal{A}), \ldots, V^m(X;\mathcal{A})$, in such a way that

$$V^m(X;\mathcal{A}) \hookrightarrow V^{m-1}(X;A) \hookrightarrow \ldots \hookrightarrow V^1(X;\mathcal{A}) \hookrightarrow X, \qquad m \geq 1.$$

Let

$$D_j^m = \{x \in V^m(X;\mathcal{A}) : A_j x \in V^m(X;\mathcal{A})\}, \qquad j = 1, \ldots, n,$$

and

$$A_j^m = A_j | D_j^m, \qquad j = 1, \ldots, n.$$

Next, consider the family of linear operators

$$\mathcal{A}^m = \{A_j^m : j = 1, \dots, n\}.$$

We then define

$$V^{m+1}(X;\mathcal{A}) = V(V^m(X;\mathcal{A});\mathcal{A}^m). \tag{1.3.1}$$

That is, $V^{m+1}(X;\mathcal{A})$ is the space induced on $V^m(X;\mathcal{A})$ by the family $\mathcal{A}^m$. From (1.2.4) and (1.2.5) we obtain

$$V^{m+1}(X;\mathcal{A}) \hookrightarrow V^m(X;\mathcal{A}) \hookrightarrow \dots \hookrightarrow X, \quad m = 0, 1, \dots \tag{1.3.2}$$

and

$$A_j : V^{m+1}(X;\mathcal{A}) \longrightarrow V^m(X;\mathcal{A}) \text{ is continuous, } j = 1, \dots, n. \tag{1.3.3}$$

LEMMA 1.3.1. If each A_j, $j = 1, \dots, n$, is a closed operator, then

$$A_j^m : D_j^m \subseteq V^m(X;\mathcal{A}) \longrightarrow V^m(X;\mathcal{A}), \quad j = 1, \dots, n,$$

is a closed linear operator.

PROOF. Let: $\{x_k\} \subseteq D_j^m$, x, $y \in V^m(X;\mathcal{A})$, such that

$$x_k \longrightarrow x, \ A_j^m x_k \longrightarrow y, \text{ in } V^m(X;\mathcal{A}).$$

Since $V^m(X;\mathcal{A}) \hookrightarrow X$, and being each A_j closed, we have $x \in D_j^m$ and $y_j = A_j x$. Therefore A_j^m is closed.

From propositions 1.2.2, 1.2.3, 1.2.4 and Lemma 1.3.1 we obtain the following result

COROLLARY 1.3.2. Let X be a locally convex metrizable space, and $\mathcal{A}$ be a family of linear operators as in (1.2.1).

(i) If X is a Frechet space, and if the operators A_j, $j = 1, \dots, n$, are closed, then $V^m(X;\mathcal{A})$ is also a Frechet space, $m = 0, 1, \dots$.

(ii) If X is separable, then so it is $V^m(X;\mathcal{A})$, $m = 0, 1, \dots$.

(iii) If X is reflexive, and if the operators A_j, $j = 1, \dots, n$, are closed, then $V^m(X;\mathcal{A})$ is also reflexive, $m = 0, 1, \dots$.

In order to describe the spaces $V^m(X;\mathcal{A})$ in a concise manner, we introduce the following notation. Let $I_n = \{1, 2, \dots, n\}$. Taking $I_n^{(0)} = \{0\}$, for $\gamma = (\gamma_1, \dots, \gamma_\ell) \in I_n^{(\ell)}$, $\ell = 0, 1, \dots,$ we define:

$$[\gamma] = \ell, \tag{1.3.4}$$

$$A_\gamma = A_{\gamma_1} \dots A_{\gamma_\ell}, \tag{1.3.5}$$

where A_0 is the identity operator in X. In this context γ will be called a *subindex*, and $[\gamma]$ is its *length*.

The following result is obvious:

PROPOSITION 1.3.3. Under the same hypothesis as above we have:

(i) $V^m(X;\mathcal{A}) = \bigcap \{D(A_\gamma) : [\gamma] \le m\}$.

(ii) $x_k \longrightarrow x$ in $V^m(X;\mathcal{A})$ if and only if $A_\gamma x_k \longrightarrow A_\gamma x$, for every γ with $[\gamma] \le m$.

(iii) If $[\gamma] \le \ell \le m$, then $A_\gamma : V^m(X;\mathcal{A}) \longrightarrow V^{m-\ell}(X;\mathcal{A})$, is a continuous linear operator.

If X is a normed space, the corresponding norm on $V^m(X;\mathcal{A})$ is given by

$$\|x\|_{\mathcal{A},m} = \max \{\|A_\gamma x\| : [\gamma] \le m\}. \tag{1.3.6}$$

Also, when H is a inner product space, then the corresponding inner product on $V^m(H;\mathcal{A})$ is given by

$$(x,x')_{\mathcal{A},m} = \sum_{[\gamma]\le m} (A_\gamma x, A_\gamma x'), \tag{1.3.7}$$

and the associated norm is

$$\|x\|_{\mathcal{A},m} = \left\{ \sum_{[\gamma]\le m} \|A_\gamma x\|^2 \right\}^{1/2}. \tag{1.3.8}$$

Let us note, that in this case the norm (1.3.8) is equivalent to the norm (1.3.6). Furthermore, when $m = 1$, (1.3.7) and (1.3.8) coincide with (1.2.9) and (1.2.10) respectively.

From Corollary 1.3.2 we get immediately the following result.

COROLLARY 1.3.4. Supose that each A_j, $j = 1, \dots, n$, is a closed linear operator.

(i) If X is a Banach space, then so it is $V^m(X;\mathcal{A})$.

(ii) If H is a Hilbert space, then so it is $V^m(H;\mathcal{A})$.

To end this section we define the space

$$V^{+\infty}(X;\mathcal{A}) = \bigcap_{m=1}^{+\infty} V^m(X;\mathcal{A}). \tag{1.3.9}$$

In order to define a topology on $V^{+\infty}(X;\mathcal{A})$, we consider the increasing family of semi-norms $\mathcal{P} = \{p_\ell\}$ defining the topology on X. Then, a locally convex topology on $V^{+\infty}(X;\mathcal{A})$ is defined by the family of semi-norms

$$Q = \{p_\ell \circ A_\gamma \;:\; \ell = 1, 2, \dots;\; \gamma \in I_n^{(m)},\; m = 0, 1, \dots \} \tag{1.3.10}$$

In this way, $V^{+\infty}(X;\mathcal{A})$ results a locally convex metrizable space. The convergence in $V^{+\infty}(X;\mathcal{A})$ is characterized as follows:

LEMMA 1.3.5. Let $\{x_k\}$ be a sequence in $V^{+\infty}(X;\mathcal{A})$. Then

$$x_k \longrightarrow x \text{ in } V^{+\infty}(X;\mathcal{A})$$

if and only if

$$A_\gamma x_k \longrightarrow A_\gamma x \text{ in } X,$$

for all $\gamma \in I_n^{(\ell)}$, $\ell = 0, 1, \dots$.

From the previous lemma is clear that

$$V^{+\infty}(X;\mathcal{A}) \hookrightarrow V^{m}(X;\mathcal{A}), \qquad m = 0,\ 1,\ \dots, \tag{1.3.11}$$

and

$$A_{\gamma} : V^{+\infty}(X;\mathcal{A}) \longrightarrow V^{+\infty}(X;\mathcal{A}) \text{ is continuous,} \tag{1.3.12}$$

for $\gamma \in I_n(\ell)$, $\ell = 0,\ 1,\ \dots$.

Also we have the analogous result to Corollary 1.3.2 for this case.

COROLLARY 1.3.6. Let X be a locally convex metrizable space, and $\mathcal{A}$ a be a family of operators as in (1.2.1).

(i) If X is a Frechet space, and if the operators A_j, $j = 1, \dots, n$, are closed, then $V^{+\infty}(X;\mathcal{A})$ is a Frechet space.

(ii) If X is separable, then so it is $V^{+\infty}(X;\mathcal{A})$.

(iii) If X is reflexive, and if the operators A_j, $j = 1, \dots, n$, are closed, then $V^{+\infty}(X;\mathcal{L})$ is also reflexive.

Finally, we want to remark that if X is normed, then it not necessarily follows that $V^{+\infty}(X;\mathcal{A})$ is normed.

2. Test Spaces, and Maximal Closed Extensions of Linear Operators.

2.1 P-spaces.

Let X and Y be vector spaces over $\mathbb{K}$. Suppose that

$$[\cdot,\cdot] : X \times Y \longrightarrow \mathbb{K},$$

is a sesquilineal form; that is, in the first variable is linear, while in the second is conjugate linear. We say that $(X,Y;[\cdot,\cdot])$ is a *pairing*, if it is also nondegenerate. That is:

(a) If $y \in Y$, and if $[x,y] = 0$ for all $x \in X$, then $y = 0$.

(b) If $x \in X$, and if $[x,y] = 0$ for all $y \in Y$, then $x = 0$.

Let Φ be a vector subspace of X. We say that Φ is a *test space* for Y, if $[\cdot,\cdot] : \Phi \times Y \longrightarrow \mathbb{K}$ still is a nondegenerate sesquilinear form, i.e., if $(\Phi, Y, [\cdot,\cdot])$ is a pairing. We have an analogous definition for $\Psi \subseteq Y$ to be a test space for X.

Let $(X, Y, [\cdot,\cdot])$ be a pairing. Consider a linear operator L on X, such that $D(L)$ is a test space for Y. Then, given $y \in Y$ there exists at most one $z \in Y$ such that

$$[Lx, y] = [x, z], \text{ for all } x \in D(L). \tag{2.1.1}$$

If such a z exists we denote it by $z = L^* y$, and (2.1.1) takes the form

$$[Lx, y] = [x, L^* y], \quad x \in D(L). \tag{2.1.2}$$

Let $D(L^*)$ be the set of all $y \in Y$ for which there is a z satisfying condition (2.1.1). It is clear that $D(L^*)$ is a vector subspace of Y, and that $L^* : D(L^*) \subseteq Y \longrightarrow Y$ is a linear operator, called the *adjoint* of L (with respect to the pairing). The following property of the adjoint is clear.

LEMMA 2.1.1. Let L and M be linear operators on X. If $D(L)$ is a test space for Y, and if $L \subseteq M$, then $M^* \subseteq L^*$.

Given the pairing $(X, Y; [\cdot,\cdot])$, we define

$$[y, x]^* = \overline{[x, y]}, \quad y \in Y,\ x \in X.$$

In this way we obtain a pairing $(Y, X, [\cdot,\cdot]^*)$, called the *adjoint pairing*.

Let $M : D(M) \subseteq Y \longrightarrow Y$ be a linear operator such that $D(M)$ is a test space for X. Then, its adjoint operator (with respect to the adjoint pairing) $M^* : D(M^*) \subseteq X \longrightarrow X$, is determined by the condition

$$[My,x]^* = [y,M^*x]^*, \quad y \in D(M),\ x \in D(M^*),$$

or equivalently

$$[x,My] = [M^*x,y], \quad x \in D(M^*),\ y \in D(M). \tag{2.1.3}$$

Let us consider a pairing $(X,Y,[\cdot,\cdot])$. For every $y \in Y$, we denote by Jy the linear functional on X given by

$$<x,Jy> = [x,y], \quad x \in X, \tag{2.1.4}$$

i.e., $Jy = [\cdot,y]$, $y \in Y$.

We say that the pairing $(X,Y;[\cdot,\cdot])$ is a *P-space*, if X is a locally convex metrizable space, and if the linear functional Jy is continuous, for all $y \in Y$, i.e., $J(Y) \subseteq X'$.

With the purpose of working more easily, according to our development, in the space X' we define the multiplication by scalars in the following way:

$$<x,k\phi> = \bar{k}<x,\phi>, \quad x \in X,\ \phi \in X',\ k \in \mathbb{K}. \tag{2.1.5}$$

Given a P-space $(X,Y;[\cdot,\cdot])$, from (2.1.4) and (2.1.5) it follows that $J : Y \longrightarrow X'$ is a linear operator. Furthermore, it is easy to see that J is also one-to-one. This will allow us to identify Y with the subspace $J(Y)$ of X'. From now on, we call the map J, the *canonical identification* of Y in X'.

The following result is immediate from the definition of P-space.

LEMMA 2.1.2. Let $(X,Y;[\cdot,\cdot])$ be a P-space, and Φ a linear subspace of X. If Φ is dense in X, then Φ is a test space for Y.

If the canonical identification J is onto, then making use of the Hahn-Banach theorem, we can verify that the converse of the previous result also holds. That is, if $\Phi \subseteq X$ is a test space for Y, then Φ is dense in X.

PROPOSITION 2.1.3. Let $(X,Y;[\cdot,\cdot])$ be a P-space, and L a linear operator on X. If $D(L)$ is a test space for Y, then $\tilde{L}^{*} = L^{*}$.

PROOF. Since $L \subseteq \tilde{L}$, then from Lemma 2.1.1 it follows that $\tilde{L}^{*} \subseteq L^{*}$. Now, let $y \in D(L^{*})$, and take any $x \in D(\tilde{L})$. Then, there exists a sequence $\{x_k\}$ in $D(L)$ such that $x_k \longrightarrow x$ and $Lx_k \longrightarrow \tilde{L}x$. Thus

$$[\tilde{L}x,y] = \lim_{k\longrightarrow\infty} [Lx_k,y] = \lim_{k\longrightarrow\infty} [x_k,L^{*}y] = [x,L^{*}y].$$

Therefore, $y \in D(\tilde{L}^{*})$ and $\tilde{L}^{*}y = L^{*}y$. This proves that $L^{*} \subseteq \tilde{L}^{*}$.

2.2 The Maximal Closed Extension of a Linear Operator.

Let $(X,Y,[\cdot,\cdot])$ be a P-space, and suppose that Φ and Ψ are test spaces for Y and X respectively. We say that a linear operator L on X belongs to the class $\mathscr{C}_1(\Phi,\Psi)$, and we write $L \in \mathscr{C}_1(\Phi,\Psi)$, if satisfies

$$\Phi \subseteq D(L) \text{ and } \Psi \subseteq D(L^{*}). \tag{2.2.1}$$

Given an operator $L \in \mathscr{C}_1(\Phi,\Psi)$, one has defined its adjoint $L^{*} : D(L^{*}) \subseteq Y \longrightarrow Y$. We denote by L_0^{*} the restriction of L^{*} to Ψ. Next, we let $\mathrm{L} = (L_0^{*})^{*}$ (cf.(2.1.3)). Thus, $\mathrm{L} : D(\mathrm{L}) \subseteq X \longrightarrow X$, is determined by the condition

$$[\mathrm{L}x,y] = [x,L^{*}y], \quad x \in D(\mathrm{L}),\ y \in \Psi. \tag{2.2.2}$$

We call the operator L the *maximal closed extension* of L. The reason for this terminology, is due to the following two results.

PROPOSITION 2.2.1. Let $L \in \mathscr{C}_1(\Phi,\Psi)$.

(i) L is a closed linear operator.

(ii) L is closable and $\tilde{L} \subseteq \mathrm{L}$.

PROOF. (i) Let us consider: a sequence $\{x_k\}$ in $D(\mathrm{L})$, $x \in X$, $y \in X$, such that $x_k \longrightarrow x$ and $\mathrm{L}x_k \longrightarrow y$ in X. Since $Jz \in X'$ for every $z \in \Psi$, from (2.2.2) we have

$$[y,z] = \lim_{k\longrightarrow\infty} [\mathrm{L}x_k, z] = \lim_{k\longrightarrow\infty} [x_k, L^* z] = [x, L^* z].$$

But this together with (2.2.2) says that $x \in D(\mathrm{L})$ and $\mathrm{L}x = y$. Therefore L is closed.

(ii) Taking into account that $L \subseteq \mathrm{L}$, the conclusion follows immediately from (i) and (1.1.2).

The next result explains the sense in which L is a maximal extension of L.

PROPOSITION 2.2.2. Let $L \in \mathcal{C}_1(\Phi,\Psi)$, and M a linear operator on X such that $L \subseteq M$. If $\Psi \subseteq D(M^*)$, then $M \subseteq \mathrm{L}$.

PROOF. From $L \subseteq M$ we get $M^* \subseteq L^*$. Hence, for $x \in D(M)$ we have

$$[Mx, y] = [x, M^* y] = [x, L^* y], \qquad y \in \Psi,$$

because $\Psi \subseteq D(M^*)$. The result follows from the definition of L.

From the closed graph theorem we obtain the following

COROLLARY 2.2.3. Let $L \in \mathcal{C}_1(\Phi,\Psi)$. If X is a Frechet space, and if $D(\mathrm{L}) = X$, then L is continuous.

REMARK 2.2.1. We want to point out, that the condition on the product $[\cdot,\cdot]$ to be conjugate linear in the second variable is not essential. All the theory developed untill now is still valid when the product is a bilinear function. We merely ask the product to be a sesquilinear mapping in order to immediately apply the results obtained so far, to the case of Hilbert spaces. Furthermore, it happens that in the spaces X appearing in the study of partial differential equations, it is allways possible give a weak formulation of the problem by means of a sesquilinear form.

2.3 Examples of P-spaces.

We start with the simplest one.

EXAMPLE 2.3.1. Let H be a Hilbert space with inner product $(\cdot,\cdot)$. Clearly $(H,H,(\cdot,\cdot))$ constitutes a P-space. By Riesz representation theorem and from definition (2.1.5), it follows that $J : H \longrightarrow H'$ is a linear isomorphism.

EXAMPLE 2.3.2. Let Ω be a nonempty open bounded subset of $\mathbb{R}^n$. Let us consider $X = C(\overline{\Omega})$, the space of all continuous functions $u : \overline{\Omega} \longrightarrow \mathbb{C}$, together with the norm

$$\|u\| = \max \{ |u(x)| : x \in \overline{\Omega} \}.$$

Let us take $X = Y$, and define

$$[u,v] = \int_{\Omega} u\overline{v}\, dx.$$

It is clear that $(C(\overline{\Omega}),C(\overline{\Omega}),[\cdot.\cdot])$ is a P-space. From the Stone-Weierstrass theorem, it follows that the space Φ $(=\Psi)$ consisting of all polynomials in n variables, is a test space for $C(\overline{\Omega})$. In this case, the canonical identification $J : C(\overline{\Omega}) \longrightarrow C(\overline{\Omega})'$ is not onto.

EXAMPLE 2.3.3. Consider the Banach space $X = C_0(\mathbb{R}^n)$, consisting of all continuous functions $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ vanishing at infinite, together with the norm

$$\|u\|_\infty = \sup \{ |u(x)| : x \in \mathbb{R}^n \}.$$

let w be a "normalized weight function", i.e., $w \in L^1(\mathbb{R}^n)$ satisfies:

$$w(x) > 0 \text{ a.e. on } \mathbb{R}^n \text{ and } \int w \, dx = 1.$$

If for $u, v \in C_0(\mathbb{R}^n)$ we define

$$[u,v]_\infty = \int u\bar{v} \, w \, dx.$$

Then, $[\cdot,\cdot]_\infty$ is a sesquilinear form on X satisfying

$$|[u,v]_\infty| \le \|u\|_\infty \|v\|_\infty , \qquad u, v \in C_0(\mathbb{R}^n).$$

Also, an immmediate application of du Bois-Reymond lemma shows that the sesquilinear form is nondegenerate. Since clearly one has $JC_0(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)'$, then $(C_0(\mathbb{R}^n), C_0(\mathbb{R}^n), [\cdot,\cdot]_\infty)$ is a P-space.

Also, if we take $\Phi = \Psi = C_c(\mathbb{R}^n)$, where $C_c(\mathbb{R}^n)$ is the subspace of $C_0(\mathbb{R}^n)$ consisting of all functions with compact support, then using the well known fact that the former is dense in the later, we see that $C_c(\mathbb{R}^n)$ is a test space for $C_0(\mathbb{R}^n)$. Let us note that in this case, the canonical identification $J : C_0(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)'$ is not onto.

EXAMPLE 2.3.4. With the aid of the previous example, we can define a strucure of P-space on $L^1(\mathbb{R}^n)$ as folllows: As is well known, the Fourier transform

$$\hat{f}(\xi) = \int e^{-i\xi\cdot x} f(x) \, dx,$$

for functions $f \in L^1(\mathbb{R}^n)$, satisfies the following properties: for all $f \in L^1(\mathbb{R}^n)$ one has

$$\text{(a)} \quad \hat{f} \in C_0(\mathbb{R}^n) \quad \text{, and} \quad \text{(b)} \quad \|\hat{f}\|_\infty \le \|f\|_1.$$

Taking into account (a), for $f, g \in L^1(\mathbb{R}^n)$ we define

$$[f,g]_1 = [\hat{f},\hat{g}]_\infty,$$

i.e.,

$$[f,g]_1 = \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\; w(\xi)\; d\xi.$$

Then, from (b) we have

$$|[f,g]_1| = |[\hat{f},\hat{g}]_\infty| \le \|\hat{f}\|_\infty \|\hat{g}\|_\infty \le \|f\|_1 \|g\|_1.$$

Hence $[\cdot,\cdot]_1$ is a continuous sesquilinear form on $L^1(\mathbb{R}^n)$. Clearly, it is not degenerate. The continuity of the canonical identification J, follows immediately from the continuity of the Fourier transform (condition (b)). Therefore, $(L^1(\mathbb{R}^n), L^1(\mathbb{R}^n), [\cdot,\cdot]_1)$ is a P-space.

In this case, $\Phi = \Psi = C_c(\mathbb{R}^n)$ is a test space for $L^1(\mathbb{R}^n)$.

EXAMPLE 2.3.5. Let Ω be an open nonempty subset of $\mathbb{R}^n$. Consider the Frechet space $X = L^p_{\ell oc}(\Omega)$, $1 \le p \le \infty$. Let us take $Y = L^{p'}_c(\Omega)$ (see the Appendix), where $1/p + 1/p' = 1$, and define

$$[u,v] = \int_\Omega u\bar{v}\; dx, \qquad u \in L^p_{\ell oc}(\Omega),\; v \in L^{p'}_c(\Omega).$$

Taking into account what we have established in the Appendix, it is easy to check that $(L^p_{\ell oc}(\Omega), L^{p'}_c(\Omega), [\cdot,\cdot])$ is a P-space. In this case, the canonical identification $J : L^{p'}_c(\Omega) \longrightarrow (L^p_{\ell oc}(\Omega))'$ is onto when $1 \le p < \infty$.

From the du Bois-Reymond lemma, it follows that $\Phi = \Psi = C^\infty_c(\Omega)$ is a test space for $L^p_{\ell oc}(\Omega)$ and $L^{p'}_c(\Omega)$ respectively.

EXAMPLE 2.3.6. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Consider the Banach space $X = L^p(\Omega)$, $1 \le p \le \infty$. Let us take $Y = L^{p'}(\Omega)$, where p' and $[\cdot,\cdot]$ are as in the previous example. From the well known properties of the L^p-spaces it follows immediately that $(L^p(\Omega), L^{p'}(\Omega), [\cdot,\cdot])$ is a

P-space, and the canonical identification $J : L^{p'}(\Omega) \longrightarrow (L^{p}(\Omega))'$ is onto if and only if $1 \le p < \infty$. Also, in this case, $\Phi = \Psi = C_c^{\infty}(\Omega)$ is a test space for $L^{p}(\Omega)$ and $L^{p'}(\Omega)$ respectively.

It is of special interest to note, that in the case $p = 1$, $p' = \infty$; $C_c^{\infty}(\Omega) \subseteq L^{\infty}(\Omega)$ is a test space for $L^{1}(\Omega)$ which is not dense in $L^{\infty}(\Omega)$.

EXAMPLE 2.3.7. Given Ω, an open nonempty subset of $\mathbb{R}^n$. Consider the P-space $(L^1_{\ell oc}(\Omega), L^{\infty}_c(\Omega), [\cdot,\cdot])$ described in Example 2.3.5., together with the test spaces $\Phi = \Psi = C_c^{\infty}(\Omega)$. Let α be a multi-index and

$$\partial^{\alpha} : C_c^{\infty}(\Omega) \subseteq L^1_{\ell oc}(\Omega) \longrightarrow L^1_{\ell oc}(\Omega),$$

the corresponding differential operator. Since

$$[\partial^{\alpha}\phi, \psi] = [\phi, (-1)^{|\alpha|}\partial^{\alpha}\psi], \qquad \phi,\ \psi \in C_c^{\infty}(\Omega),$$

it follows from (2.1.2) that $C_c^{\infty}(\Omega) \subseteq D((\partial^{\alpha})^{*})$. Thus we see that $\partial^{\alpha} \in \mathscr{C}_1(C_c^{\infty}(\Omega), C_c^{\infty}(\Omega))$, and accordingly to what we have established in Section 2.2., its maximal closed extension is defined, which is precisely the derivative ∂^{α} in the sense of distributions.

More generally, we can verify that for the differential operator

$$P = \sum_{|\alpha| \le m} a_{\alpha}(x)\partial^{\alpha},$$

where $a_{\alpha} \in C^{|\alpha|}(\Omega)$, $|\alpha| \le m$, its maximal closed extension is also defined in the sense of distributions.

CHAPTER II : ABSTRACT SPACES OF DISTRIBUTIONS.

Given a P-space $(X,Y;[\cdot,\cdot])$, in Section 2.2 we defined the maximal closed extension L of an operator $L \in \mathscr{C}_1(\Phi,\Psi)$. In general, according to Proposition 2.2.3, this extension is not defined in all of X. In this chapter we will see how it is possible to extend L, in a weak sense, to a space that in certain sense includes X; which following Scwartz [13], we have called "space of distributions".

3. Weak extension of Operators in the Class $\mathscr{C}_1(\Phi,\Psi)$: The Normed Case.

3.1 The Negative Norm.

Let $(X,Y;[\cdot,\cdot])$ be a P-space, where X is a normed space, with norm $\|\cdot\|$. Using the canonical identification $J : Y \longrightarrow X'$, for $y \in Y$ we define

$$\|y\|^{-} = \|Jy\| = \sup \{ |[x,y]| : \|x\| \le 1 \}. \tag{3.1.1}$$

Since J is a one-to-one linear transformation, (3.1.1) defines a norm on Y, called the *negative norm on* Y *induced by* X. In this manner J results an isometry, and

$$|[x,y]| \le \|x\| \, \|y\|^{-}, \quad x \in X, \; y \in Y, \tag{3.1.2}$$

i.e., $[\cdot,\cdot] : X \times Y \longrightarrow \mathbb{K}$ is a continuous sesquilinear form.

Considering Y together with its negative norm $\|\cdot\|^{-}$, then it is immediate to check that the adjoint pairing $(Y,X;[\cdot,\cdot]^{*})$ constitutes a P-space, called the *adjoint P-space.* Also, it is clear that Y is a Banach space if and only if $J(Y)$ is a closed subspace of X'.

For $x \in X$, let $Ix = [\cdot,x]^{*}$. That is

$$\langle y, Ix\rangle = \overline{[x,y]}, \qquad y \in Y. \tag{3.1.3}$$

Since $(Y,X;[\cdot,\cdot]^{*})$ is a P-space, then it is clear that $Ix \in Y'$ for every $x \in X$. Furthermore, $I : X \longrightarrow Y'$ is a one-to-one bounded linear operator, called the *canonical identification* of X in Y'.

The following result is clear

LEMMA 3.1.1. If $L : D(L) \subseteq X \longrightarrow X$ is a linear operator and $D(L)$ is a test space for Y, Then $L^{*} : D(L^{*}) \subseteq Y \longrightarrow Y$ is closed.

REMARK 3.1.1. In the general case, when X is a locally convex metrizable space, if we equipp Y with the strong topology by means of the canonical identification $J : Y \longrightarrow X'$, one could think of defining the adjoint P-space $(Y,X;[\cdot,\cdot]^{*})$. The difficulty here lies in that, given that X is metrizable, X'_{s} will be metrizable if and only if X is normable (Schafer, p. 152).

From Lemma 1.1.1 we immediately obtain the following

LEMMA 3.1.2. If X is a reflexive space and $\Psi \subseteq Y$ is a test space for X, then Ψ is dense in Y.

REMARK 3.1.2. Let $(X,Y;[\cdot,\cdot])$ be a P-space, where X is a Banach space. Let us assume that $\Phi \subseteq X$ is a test space for Y. In Section 6.1 we are interested in the case when Φ is a dense subspace of X. If this is not the case, we can think of modifying the original P-space and consider in its place the P-space $(\overline{\Phi},Y;[\cdot,\cdot])$, where $\overline{\Phi}$ is the closure of Φ in X. However, this procedure can modify the corresponding negative norm

on Y. Next we present two examples; in the first one the norm does not change, while in the second it is altered.

EXAMPLE 3.1.1. Let Ω be an open nonempty subset of $\mathbb{R}^n$. Let us consider the P-space over $\mathbb{R}$, $X = L^\infty(\Omega)$, $Y = L^1(\Omega)$ (cf. Example 2.3.6), and let $\Phi = C_c^\infty(\Omega)$ be the test space for Y. The negative norm on $L^1(\Omega)$ induced by $L^\infty(\Omega)$ is given by

$$\|g\|_X^- = \sup \{ |\int_\Omega fg\ dx| \ :\ \|f\|_X \le 1 \}, \qquad g \in L^1(\Omega). \tag{3.1.4}$$

Since $(L^1(\Omega))' = L^\infty(\Omega)$, then it follows that

$$\|g\|_X^- = \|g\|_Y = \int_\Omega |g|\ dx. \tag{3.1.5}$$

On the other hand, the negative norm on $L^1(\Omega)$ induced by $C_c^\infty(\Omega)$ is given by

$$\|g\|_\Phi^- = \sup \{ |\int_\Omega fg\ dx| \ :\ \|f\|_X \le 1,\ f \in C_c^\infty(\Omega) \}, \quad g \in L^1(\Omega). \tag{3.1.6}$$

Even though $C_c^\infty(\Omega)$ is not dense in $L^\infty(\Omega)$, next we will see that (3.1.5) is equal to (3.1.6).

PROPOSITION 3.1.3. If $g \in L^1(\Omega)$, then $\|g\|_\Phi^- = \|g\|_X^-$.

PROOF. Let $g \in L^1(\Omega)$, $g \ne 0$. It is clear that $\|g\|_X^- \ge \|g\|_\Phi^-$.

Now, let $\varepsilon > 0$ be given. Pick an open set ω such that $\bar{\omega}$ is a compact subset of Ω, and

$$\int_{\Omega\setminus\omega} |g|\ dx \le \varepsilon. \tag{3.1.7}$$

Next, let B a measurable subset of ω and $C > 0$, such that $|g(x)| \le C$ on B and

$$\int_{\omega\setminus B} |g|\ dx \le \varepsilon. \tag{3.1.8}$$

Define

$$\alpha(x) = \begin{cases} 1, & \text{if } g(x) > 0 \text{ and } x \in B \\ -1, & \text{if } g(x) < 0 \text{ and } x \in B \\ 0, & \text{if } x \in \Omega\backslash B . \end{cases}$$

From Luzin's theorem, there exists a continuous function $h : \Omega \longrightarrow \mathbb{R}$ with compact support contained in B, such that $\|h\|_{L^\infty(\Omega)} \le 1$ and the set

$$A = \{x \in \Omega : h(x) \ne \alpha(x) \}$$

has measure less than $\varepsilon/2C$. Thus

$$\int_B |g| \, dx \le \int_B gh \, dx + \varepsilon. \tag{3.1.9}$$

Since h has compact support, from the Stone-Weierstrass theorem it follows that the exists a polynomial P such that

$$\|P - h\|_{L^\infty(\omega)} \le \varepsilon', \tag{3.1.10}$$

where $\varepsilon' = \varepsilon/\|g\|_{L^1(\Omega)}$. From (3.1.9) we have

$$|P(x)| \le \varepsilon', \quad x \in \omega\backslash \text{supp}(h), \tag{3.1.11}$$

and

$$\|P\|_{L^\infty(\Omega)} \le 1 + \varepsilon'. \tag{3.1.12}$$

Let us consider a function $\phi \in C_c^\infty(\Omega)$, such that $0 \le \phi \le 1$ and $\phi(x) = 1$ for $x \in \text{supp}(h)$. Define $\psi = \phi P$. Then $\psi \in C_c^\infty(\omega)$, and from (3.1.12) we have

$$\|\psi\|_{L^\infty(\Omega)} \le 1 + \varepsilon'. \tag{3.1.13}$$

Furthermore, using (3.1.11) we get

$$\|\psi - h\|_{L^\infty(\omega)} \le \varepsilon'.$$

But this implies that

$$\int_B gh \, dx \le \int_\omega g\psi \, dx + \varepsilon.$$

This last inequality together with (3.1.7), (3.1.8), (3.1.9) and (3.1.13) give

$$\int_\Omega |g| \, dx \le 4\varepsilon + (1 + \varepsilon')\|g\|_{\bar{\Phi}} .$$

Finally, letting $\varepsilon \longrightarrow 0$, we obtain

$$\|g\|_X^- \leq \int_\Omega |g| \; dx \leq \|g\|_\Phi^-,$$

and this completes the proof.

EXAMPLE 3.1.2. Let X be the Banach space over $\mathbb{R}$ consisting of all functions $f \in C^1(\mathbb{R})$ such that f and f' are bounded on $\mathbb{R}$, together with the norm

$$\|f\|_X = \max \{\|f\|_{L^\infty(\mathbb{R})} \; , \; \|f'\|_{L^\infty(\mathbb{R})} \}.$$

Taking $Y = L^1(\mathbb{R})$ and the product

$$[f,g] = \int fg \; dx,$$

it is a simple matter to verify that $(X,Y;[\cdot,\cdot])$ is a P-space.

Let us consider the points in $\mathbb{R}$

$\alpha_0 = 0$, $\alpha_k = 1 + (1/2) + \ldots + (1/k)$, and $\alpha_{-k} = -\alpha_k$, $k = 1, 2, \ldots$;

and let Φ be the closed subspace of X consisting of all $f \in X$ such that $f(\alpha_k) = 0$, $k \in \mathbb{Z}$.

LEMMA 3.1.4. The following holds:

(i) Φ is a test space for $L^1(\mathbb{R})$.

(ii) There is a positive constant C such that

$$\|f\|_Y \leq C\|f\|_X \; , \text{ for all } f \in \Phi.$$

PROOF. (i) Let $g \in L^1(\mathbb{R})$ be such that $[f,g] = 0$ for all $f \in \Phi$. Letting $I_k = (\alpha_{k-1}, \alpha_k)$, then we have $[\phi, g] = 0$ for all $\phi \in C_c^\infty(I_k)$. From du Bois-Reymond lemma it follows that $g = 0$ a.e. on I_k. Therefore $g = 0$.

(ii) Take an $f \in \Phi$, and let $x \in I_k$. Since $f(\alpha_k) = 0$, from the mean value theorem we have

$$|f(x)| \leq k^{-1}\|f\|_X \; , \qquad x \in I_k,$$

and hence

$$\int |f|\, dx = \sum_{k\in\mathbb{Z}} \int_{I_k} |f|\, dx \le 2 \sum_{k=1}^{\infty} k^{-2} \|f\|_X .$$

PROPOSITION 3.1.5. The norms $\|\cdot\|_X^-$ and $\|\cdot\|_\Phi^-$ are not equivalent.

PROOF. Consider the sequence $\{g_k\}$ in $L^1(\mathbb{R})$ defined by

$$g_k(x) = 1, \text{ if } |x| \le k,\ g_k(x) = 0 \text{ if } |x| > k.$$

From (ii) of the previous lemma, this sequence is bounded under the norm $\|\cdot\|_\Phi^-$. Nevertheless, we are going to show that it is not bounded under the norm $\|\cdot\|_X^-$.

Fix an $\phi \in C_c^\infty(\mathbb{R})$ such that, $0 \le \phi \le 1$ and $\phi \equiv 1$ on $[-1,1]$. Let $\phi_j(x) = \phi(x/j)$, $j = 1, 2. \ldots$; and observe that

$$\|\phi_j\|_X \le \|\phi\|_X ,\quad j = 1, 2, \ldots .$$

If $\{g_k\}$ were bounded under the norm $\|\cdot\|_X^-$, then from the previous inequality it would follow that the set

$$\left\{ \left| \int \phi_j g_k\, dx \right| : j,\ k \in \mathbb{N} \right\}$$

is bounded. But, this is not the case, since

$$\int \phi_j g_j\, dx = \int_{-j}^{j} \phi_j\, dx = 2j.$$

3.2 Weak Extension of Operators in the Class $\mathscr{C}_1(\Phi,\Psi)$: The Normed Case.

Let $(X,Y;[\cdot,\cdot])$ be a P-space, where X is a normed space with norm $\|\cdot\|$. In Y we consider the negative norm $\|\cdot\|^-$ induced by X. Let $L \in \mathscr{C}_1(\Phi,\Psi)$, where $\Phi \subseteq X$ and $\Psi \subseteq Y$ are test spaces for Y and X respectively. Then we have

$$[L\phi,\psi] = [\phi,L^*\psi], \qquad \phi \in \Phi,\ \psi \in \Psi. \tag{3.2.1}$$

Using the definition of $I : X \longrightarrow Y'$, we can rewrite identity (3.2.1) as

$$\langle\psi, IL\phi\rangle = \langle L^*\psi, I\phi\rangle, \qquad \phi \in \Phi,\ \psi \in \Psi.$$

Taking into account the identification $I : X \longrightarrow Y'$, we write the previous equality simply as

$$\langle\psi, L\phi\rangle = \langle L^*\psi, \phi\rangle, \qquad \phi \in \Phi,\ \psi \in \Psi. \tag{3.2.2}$$

Formula (3.2.2) provides us with a way of "extending" L to the whole dual Y'. Given $y' \in Y'$, we define the linear functional

$$L_w y' : \Psi \longrightarrow \mathbb{K}$$

by

$$\langle\psi, L_w y'\rangle = \langle L^*\psi, y'\rangle, \qquad \psi \in \Psi. \tag{3.2.3}$$

From (3.2.3) it follows that

$$|\langle\psi, L_w y'\rangle| \le \|L^*\psi\|^- \|y'\|, \qquad \psi \in \Psi,\ y' \in Y'. \tag{3.2.4}$$

Hence, if we denote by L_0^* the restiction of L^* to Ψ, and if we take into account that $L_0^* : V(Y;L_0^*) \longrightarrow Y$ is a continuous linear operator (e.g.(1.2.5)), from (3.2.3) and (3.2.4) we have

$$L_w y' = (L_0^*)'y', \qquad y' \in Y'.$$

Therefore, the linear operator

$$L_w = (L_0^*)' : Y' \longrightarrow [V(Y;L_0^*)]' \tag{3.2.5}$$

is also continuous. The operator L_w is called the *weak extension* of L.

To see that indeed L_w is an extension of L via the identification $x \longrightarrow Ix$, we must check that the following holds:

$$L_w Ix = ILx, \qquad x \in D(L).$$

More generally, we have the following result.

LEMMA 3.2.1. $L_w Ix = ILx$, for all $x \in D(L)$.

PROOF. Let $x \in D(L)$. From (2.2.2), (3.1.3) and (3.2.3) we obtain

$$\langle\psi, L_w Ix\rangle = \langle L^*\psi, Ix\rangle = \overline{[x, L^*\psi]} = \overline{[Lx, \psi]} = \langle\psi, ILx\rangle, \qquad \psi \in \Psi.$$

PROPOSITION 3.2.2. Given $f \in [V(Y;L_0^*)]'$, there exists $y' \in Y'$ such that

$$L_w y' = f,$$

if and only if there is a positive constant C such that

$$\|L^*\psi\|^- \geq C|\langle\psi, f\rangle| \text{ for all } \psi \in \Psi. \tag{3.2.6}$$

PROOF. Let $y' \in Y'$ be such that $L_w y' = f$. Then

$$\langle\psi, L_w y'\rangle = \langle L^*\psi, y'\rangle = \langle\psi, f\rangle, \qquad \psi \in \Psi,$$

and so

$$|\langle\psi, f\rangle| \leq \|L^*\psi\|^- \|y'\|, \qquad \psi \in \Psi.$$

Let us assume now that there exists a positive constant C such that (3.2.6) holds. Define the linear functional

$$y' : R(L_0^*) \subseteq Y \longrightarrow \mathbb{K}$$

as

$$\langle L^*\psi, y'\rangle = \langle\psi, f\rangle, \qquad \psi \in \Psi. \tag{3.2.7}$$

From (3,2,6) it follows that y' is well defined, and it is continuos on $R(L_0^*)$. By Hahn-Banach theorem, we can extend y' to all of Y in a continuos way, and from (3.2.7) it is clear that $L_w y' = f$.

In order to keep the notation as simple as possible, from now on we will use the same symbol L to denote the operator and its weak extension L_w as well. It will be clear in the context to which operator we are reffering to.

Using the fact that the weak extension L_w of L, is the dual of L_0^* (cf. (3.2.5)), it is interesting to note that several properties of L (i.e., one-to-one, onto, closed range, etc.) can be formulated in terms of L_0^* (cf. Dieudonné and Schwartz, pp. 90-93).

REMARK 3.2.1. Let $b \in X$. Using Proposition 3.2.2, we see that if there exists $C > 0$ such that $\|L^* \psi\|^- \geq C|[b,\psi]|$, for all $\psi \in \Psi$, then the equation $Lx = b$ has a weak solution in Y'. This fact suggests the convenience of considering in the test space Ψ "very strong" norms, and also how suitable is to have "very small" test spaces. In sections 4.2 and 4.3 we will see that, under certain conditions, it is possible to equip the test space Ψ with an increasing family of norms.

4. Spaces of Distributions Associated to Operators in the Class $\mathscr{E}_2(\Phi,\Psi)$.

In the previous section, we defined the weak extension L_w for a linear operator $L \in \mathscr{E}_1(\Phi,\Psi)$. A characteristic feature of this situation is that, in general, L_w cannot be iterated. To have the advantage that L_w and all of its iterates be defined in certain space, which we want to include X, we must impose certain additional properties on the operator L. This is the object of study in the remaining sections of this chapter.

4.1 The Dual of a Locally Convex Metrizable Space.

Consider a locally convex metrizable space X. Let $\mathcal{P} = \{p_\ell\}$ be an increasing family of semi-norms generating the topology of X. Denote by $X_\ell = (X, p_\ell)$, the space X together with the locally convex (not necessarily Hausdorff) topology generated by the semi-norm p_ℓ. The next result follows immediately from the definitions.

LEMMA 4.1.1. Under the situation described above, we have

$$X' = \bigcup_{\ell} X'_{\ell}.$$

Proceeding as in the case of normed spaces, it results that

$$\|\phi\|_{\ell} = \sup \{ |\langle x,\phi\rangle| : p_{\ell}(x) \le 1 \}, \qquad \phi \in X'_{\ell} ;$$

defines a norm on X'_{ℓ}. Also, under this norm, X'_{ℓ} is a Banach space, and

$$|\langle x,\phi\rangle| \le p_{\ell}(x) \, \|\phi\|_{\ell} , \quad x \in X, \ \phi \in X'_{\ell}. \tag{4.1.1}$$

LEMMA 4.1.2. We have the following continuous inclusions

$$X'_1 \hookrightarrow \dots \hookrightarrow X'_{\ell} \hookrightarrow X'_{\ell+1} \hookrightarrow \dots \hookrightarrow X'_s .$$

PROOF. From condition $p_{\ell} \le p_{\ell+1}$, it follows that $X_{\ell+1} \hookrightarrow X_{\ell}$, and hence $X'_{\ell} \hookrightarrow X'_{\ell+1}$. Now we fix ℓ , and we are going to show that $X'_{\ell} \hookrightarrow X'_s$. Let $\{\phi_k\}$ be a sequence in X'_{ℓ} such that

$$\phi_k \longrightarrow 0 \text{ in } X'_{\ell}. \tag{4.1.2}$$

Let B be a bounded subset of X. Then, from (4.1.1) and (4.1.2) we have $\phi_k \longrightarrow 0$ uniformly on B. This shows that $\phi_k \longrightarrow 0$ in X'_s.

EXAMPLE 4.1.1. Consider the P-space given in Example 2.3.5 with $1 \le p < \infty$. Let $\{K_{\ell}\}$ be a sequence of compact sets such that

$$K_{\ell} \subseteq \operatorname{int} K_{\ell+1} \text{ and } \Omega = \bigcup_{\ell} K_{\ell} . \tag{4.1.3}$$

For $u \in L^p_{\ell oc}(\Omega)$ consider the semi-norms

$$p_{\ell}(u) = \left\{ \int_{K_{\ell}} |u|^p \, dx \right\}^{1/p}, \tag{4.1.4}$$

and let $X_{\ell} = (X, p_{\ell})$, $\ell = 1, 2, \dots$.

Let

$$Y_{\ell} = \{ v \in L^{p'}(\Omega) : \operatorname{supp}(v) \subseteq K_{\ell} \}.$$

Then, Y_{ℓ} is a Banach space. Given $v \in Y_{\ell}$, it is clear that $Jv \in X'_{\ell}$.

Using the usual identification between $L^{p}(K_\ell)$ and $L^{p'}(K_\ell)$, one verifies that $J : Y_\ell \longrightarrow X'_\ell$ is an isometric isomorphism (see the Appendix).

4.2 Spaces of Distributions for Operators in the Class $\mathscr{C}_2(\Phi,\Psi)$.

Let $(X,Y;[\cdot,\cdot])$ be a P-space, and let $\mathcal{P} = \{p_\ell\}$ be an increasing sequence of seminorms generating the topology of X. Let $X_\ell = (X,p_\ell)$, and $Y_\ell = J^{-1}(X'_\ell)$, where $J : Y \longrightarrow X'$ is the canonical identification of Y in X'. We equip the space Y_ℓ with the negative norm

$$\|y\|^{-}_{\ell} = \|Jy\|_{\ell} = \sup \{ |[x,y]| \ : \ p_\ell(x) \le 1 \}, \qquad y \in Y_\ell .$$

Then $J_\ell = J|Y_\ell : Y_\ell \longrightarrow X'_\ell$ is an isometry, and

$$|[x,y]| \le p_\ell(x)\|y\|^{-}_{\ell}, \qquad x \in X, \ y \in Y_\ell. \tag{4.2.1}$$

Furthermore, when $J(Y) = X'$, Y_ℓ together with the norm $\|\cdot\|^{-}_{\ell}$ results a Banach space.

We say that a linear operator L on X belongs to the class $\mathscr{C}_2(\Phi,\Psi)$, and we write $L \in \mathscr{C}_2(\Phi,\Psi)$, if $L \in \mathscr{C}_1(\Phi,\Psi)$, and also satisfies

$$L(\Phi) \subseteq \Phi, \ L^{*}(\Psi \cap Y_\ell) \subseteq \Psi \cap Y_\ell, \qquad \ell = 1,\ 2,\ \dots\ . \tag{4.2.2}$$

REMARK 4.2.1. Given a family of linear operators $\mathscr{L} = \{L_1, \dots, L_n\}$ on X, there is a natural way of looking for spaces Φ and Ψ satisfying condition (4.2.2). For example, we can take

$$\Phi = \bigcap_{\gamma} D(L_\gamma).$$

It is clear that Φ in an $\mathscr{L}$-invariant subspace of X. If Φ is a test space for Y, then

$$\Psi = \bigcap_{\gamma} D(L^{*}_{\gamma}),$$

is an $\mathscr{L}^{*}$-invariant subspace of Y.

EXAMPLE 4.2.1. Let us consider the P-space $X = L^1_{\ell oc}(\mathbb{R}^n)$, $Y = L^\infty_c(\mathbb{R}^n)$ (cf. Example 2.3.5), and let $\Phi = \Psi = C^\infty_c(\mathbb{R}^n)$. Then, it is a simple matter to verify that the linear operators of the form $L = a(x)\partial^\alpha$, where $a \in C^\infty(\mathbb{R}^n)$, belong to $\mathscr{C}_2(\Phi, \Psi)$.

Returning to our original discussion, let $\mathscr{L} = \{L_1, \ldots, L_n\}$ be a family in the class $\mathscr{C}_2(\Phi,\Psi)$. For ℓ fixed, let us denote by $\mathscr{L}^*_\ell$ the family of operators

$$L^*_j : \Psi \cap Y_\ell \subseteq Y_\ell \longrightarrow Y_\ell, \quad j = 1, 2, \ldots, n.$$

Consider the family of spaces

$$V^{+\infty}(Y_\ell; \mathscr{L}^*_\ell), \quad \ell = 1, 2, \ldots ;$$

and let

$$\Psi_\ell = \Psi \cap Y_\ell.$$

In Ψ_ℓ we consider the topology induced by the locally convex metrizable space $V^{+\infty}(Y_\ell; \mathscr{L}^*_\ell)$.

From Lemma 4.1.2. we have $\Psi_\ell \hookrightarrow \Psi_{\ell+1}$, and from (1.3.11) it follows that

$$\Psi_\ell \hookrightarrow Y_\ell, \quad \ell = 1, 2, \ldots . \tag{4.2.4}$$

Also, from Lemma 4.1.1 we have

$$\Psi = \bigcup_\ell \Psi_\ell .$$

Let us consider in Ψ the inductive limit topology defined by the incerasing sequence of locally convex Hausdorff spaces $\Psi_\ell \hookrightarrow \Psi_{\ell+1}$. The next result will show that this topology turns Ψ into a locally convex Hausdorff space, which is simply referred as *the test space*. This test space will be denoted by $\Psi(X;\mathscr{L})$ or, when the context is clear, simply by Ψ.

LEMMA 4.2.1. Let $\mathcal{L}$ be a finite family of operators in the class $\mathcal{C}_2(\Phi,\Psi)$. Then

(i) The canonical identification $J : \Psi(X;\mathcal{L}) \longrightarrow X'_s$ is continuous.

(ii) $\Psi(X;\mathcal{L})$ is a Hausdorff space.

PROOF. (i) From the properties of the inductive limit, it is sufficient to show that $J : \Psi_\ell \longrightarrow X'_s$ is continuous for every ℓ. For this, let us note that J can be expressed as the composition

$$\Psi_\ell \hookrightarrow Y_\ell \xrightarrow{J_\ell} X'_\ell \hookrightarrow X'_s .$$

From (4.2.4), the definition of $\|\cdot\|_\ell$, and Lemma 4.1.2, it follows that the composition is continuous.

(ii) It is immediate from the fact that J is one-to-one , and X'_s is Hausdorff.

REMARK 4.2.2. We want to point out that condition (4.2.2) depends on the family of seminorms defining the locally convex topology of X. Indeed, it is enough to recall that $X_\ell = (X,p_\ell)$, and that $Y_\ell = J^{-1}(X'_\ell)$. Nevertheless, if another family of seminorms generate the same locally convex topology on X, under which also $\mathcal{L} \subseteq \mathcal{C}_2(\Phi,\Psi)$, then it is not difficult to check that the corresponding test spaces coincide.

PROPOSITION 4.2.2. If $J(Y) = X'$, and $\Psi = \cap_\gamma D(L^*_\gamma)$, then each Ψ_ℓ is a Frechet space.

PROOF. Let $\{\psi_k\}$ be a Cauchy sequence in Ψ_ℓ. Then $\{L_\gamma\psi_k\}$ is a Cauchy sequence in Y_ℓ, for each subindex γ. Since $J(Y) = X'$, each Y_ℓ is complete, and hence, there exist $y \in Y_\ell$, $y_\gamma \in Y_\ell$, such that

$$\psi_k \longrightarrow y, \; L_\gamma^* \psi_k \longrightarrow y_\gamma .$$

Being each $L_j^* : D(L_j^*) \subseteq Y_\ell \longrightarrow Y_\ell$ closed, we have $y \in D(L_\gamma^*)$ and $L_\gamma^* y = y_\gamma$, for each γ. Thus, $y \in \bigcap_\gamma D(L_\gamma^*) = \Psi$, and $L_\gamma^* \psi_k \longrightarrow L_\gamma^* y$. Therefore, $y \in \Psi_\ell$ and $\psi_k \longrightarrow y$ in Ψ_ℓ.

Later, we are going to give examples which show that the test space $\Psi(X;\mathcal{L})$ does not necessarily has to be complete. Applying the theory of inductive limits, we can see that several properties of the spaces Ψ_ℓ are inherited to the test space $\Psi(X;\mathcal{L})$. In particular, since each Ψ_ℓ is a locally convex metrizable space, we have the following

PROPOSITION 4.2.3. The test space $\Psi(X;\mathcal{L})$ is bornological.

Given a nonnegative integer m and $\gamma = (\gamma_1, \ldots, \gamma_m) \in I_n^{(m)}$, as in Section 1.3 , we will use the notation $L_\gamma^* = L_{\gamma_1}^* \ldots L_{\gamma_m}^*$.

PROPOSITION 4.2.4. For each positive integer m and $\gamma \in I_n^{(m)}$, the linear operator

$$L_\gamma^* : \Psi(X;\mathcal{L}) \longrightarrow \Psi(X;\mathcal{L})$$

is continuous.

PROOF. From the properties of the inductive limit, it suffices to show that each $L_\gamma^* : \Psi_\ell \longrightarrow \Psi(X;\mathcal{L})$ is continuous. Since $\Psi_\ell \hookrightarrow \Psi(X;\mathcal{L})$, from condition (4.2.2), it suffices to prove the continuity of each $L_\gamma^* : \Psi_\ell \longrightarrow \Psi_\ell$. But this is immediate from the definition of Ψ_ℓ and (1.3.12).

The dual of the test space $\Psi(X;\mathcal{L})$, is called the *space of distributions* corresponding to the family $\mathcal{L}$ and the space X.

PROPOSITION 4.2.5. The space of distributions $\Psi(X;\mathcal{L})'$ is complete.

PROOF. It is immediate from Proposition 4.2.3, and the fact that the strong dual of a bornological space is always complete (Bourbaki, p. 12).

PROPOSITION 4.2.6. Let $u : \Psi \longrightarrow \mathbb{K}$ be a linear functional. Then, u is a distribution if and only if for every $\ell = 1, 2, \ldots,$ there exist C_ℓ and $m_\ell \in \mathbb{N}$, such that

$$|\langle\psi, u\rangle| \leq C_\ell \max \{\|L_\gamma^* \psi\|^- : [\gamma] \leq m_\ell\}, \qquad \psi \in \Psi_\ell . \tag{4.2.5}$$

PROOF. From the properties of the inductive limit, u is continuous if and only if each restriction $u : \Psi_\ell \longrightarrow \mathbb{K}$ is continuous. Since Ψ_ℓ is a locally convex metrizable space, whose topology is generated by the increasing sequence of norms

$$\|\psi\|_{\mathcal{L}^*, m} = \max \{ \|L_\gamma^* \psi\|^- : [\gamma] \leq m \},$$

the continuity of $u : \Psi_\ell \longrightarrow \mathbb{K}$ is equivalent to the condition (4.2.5).

Let us recall that in Section 3.1 we defined for each $x \in X$, the linear functional $Ix : Y \longrightarrow \mathbb{K}$ by

$$\langle y, Ix\rangle = \overline{[x, y]}, \qquad y \in Y.$$

PROPOSITION 4.2.7. The following holds:

(i) $Ix \in \Psi(X;\mathcal{L})'$, for each $x \in X$.

(ii) $I : X \longrightarrow \Psi(X;\mathcal{L})'_S$ is continuous.

PROOF. (i) This is an immediate consequence of (4.2.1), and the previous proposition.

(ii) Let $\{x_k\}$ be a sequence such that $x_k \longrightarrow 0$ in X. We must show that $Ix_k \longrightarrow 0$ in $\Psi(X;\mathcal{L})'_s$. From the definition of strong topology, this is equivalent to show that $Ix_k \longrightarrow 0$ uniformly on each bounded subset of $\Psi(X;\mathcal{L})$. Let then B be a bounded subset of $\Psi(X;\mathcal{L})$. From Lemma 4.2.1, $J(B)$ is a bounded subset of X'_s. Let us pick a sequence $\{r_k\}$ of positive numbers satisfying $r_k \longrightarrow +\infty$ and $r_k x_k \longrightarrow 0$ in X. Then, by definition of the strong topology in X'_s, there is a $C > 0$ such that

$$C \geq |\langle r_k x_k, J\psi\rangle| = r_k|[x_k,\psi]| = r_k|\langle\psi, Ix_k\rangle| \quad (k = 1, 2, \ldots),$$

for all $\psi \in B$. This implies that $Ix_k \longrightarrow 0$ uniformly on B.

Since Ψ is a test space for X, the correspondence $I : X \longrightarrow Y'$ is one-to-one. This will allow us to identify X with $I(X) \subseteq \Psi(X;\mathcal{L})'$. As in the classic case, the distributions of the form Ix where $x \in X$, will be called *regular distributions.*

The next result shows that in most cases of interest, there allways are distributions which are non regular.

PROPOSITION 4.2.8. Let $(X,Y;[\cdot,\cdot])$ be a P-space, where X is a Frechet space, and $\mathcal{L} = \{L_1, \ldots, L_n\}$ is a family of operators in $\mathcal{C}_2(\Phi,\Psi)$. If the identification $I : X \longrightarrow \Psi(X;\mathcal{L})'$ is onto, then each of the linear operators $L_1, \ldots, L_n$ is continuous.

PROOF. Let $x \in X$. From propositions 4.2.4 and 4.2.7 it follows that the linear functional $u : \Psi(X;\mathcal{L}) \longrightarrow \mathbb{K}$, given by

$$\langle\psi,u\rangle = \langle L_j^*\psi, Ix\rangle, \qquad \psi \in \Psi,$$

is continuous. By our hypothesis, there exists a $z \in X$ such that

$$\langle L_j^* \psi, Ix\rangle = \langle \psi, Iz\rangle, \qquad \psi \in \Psi,$$

or equivalently,

$$[x, L_j^* \psi] = [z, \psi], \qquad \psi \in \Psi.$$

This last fact, together with (2.2.2) show that $x \in D(L_j)$ and $L_j x = z$. Thus $D(L_j) = X$, and applying Proposition 2.2.3 we obtain the result.

From conditions (4.2.2), it is easy to check that for $\mathcal{L} = \{L_1, \ldots, L_n\}$ in $\mathscr{C}_2(\Phi, \Psi)$ and $\gamma = (\gamma_1, \ldots, \gamma_m) \in I_n^{(m)}$ we have

$$[L_\gamma x, y] = [x, L_{\gamma_{op}}^* y], \qquad x \in \Phi, \; y \in \Psi, \tag{4.2.6}$$

where

$$\gamma_{op} = (\gamma_m, \ldots, \gamma_1) \tag{4.2.7}$$

If $u \in \Psi'$, we define the linear functional

$$L_\gamma u : \Psi \longrightarrow \mathbb{K},$$

in the *weak sense* (or in the *sense of distributions*), as

$$\langle \psi, L_\gamma u\rangle = \langle L_{\gamma_{op}}^* \psi, u\rangle, \qquad \psi \in \Psi. \tag{4.2.8}$$

Proposition 4.2.4 implies that $L_\gamma u \in \Psi(X;\mathcal{L})'$.

If $x \in X$, then from Proposition 4.2.7 we know that $Ix \in \Psi(X;\mathcal{L})'$. In this case we write $L_\gamma x$ instead of $L_\gamma Ix$, and we interpret $L_\gamma x$ in the weak sense. Thus we have:

$$\langle \psi, L_\gamma x\rangle = \overline{[x, L_{\gamma_{op}}^* \psi]}, \qquad \psi \in \Psi. \tag{4.2.9}$$

The next result is clear.

PROPOSITION 4.2.9. The following holds:

(i) The weak extension $L_\gamma : \Psi(X;\mathcal{L})' \longrightarrow \Psi(X;\mathcal{L})'$ is the dual operator of $L_{\gamma_{op}}^* : \Psi(X;\mathcal{L}) \longrightarrow \Psi(X;\mathcal{L})$.

(ii) If $x \in D(\mathrm{L}_\gamma)$, then $L_\gamma Ix = I\mathrm{L}_\gamma x$, where L_γ is the maximal closed extension of L_γ.

(iii) If $\gamma = (\gamma_1, \ldots, \gamma_m) \in I_n^{(m)}$, then $L_\gamma = L_{\gamma_1} \ldots L_{\gamma_m}$ in the weak sense.

Let $u \in \Psi(X;\mathcal{L})'$ and $\gamma \in I_n(m)$. We say that $L_\gamma u$ *belongs to* X, and we write $L_\gamma u \in X$, if there exists a $z \in X$ such that $L_\gamma u = Iz$. This is equivalent to the condition

$$\langle L^*_{\gamma_{op}} \psi, u\rangle = [z,\psi], \qquad \psi \in \Psi. \tag{4.2.10}$$

If this is the case, we write $L_\gamma u = z$.

LEMMA 4.2.10. If $x \in X$ and $L_j x \in X$, where L_j is the weak extension, then $x \in D(\mathrm{L}_j)$ and $L_j x = \mathrm{L}_j x$.

PROOF. From (4.2.9) we have $L_j x \in X$ if and only if there is a $z \in X$ such that

$$[x, L^*_j \psi] = [z,\psi], \qquad \psi \in \Psi.$$

From (2.2.2) we obtain the desired result.

PROPOSITION 4.2.11. If the test space $\Psi(X;\mathcal{L})$ is reflexive, then Φ is dense in the space of distributions $\Psi(X;\mathcal{L})'_s$.

PROOF. According to Lemma 1.1.1, it is enough to note that Φ is total in $\Psi(X;\mathcal{L})'_s$.

Using the same argument as in the proof of the previous proposition, we see that Φ is allways weak*-dense in $\Psi(X;\mathcal{L})'_s$.

EXAMPLE 4.2.2. Let Ω be an open nonempty subset of $\mathbb{R}^n$. According with Example 2.3.5, let us consider the P-space $X = L^1_{\ell oc}(\Omega)$, $Y = L^\infty_c(\Omega)$. Using the same notation as in Example 4.1.1 with $p = 1$, $p' = \infty$, we assume that X is equipped with the family of semi-norms $\{p_\ell\}$ given by (4.1.4).

Then, we know that $J : Y_\ell \longrightarrow X'_\ell$ is an isometric isomorphism, where

$$Y_\ell = J^{-1}(X'_\ell) = \{\ v \in L^\infty(\Omega)\ :\ \mathrm{supp}(v) \subseteq K_\ell\ \}.$$

Thus, for $\Phi = \Psi = C^\infty_c(\Omega)$, we have

$$\Psi_\ell = \{\ \psi \in C^\infty_c(\Omega)\ :\ \mathrm{supp}(\psi) \subseteq K_\ell\ \}.$$

From this it follows immediately that the family of operators

$$\partial = \left\{ \frac{\partial}{\partial x_1},\ \ldots,\ \frac{\partial}{\partial x_n} \right\},$$

belongs to the class $\mathscr{E}_2(C^\infty_c(\Omega), C^\infty_c(\Omega))$.

Also, $\psi_k \longrightarrow \psi$ in Ψ_ℓ is equivalent to $\partial^\alpha\psi_k \longrightarrow \partial^\alpha\psi$ uniformly on the compact set K_ℓ, for every multi-index α. Therefore

$$\Psi(L^1_{\ell oc}(\Omega);\partial) = \mathcal{D}(\Omega),$$

the space of test functions on Ω with the Schwartz topology; and

$$\Psi(L^1_{\ell oc}(\Omega);\partial)' = \mathcal{D}(\Omega)',$$

the space of distributions on Ω.

REMARK 4.2.3. The previous result, naturally leads us to ask the question that if changing the space $L^1_{\ell oc}(\Omega)$ by $L^p_{\ell oc}(\Omega)$, $1 < p \le \infty$, this will yield another space of distributions. Surprisingly, it happens that

$$\Psi(L^1_{\ell oc}(\Omega);\partial) = \Psi(L^p_{\ell oc}(\Omega);\partial), \qquad 1 \le p \le \infty.$$

Indeed, from $L^p_{\ell oc}(\Omega) \hookrightarrow L^1_{\ell oc}(\Omega)$ it follows that

$$\Psi(L^1_{\ell oc}(\Omega);\partial) \hookrightarrow \Psi(L^p_{\ell oc}(\Omega);\partial).$$

Now, being $\Psi(L^p_{\ell oc}(\Omega);\partial)$ the inductive limit of the spaces Ψ_ℓ, in order to obtain the other continuous inclusion it is enough to show that $\Psi_\ell \hookrightarrow \Psi(L^1_{\ell oc}(\Omega);\partial)$. According with Example 4.1.1, we know that $\psi_k \longrightarrow \psi$ in Ψ_ℓ is equivalent to $\partial^\alpha\psi_k \longrightarrow \partial^\alpha\psi$ in the $L^p(K_\ell)$-norm for every multi-index α. But the well known Sobolev's immersion theorem (Adams [1], p. 97) says that $W^{m,p}(\mathbb{R}^n) \hookrightarrow BC(\mathbb{R}^n)$ if $mp > n$; where $BC(\mathbb{R}^n)$ is the space of bounded continuous functions on $\mathbb{R}^n$. From this it follows that $\psi_k \longrightarrow \psi$ uniformly on K_ℓ.

Going back to our initial context, let $\mathcal{L} = \{L_1, \ldots, L_n\}$ be a family of operators in the class $\mathcal{C}_2(\Phi,\Psi)$. Let us consider a linear operator $L : D(L) \subseteq X \longrightarrow X$, $D(L) = D(L_1) \cap \ldots \cap D(L_n)$, of the form

$$L = \sum_{[\gamma]\le m} a_\gamma L_\gamma \,, \tag{4.2.11}$$

where $a_\gamma \in \mathbb{K}$. Then $L \in \mathcal{C}_2(\Phi,\Psi)$, and its maximal closed extension is defined in the obvious way.

PROPOSITION 4.2.12. Let L be an operator as in (4.2.11). Then:

(i) $\Psi(X;\mathcal{L}) \hookrightarrow \Psi(X;L)$,

(ii) $\Psi(X;L)'_s \hookrightarrow \Psi(X;\mathcal{L})'_s$.

PROOF. (i) Since $\Psi(X;\mathcal{L})$ is the inductive limit of the increasing sequence of spaces $\{\Psi_\ell(X;\mathcal{L})\}$, it is enough to show that each inclusion $\Psi_\ell(X;\mathcal{L}) \longrightarrow \Psi(X;L)$ is continuous. But since we have the continuous inclusions $\Psi_\ell(X;L) \hookrightarrow \Psi(X;L)$, then it is sufficient to establish the continuity for each one of the inclusions $\Psi_\ell(X;\mathcal{L}) \longrightarrow \Psi_\ell(X;L)$.

So let $\{y_k\}$ be a sequence such that $y_k \longrightarrow 0$ in $\Psi_\ell(X;\mathcal{L})$. Then

$$L^*_\gamma y_k \longrightarrow 0 \text{ in } \Psi_\ell(X;\mathcal{L}), \text{ for every subindex } \gamma. \tag{4.2.12}$$

Taking into account that $(L^{*})^{m}$, $m = 0, 1, \ldots$, is a linear combination of operators of the form L^{*}_{γ}, from (4.2.12) it follows that

$$(L^{*})^{m} y_k \longrightarrow 0 \text{ in } \Psi_{\ell}(X;L), \quad m = 0, 1, \ldots ;$$

and hence that $y_k \longrightarrow 0$ in $\Psi_{\ell}(X;L)$.

(ii) It is immediate from (i).

4.3 Spaces of Distributions for Operators in the Class $\mathscr{C}_2(\Phi,\Psi)$: The Normed Case.

In this section we want to discuss some questions related to the space of distributions $\Psi(X;\mathscr{L})'$, when X is a normed space.

Let $(X,Y,[\cdot,\cdot])$ be a P-space, where X is a normed space with norm $\|\cdot\|$. In Y we consider the negative norm $\|\cdot\|^{-}$ given by (3.1.1). Let Φ and Ψ be test spaces for Y and X respectively, and let $\mathscr{L} = \{L_1, \ldots, L_n\}$ be a family of operators in the class $\mathscr{C}_2(\Phi,\Psi)$. In this context, condition (4.2.2) assumes the simpler form

$$L_j(\Phi) \subseteq \Phi, \ L^{*}_j(\Psi) \subseteq \Psi, \quad j = 1, \ldots, n. \tag{4.3.1}$$

Let us denote by $\mathscr{L}^{*}$ the family of closed operators (cf. Lemma 3.1.1)

$$L^{*}_j : D(L^{*}_j) \subseteq Y \longrightarrow Y, \quad j = 1, \ldots, n.$$

Next we form the family of spaces

$$V^{m}(Y;\mathscr{L}^{*}), \quad m = 0, 1, \ldots, +\infty.$$

According to (1.3.6) the norm on $V^{m}(Y;\mathscr{L}^{*})$, $m < +\infty$, is given by

$$\|y\|_{\mathscr{L}^{*},m} = \max\{\|L^{*}_{\gamma}y\|^{-} : [\gamma] \le m\}.$$

Then

$$\|y\|_{\mathscr{L}^{*},m} \le \|y\|_{\mathscr{L}^{*},m+1}, \quad y \in V^{m+1}(Y;\mathscr{L}^{*}).$$

From condition (4.3.1) we have

$$\Psi \subseteq V^{m}(Y;\mathscr{L}^{*}), \quad m = 0, 1, \ldots, +\infty;$$

so we can define for $m = 0, 1, \ldots, +\infty$,

$$V_0^m(Y;\mathcal{L}^*) = \text{closure of } \Psi \text{ in } V^m(Y;\mathcal{L}^*).$$

In this case, the test space $\Psi \subseteq V^{+\infty}(Y;\mathcal{L}^*)$ is metrizable, and its topology is generated by the increasing sequence of norms

$$\{\|\cdot\|_{\mathcal{L}^*,m} : m = 0, 1, \ldots\}.$$

Since Ψ is dense in $V^{+\infty}(Y;\mathcal{L}^*)$, fron Hahn-Banach theorem it follows that

$$\Psi(X;\mathcal{L})' = V_0^{+\infty}(Y;\mathcal{L}^*)'. \tag{4.3.2}$$

REMARK 4.3.1. If the locally convex metrizable space $V_0^{+\infty}(Y;\mathcal{L}^*)$ is separable (from Corollary 1.3.6 (ii), this will happen if Y is separable), then

$$\Psi(X;\mathcal{L})'_s = V_0^{+\infty}(Y;\mathcal{L}^*)'_s ,$$

(Grothendieck, p.62, Corollary 4).

REMARK 4.3.2. Let us suppose that Y, under the negative norm, is a Banach space. Consider the increasing family of norms $\|\cdot\|_{\mathcal{L}^*,m}$, $m \in \mathbb{N}$, on the space $V_0^{\infty}(Y;\mathcal{L}^*)$. Since each of the spaces $V^m(Y;\mathcal{L}^*)$ is complete, and $V^m(Y;\mathcal{L}^*) \hookrightarrow Y$, it is a simple matter to check that $V_0^{\infty}(Y;\mathcal{L}^*)$ is a "countably normed" space in the sense of Gelfand-Shilov ([6], p.12). Therefore, $\Psi(X;\mathcal{L})' = V_0^{\infty}(Y;\mathcal{L}^*)'$ is also a space of distributions in the sense of these authors ([6], p.82).

We define

$$V^{-m}(Y;\mathcal{L}^*) = [V_0^m(Y;\mathcal{L}^*)]', \quad m = 0, 1, \ldots ; \tag{4.3.3}$$

and denote the norm on this Banach space by $\|\cdot\|_{\mathcal{L}^*,-m}$.

From (1.3.2) and the definition of the topology on $\Psi(X;\mathcal{L})$ we have

$$\Psi(X;\mathcal{L}) \hookrightarrow \ldots \hookrightarrow V_0^{m+1}(Y;\mathcal{L}^*) \hookrightarrow V_0^m(Y;\mathcal{L}^*) \hookrightarrow \ldots \hookrightarrow V_0^0(Y;\mathcal{L}^*), \qquad (4.3.4)$$

each inclusion being dense (cf. (4.3.2)). When Ψ is dense in Y, we have $V_0^0(Y;\mathcal{L}^*) = Y$. From (4.3.4) it follows that

$$V^{-0}(Y;\mathcal{L}^*) \hookrightarrow \ldots \hookrightarrow V^{-m}(Y;\mathcal{L}^*) \hookrightarrow V^{-m-1}(Y;\mathcal{L}^*) \hookrightarrow \ldots \hookrightarrow \Psi(X;\mathcal{L})'_s. \quad (4.3.5)$$

PROPOSITION 4.3.1. Under the above hypothesis we have

$$\Psi(X;\mathcal{L})' = \bigcup_{m=0}^{\infty} V^{-m}(Y;\mathcal{L}^*).$$

PROOF. Let $\Psi_m = (\Psi, \|\cdot\|_{\mathcal{L}^*,m})$, $m = 0, 1, \ldots$. Since Ψ is dense in $V_0^m(Y;\mathcal{L}^*)$, we have $\Psi'_m = V^{-m}(Y;\mathcal{L}^*)$, and the conclusion follows from Lemma 4.1.1.

Proceeding as in the proof of Proposition 3.2.2 and using the previous result we obtain the following

PROPOSITION 4.3.2. Let L be a linear operator on X of the form (4.2.11). Given $b \in X$, the equation $Lu = b$ has a solution in the sense of distributions $u \in \Psi(X;\mathcal{L})'$ if and only if there exists a nonnegative integer m and a positive constant C, such that

$$\|L^*\psi\|_{\mathcal{L}^*,m} \geq C|[b,\psi]|, \qquad \psi \in \Psi.$$

In this case we have $u \in V^{-m}(Y;\mathcal{L}^*)$.

PROPOSITION 4.3.3. If $u \in V^{-m}(Y;\mathcal{L}^*)$ and $[\gamma] = k$, then $L_\gamma u \in V^{-m-k}(Y;\mathcal{L}^*)$.

PROOF. Since for every $\psi \in \Psi$ we have $\langle\psi, L_\gamma u\rangle = \langle L^*_{\gamma_{op}}\psi, u\rangle$, then

$$|\langle\psi, L_\gamma u\rangle| \leq \|L^*_{\gamma_{op}}\psi\|_{\mathcal{L}^*,m}\|u\|_{\mathcal{L}^*,-m} \leq \|\psi\|_{\mathcal{L}^*,m+k}\|u\|_{\mathcal{L}^*,-m},$$

for all $\psi \in \Psi$. The result follows from the fact that Ψ is dense in

$V^{-m-k}(Y;\mathcal{L}^*)$.

EXAMPLE 4.3.1. Let Ω be a nonempty open subset of $\mathbb{R}^n$, and consider the P-space described in Example 2.3.6. Then, it is clear that the family $\partial = \{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ belongs lo the class $\mathscr{C}_2(\Phi,\Psi)$.

If Ω has finite measure, then the same argument given at Remark 4.2.3, shows that

$$\Psi(L^p(\Omega);\partial) = \Psi(L^1(\Omega);\partial), \quad 1 \le p \le \infty, \tag{4.3.6}$$

algebraically and topologically.

If $\Omega = \mathbb{R}^n$, then (4.3.6) is not valid, i.e., the corresponding topologies do not coincide. To see this, fix a $\psi \in C_c^\infty(\mathbb{R}^n)$, $\psi \ne 0$, and consider the sequence

$$\psi_k(x) = k^{-n}\psi(x/k).$$

Then, $\{\psi_k\}$ converges to 0 in $\Psi(L^1(\mathbb{R}^n);\partial)$, but does not converges to 0 in $\Psi(L^\infty(\mathbb{R}^n);\partial)$.

Next we are going to see that every distribution in $\Psi(L^p(\mathbb{R}^n),\partial)'$, $1 \le p \le \infty$, is a tempered distribution.

PROPOSITION 4.3.4. Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$, and $V_0^{+\infty}(L^{p'}(\mathbb{R}^n);\partial)$, $1 \le p \le \infty$, $1/p+1/p' = 1$, be the closure of $\Psi = C_c^\infty(\mathbb{R}^n)$ in $V^{+\infty}(L^{p'}(\mathbb{R}^n);\partial)$. Then:

(i) $\mathscr{S} \hookrightarrow V_0^{+\infty}(L^{p'}(\mathbb{R}^n);\partial)$, and the inclusion is dense.

(ii) $\Psi(L^p(\mathbb{R}^n);\partial)' \subseteq \mathscr{S}'$.

PROOF. (i) Let $\psi \in \mathscr{S}$. Then, there exsits a sequence $\{\psi_k\}$ in $C_c^\infty(\mathbb{R}^n)$, such that $\psi_k \longrightarrow \psi$ in $\mathscr{S}$. Since $\mathscr{S} \hookrightarrow L^{p'}(\mathbb{R}^n)$ and $\partial^\alpha : \mathscr{S} \longrightarrow \mathscr{S}$ is continuous, we have

$$\partial^\alpha \psi_k \longrightarrow \partial^\alpha \psi \text{ in } L^{p'}(\mathbb{R}^n), \text{ for every multi-index } \alpha.$$

This shows that $\psi \in V_0^\infty(L^{p'}(\mathbb{R}^n);\partial)$. Analogously, one shows that the inclusion is continuous, which also is clearly dense.

(ii) It is an immediate consequence of (i) and (4.3.2).

EXAMPLE 4.3.2. Consider the P-space described in Example 2.3.6, with $\Omega = \mathbb{R}^n$, $p = 1$ and $p' = \infty$. Then, the family of operators

$$\mathcal{L} = \{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, x_1, \dots, x_n \},$$

where x_j represents the multiplication by the monomial x_j, belongs to the class $\mathcal{C}_2(\Phi,\Phi)$, where $\Phi = \Psi = C_c^\infty(\mathbb{R}^n)$.

Within the context of the previous example, we have the following

PROPOSITION 4.3.5. The topology of the test space $\Psi(L^1(\mathbb{R}^n);\mathcal{L})$ coincides with the topology of $\Psi = C_c^\infty(\mathbb{R}^n)$ as a subspace of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

PROOF. Let us denote by $\Psi_{\mathcal{S}}$, the space Ψ together with the topology induced as a subspace of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Then, the topology of $\Psi_{\mathcal{S}}$ is generated by the family of seminorms

$$\|\psi\|_{\alpha,\beta} = \sup \{ |x^\alpha \partial^\beta \psi(x)| : x \in \mathbb{R}^n \},$$

where α and β are arbitrary multi-indices. On the other hand, the topology of $\Psi = \Psi(L^1(\mathbb{R}^n);\mathcal{L})$ is generated by the family of norms

$$\|\psi\|_m = \max \{ \|L_\gamma^* \psi\|_{L^\infty(\mathbb{R}^n)} : [\gamma] \le m \},$$

where

$$\mathcal{L}^* = \{- \frac{\partial}{\partial x_1}, \dots, - \frac{\partial}{\partial x_n}, x_1, \dots, x_n \}.$$

It is clear now that $\Psi \hookrightarrow \Psi_{\mathcal{S}}$. The other continuous inclusion

$\Psi_{\varphi} \hookrightarrow \Psi$, is established just by observing that for any sub-index γ, $L_{\gamma}^{*}\psi$ can be expressed a a sum of terms of the form $x^{\alpha}\partial^{\beta}\psi$.

COROLLARY 4.3.6. Under the same hypothesis as above we have:

(i) $\Psi(L^{1}(\mathbb{R}^{n});\mathcal{L})'$ is the space of tempered distributions $\mathcal{S}(\mathbb{R}^{n})'$.

(ii) $\Psi(L^{1}(\mathbb{R}^{n});\mathcal{L})'_{s} = \mathcal{S}(\mathbb{R}^{n})'_{s}$.

PROOF. (i) Immediate from Hahn-Banach theorem, since $C_{c}^{\infty}(\mathbb{R}^{n})$ is dense in $\mathcal{S}(\mathbb{R}^{n})$.

(ii) Since $\mathcal{S}(\mathbb{R}^{n})$ has the Heine-Borel property, it is separable (Gelfand-Shilov, p. 58). The result follows from Remark 4.3.1.

In this chapter we are going to discuss with detail the spaces of the type $V^m(Y;\mathcal{L}^*)$ and $V_0^m(Y;\mathcal{L}^*)$, which appeared during the construction of the space of distributions corresponding to a family $\mathcal{L}$ of operators in the class $\mathscr{C}_2(\Phi,\Psi)$.

5. The Sobolev Spaces $W^m(X;\mathcal{L})$ and $W_0^m(X;\mathcal{L})$.

5.1 Definitions and Basic Properties.

Let $(X,Y,[\cdot,\cdot])$ be a P-space, where X is a Frechet space. Suppose that Φ and Ψ are test spaces for Y and X respectively, and let $\mathcal{L} = \{L_1, \ldots, L_n\}$ be a finite family of operators in the class $\mathscr{C}_2(\Phi,\Psi)$. Then, the following holds:

$$\Psi \subseteq D(L_j), \quad \Psi \subseteq D(L_j^*), \quad j = 1, \ldots, n, \tag{5.1.1}$$

and

$$L_j(\Phi) \subseteq \Phi, \quad L_j^*(\Psi) \subseteq \Psi, \quad j = 1, \ldots, n. \tag{5.1.2}$$

Let

$$\mathcal{A} = \{\mathbf{L}_1, \ldots, \mathbf{L}_n\},$$

where $\mathbf{L}_j$ is the maximal closed extension of L_j, $j = 1, \ldots, n$. According to our development in Section 1.3, we define *the Sobolev space of order m induced by $\mathcal{L}$ in X*, as

$$W^m(X;\mathcal{L}) = V^m(X;\mathcal{A}), \quad m = 0, 1, \ldots, +\infty. \tag{5.1.3}$$

From Proposition 2.2.1 we know that each $\mathbf{L}_j$, $j = 1, \ldots, n$, is a closed linear operator. Thus, from corollaries 1.3.2 and 1.3.6 we obtain the following

PROPOSITION 5.1.1. Let $m = 0, 1, \ldots, +\infty$. Then we have:

(i) $W^m(X;\mathcal{L})$ is a Frechet space.

(ii) If X is separable, then so it is $W^m(X;\mathcal{L})$.

(ii) If X is reflexive, then so it is $W^m(X;\mathcal{L})$.

PROPOSITION 5.1.2. Let us assume that Φ is dense in X. Then $W^1(X;\mathcal{L}) = X$ if and only if each L_j is continuous.

PROOF. It is an immediate consequence of Proposition 2.2.3.

From (1.3.2), (1.3.11), Proposition 1.3.3, and (1.3.12) we see that

$$W^{+\infty}(X;\mathcal{L}) \hookrightarrow \ldots \hookrightarrow W^{m+1}(X;\mathcal{L}) \hookrightarrow W^m(X;\mathcal{L}) \hookrightarrow \ldots \hookrightarrow X, \qquad (5.1.4)$$

and

$$L_\gamma : W^m(X;\mathcal{L}) \longrightarrow W^{m-k}(X;\mathcal{L}) \text{ is continuous, } [\gamma] = k \le m. \qquad (5.1.5)$$

If we interpret $L_\gamma x$, $x \in X$, in the sense of distributions (cf. Section 4.2), we obtain the following description for the Sobolev spaces.

PROPOSITION 5.1.3. We have

(i) $W^m(X;\mathcal{L}) = \{ x \in X : L_\gamma x \in X, [\gamma] \le m \}$, $m = 0, 1, \ldots$.

(ii) $W^{+\infty}(X;\mathcal{L}) = \{ x \in X : L_\gamma x \in X, [\gamma] = 0, 1, 2, \ldots \}$.

PROOF. (i) By induction on m. If $m = 1$, the conclusion follows from Lemma 4.2.12. Suppose now that $m \ge 1$, and the conclusion is true for m. Then, from definiton (1.3.1) we see that

$$W^{m+1}(X;\mathcal{L}) = \{ x \in W^m(X;\mathcal{L}) : L_j x \in W^m(X;\mathcal{L}),\ j = 1, \ldots, n \}.$$

From our induction hypothesis, this implies that

$$W^{m+1}(X;\mathcal{L}) = \{\ x \in X : L_\gamma x \in X \text{ and } L_\gamma \mathrm{L}_j x \in X,\ [\gamma] \le m\ \}$$
$$= \{\ x \in X : L_\gamma x \in X,\ [\gamma] \le m+1\ \}.$$

(ii) It follows immediately from (i) and the definition of $W^{+\infty}(X;\mathcal{L})$.

LEMMA 5.1.4. Let $x \in W^m(X;\mathcal{L})$, and $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ be a subindex with $[\gamma] \le m$. Then $x \in D(\mathrm{L}_\gamma)$ and

$$\mathrm{L}_\gamma x = \mathrm{L}_{\gamma_1} \circ \ldots \circ \mathrm{L}_{\gamma_\ell} x,$$

where L_γ is the maximal closed extension of L_γ.

PROOF. Let us note first that, due to conditions (5.1.1) and (5.1.2), we have $L_\gamma \in \mathcal{C}_1(\Phi,\Psi)$, an hence its maximal closed extension L_γ is defined. Since $L_j^*(\Psi) \subseteq \Psi \subseteq D(L_j^*)$, we have for $\psi \in \Psi$:

$$[\mathrm{L}_{\gamma_1} \circ \ldots, \circ \mathrm{L}_{\gamma_\ell} x, \psi] = [x,\ L^*_{\gamma_\ell} \circ \ldots \circ L^*_{\gamma_1} \psi] = [x, L^*_{\gamma_{op}} \psi].$$

This gives the desired result.

From conditions (5.1.2) we see that

$$\Phi \subseteq W^m(X;\mathcal{L}), \qquad m = 0,\ 1,\ \ldots,\ +\infty.$$

Thus, we can define

$$W^m_0(X;\mathcal{L}) = \text{closure of } \Phi \text{ in } W^m(X;\mathcal{L}), \qquad m = 0,\ 1,\ \ldots,\ +\infty.$$

The following properties of the spaces $W^m_0(X;\mathcal{L})$ are clear.

PROPOSITION 5.1.5. The following holds:

(i) $W^m_0(X;\mathcal{L})$ is a Frechet space, $m = 0,\ 1,\ \ldots,\ +\infty$.

(ii) If X is reflexive, then so it is $W_0^m(X;\mathcal{L})$.

(iii) Let $x \in W^m(X;\mathcal{L})$, $m = 0, 1, \dots, +\infty$. Then, $x \in W_0^m(X;\mathcal{L})$ if and only if there exists a sequence $\{\phi_k\}$ in Φ, such that

$$L_\gamma \phi_k \longrightarrow L_\gamma x \text{ in } X, \ [\gamma] \le m.$$

If in the previous result we let $m = 1$, then we see that

$$W_0^1(X;\mathcal{L}) \subseteq D(\tilde{L}_j), \ j = 1, \dots, n; \tag{5.1.6}$$

where $\tilde{L}_j$ is the minimal closed extension of L_j.

From (5.1.2) and (5.1.4), we obtain:

$$W_0^{+\infty}(X;\mathcal{L}) \hookrightarrow \dots \hookrightarrow W_0^{m+1}(X;\mathcal{L}) \hookrightarrow W_0^m(X;\mathcal{L}) \hookrightarrow \dots \hookrightarrow W_0^0(X;\mathcal{L}), \tag{5.1.7}$$

each inclusion being dense. When Φ is dense in X, then $W_0^0(X;\mathcal{L}) = X$.

Conditions (5.1.2) and (5.1.5) imply that

$$L_\gamma : W_0^m(X;\mathcal{L}) \longrightarrow W_0^{m-k}(X;\mathcal{L}) \text{ is continuous, } [\gamma] = k \le m. \tag{5.1.8}$$

Furthermore, making use of Proposition 4.2.7 and (5.1.7), we see that

$$W_0^m(X;\mathcal{L}) \hookrightarrow \Psi(X;\mathcal{L})'_s \ , \ m = 0, 1, \dots, +\infty. \tag{5.1.9}$$

PROPOSITION 5.1.6. If the inclusion $i : W_0^1(X;\mathcal{L}) \longrightarrow X$ is compact, then each of the inclusions

$$i : W_0^{m+1}(X;\mathcal{L}) \longrightarrow W_0^m(X;\mathcal{L}), \quad m = 0, 1, \dots ,$$

is compact.

PROOF. Let $\{x_k\}$ be a bounded sequence in $W_0^{m+1}(X;\mathcal{L})$. From (5.1.8), it follows that $\{L_\gamma x_k\}$ is a bounded sequence in $W_0^1(X;\mathcal{L})$, for each subindex γ with $[\gamma] \le m$. From our hypothesis, we can find a subsequence $\{x_{k_\ell}\}$ such that $\{L_\gamma x_{k_\ell}\}$ converges in X for every subindex

$[\gamma] \le m$. Therefore, $\{x_{k_\ell}\}$ converges in $W_0^m(X;\mathcal{L})$.

PROPOSITION 5.1.7. Suppose that the inclusion $i : W_0^1(X;\mathcal{L}) \longrightarrow X$ is compact. Then:

(i) $W_0^{+\infty}(X;\mathcal{L})$ has the Heine-Borel property, i.e., every closed bounded subset of $W_0^{+\infty}(X;\mathcal{L})$ is compact.

(ii) $W_0^{+\infty}(X;\mathcal{L})$ is reflexive.

PROOF. (i) Since $W_0^{+\infty}(X;\mathcal{L})$ is a Frechet space, it is enough to show that every bounded sequence in $W_0^{+\infty}(X;\mathcal{L})$ has a convergent subsequence. So let $\{x_k\}$ be a bounded sequence. According to the previous proposition, we can find a family of subsequences $\{x_k^{(m)}\}$ of $\{x_k\}$ such that: $\{x_k^{(m+1)}\}$ is a subsequence of $\{x_k^{(m)}\}$, and $\{x_k^{(m)}\}$ converges in $W_0^{+\infty}(X;\mathcal{L})$. Now, it is clear that the diagonal sequence $u_m = x_m^{(m)}$, converges in $W_0^{+\infty}(X;\mathcal{L})$.

(ii) Immediate from the fact that every Frechet space with the Heine-Borel property is reflexive (Dieudonné-Schwartz, p.79).

5.2 The Commutative Case.

Now we are interested in the case when $\mathcal{L}$ is a *commutative family* of linear operators on Φ, in the class $\mathscr{C}_2(\Phi,\Psi)$,i.e., here we assume that

$$L_iL_j\phi = L_jL_i\phi \;, \quad \phi \in \Phi, \quad i,\ j = 1,\ \dots,\ n.$$

LEMMA 5.2.1. If $\mathcal{L}$ is a commutative family, then

$$L_i^*L_j^* = L_j^*L_i^* \text{ on } \Psi, \quad i,\ j = 1,\ \dots,\ n.$$

PROOF. Let $\psi \in \Psi$. From (5.1.1) and (5.1.2) we get

$$[L_iL_j\phi,\psi] = [L_j\phi, L_i^*\psi] = [\phi, L_j^*L_i^*\psi], \qquad \phi \in \Phi.$$

Interchanging the roles of i and j, from the previous we obtain

$$[\phi, L_i^*L_j^*\psi - L_j^*L_i^*\psi] = 0, \qquad \phi \in \Phi.$$

Since Φ is a test space for Y, the result follows.

Given a commutative family of operators $\mathcal{L} = \{L_1, \ldots, L_n\}$, it is convenient to employ the multi-index notation as follows: Letting $L = (L_1, \ldots, L_n)$, if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, then we define

$$L^\alpha = L_1^{\alpha_1} \ldots L_n^{\alpha_n}, \tag{5.2.1}$$

whenever the composition is defined. If α is a multi-index, then as is usual, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, will denote its height.

Returning to our original problem, let $L = (L_1, \ldots, L_n)$. Then, from (5.1.1) and (5.1.2) we have

$$\Phi \subseteq D(L^\alpha) \text{ and } \Psi \subseteq D((L^*)^\alpha).$$

From the previous lemma we see that

$$(L^\alpha)^*\psi = (L^*)^\alpha\psi, \qquad \psi \in \Psi, \tag{5.2.2}$$

where $L^* = (L_1^*, \ldots, L_n^*)$. From this it follows that $L^\alpha \in \mathcal{C}_1(\Phi,\Psi)$, and hence that its maximal closed extension L^α is defined.

LEMMA 5.2.2. Let $x \in D(\mathrm{L}^\beta) \subseteq X$. Then, $x \in D(\mathrm{L}^\alpha\mathrm{L}^\beta)$ if and only if $x \in D(\mathrm{L}^{\alpha+\beta})$, and in this case $\mathrm{L}^\alpha\mathrm{L}^\beta x = \mathrm{L}^{\alpha+\beta}x$.

PROOF. According to (5.2.2) and the previous lemma, we observe first that

$$(L^{\alpha+\beta})^* = (L^*)^{\alpha+\beta} = (L^*)^\beta(L^*)^\alpha = (L^\beta)^*(L^\alpha)^* \text{ on } \Psi.$$

Suppose now that $x \in D(\mathrm{L}^\alpha\mathrm{L}^\beta)$. Then, for $\psi \in \Psi$ we have

$$[x,(L^{\alpha+\beta})^*\psi] = [x,(L^\beta)^*(L^\alpha)^*\psi] = [L^\beta x,(L^\alpha)^*\psi] = [L^\alpha L^\beta x,\psi].$$

Thus, $x \in D(L^{\alpha+\beta})$ and $L^{\alpha+\beta}x = L^\alpha L^\beta x$.

If $x \in D(L^{\alpha+\beta})$, then for $\psi \in \Psi$ we have

$$[L^\beta x,(L^\alpha)^*\psi] = [x,(L^\beta)^*(L^\alpha)^*\psi] = [x,(L^{\alpha+\beta})^*\psi] = [L^{\alpha+\beta}x,\psi].$$

Therefore $x \in D(L^\alpha L^\beta)$ and $L^\alpha L^\beta x = L^{\alpha+\beta}x$.

Letting $|\alpha| = |\beta| = 1$ in the previous lemma, we find:

$$L_iL_jx = L_jL_ix, \quad x \in D(L_iL_j) \cap D(L_jL_i). \tag{5.2.3}$$

Making use of (5.1.3), Lemma 5.2.2, and (5.2.3) we obtain the following characterization of the Sobolev spaces when $\mathcal{L}$ is a commutative family (compare with Proposition 1.3.3 (i)).

PROPOSITION 5.2.3. If $\mathcal{L}$ is a commutative family, then:

(i) $W^m(X;\mathcal{L}) = \bigcap \{ D(L^\alpha) : |\alpha| \le m \}, \quad m = 0, 1, \ldots .$

(ii) $W^{+\infty}(X;\mathcal{L}) = \bigcap \{D(L^\alpha) : |\alpha| = 0, 1, \ldots \}.$

Utilizing the notation of Section 1.3, we have

$$W^m(X;\mathcal{L}) = V(X;\mathcal{A}_m), \tag{5.2.4}$$

where

$$\mathcal{A}_m = \{ L^\alpha : |\alpha| \le m \}, \quad m = 0, 1, \ldots .$$

Finally, from Lemma 5.2.2 we see that

$$L^\alpha L^\beta x = L^{\alpha+\beta}x, \quad x \in W^m(X;\mathcal{L}), \tag{5.2.5}$$

where $|\alpha+\beta| \le m$.

EXAMPLE 5.2.1. Consider the P-space described in Example 2.3.5. Then

$\partial = \{\partial/\partial x_1, \ldots, \partial/\partial x_n\}$ is a commutative family in the class $\mathscr{C}_2(\Phi,\Phi)$. Also, it is easy to see that

$$W^m(L^p_{loc}(\Omega);\partial) = W^{m,p}_{loc}(\Omega).$$

6. The Banach-Sobolev Spaces.

In this section we continue the study of the abstract Sobolev spaces, under the aditional hypothesis that the underlying space X is Banach. Such spaces will be called Banach-Sobolev spaces.

6.1 The Spaces $W^{-m}(X;\mathscr{L})$.

Let X be a Banach space with norm $\|\cdot\|$, and let $(X,Y,[\cdot,\cdot])$ be a P-space. Suppose that $\Phi \subseteq X$ and $\Psi \subseteq Y$ are test spaces for Y and X respectively, and let $\mathscr{L} = \{L_1, \ldots, L_n\}$ be a family of operators in the class $\mathscr{C}_2(\Phi,\Psi)$.

Since X is a Banach space, from (1.3.6) we see that the norm on the Banach space $W^m(X;\mathscr{L})$ is given by

$$\|x\|_{\mathscr{L},m} = \max\{\ \|L_\gamma x\| : [\gamma] \le m\ \}, \quad m = 0, 1, \ldots . \tag{6.1.1}$$

According with (5.2.4), when $\mathscr{L}$ is a commutative family, the norm

$$\|x\|_{\mathscr{L},m} = \max\{\ \|L^\alpha x\| : |\alpha| \le m\ \}, \quad m = 0, 1, \ldots , \tag{6.1.2}$$

is equivalent to the norm (6.1.1).

EXAMPLE 6.1.1. Consider the P-space described in Example 2.3.6. Then $\partial = \{\ \partial/\partial x_1, \ldots, \partial/\partial x_n\ \}$ is a family of operators in the class $\mathscr{C}_2(\Phi,\Phi)$, where $\Phi = \Psi = C_c^\infty(\Omega)$. It is easy to check that

$$W^m(L^p(\Omega);\partial) = W^{m,p}(\Omega),$$

where $W^{m,p}(\Omega)$ is the usual Sobolev space of order m. According to

Proposition 5.1.1, $W^{m,p}(\Omega)$ is a Banach space. Also, from Corollary 1.3.2 (iii), $W^{m,p}(\Omega)$ is reflexive when $1 < p < \infty$.

In this case we have $\partial^* = \{ -\partial/\partial x_1, \ldots, -\partial/\partial x_n \}$. Thus, it is clear that

$$W^m(L^{p'}(\Omega);\partial^*) = W^{m,p'}(\Omega).$$

For $m = 0, 1, \ldots$, we define $W^{-m}(X;\mathcal{L})$ as the dual space of $W_0^m(X;\mathcal{L})$. That is,

$$W^{-m}(X;\mathcal{L}) = W_0^m(X;\mathcal{L})',$$

together with the norm

$$\|u\|_{\mathcal{L},-m} = \sup \{ \ |\langle x,u\rangle| \ : \ \|x\|_{\mathcal{L},m} \le 1 \ \}, \quad u \in W^{-m}(X;\mathcal{L}). \tag{6.1.3}$$

It will be convenient to consider also the space

$$W^{-\infty}(X;\mathcal{L}) = (W_0^{+\infty}(X;\mathcal{L}))'_s. \tag{6.1.4}$$

If we identify Y with $J(Y) \subseteq W^{-0}(X:\mathcal{L})$ $(= W_0^0(X;\mathcal{L})')$, from (3.1.2), and (5.1.7), we obtain the following result.

PROPOSITION 6.1.1. For $m = 0, 1, \ldots,$ we have

$$Y \hookrightarrow \ldots \hookrightarrow W^{-m}(X;\mathcal{L}) \hookrightarrow W^{-m-1}(X;\mathcal{L}) \hookrightarrow \ldots \hookrightarrow W^{-\infty}(X;\mathcal{L}).$$

Now, consider the closed linear operators (cf. Lemma 3.1.1) $L_j^* : D(L_j^*) \subseteq Y \longrightarrow Y$, $j = 1, \ldots, n$, and let $\mathcal{L}^* = \{ L_1^*, \ldots, L_n^* \}$. Then, the family of operators $\mathcal{L}^*$ belongs to the class $\mathcal{C}_2(\Psi,\Phi)$, with respect to the adjoint P-space $(Y,X,[\cdot,\cdot]^*)$. If Y together with its negative norm $\|\cdot\|^-$ were a Banach space (this will be the case if $J(Y)$ is a closed subspace of X′), then we have defined the Sobolev spaces $W^m(Y;\mathcal{L}^*)$ and $W_0^m(Y;\mathcal{L}^*)$. So, throughout the rest of this section we shall assume that $J(Y)$ is a closed subspace of X'.

From (1.3.2), (1.3.11) and (3.1.2) we obtain the following

PROPOSITION 6.1.2. If $J(Y)$ is a closed subspace of X', then we have

$$X \hookrightarrow \dots \hookrightarrow W^{-m}(Y;\mathcal{L}^*) \hookrightarrow W^{-m-1}(Y;\mathcal{L}^*) \hookrightarrow \dots \hookrightarrow \Psi(X;\mathcal{L}^*)'_s.$$

From (5.1.7) and the previous proposition, we see that we have constructed a family of Banach spaces with continuous inclusions

$$\begin{aligned} \dots \hookrightarrow W_0^{m+1}(X;\mathcal{L}) \hookrightarrow W_0^{m}(X;\mathcal{L}) \hookrightarrow \dots \hookrightarrow X \\ X \hookrightarrow \dots \hookrightarrow W^{-m}(Y;\mathcal{L}^*) \hookrightarrow W^{-m-1}(Y;\mathcal{L}^*) \hookrightarrow \dots . \end{aligned} \tag{6.1.5}$$

If X is a reflexive space, then from Lemma 1.1.1 it follows that Φ is dense in each of the spaces appearing in (6.1.5).

Let us consider a linear operator on X of the form

$$L = \sum_{[\gamma]\le m} a_\gamma L_\gamma, \qquad a_\gamma \in \mathbb{K}, \quad [\gamma] \le m. \tag{6.1.6}$$

An advantage of having at one's dispossal a family of spaces given by (6.1.5), is that this will allow us to interpret L in different ways, as the following result shows.

PROPOSITION 6.1.3. Suppose that $J(Y)$ is a closed subspace of X'. If L is as in (6.1.6), and $\mu \in \{0, 1, \dots, m\}$, $m = 0, 1, \dots$, then

$$L : W_0^{\mu}(X;\mathcal{L}) \longrightarrow W^{-m+\mu}(Y;\mathcal{L}^*),$$

is a continuous linear operator.

PROOF. Let $\phi \in \Phi$. Then, for every $\psi \in \Psi$, we have in the sense of distributions

$$\overline{\langle\psi, L\phi\rangle} = [L\phi, \psi] = [\sum a_\gamma L_\gamma \phi, \psi].$$

Using the adjoint, we can "move to the right" some of the operators L_j appearing in the compositions defining L_γ, obtaining a expression of the form

$$\sum a_\gamma [L_{\gamma_1} \phi, L^*_{\gamma_2} \psi],$$

where $[\gamma_1] \leq \mu$ and $[\gamma_2] \leq m-\mu$. Thus we have

$$|\langle\psi, L\phi\rangle| \leq NC\|\phi\|_{\mathcal{L},\mu}\|\psi\|_{\mathcal{L}^*,m-\mu}, \tag{6.1.7}$$

where $C = \max\{ |a_\gamma| : [\gamma] \leq m \}$ and N is the number of subindices γ such that $[\gamma] \leq m$. The density of Ψ in $W_0^{m-\mu}(Y;\mathcal{L}^*)$, together with (6.1.7) show that we can extend $L\phi$ to all of $W_0^{m-\mu}(Y;\mathcal{L}^*)$ in a unique way. Furthermore, we have

$$\|L\phi\|_{\mathcal{L}^*,-m+\mu} \leq CN\|\phi\|_{\mathcal{L},m}. \tag{6.1.8}$$

We obtain in this way, a continuous linear transformation

$$L : \Phi \longrightarrow W^{-m+\mu}(Y;\mathcal{L}^*).$$

Finally, the density of Φ in $W_0^{\mu}(X;\mathcal{L})$ allow us to extend L in a unique and continuos fashion to all of $W_0^{\mu}(X;\mathcal{L})$.

For an operator of the form (6.1.7), the last result makes evident the importance of the sesquilinear form associated to L:

$$(\phi,\psi) \longrightarrow [L\phi,\psi].$$

When dealing with differential operators, this is called the Dirichlet form. The point here, is to study this form and to find conditions which allow us to make statements about the operators described in the previous proposition. The next result is a simple example.

PROPOSITION 6.1.4. Let $m \geq 0$, and $\mu \in \{0, 1, \ldots, m\}$. Suppose that there is a constant $C > 0$, such that for every $\phi \in \Phi$, $\phi \neq 0$, there exists $\psi \in \Psi$, $\psi \neq 0$, with

$$|[L\phi,\psi]| \geq C\|\phi\|_{\mathcal{L},m} \|\psi\|_{\mathcal{L}^*,m-\mu}. \tag{6.1.9}$$

Then, the operator

$$L : W_0^{\mu}(X;\mathcal{L}) \longrightarrow W^{-m+\mu}(Y;\mathcal{L}^*)$$

is a linear isomorphism onto its image.

PROOF. From (6.1.3) and inequality (6.1.9) we obtain

$$\|L\phi\|_{\mathcal{L}^*,-m+\mu} \geq C\|\phi\|_{\mathcal{L},\mu}, \qquad \phi \in \Phi.$$

Since L is continuous, the density of Φ in $W_0^{\mu}(X;\mathcal{L})$ and the last inequality imply that L^{-1} is continuous.

6.2 A Characterization of $W^{-m}(X;\mathcal{L})$.

Let X be a Banach space, and $(X,Y,[\cdot,\cdot])$ be a P-space. Throughout this section we shall assume that $J(Y) = X'$. This, in particular, implies that Y together with its negative norm $\|\cdot\|^-$ is a Banach space, and that the negative norm on X induced by Y with respect to the adjoint $(Y,X,[\cdot,\cdot]^*)$ pairing, coincides with the original norm $\|\cdot\|$ on X. Thus, if the family $\mathcal{L} = \{ L_1, \ldots, L_n \}$ belongs to $\mathcal{E}_2(\Phi,\Psi)$, then the topology of the test space $\Phi(Y;\mathcal{L}^*)$ is simply the topology induced as a subspace of $W_0^{+\infty}(X;\mathcal{L})$.

Given $m \in \mathbb{N}$, we define the subspace $W^{-m}(X;\mathcal{L})$ of the space of distributions $\Phi' = \Phi(Y;\mathcal{L}^*)'$ as follows: $W^{-m}(X;\mathcal{L})$ consists of all $v \in \Phi'$ that can be written in the form

$$v = \sum_{[\gamma]\leq m} L_\gamma^* z_\gamma, \qquad z_\gamma \in Y, \tag{6.2.1}$$

where $L_\gamma^* z_\gamma$ is defined in the sense of distributions (cf. (4.2.8)). In $W^{-m}(X;\mathcal{L})$ we consider

$$\|v\|_{W^{-m}} = \inf \sum_{[\gamma]\le m} \|z_\gamma\|^-, \qquad (6.2.2)$$

where the infimum is taken over all collections $\{z_\gamma\}$ satisfying (6.2.1).

Clearly, $\|\cdot\|_{W^{-m}}$ is a seminorm on $W^{-m}(X;\mathcal{L})$. For $v \in W^{-m}(X;\mathcal{L})$, we have

$$<\phi, v> = <\phi, \sum_{[\gamma]\le m} L^*_\gamma z_\gamma> = \sum_{[\gamma]\le m} [L_{\gamma_{op}} \phi, z_\gamma], \qquad \phi \in \Phi.$$

Thus,

$$|<\phi, v>| \le \|\phi\|_{\mathcal{L},m} \sum_{[\gamma]\le m} \|z_\gamma\|^-,$$

and hence

$$|<\phi, v>| \le \|\phi\|_{\mathcal{L},m} \|v\|_{W^{-m}}, \qquad \phi \in \Phi.$$

This inequality implies that $\|\cdot\|_{W^{-m}}$ is a norm on $W^{-m}(X;\mathcal{L})$. Also, from this it follows we can extend v in a unique way to a continuous linear functional on $W_0^m(X;\mathcal{L})$, which we denote by $\tilde{v}$. Furthermore, it is clear that

$$\|\tilde{v}\|_{\mathcal{L},-m} \le \|v\|_{W^{-m}}. \qquad (6.2.3)$$

Now, let $\tilde{v} \in W^{-m}(X;\mathcal{L})$, and let v denote its restriction to Φ. Let N be the number of subindices γ with $[\gamma] \le m$, and consider the natural embedding

$$i : W_0^m(X;\mathcal{L}) \longrightarrow X^{(N)},$$

given by $i(x) = (L_\gamma x;\ [\gamma]\le m)$. Since the linear functional v' defined on $i(W_0^m(X;\mathcal{L})) \subseteq X^{(N)}$ by

$$<w, v'> = <i^{-1}w, v>,$$

is continuous, from Hahn-Banach theorem, we can extend v' to all of $X^{(N)}$ in a continuous way. Now, being the dual of $X^{(N)}$ a direct sum of the duals of X, it follows that for every subindex γ with $[\gamma] \le m$, there is a $z_\gamma \in Y$ such that

$$\langle \phi, v \rangle = \sum_{[\gamma] \le m} [L_\gamma \phi, z_\gamma], \qquad \phi \in \Phi, \tag{6.2.4}$$

and

$$\| v \|_{\mathcal{L}, -m} = \sum_{[\gamma] \le m} \| z_\gamma \|^- . \tag{6.2.5}$$

From (6.2.4) we have

$$v = \sum_{[\gamma] \le m} L^*_{\gamma_{op}} z_\gamma , \tag{6.2.6}$$

and from (6.2.5)

$$\| v \|_{\mathcal{W}^{-m}} \le \| \tilde{v} \|_{\mathcal{L}, -m} . \tag{6.2.7}$$

In this way, from (6.2.3) and (6.2.7), we obtain the following

PROPOSITION 6.2.1. If $J(Y) = X'$, then $\| \cdot \|_{\mathcal{W}^{-m}}$ is a norm on $\mathcal{W}^{-m}(X;\mathcal{L})$, and under this norm, the linear correspondence

$$\mathcal{W}^{-m}(X;\mathcal{L}) \ni v \longleftrightarrow \tilde{v} \in W^{-m}(X;\mathcal{L}),$$

is an isometric isomorphism.

6.3 The Reflexive Case.

In this section we are going to study some questions related to the Banach-Sobolev spaces $W^m_0(X;\mathcal{L})$, when the base space X is reflexive.

PROPOSITION 6.3.1. $W^{-m}(X;\mathcal{L})$ can be identified with the completion of Ψ with respect to the (negative) norm

$$\| \psi \|_{\mathcal{L}, -m} = \sup \{ |[x, \psi]| : \| x \|_{\mathcal{L}, m} \le 1 \}, \qquad m = 0, 1, \ldots . \tag{6.3.1}$$

PROOF. Given $\psi \in \Psi$, we define $J_m \psi : W^m_0(X;\mathcal{L}) \longrightarrow \mathbb{K}$ by

$$\langle x, J_m \psi \rangle = [x, \psi].$$

If we take into account that $J_m(\Psi) \subseteq W^{-m}(X;\mathcal{L})$ is total, and that $W^m_0(X;\mathcal{L})$ reflexive, then from Lemma 1.1.1 it follows that $J_m(\Psi)$ is dense in $W^{-m}(X;\mathcal{L})$. Finally, from (6.1.3) and (6.3.1) it is clear that

$J_m : \Psi \longrightarrow J_m(\Psi) \subseteq W^{-m}(X;\mathcal{L})$, is a linear isometry.

PROPOSITION 6.3.2. Suppose that $J(Y) = X'$. Let L a linear operator as in (6.1.6) amd $\mu \in \{ 0, 1, \ldots, m \}$. Assume that there are positive constants C_1 and C_2, such that

For every $\phi \in \Phi$, $\phi \neq 0$, there exists $\psi \in \Psi$, $\psi \neq 0$, with

$$|[L\phi,\psi]| \geq C_1 \|\phi\|_{\mathcal{L},\mu} \|\psi\|_{\mathcal{L}^*,m-\mu}. \tag{6.3.2}$$

For every $\psi \in \Psi$, $\psi \neq 0$, there exists $\phi \in \Phi$, $\phi \neq 0$, with

$$|[L\phi,\psi]| \geq C_2 \|\phi\|_{\mathcal{L},\mu} \|\psi\|_{\mathcal{L}^*,m-\mu}. \tag{6.3.3}$$

Then,

$$L : W_0^{\mu}(X;\mathcal{L}) \longrightarrow W^{-m+\mu}(Y;\mathcal{L}^*)$$

is a linear isomorphism.

PROOF. In view of Proposition 6.1.4, it is enough to show that L is onto. Now, applying (6.3.3) and Proposition 6.1.3 to the operator

$$L^* : W_0^{m-\mu}(Y;\mathcal{L}^*) \longrightarrow W^{-\mu}(X;\mathcal{L}),$$

we obtain

$$\|L^*\psi\|_{\mathcal{L},-\mu} \|\phi\|_{\mathcal{L},\mu} \geq |[\phi, L^*\psi]| \geq C_2 \|\phi\|_{\mathcal{L},\mu} \|\psi\|_{\mathcal{L}^*,m-\mu}.$$

Thus

$$\|L^*\psi\|_{\mathcal{L},-\mu} \geq C_2 \|\psi\|_{\mathcal{L}^*,m-\mu}, \qquad \psi \in \Psi.$$

Since Ψ is dense in $W_0^{m-\mu}(Y;\mathcal{L}^*)$, from Proposition 6.1.3, we obtain

$$\|L^* y\|_{\mathcal{L},-\mu} \geq C_2 \|y\|_{\mathcal{L}^*,m-\mu}, \qquad y \in W_0^{m-\mu}(Y;\mathcal{L}^*). \tag{6.3.4}$$

Being X a reflexive space, we can identify the operator

$$L^* : W_0^{m-\mu}(Y;\mathcal{L}^*) \longrightarrow W_0^{\mu}(X;\mathcal{L}),$$

with the dual operator of

$$L : W_0^{\mu}(X;\mathcal{L}) \longrightarrow W^{-m+\mu}(Y;\mathcal{L}^*)$$

Thus, from (6.3.4) the dual of L has a continuous inverse. Therefore, L is onto (Rudin p.97).

COROLLARY 6.3.3. Under the same hypothesis as above we have:

(i) For every $b \in X$, there exists a unique $x \in W_0^{\mu}(X;\mathcal{L})$ such that $Lx = b$.

(ii) $L^{-1} : X \longrightarrow W_0^{\mu}(X;\mathcal{L})$ is a continuous linear operator.

PROOF. (i) From Proposition 6.1.2, we have $X \hookrightarrow W^{-m+\mu}(Y;\mathcal{L}^*)$, and the result follows from the previous proposition.

(ii) Since X and $W_0^{\mu}(X;\mathcal{L})$ are Banach spaces, it is sufficient to prove that L^{-1} is closed. So let $\{x_k\}$ be a sequence in X, and x, u in X such that

$$x_k \longrightarrow x \text{ and } L^{-1}x_k \longrightarrow u. \tag{6.3.5}$$

From the definition of L_w we have

$$\overline{[x_k,\psi]} = \langle\psi, LL^{-1}x_k\rangle = \overline{[L^{-1}x_k, L^*\psi]}, \qquad \psi \in \Psi.$$

Letting $k \longrightarrow \infty$, from (3.2.4) and (6.3.5) we obtain

$$[x,\psi] = [u, L^*\psi], \qquad \psi \in \Psi.$$

From this it follows that $x = Lu$, that is, $L^{-1}x = u$.

CHAPTER IV: THE HILBERT-SOBOLEV SPACES.

In this chapter we are interested in the study of the Sobolev spaces $W^m(H;\mathcal{L})$ and $W_0^m(H;\mathcal{L})$, in the very important case, when the P-space is given by a Hilbert space H (e.g., Example 2.3.1). Here, we assume that $\Phi = \Psi$, where Φ and Ψ are the test spaces related with the family of linear operators $\mathcal{L}$. Under these circumstances, the spaces $W^m(H;\mathcal{L})$ and $W_0^m(H;\mathcal{L})$ will be called *Hilbert-Sobolev spaces.*

7. The Gradient, Divergence, and Laplace Operators.

7.1 $grad_{\mathcal{L}}$, $div_{\mathcal{L}}$, and $\Delta_{\mathcal{L}}$.

We start this section, by pointing out the form some of the previous results take in the present context.

Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$. If $A \subseteq H$, we denote the orthogonal complement of A in H by $H \ominus A$.

According with Example 2.3.1, from now on we treat the Hilbert space H as a P-space. From the properties of the inner product, it is clear that the P-space H coincides with its adjoint P-space. Also, in this case, the canonical identification of H in H' is given by Riesz canonical identification $I : H \longrightarrow H'$,i.e.,

$$\langle x, Iy\rangle = (x,y), \qquad x,\ y \in H.$$

Since I is an isometric isomophism, the negative norm on H coincides with the original norm.

Given $n \in \mathbb{N}$, the inner product on

$$H^{(n)} = H \times \dots \times H \qquad (n \text{ times}),$$

and its corresponding norm, will also be denoted by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. If $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n) \in H^{(n)}$, then we have

$$(\vec{x},\vec{y}) = \sum_{j=1}^{n} (x_j, y_j), \tag{7.1.1}$$

and

$$\|\vec{x}\| = \left\{ \sum_{j=1}^{n} \|x_j\|^2 \right\}^{1/2}. \tag{7.1.2}$$

Let Φ be a dense subspace of H, and consider a family of linear operators $\mathcal{L} = \{ L_1, \dots, L_n \}$ on H, such that

$$L_j \in \mathcal{C}_2(\Phi,\Phi), \qquad j = 1, \dots, n. \tag{7.1.3}$$

Then, we have defined the Hilbert-Sobolev spaces

$$W^m(H;\mathcal{L}) \text{ and } W_0^m(H;\mathcal{L}), \qquad m = 0, 1, \dots . \tag{7.1.4}$$

From (1.3.7), (1.3.8) and Corollary 1.3.4, $W^m(H;\mathcal{L})$ and $W_0^m(H;\mathcal{L})$ are Hilbert spaces under the inner product

$$(u,v)_{\mathcal{L},m} = \sum_{[\gamma]\le m} (L_\gamma u, L_\gamma v), \tag{7.1.5}$$

and corresponding norm

$$\|u\|_{\mathcal{L},m} = \left\{ \sum_{[\gamma]\le m} \|L_\gamma u\|^2 \right\}^{1/2}. \tag{7.1.6}$$

Finally, let us recall that $\|\cdot\|_{\mathcal{L},-m}$ denotes the norm on

$$W^{-m}(H;\mathcal{L}) = W_0^m(H;\mathcal{L})', \qquad m = 1, 2, \dots . \tag{7.1.7}$$

Observe that condition (7.1.3) implies that

$$L_j^* \in \mathcal{C}_2(\Phi,\Phi), \qquad j = 1, \dots, n. \tag{7.1.8}$$

Thus, all the previous concepts are also defined for the family of operators $\mathcal{L}^* = \{ L_1^*, \dots, L_n^* \}$.

PROOF. Let us start by noting that according with (4.2.9), we have

$$<\phi, L_j^* x> = (L_j \phi, x), \qquad \phi \in \Phi, \; x \in H. \tag{7.1.15}$$

Since $L_j^* x$ is continuous on Φ with respect to the norm $\|\cdot\|_{\mathcal{L},1}$, and since Φ is dense in $W_0^1(H;\mathcal{L})$, then there is a unique continuous extension of $L_j^* x$ to the whole of $W_0^1(H;\mathcal{L})$. Such an extension will be identified with $L_j^* x$.

(i) For $\vec{x} \in H^{(n)}$ and $u \in W_0^1(H;\mathcal{L})$, we have from (7.1.15)

$$<u, \, div_{\mathcal{L}}\vec{x}> = (grad_{\mathcal{L}} u, \vec{x}) = \sum_{j=1}^{n} (L_j u, x_j)$$

$$= \sum_{j=1}^{n} <u, L_j^* x_j> = <u, \sum_{j=1}^{n} L_j^* x_j>.$$

(ii) It is an immediate consequence of (i) and the definition of $\Delta_{\mathcal{L}}$.

7.2 Strong Divergence and Laplacian.

Let $\vec{v} \in H^{(n)}$. We will write $div_{\mathcal{L}}\vec{v} \in H$, if there exists $w \in H$ such that

$$(grad_{\mathcal{L}} u, \vec{v}) = <u, div_{\mathcal{L}}\vec{v}> = (u, w), \qquad u \in W_0^1(H;\mathcal{L}). \tag{7.2.1}$$

Since $W_0^1(H;\mathcal{L})$ is dense in H, such a w is unique. Thus, to express that condition (7.2.1) holds, we write $div_{\mathcal{L}}\vec{v} = w$, and we say that *$div_{\mathcal{L}}$ is strong on $\vec{v}$.* Next, we define

$$\mathcal{E} = \{ \; \vec{v} \in H^{(n)} \; : \; div_{\mathcal{L}}\vec{v} \in H \; \}. \tag{7.2.2}$$

Then, $\mathcal{E}$ is a vector subspace of $H^{(n)}$. We consider on $\mathcal{E}$ the inner product

$$(\vec{u}, \vec{v})_{\mathcal{E}} = (\vec{u}, \vec{v}) + (div_{\mathcal{L}}\vec{u}, div_{\mathcal{L}}\vec{v}), \tag{7.2.3}$$

and its associated norm

$$\|\vec{u}\|_{\mathcal{E}} = (\|\vec{u}\|^2 + \|div_{\mathcal{L}}\vec{u}\|^2)^{1/2}. \tag{7.2.4}$$

PROPOSITION 7.2.1. $\mathcal{E}$ is a Hilbert space.

PROOF. First, let us note that $\mathcal{E} = V(H^{(n)}, H; div_{\mathcal{L}})$ is the space induced on $H^{(n)}$ by the linear operator $div_{\mathcal{L}} : \mathcal{E} \subseteq H^{(n)} \longrightarrow H$ (cf. Remark 1.2.1). Since H and $H^{(n)}$ are complete, from Proposition 1.2.2 and Remark 1.2.1, we see that only rests to show that $div_{\mathcal{L}}$ is closed. So, let us consider a sequence $\{\vec{v}_k\}$ in $\mathcal{E}$, such that

$$\vec{v}_k \longrightarrow \vec{v} \text{ in } H^{(n)} \text{ and } div_{\mathcal{L}}\vec{v}_k \longrightarrow w \text{ in } H. \tag{7.2.5}$$

Let $u \in W_0^1(H;\mathcal{L})$. From (7.2.1) we have

$$(grad_{\mathcal{L}}u, \vec{v}_k) = (u, div_{\mathcal{L}}\vec{v}_k).$$

If we let $k \longrightarrow \infty$, and take into account (7.2.5) we obtain

$$(grad_{\mathcal{L}}u, \vec{v}) = (u, w).$$

Therefore, $\vec{v} \in \mathcal{E}$ and $div_{\mathcal{L}}\vec{v} = w$.

LEMMA 7.2.2. We have

$$D(L_1^*) \times \dots \times D(L_n^*) \subseteq \mathcal{E}.$$

Also, if $\vec{v} = (v_1, \dots, v_n) \in D(L_1^*) \times \dots \times D(L_n^*)$, then

$$div_{\mathcal{L}}\vec{v} = L_1^*v_1 + \dots + L_n^*v_n. \tag{7.2.6}$$

PROOF. Let $\phi \in \Phi$. If $\vec{v} = (v_1, \dots, v_n) \in D(L_1^*) \times \dots \times D(L_n^*)$, then

$$\begin{aligned} \langle\phi, div_{\mathcal{L}}\vec{v}\rangle = (grad_{\mathcal{L}}\phi, \vec{v}) &= \sum_{j=1}^{n} (L_j\phi, v_j) \\ &= \sum_{j=1}^{n} (\phi, L_j^*v_j) = (\phi, \sum_{J=1}^{n} L_j^*v_j). \end{aligned}$$

Since Φ is dense in $W_0^1(H;\mathcal{L})$ and $W_0^1(H;\mathcal{L}) \hookrightarrow H$, we obtain the desired result.

COROLLARY 7.2.3. We have

$$[W^1(H;\mathcal{L}^*)]^{(n)} \subseteq \mathcal{E}.$$

Let $\mathcal{E}_0$ be the closure of $\Phi^{(n)}$ in $\mathcal{E}$. Then, it is clear that $\mathcal{E}_0$ is also a Hilbert space.

PROPOSITION 7.2.4. We have

$$[W_0^1(H;\mathcal{L}^*)]^{(n)} \subseteq \mathcal{E}_0.$$

Also, if $\vec{v} = (v_1, \ldots, v_n) \in [W_0^1(H;\mathcal{L}^*)]^{(n)}$, then

$$div_{\mathcal{L}}\vec{v} = L_1^* v_1 + \ldots + L_n^* v_n.$$

PROOF. The second assertion follows from Lemma 7.2.2 and Corollary 7.2.3.

Now, let $\vec{v} = (v_1, \ldots, v_n) \in [W_0^1(H;\mathcal{L}^*)]^{(n)}$. Then, there is a sequence $\{\vec{\phi}_k\}$ in $\Phi^{(n)}$, $\vec{\phi}_k = (\phi_{k1}, \ldots, \phi_{kn})$, such that

$$\vec{\phi}_k \longrightarrow \vec{v} \text{ and } L_j^*(\phi_{k\ell}) \longrightarrow L_j^*(v_\ell);\ j,\ \ell = 1, \ldots, n, \tag{7.2.7}$$

as $k \longrightarrow \infty$. From Lemma 7.2.2, and (7.2.7) we see that $\vec{\phi}_k \longrightarrow \vec{v}$ in $\mathcal{E}$.

Just as $div_{\mathcal{L}}$ is orginally defined in a weakly manner, then so it is the operator $\Delta_{\mathcal{L}}$. Let $v \in W^1(H;\mathcal{L})$. We will write $\Delta_{\mathcal{L}}v \in H$, if there exists $w \in H$, such that

$$\langle u, \Delta_{\mathcal{L}}v\rangle = (u,w), \qquad u \in W_0^1(H;\mathcal{L}). \tag{7.2.8}$$

From (7.1.14), this is equivalent to the condition

$$(grad_{\mathcal{L}}u, grad_{\mathcal{L}}v) = (u,w), \qquad u \in W_0^1(H;\mathcal{L}). \tag{7.2.9}$$

Thus, to express that condition (7.2.8) holds, we write $\Delta_{\mathcal{L}}v = w$, and we say that $\Delta_{\mathcal{L}}$ is *strong* on v.

Next, we give conditions under which the laplacian $\Delta_{\mathcal{L}}$ assumes the strong form $L_1^*L_1 + \ldots + L_n^*L_n$. For this, we impose a new condition on the family $\mathcal{L} = \{L_1, \ldots, L_n\}$.

We say that the operator L_j^* is L_j-*bounded*, if

$$L_j^* : \Phi \subseteq W_0^1(H;L_j) \longrightarrow H, \qquad (7.2.10)$$

is continuous.

LEMMA 7.2.5. If L_j^* is L_j-bounded, $j = 1, \dots, n$, then

$$W_0^1(H;\mathcal{L}) \subseteq W_0^1(H;\mathcal{L}^*).$$

PROOF. Let $u \in W_0^1(H;\mathcal{L})$. Then, there is a sequence $\{\phi_k\} \subseteq \Phi$, such that

$$\phi_k \longrightarrow u \text{ and } L_j\phi_k \longrightarrow L_ju, \quad j = 1, \dots, n.$$

Since L_j^* is L_j-bounded, $\{L_j^*\phi_k\}$ is a Cauchy sequence in H. Thus, there is a w_j such that

$$L_j^*\phi_k \longrightarrow w_j \text{ in } H, \quad k \longrightarrow \infty.$$

Being each $L_j^* : D(L_j^*) \subseteq H \longrightarrow H$ a closed linear operator, from all the previous we conclude that $u \in W_0^1(H;\mathcal{L}^*)$ and $L_j^*u = w_j$.

PROPOSITION 7.2.6. If L_j^* is L_j-bounded, $j = 1, \dots, n$, then, in the strong sense we have

$$\Delta_{\mathcal{L}}u = L_1^*L_1u + \dots + L_n^*L_nu, \quad u \in W_0^2(H;\mathcal{L}),$$

PROOF. Let $u \in W_0^2(H;\mathcal{L})$. Then, $grad_{\mathcal{L}}u \in [W_0^1(H;\mathcal{L})]^{(n)}$, and the result follows from the previous lemma and Proposition 7.2.4.

8. The Dirichlet Problem.

Let H be a Hilbert space, Φ a dense subspace of H, and $\mathcal{L}$ a family of n linear operators on H in the class $\mathcal{C}_2(\Phi,\Phi)$. By the "Dirichlet problem", we understand to find a solution in $W_0^1(H;\mathcal{L})$ of an equation of the form $Lu = b$, where L is a linear operator on H and $b \in H$.

8.1 The Dirichlet Problem for $\Delta_{\mathcal{L}} - \lambda$, $\lambda < 0$.

PROPOSITION 8.1.1. Let $\lambda < 0$. Then, given $b \in W^1(H;\mathcal{L})$, there is a unique $u \in W^1(H;\mathcal{L})$ such that

$$(\Delta_{\mathcal{L}} - \lambda)u = 0 \text{ and } u - b \in W_0^1(H;\mathcal{L}). \tag{8.1.1}$$

PROOF. On $W^1(H;\mathcal{L})$, let us consider the inner product

$$(u,v)_\lambda = (grad_{\mathcal{L}}u, grad_{\mathcal{L}}v) - \lambda(u,v). \tag{8.1.2}$$

Then, its corresponding norm is equivalent to the norm $\|\cdot\|_{\mathcal{L},1}$. Hence, $W^1(H;\mathcal{L})$ together with the inner product $(\cdot,\cdot)_\lambda$, is a Hilbert space.

For $u \in W^1(H;\mathcal{L})$ and $v \in W_0^1(H;\mathcal{L})$ we have

$$\langle v, (\Delta_{\mathcal{L}} - \lambda)u\rangle = (u,v)_\lambda . \tag{8.1.3}$$

According with definition (8.1.2), from (7.1.14) we see that condition (8.1.1) is equivalent to find $u \in b + W_0^1(H;\mathcal{L})$, orthogonal to $W_0^1(H;\mathcal{L})$ with respect to the inner product $(\cdot,\cdot)_\lambda$. Thus, if we let $b = u - u_0$, where u is orthogonal (with respect to the inner product $(\cdot,\cdot)_\lambda$) to $W_0^1(H;\mathcal{L})$ and $u_0 \in W_0^1(H;\mathcal{L})$, we obtain the desired result.

From (8.1.3) we obtain immediately the following

PROPOSITION 8.1.2. Let $\lambda < 0$. Then, the Riesz canonical representation for the dual of $W_0^1(H;\mathcal{L})$ with respect to the inner product (8.1.2) is given by

$$\Delta_{\mathcal{L}} - \lambda : W_0^1(H;\mathcal{L}) \longrightarrow W^{-1}(H;\mathcal{L}).$$

In particular, for every $b \in W^{-1}(H;\mathcal{L})$ there is a unique $u \in W_0^1(H;\mathcal{L})$ such that $(\Delta_{\mathcal{L}} - \lambda)u = b$.

8.2 Adjoints of $grad_{\mathcal{L}}$, $div_{\mathcal{L}}$ and $\Delta_{\mathcal{L}}$.

PROPOSITION 8.2.1. The adjoint of $grad_{\mathcal{L}} : W_0^1(H;\mathcal{L}) \longrightarrow H^{(n)}$ is given by

$$grad_{\mathcal{L}}^* = (1 + \Delta_{\mathcal{L}})^{-1} div_{\mathcal{L}} . \qquad (8.2.1)$$

PROOF. Let $\vec{x} \in H^{(n)}$ and $u \in W_0^1(H;\mathcal{L})$. Since $div_{\mathcal{L}}\vec{x} \in W^{-1}(H;\mathcal{L})$, from Proposition 8.1.2, we see that there is a unique $w \in W_0^1(H;\mathcal{L})$ such that $div_{\mathcal{L}}\vec{x} = (1 + \Delta_{\mathcal{L}})w$. Thus,

$$(grad_{\mathcal{L}}u, \vec{x}) = \langle u, div_{\mathcal{L}}\vec{x}\rangle = \langle u, (1 + \Delta_{\mathcal{L}})w\rangle = (u,w)_{\mathcal{L},1} .$$

Since $w = (1 + \Delta_{\mathcal{L}})^{-1} div_{\mathcal{L}}\vec{x}$, we obtain the desired result.

PROPOSITION 8.2.2. The adjoint of $div_{\mathcal{L}} : H^{(n)} \longrightarrow W^{-1}(H;\mathcal{L})$ is given by

$$div_{\mathcal{L}}^* = grad_{\mathcal{L}}(1 + \Delta_{\mathcal{L}})^{-1}. \qquad (8.2.2)$$

PROOF. From (8.2.1), and the fact that $1 + \Delta_{\mathcal{L}}$ is a unitary operator (Proposition 8.1.2), we obtain

$$\begin{aligned} grad_{\mathcal{L}} = grad_{\mathcal{L}}^{**} &= [(1 + \Delta_{\mathcal{L}})^{-1} div_{\mathcal{L}}]^* = div_{\mathcal{L}}^*[(1 + \Delta_{\mathcal{L}})^{-1}]^* \\ &= div_{\mathcal{L}}^*[(1 + \Delta_{\mathcal{L}})^*]^{-1} = div_{\mathcal{L}}^*(1 + \Delta_{\mathcal{L}}). \end{aligned}$$

PROPOSITION 8.2.3. The adjoint of $\Delta_{\mathcal{L}} : W_0^1(H;\mathcal{L}) \longrightarrow W^{-1}(H;\mathcal{L})$ is given by

$$\Delta_{\mathcal{L}}^* = (1 + \Delta_{\mathcal{L}})^{-1}\Delta_{\mathcal{L}}(1 + \Delta_{\mathcal{L}})^{-1}. \qquad (8.2.3)$$

PROOF. Since $\Delta_{\mathcal{L}} = div_{\mathcal{L}} \circ grad_{\mathcal{L}}$, from (8.2.1) and (8.2.2) we get

$$\begin{aligned} \Delta_{\mathcal{L}}^* &= (div_{\mathcal{L}} \circ grad_{\mathcal{L}})^* = grad_{\mathcal{L}}^* \circ div_{\mathcal{L}}^* \\ &= (1 + \Delta_{\mathcal{L}})^{-1} \circ div_{\mathcal{L}} \circ grad_{\mathcal{L}} \circ (1 + \Delta_{\mathcal{L}})^{-1} = (1 + \Delta_{\mathcal{L}})^{-1}\Delta_{\mathcal{L}}(1 + \Delta_{\mathcal{L}})^{-1}. \end{aligned}$$

8.3 The Dirichlet Problem for $\Delta_{\mathcal{L}}$, and Friedrichs' Inequality.

Given $b \in H$, we are interested in the problem of finding a solution $v \in W_0^1(H;\mathcal{L})$ of the equation $\Delta_{\mathcal{L}}v = b$. That is, we are looking for a $v \in W_0^1(H;\mathcal{L})$ satisfying

$$(grad_{\mathcal{L}}u, grad_{\mathcal{L}}v) = (u,b), \text{ for all } u \in W_0^1(H;\mathcal{L}). \tag{8.3.1}$$

For this, let us define

$$(u,v)_E = (grad_{\mathcal{L}}u, grad_{\mathcal{L}}v), \quad u,\ v \in W_0^1(H;\mathcal{L}). \tag{8.3.2}$$

Then, $(\cdot,\cdot)_E$ satisfies all the conditons for an inner product, except that is not positive definite, but only positive semidefinite.

Given $b \in H$, consider the linear functional $Ib : W_0^1(H;\mathcal{L}) \longrightarrow \mathbb{K}$, given by

$$<u, Ib> = (u,b), \quad u \in W_0^1(H;\mathcal{L}). \tag{8.3.3}$$

From Schwarz inequality we have

$$\|Ib\|_{\mathcal{L},-1} \le \|b\|. \tag{8.3.4}$$

Note that we can solve equation (8.3.1), if we can represent the linear functional Ib by means the "inner product" (8.3.2). Now, since $Ib \in W^{-1}(H;\mathcal{L})$, this will be possible if the norms associated with the inner products $(\cdot,\cdot)_E$ and $(\cdot,\cdot)_{\mathcal{L},1}$ are equivalent on $W_0^1(H;\mathcal{L})$. This naturally lead us to the following concept.

We say that the family $\mathcal{L}$ satisfies *Friedrichs' inequality*, if there is a positive constant C such that

$$(\phi,\phi) \le C^2(grad_{\mathcal{L}}\phi, grad_{\mathcal{L}}\phi), \quad \phi \in \Phi. \tag{8.3.5}$$

Using the fact that Φ is dense in $W_0^1(H;\mathcal{L})$, we see that (8.3.5) holds for all $u \in W_0^1(H;\mathcal{L})$. Thus, $(\cdot,\cdot)_E$ is an inner product on $W_0^1(H;\mathcal{L})$ whose norm $\|\cdot\|_E$ is equivalent to the norm $\|\cdot\|_{\mathcal{L},1}$. From this we obtain the following

PROPOSITION 8.3.1. If $\mathcal{L}$ satisfies Friedrichs' inequality, then

$$\Delta_{\mathcal{L}} : W_0^1(H;\mathcal{L}) \longrightarrow W^{-1}(H;\mathcal{L}),$$

is a linear isomorphism.

COROLLARY 8.3.2. Under the same hypothesis as in the previous proposition we have:

(i) Given $b \in H$, there is a unique $v \in W_0^1(H;\mathcal{L})$ such that $\Delta_{\mathcal{L}} v = b$.

(ii) $\Delta_{\mathcal{L}}^{-1} : H \longrightarrow W_0^1(H;\mathcal{L})$ is continuous.

PROOF. (i) Given $b \in H$, observe that the equality $\Delta_{\mathcal{L}} v = b$ is in the strong sense. Since $Ib \in W^{-1}(H;\mathcal{L})$, the result follows from the previous proposition.

(ii) From (8.3.4) we see that $I : H \longrightarrow W^{-1}(H;\mathcal{L})$ is continuous. Hence,

$$\Delta_{\mathcal{L}}^{-1} \circ I : H \longrightarrow W_0^1(H;\mathcal{L}),$$

is also continuous, where $\Delta_{\mathcal{L}}$ is the weak laplacian (7.1.13).

PROPOSITION 8.3.3. Suppose that $\mathcal{L}$ satisfies Friedrichs' inequality, and let $b \in H$. Then, the solution of the equation $\Delta_{\mathcal{L}} v = b$, is given by the point $u \in W_0^1(H;\mathcal{L})$ where the real functional

$$\lambda(v) = (\mathit{grad}_{\mathcal{L}} v, \mathit{grad}_{\mathcal{L}} v) - (v,b) - (b,v), \quad v \in W_0^1(H;\mathcal{L}),$$

attains its minimum value.

PROOF. Let $u \in W_0^1(H;\mathcal{L})$ be a solution of $\Delta_{\mathcal{L}} u = b$. Define

$$\Lambda(v) = \|v-u\|_E^2 = (v-u, v-u)_E, \quad v \in W_0^1(H;\mathcal{L}). \tag{8.3.6}$$

It is clear that Λ attains its minimum value precisely at u. Now, from (8.3.1) and (8.3.6) we obtain

$$\Lambda(v) = (\mathit{grad}_{\mathcal{L}} v, \mathit{grad}_{\mathcal{L}} v) - (v,b) - (b,v) + (u,u)_E \, .$$

From this the conclusion is clear.

Let $D(\Delta_{\mathcal{L}})$ be the subspace of $W_0^1(H;\mathcal{L})$, which consists of all u such that $\Delta_{\mathcal{L}}u \in H$. If the hypothesis in Proposition 7.2.6 are satisfied, then we have

$$W_0^2(H;\mathcal{L}) \subseteq D(\Delta_{\mathcal{L}}).$$

PROPOSITION 8.3.4. The following properties are equivalent:

(i) $\mathcal{L}$ satisfies Friedrichs' inequality.

(ii) The strong laplacian $\Delta_{\mathcal{L}} : D(\Delta_{\mathcal{L}}) \subseteq W_0^1(H;\mathcal{L}) \longrightarrow H$ is onto.

(iii) $grad_{\mathcal{L}} : W_0^1(H;\mathcal{L}) \longrightarrow H^{(n)}$ is one-to-one, and has closed range.

PROOF. First we prove the equivalence of (i) and (ii). If $\mathcal{L}$ satisfies Friedrichs' inequality, then from Corollary 8.3.2. it follows that $\Delta_{\mathcal{L}}$ is onto H. To establish the converse, first we check that $(\cdot,\cdot)_E$ is positive definite on $W_0^1(H;\mathcal{L})$. Suppose that $v \in W_0^1(H;\mathcal{L})$ is such that $grad_{\mathcal{L}}v = 0$. Let $b \in H$. Since $\Delta_{\mathcal{L}}$ is onto, there is a $u \in W_0^1(H;\mathcal{L})$ with $\Delta u_{\mathcal{L}} = b$. Hence

$$(v,b) = (v,\Delta_{\mathcal{L}}u) = (grad_{\mathcal{L}}v, grad_{\mathcal{L}}u) = 0,$$

for all $b \in H$, i.e., $v = 0$. Next, we will establish Friedrichs' inequality, by showing that the inclusion $i : (W_0^1(H;\mathcal{L}), \|\cdot\|_E) \longrightarrow H$ is continuous. For this, it is enough to show that such inclusion is weakly continuous. Let $b \in H$, and pick a $v \in W_0^1(H;\mathcal{L})$ with $\Delta_{\mathcal{L}}v = b$. Then

$$(u,b) = (grad_{\mathcal{L}}u, grad_{\mathcal{L}}v), \qquad u \in W_0^1(H;\mathcal{L}),$$

it is continuous, since $grad_{\mathcal{L}} : (W_0^1(H;\mathcal{L}), \|\cdot\|_E) \longrightarrow H$ is continuous.

Now we show that (i) and (iii) are equivalent. If $\mathcal{L}$ satisfies Friedrichs' inequality, then using the fact that the norms $\|\cdot\|_{\mathcal{L},1}$ and $\|\cdot\|_E$ are equivalent on $W_0^1(H;\mathcal{L})$, we obtain (iii). If (iii) holds, then from the open mapping theorem applied to $grad_{\mathcal{L}}$, it follows that there is a positive constant C such that

$$\|u\|_{\mathcal{L},1} \leq C\|grad_{\mathcal{L}} u\|, \qquad u \in W_0^1(H;\mathcal{L}).$$

From this, we obtain (i).

LEMMA 8.3.5. If $\mathcal{L}$ satisfies Friedrichs' inequality, then

$$|u|_{\mathcal{L},m} = \left\{ \sum_{[\gamma]=m} \|L_\gamma u\|^2 \right\}^{1/2}, \qquad u \in W_0^1(H;\mathcal{L}), \tag{8.3.7}$$

is a norm equivalent with $\|\cdot\|_{\mathcal{L},m}$ on $W_0^m(H;\mathcal{L})$.

PROOF. Clearly we have $|u|_{\mathcal{L},m} \leq \|u\|_{\mathcal{L},m}$, $u \in W_0^m(H;\mathcal{L})$. From the density of Φ in $W_0^m(H;\mathcal{L})$, it is enough to show that there is a positive constant K such that

$$\|\phi\|_{\mathcal{L},m} \leq K|\phi|_{\mathcal{L},m}, \qquad \phi \in \Phi.$$

For this, it is sufficient to prove that for every γ, $[\gamma] < m$, there is a K_γ, such that

$$\|L_\gamma \phi\| \leq K_\gamma |\phi|_{\mathcal{L},m}.$$

But this is an immediate consequence of Friedrichs' inequality (8.3.5).

PROPOSITION 8.3.6. Suppose that $\mathcal{L}$ satisfies Friedrichs' inequality. If the operators in the family $\mathcal{L} \cup \mathcal{L}^*$ satisfy:

$$LM\phi = ML\phi, \qquad \phi \in \Phi;\ L,\ M \in \mathcal{L} \cup \mathcal{L}^*.$$

Then

$$\Delta_{\mathcal{L}}^m : W_0^m(H;\mathcal{L}) \longrightarrow W^{-m}(H;\mathcal{L}),$$

is a linear isomorphism.

PROOF. From the previous lemma we see that it is enough to show that

$$\Delta_{\mathcal{L}}^m : (W_0^m(H;\mathcal{L}), |\cdot|_{\mathcal{L},m}) \longrightarrow (W_0^m(H:\mathcal{L}), |\cdot|_{\mathcal{L},m})', \tag{8.3.8}$$

is a linear isomorphism. For this, let us note that for ϕ, $\psi \in \Phi$, we have

$$\begin{aligned}(\Delta_{\mathcal{L}}{}^{m}\phi,\psi) &= \sum (L_{j_1}^{*}L_{j_1} \;\dots\; L_{j_m}^{*}L_{j_m}\phi,\psi) \\ &= \sum (L_{j_1} \;\dots\; L_{j_m}\phi, L_{j_1} \;\dots\; L_{j_m}\psi) \\ &= (\phi,\psi)_{\mathcal{L},m}^{0} ,\end{aligned}$$

where

$$(u,v)_{\mathcal{L},m}^{0} = \sum_{[\gamma]=m} (L_{\gamma}u, L_{\gamma}v),$$

is the inner product on $W_0^m(H;\mathcal{L})$ defining the norm (8.3.7). From the density of Φ in $(W_0^m(H;\mathcal{L}), |\cdot|_{\mathcal{L},m})$ we obtain

$$\langle u, \Delta_{\mathcal{L}}{}^{m}v\rangle = (u,v)_{\mathcal{L},m}^{0} , \qquad u,\ v \in W_0^m(H;\mathcal{L}).$$

Thus, (8.3.8) is the Riesz canonical representation, and this proves the result.

8.4 The Energy Space of $\Delta_{\mathcal{L}}$.

Let $L \in \mathscr{C}_2(\Phi,\Phi)$. Assume that L is symmetric:

$$(L\phi,\psi) = (\phi,L\psi), \qquad \phi,\ \psi \in \Phi; \tag{8.4.1}$$

and positive:

$$(L\phi,\phi) > 0 \text{ if } \phi \in H,\ \phi \neq 0. \tag{8.4.2}$$

Then,

$$(\phi,\psi)_E = (L\phi,\psi), \qquad \phi,\ \psi \in \Phi, \tag{8.4.3}$$

is an inner product on Φ. We define the *energy space* $E(L)$ of L, as the completion of $(\Phi, \|\cdot\|_E)$, where

$$\|\phi\|_E = (L\phi,\phi)^{1/2}, \tag{8.4.4}$$

is the *energy norm* (Mikhlin, p.91).

Now, let $\mathcal{L} = \{L_1, \dots, L_n\}$ be a family of operators on H in the class $\mathscr{C}_2(\Phi,\Phi)$. Since $L_j(\Phi) \subseteq \Phi \subseteq D(L_j^{*})$, it is a simple matter to check that the strong laplacian

$$\Delta_{\mathscr{L}}\phi = L_1^* L_1\phi + \ldots + L_n^* L_n\phi , \qquad \phi \in \Phi,$$

is symmetric. Furthermore, we have

$$(\phi,\phi)_E = (\Delta_{\mathscr{L}}\phi,\phi) = (\mathrm{grad}_{\mathscr{L}}\phi, \mathrm{grad}_{\mathscr{L}}\phi) \geq 0.$$

Assume for the moment that the strong laplacian $\Delta_{\mathscr{L}}$ is positive. Given that

$$\langle\psi, \Delta_{\mathscr{L}}\phi\rangle = (\psi,\phi)_E , \qquad \phi,\ \psi \in \Phi,$$

using the density of Φ in the energy space $E(\Delta_{\mathscr{L}})$, we can interpret $\Delta_{\mathscr{L}}$ as the Riesz canonical identification for the Hilbert space $E(\Delta_{\mathscr{L}})$. Thus, we conclude that

$$\Delta_{\mathscr{L}} : E(\Delta_{\mathscr{L}}) \longrightarrow E(\Delta_{\mathscr{L}})', \tag{8.4.5}$$

is an isometric isomorphism. Hence, we can say that the study of the Dirichlet problem for $\Delta_{\mathscr{L}}$ is reduced to find its energy space.

If $\mathscr{L}$ satisfies Friedrichs' inequality, then $\Delta_{\mathscr{L}}$ is positive and the discussion previous to Proposition 8.3.1 establishes that

$$E(\Delta_{\mathscr{L}}) = W_0^1(H;\mathscr{L}), \tag{8.4.6}$$

together with the inner product

$$(u,v)_E = (\mathrm{grad}_{\mathscr{L}}u, \mathrm{grad}_{\mathscr{L}}v). \tag{8.4.7}$$

Nevertheless, in some cases, it can happen that $\mathscr{L}$ instead of satisfying Friedrichs' inequality, satisfies a similar but weaker condition. In such a situation, it is still possible to give a characterization of the energy space $E(\Delta_{\mathscr{L}})$ in a similar way to a Hilbert-Sobolev space.

An examination of Friedrichs' inequality (8.3.5), suggests the convenience of replacing there the norm on H by a weaker norm. This lead us, in a natural way, to consider the spaces $W^{-m}(H;\mathscr{L}^*)$, taking into account that we have for $\phi \in \Phi$:

$$\|\phi\| \geq \|\phi\|_{\mathcal{L}^*,-1} \geq \dots \geq \|\phi\|_{\mathcal{L}^*,-m} \geq \|\phi\|_{\mathcal{L}^*,-(m+1)} \geq \dots .$$

We say that the family $\mathcal{L}$ satisfies *Friedrichs' inequality of order* m, if there is a positive constant C such that

$$\|\phi\|^2_{\mathcal{L}^*,-m} \leq C^2(grad_{\mathcal{L}}\phi, grad_{\mathcal{L}}\phi) , \quad \phi \in \Phi. \tag{8.4.8}$$

In this context it will be useful to consider the spaces

$$X_j = \{ u \in W^{-m}(H;\mathcal{L}^*) : L_j u \in H \}, \quad j = 1, \dots, n.$$

LEMMA 8.4.1. Each

$$L_j : X_j \subseteq W^{-m}(H;\mathcal{L}^*) \longrightarrow H, \quad j = 1, \dots, n,$$

is a closed linear operator.

PROOF. Let $\{u_k\} \subseteq X_j$, $u \in W^{-m}(H;\mathcal{L}^*)$, and $v \in H$. Suppose that

$$u_k \longrightarrow u \text{ in } W^{-m}(H;\mathcal{L}^*) \text{ and } L_j u_k \longrightarrow v \text{ in } H.$$

Then, for $\phi \in \Phi$ we have

$$\langle\phi, L_j u\rangle = \langle L_j^*\phi, u\rangle = \lim_{k\to\infty} \langle L_j^*\phi, u_k\rangle = \lim_{k\to\infty} \langle\phi, L_j u_k\rangle = \langle\phi, v\rangle.$$

This says precisely that $u \in X_j$ and $L_j u = v$. Therefore, L_j is closed.

Let $\mathcal{A} = \{A_1, \dots, A_n\}$, where A_j is the restriction of L_j to X_j, and let $W^{-m} = W^{-m}(H;\mathcal{L}^*)$. From Corollary 1.2.5, Remark 1.2.1 and the previous lemma, $V(W^{-m}, H;\mathcal{A})$, which is the space induced on W^{-m} by the family $\mathcal{A}$, is a Hilbert space.

Since $\Phi \subseteq V(W^{-m}, H;\mathcal{A})$, we can consider the space

$$V_0(W^{-m}, H;\mathcal{A}) = \text{closure of } \Phi \text{ in } V(W^{-m}, H;\mathcal{A}).$$

Proceeding as in the previous section, we obtain the following results.

PROPOSITION 8.4.2. If $\mathscr{L}$ satisfies Friedrichs' inequality of order m, then

$$E(\Delta_{\mathscr{L}}) = V_0(W^{-m}, H; \mathscr{A}),$$

together with the inner product

$$(u,v)_E = (\mathscr{grad}_{\mathscr{L}} u, \mathscr{grad}_{\mathscr{L}} v).$$

In particular,

$$\Delta_{\mathscr{L}} : V_0(W^{-m}, H; \mathscr{A}) \longrightarrow (V_0(W^{-m}, H; \mathscr{A}))'$$

is a linear isomorphism.

COROLLARY 8.4.3. Under the same hypothesis as in the previous proposition we have:

(i) Given $b \in W_0^m(H; \mathscr{L}^*)$, there is a unique $v \in V_0(W^{-m}, H; \mathscr{A})$, such that $\Delta_{\mathscr{L}} v = b$.

(ii) $\Delta_{\mathscr{L}}^{-1} : W_0^m(H; \mathscr{L}^*) \longrightarrow V_0(W^{-m}, H; \mathscr{A})$ is continuous.

PROPOSITION 8.4.4. Suppose that $\mathscr{L}$ satisfies Friedrichs' inequality of order m. If $b \in W_0^m(H, \mathscr{L}^*)$, then the solution $u \in V_0(W^{-m}, H; \mathscr{A})$ of $\Delta_{\mathscr{L}} v = b$, is the point where the real functional

$$\lambda(v) = (\mathscr{grad}_{\mathscr{L}} v, \mathscr{grad}_{\mathscr{L}} v) - (v,b) - (b,v),$$

attains it minimum value.

PROPOSITION 8.4.5. $\mathscr{L}$ satisfies Friedrichs' inequality of order m if and only if, for every $u \in W_0^m(H; \mathscr{L}^*)$, the linear functional

$$\phi \longrightarrow (\phi, u), \qquad \phi \in \Phi,$$

is continuous in $\Phi_E = (\Phi, \|\bullet\|_E)$.

PROOF. Suppose that $\mathscr{L}$ satisfies Friedrichs' inequality of order m. Let $u \in W_0^m(H; \mathscr{L}^*)$, $u \neq 0$. Then, from (8.4.8) and the definition of the norm

$\|\cdot\|_{\mathcal{L}^*,-m}$, we have

$$C\|grad_{\mathcal{L}}\phi\| \geq \|\phi\|_{\mathcal{L}^*,-m} \geq |(u/\|u\|_{\mathcal{L}^*,m}, \phi)|, \qquad \phi \in \Phi.$$

Therefore, the linear functional $\phi \longrightarrow (\phi, u)$ is continuous in Φ_E.

To prove the converse, first we check that $\|\cdot\|_E$ is a norm on Φ. Let $\phi \in \Phi$, $\phi \neq 0$. Since $\Phi \subseteq W_0^m(H;\mathcal{L}^*)$, the linear functional

$$\psi \longrightarrow (\psi, \phi), \qquad \psi \in \Phi,$$

is continuous in Φ_E. Hence, $\|\phi\| > 0$ implies that $\|\phi\|_E > 0$.

Next, let us consider the sesquilinear form

$$B : W_0^m(H;\mathcal{L}^*) \times \Phi_E \longrightarrow \mathbb{K},$$

given by $B(u,\phi) = (u,\phi)$. Being $W_0^m(H;\mathcal{L}^*)$ complete, from a well known result by Mazur and Orlicz, to establish the continuity of B, it is enough to verify that B is separately continuous. The continuity of $u \longrightarrow (u,\phi)$, for each $\phi \in \Phi$, is an immediate consecuence of the continuous inclusion $W_0^m(H;\mathcal{L}^*) \hookrightarrow H$. The continuity of $\phi \longrightarrow (u,\phi)$, is part of our hypothesis. Being B continuous, there is a $C > 0$, such that

$$C\|\phi\|_E\|u\|_{\mathcal{L}^*,m} \geq |(u,\phi)|, \qquad u \in W_0^m(H;\mathcal{L}^*), \ \phi \in \Phi.$$

This last fact, together with the definition of $\|\phi\|_{\mathcal{L}^*,-m}$ gives the desired result.

9. The Neumann Problem.

9.1 The Space $\mathcal{N}$.

Let H be a Hilbert space, Φ a dense subspace of H, and $\mathcal{L}$ a family of n operators on H in the class $\mathscr{C}_2(\Phi,\Phi)$. Following Deny and Lions ([3], p.340), we define $\mathcal{N}$ as the set of all $v \in W^1(H;\mathcal{L})$, such that $\Delta_{\mathcal{L}}v \in H$ and satisfy

$$(u,\Delta_{\mathcal{L}}v) = (grad_{\mathcal{L}}u, grad_{\mathcal{L}}v), \quad u \in W^1(H;\mathcal{L}). \tag{9.1.1}$$

Then, it is clear that $\mathcal{N}$ is a vector space, and that $\Phi \subseteq \mathcal{N}$. On $\mathcal{N}$ we consider the inner product

$$(v,w)_{\mathcal{N}} = (v,w)_{\mathcal{L},1} + (\Delta_{\mathcal{L}}v, \Delta_{\mathcal{L}}w). \tag{9.1.2}$$

PROPOSITION 9.1.1. $\mathcal{N}$ is a Hilbert space. Also we have:

(i) $\mathcal{N} \hookrightarrow W^1(H;\mathcal{L})$.

(ii) $\Delta_{\mathcal{L}} : \mathcal{N} \longrightarrow H$, is continuous.

PROOF. Let $D(\Delta_{\mathcal{L}})$ be the subspace consisting of all $u \in W^1(H;\mathcal{L})$ such that $\Delta_{\mathcal{L}}u \in H$. According with Remark 1.2.1, let X be the space induced on $W^1(H;\mathcal{L})$ by the operator $\Delta_{\mathcal{L}} : D(\Delta_{\mathcal{L}}) \subseteq W^1(H;\mathcal{L}) \longrightarrow H$. Being $W^1(H;\mathcal{L})$ and H Hilbert spaces, from Corollary 1.2.5 and Remark 1.2.1, X will be a Hilbert space if $\Delta_{\mathcal{L}}$ is closed.

So, let us consider $\{v_k\} \subseteq D(\Delta_{\mathcal{L}})$, $v \in W^1(H;\mathcal{L})$, and $w \in H$ such that

$$v_k \longrightarrow v \text{ in } W^1(H;\mathcal{L}) \text{ and } \Delta_{\mathcal{L}}v_k \longrightarrow w \text{ in } H. \tag{9.1.3}$$

We have then

$$(u,\Delta_{\mathcal{L}}v_k) = (grad_{\mathcal{L}}u, grad_{\mathcal{L}}v_k), \quad u \in W^1(H;\mathcal{L}).$$

If we let $k \longrightarrow \infty$, from (9.1.3) we obtain

$$(u,w) = (grad_{\mathcal{L}}u, grad_{\mathcal{L}}v), \quad u \in W^1(H;\mathcal{L}).$$

In particular, from (7.2.9) we have $v \in D(\Delta_{\mathcal{L}})$ and $\Delta_{\mathcal{L}}v = w$. Therefore, X is a Hilbert space.

Now, for every $u \in W^1(H;\mathcal{L})$, the linear functional

$$v \longrightarrow (\Delta_{\mathcal{L}}v,u) - (grad_{\mathcal{L}}v, grad_{\mathcal{L}}u), \quad v \in X,$$

is continuous. From this it follows that $\mathcal{N}$ is a closed subspace of X. The other assertions are clear.

9.2 The Neumann Problem for $\Delta_{\mathcal{L}} - \lambda$, $\lambda < 0$.

PROPOSITION 9.2.1. If $\lambda < 0$, then:

(i) For every $b \in H$, there is a unique $v \in \mathcal{N}$, such that

$$(\Delta_{\mathcal{L}} - \lambda)v = b. \tag{9.2.1}$$

(ii) $(\Delta_{\mathcal{L}} - \lambda)^{-1} : H \longrightarrow \mathcal{N}$ is continuous.

PROOF. (i) Let $b \in H$. If $v \in \mathcal{N}$ and is a solution of (9.2.1), then

$$(\mathrm{grad}_{\mathcal{L}} u, \mathrm{grad}_{\mathcal{L}} v) - \lambda(u,v) = (u,b), \qquad u \in W^1(H;\mathcal{L}). \tag{9.2.2}$$

Conversely, if $v \in W^1(H;\mathcal{L})$ satisfies (9.2.2), then $v \in \mathcal{N}$ and satisfies (9.2.1). Thus, it is enough to show that equation (9.2.2) has a unique solution in $W^1(H;\mathcal{L})$.

On $W^1(H;\mathcal{L})$ we consider the inner product

$$(u,v)_\lambda = (\mathrm{grad}_{\mathcal{L}} u, \mathrm{grad}_{\mathcal{L}} v) - \lambda(u,v),$$

and let $\|\cdot\|_\lambda$ be the corresponding norm. Then, $\|\cdot\|_{\mathcal{L},1}$ and $\|\cdot\|_\lambda$ are equivalent norms. Since $W^1(H;\mathcal{L}) \hookrightarrow H$, from. Riesz representation theorem it follows that there is a unique $v \in W^1(H;\mathcal{L})$, such that

$$(u,v)_\lambda = (u,b), \qquad u \in W^1(H;\mathcal{L}).$$

Which is precisely (9.2.2).

(ii) It is clear that $\Delta_{\mathcal{L}} - \lambda : \mathcal{N} \longrightarrow H$ is continuous and one-to-one and, from what we have just seen, is also onto. The result follows from the open mapping theorem.

9.3 The Neumann Problem for $\Delta_{\mathcal{L}}$ and the Poincare' Inequality.

Consider the space

$$N(\mathrm{grad}_{\mathcal{L}}) = \{v \in W^1(H;\mathcal{L}) : \mathrm{grad}_{\mathcal{L}} v = 0\}. \tag{9.3.1}$$

LEMMA 9.3.1. We have

$$N(grad_{\mathcal{L}}) = \{v \in \mathcal{N} : \Delta_{\mathcal{L}} v = 0\}. \qquad (9.3.2)$$

PROOF. If $v \in N(grad_{\mathcal{L}})$, then $\Delta_{\mathcal{L}} v = 0$ strongly, and clearly $v \in \mathcal{N}$. If $v \in \mathcal{N}$ and $\Delta_{\mathcal{L}} v = 0$, then $(grad_{\mathcal{L}} u, grad_{\mathcal{L}} v) = 0$, for all $u \in W^1(H;\mathcal{L})$. Taking $u = v$, we conclude that $v \in N(grad_{\mathcal{L}})$.

LEMMA 9.3.2. $N(grad_{\mathcal{L}})$ is a closed subspace of H.

PROOF. Let $\{v_k\} \subseteq N(grad_{\mathcal{L}})$, $v \in H$, and assume that $v_k \longrightarrow v$ in H. Since $L_j v_k = 0$ $(k = 1, 2, \dots)$ and L_j is closed, we obtain $v \in D(L_j)$ and $L_j v = 0$. Thus, $v \in W^1(H;\mathcal{L})$ and $grad_{\mathcal{L}} v = 0$.

Let $b \in H$, and assume that there is a $v \in \mathcal{N}$, such that $\Delta_{\mathcal{L}} v = b$. Then, for every $u \in N(grad_{\mathcal{L}})$ we have $(u,b) = (grad_{\mathcal{L}} u, grad_{\mathcal{L}} v) = 0$. This shows that $\Delta_{\mathcal{L}}(\mathcal{N}) \subseteq H \ominus N(grad_{\mathcal{L}})$. Next, we are going to introduce a condition under which the equality $\Delta_{\mathcal{L}}(\mathcal{N}) = H \ominus N(grad_{\mathcal{L}})$ holds.

Since $W^1(H;\mathcal{L}) \hookrightarrow H$, from the previous lemma it follows that $N(grad_{\mathcal{L}})$ is a closed subspace of $W^1(H;\mathcal{L})$. Thus, given any $u \in W^1(H;\mathcal{L})$, we can write it in the form

$$u = c + v, \qquad c \in N(grad_{\mathcal{L}}),\ v \in W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}}).$$

We will say that $\mathcal{L}$ satisfies the *Poincare' inequality*, if there is a positive constant C, such that

$$(v,v) \le C^2 (grad_{\mathcal{L}} u, grad_{\mathcal{L}} u), \qquad u \in W^1(H;\mathcal{L}). \qquad (9.3.3)$$

REMARK 9.3.1. Suppose that the vector space $N(grad_{\mathcal{L}})$ has dimension equal to 1. Fix a $c \in N(grad_{\mathcal{L}})$, with $\|c\| = 1$. If $u \in W^1(H;\mathcal{L})$, then

$$u = (u,c)c + v, \text{ where } (c,v) = 0.$$

Hence, in this case inequality (9.3.3) reduces to

$$(u - (u,c)c, u - (u,c)c) \le C^2(grad_{\mathcal{L}}u, grad_{\mathcal{L}}u), \qquad u \in W^1(H;\mathcal{L}).$$

From this we obtain the following important particular case of the Poincaré inequality:

$$(u,u) \le |(u,c)|^2 + C^2(grad_{\mathcal{L}}u, grad_{\mathcal{L}}u), \qquad u \in W^1(H;\mathcal{L}). \tag{9.3.4}$$

As an application of (9.3.4), let Ω be an open, connected, nonempty subset of $\mathbb{R}^n$, with finite measure $m_n(\Omega) < \infty$. If $H = L^2(\Omega)$, $\Phi = C_c^\infty(\Omega)$, and $L_j = \partial/\partial x_j$, $j = 1, \ldots, n$. Then, as is well known, the space $N(grad)$ is formed by the constant functions, and has dimension equal to 1. In this case $c = \pm\, m_n(\Omega)^{-1/2}$, and (9.3.4) is precisely the classical Poincaré inequality:

$$\int_\Omega |u|^2\, dx \le \frac{1}{m_n(\Omega)} \left| \int_\Omega u\, dx \right|^2 + C^2 \int_\Omega \|grad\ u\|^2\, dx, \qquad u \in H^1(\Omega).$$

PROPOSITION 9.3.3. $\Delta_{\mathcal{L}}(N) = H \ominus N(grad_{\mathcal{L}})$ if and only if $\mathcal{L}$ satisfies the Poincaré inequality.

PROOF. Suppose that the family $\mathcal{L}$ satisfies the Poincaré inequality. Let $b \in H \ominus N(grad_{\mathcal{L}})$. To find $v \in W^1(H;\mathcal{L})$ such that $\Delta_{\mathcal{L}}v = b$, is equivalent to find $v \in W^1(H;\mathcal{L})$ satisfying

$$(grad_{\mathcal{L}}u, grad_{\mathcal{L}}v) = (u,b), \qquad u \in W^1(H;\mathcal{L}). \tag{9.3.5}$$

From Poincaré inequality it follows that

$$(u,v)_1 = (grad_{\mathcal{L}}u, grad_{\mathcal{L}}v),$$

is an inner product on the space $W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}})$, whose associated norm $|\cdot|_1$ is equivalent to the norm $\|\cdot\|_{\mathcal{L},1}$. Since $W^1(H;\mathcal{L}) \hookrightarrow H$, the linear functional $u \longrightarrow (u,b)$ is continuous on $W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}})$. Riesz representation theorem, yields a unique $v \in W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}})$ such that

$$(grad_{\mathcal{L}}u, grad_{\mathcal{L}}v) = (u,b), \quad u \in W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}}).$$

Since $b \in H \ominus N(grad_{\mathcal{L}})$, from this we obtain (9.3.5).

Suppose now that $\Delta_{\mathcal{L}}(\mathcal{N}) = H \ominus N(grad_{\mathcal{L}})$. First we are going to show that $(\cdot,\cdot)_1$ is an inner product on $W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}})$. For this, it is enough to show that if $x \in W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}})$, $x \neq 0$, then $grad_{\mathcal{L}}x \neq 0$. From our hypothesis, there is a $v \in \mathcal{N}$, such that

$$(grad_{\mathcal{L}}u, grad_{\mathcal{L}}v) = (u,x), \quad u \in W^1(H;\mathcal{L}).$$

Taking $u = x$, we obtain

$$(grad_{\mathcal{L}}x, grad_{\mathcal{L}}v) = (x,x) > 0.$$

Therefore, $grad_{\mathcal{L}}x \neq 0$.

Now, the Poincaré inequality is equivalent to the continuity of the inclusion

$$i : (W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}}),\ |\cdot|_1) \longrightarrow H \ominus N(grad_{\mathcal{L}}).$$

To establish this, it is sufficient to show that such inclusion is weakly continuous. So, let $b \in H \ominus N(grad_{\mathcal{L}})$. Then, from our hypothesis, there is a $v \in \mathcal{N}$, such that (9.3.5) holds. From this it follows that $u \longrightarrow (u,b)$ is continuous.

COROLLARY 9.3.4. Suppose that the family $\mathcal{L}$ satisfies the Poincaré inequality. Then, for every $b \in H \ominus N(grad_{\mathcal{L}})$, there exists $\vec{v} \in \mathcal{E}$ such that $div_{\mathcal{L}}\vec{v} = b$.

PROOF. Given $b \in H \ominus N(grad_{\mathcal{L}})$, from the previous proposition, there is a $w \in \mathcal{N}$ such that $\Delta_{\mathcal{L}}w = b$. If we let $\vec{v} = grad_{\mathcal{L}}w$, we have then $\vec{v} \in \mathcal{E}$ and $div_{\mathcal{L}}\vec{v} = \Delta_{\mathcal{L}}w = b$.

PROPOSITION 9.3.5. Suppose that $N(grad_{\mathcal{L}})$ is finite dimensional, and that $grad_{\mathcal{L}}$ is one-to-one on $W^1_0(H;\mathcal{L})$. If $\mathcal{L}$ satisfies the Poincaré inequality, then $\mathcal{L}$ satisfies Friedrichs' inequality.

PROOF. Suppose that $\mathcal{L}$ satisfies the Poincaré inequality, but does not satisfy Friedrichs' inequality. Hence, there is a sequence $\{\phi_k\} \subseteq \Phi$, such that

$$grad_{\mathcal{L}}\phi_k \longrightarrow 0 \text{ in } H^{(n)}, \tag{9.3.6}$$

and

$$\|\phi_k\| = 1. \tag{9.3.7}$$

We have

$$\phi_k = c_k + f_k, \tag{9.3.8}$$

where $c_k \in N(grad_{\mathcal{L}})$ and $f_k \in W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}})$ Then, $grad_{\mathcal{L}}f_k \longrightarrow 0$ in $H^{(n)}$. From Poincaré inequality, this implies that

$$f_k \longrightarrow 0 \text{ in } H. \tag{9.3.9}$$

This last fact, together with (9.3.7) and (9.3.8), imply that $\{c_k\}$ is a bounded sequence in H. But since $N(grad_{\mathcal{L}})$ is finite dimensional, there is a convergent subsequence of $\{c_k\}$, which will be denoted the same, $\{c_k\}$. Thus, there is a $c \in N(grad_{\mathcal{L}})$, such that

$$c_k \longrightarrow c, \text{ in } H. \tag{9.3.10}$$

From (9.3.6), (9.3.8), (9.3.9) and (9.3.10), it follows that

$$\phi_k \longrightarrow c \text{ in } W_0^1(H;\mathcal{L}). \tag{9.3.11}$$

Hence that

$$grad_{\mathcal{L}}c = \lim_{k\longrightarrow\infty} grad_{\mathcal{L}}\phi_k = 0$$

From our hypothesis, we must have $c = 0$. On the other hand, (9.3.7) and (9.3.11) imply that $\|c\| = 1$. Which is a contradiction.

PROPOSITION 9.3.6. If the inclusion $W^1(H;\mathcal{L}) \longrightarrow H$ is compact, then $\mathcal{L}$ satisfies the Poincaré inequality.

PROOF. If $\mathcal{L}$ does not satisfy the Poincaré inequality, then there is a sequence $\{u_k\} \subseteq W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}})$, such that

$$\|grad_{\mathcal{L}} u_k\| \longrightarrow 0 \qquad (9.3.12)$$

and

$$\|u_k\| = 1. \qquad (9.3.13)$$

Since $\{u_k\}$ is a bounded sequence in $W^1(H;\mathcal{L})$, there is a $u \in H$, and a subsequence of $\{u_k\}$, which will be denoted also by $\{u_k\}$, such that

$$u_k \longrightarrow u \text{ in } H. \qquad (9.3.14)$$

From (9.3.12) and (9.3.14) it follows that $\{u_k\}$ is a Cauchy sequence in $W^1(H;\mathcal{L})$. Hence, there is a $v \in W^1(H;\mathcal{L})$, such that

$$u_k \longrightarrow v \text{ in } W^1(H;\mathcal{L}). \qquad (9.3.15)$$

Now, (9.3.14) and (9.3.15) imply $u = v$, and hence, that $u \in W^1(H;\mathcal{L})$. Thus,

$$u_k \longrightarrow u \text{ in } W^1(H;\mathcal{L}),$$

and from (9.3.12), $u \in N(grad_{\mathcal{L}})$. But from the way the sequence $\{u_k\}$ was chosen, we also have $u \in W^1(H;\mathcal{L}) \ominus N(grad_{\mathcal{L}})$. Therefore, $u = 0$, which is in contradiction with (9.3.13) and (9.3.14).

PROPOSITION 9.3.7. If the inclusion $W^1(H;\mathcal{L}) \longrightarrow H$ is compact, then $N(grad_{\mathcal{L}})$ is finite dimensional.

PROOF. It is enough to show that every sequence in $N(grad_{\mathcal{L}})$ which is bounded in H, has a convergent subsequence in H. From our hypothesis, we see that this is the case, by observing that if $\{u_k\} \subseteq N(grad_{\mathcal{L}})$ is bounded in H, then also is bounded in $W^1(H;\mathcal{L})$.

APPENDIX: THE SPACES $L^p_{\ell oc}(\Omega)$.

In this part we establish those properties of the spaces $L^p_{\ell oc}(\Omega)$, utilized in the examples illustrating the theory that we have developed.

Let Ω be a open nonempty subset of $\mathbb{R}^n$. To indicate that K is a compact set contained in Ω we write

$$K \subset\subset \Omega. \tag{A.1}$$

Let $1 \le p \le \infty$. Then, $L^p_{\ell oc}(\Omega)$ consists of all the (equivalence classes of) complex or real extended valued measurable functions u on Ω, such that

$$\|u\|_{p,K} < \infty, \text{ for every } K \subset\subset \Omega; \tag{A.2}$$

where

$$\|u\|_{p,K} = \left\{ \int_K |u|^p \, dx \right\}^{1/p}, \quad 1 \le p < \infty, \tag{A.3}$$

and

$$\|u\|_{\infty,K} = \inf \{ C : |u(x)| \le C \text{ a.e. on } K \}. \tag{A.4}$$

Clearly, $\|\cdot\|_{p,K}$ is a seminorm on $L^p_{\ell oc}(\Omega)$, for every $K \subset\subset \Omega$. Hence, the family consisting of all such seminorms determines a locally convex topology on $L^p_{\ell oc}(\Omega)$.

Given a subset A of $\mathbb{R}^n$, we denote its closure and boundary, by $\overline{A}$ and ∂A respectively.

Let $\{ U_k : k \in \mathbb{N} \}$ be a family of open subsets of Ω satisfying:

$$\overline{U}_k \text{ is compact, } \overline{U}_k \subset U_{k+1}, \text{ and } \Omega = \bigcup_{k=1}^{\infty} U_k. \tag{A.5}$$

Furthermore, each U_k can be chosen in such a way that

$$m_n(\partial U_k) = 0, \quad k = 1, 2, \dots ,$$

where m_n is the n-dimensional Lebesgue measure.

Then, the locally convex topology on $L^p_{\ell oc}(\Omega)$ is generated by the family of seminorms $\{ \|\cdot\|_{p,U_k} : k \in \mathbb{N} \}$. In particular, $L^p_{\ell oc}(\Omega)$ is a metrizable space.

LEMMA A.1. Let $\Omega_k \subseteq \Omega$, $k = 1, 2, \dots,$ be given by

$$\Omega_1 = U_1, \quad \Omega_k = U_k \setminus \overline{U}_{k-1}, \quad k = 2, 3, \dots . \tag{A.6}$$

Then,

$$L^p_{\ell oc}(\Omega) \longleftrightarrow L^p(\Omega_1) \times \dots \times L^p(\Omega_k) \times \dots ,$$

i.e., these spaces are linearly isomorphic.

PROOF. Since $U_k \subseteq U_{k+1}$, then

$$\Omega_k \cap \Omega_\ell = \emptyset, \; k \neq \ell. \tag{A.7}$$

Also, recall that we have chosen the U_k in such a way that

$$m_n(\partial U_k) = 0, \quad k = 1, 2, \dots . \tag{A.8}$$

From (A.6) it follows easily that

$$U_k \setminus (\Omega_1 \cup \dots \cup \Omega_k) \subseteq \partial U_1 \cup \dots \cup \partial U_{k-1}, \quad k = 2, 3, \dots . \tag{A.9}$$

Also, (A.5) implies

$$\Omega \setminus \bigcup_{k=1}^{\infty} \Omega_k \subseteq \bigcup_{k=1}^{\infty} (U_k \setminus (\Omega_1 \cup \dots \cup \Omega_k)). \tag{A.10}$$

Putting together (A.8), (A.9) and (A.10), we obtain

$$m_n(\Omega \setminus \bigcup_{k=1}^{\infty} \Omega_k) = 0. \tag{A.11}$$

If $u \in L^p_{\ell oc}(\Omega)$, let $u_k = u|\Omega_k$. From (A.5) and (A.6), $u_k \in L^p(\Omega_k)$. Thus, we have the linear correspondence

$$L^p_{\ell oc}(\Omega) \ni u \longrightarrow Tu = (u_1, u_2, \dots) \in \mathcal{X},$$

where $\mathcal{X} = L^p(\Omega_1) \times \dots \times L^p(\Omega_k) \times \dots$. From (A.7) and (A.11) it follows that T is one-to-one and onto. Also, from (A.8) and (A.9) we see that $u_k \longrightarrow 0$ in $L^p_{\ell oc}(\Omega)$ if and only if $Tu_k \longrightarrow 0$ in $\mathfrak{X}$.

PROPOSITION A.2.

(i) $L^p_{\ell oc}(\Omega)$ is a Frechet space $1 \le p \le \infty$.

(ii) $L^p_{\ell oc}(\Omega)$ is reflexive, if $1 < p < \infty$.

(iii) $L^p_{\ell oc}(\Omega)$ is separable, if $1 \le p < \infty$.

PROOF. Let $\{ \Omega_k : k \in \mathbb{N} \}$ be the family defined in Lemma A.1. As the case may be, let us note that each of the spaces $L^p(\Omega_k)$ has the corresponding property (completness, reflexivity or separability). Hence, the product space $L^p(\Omega_1) \times \dots \times L^p(\Omega_k) \times \dots$ will have the same property (Concerning the reflexivity, see e.g., Köthe [8]. p.304). Applying the previous lemma we obtain the desired result.

The following properties are clear:

$$L^p(\Omega) \hookrightarrow L^p_{\ell oc}(\Omega), \qquad 1 \le p \le \infty. \tag{A.12}$$

$$L^p_{\ell oc}(\Omega) \hookrightarrow L^q_{\ell oc}(\Omega), \qquad 1 \le q \le p \le \infty. \tag{A.13}$$

In particular

$$L^p_{\ell oc}(\Omega) \hookrightarrow L^1_{\ell oc}(\Omega), \qquad 1 \le p \le \infty. \tag{A.14}$$

PROPOSITION A.3. $C^\infty_c(\Omega)$ is dense in $L^p_{\ell oc}(\Omega)$, $1 \le p < \infty$.

PROOF. Let $u \in L^p_{\ell oc}(\Omega)$, $1 \le p < \infty$. Consider the family $\{ U_k : k \in \mathbb{N} \}$ of open sets given in (A.5). Define

$$u_k = u \text{ in } U_k \text{ and } u_k = 0 \text{ in } \Omega \setminus U_k.$$

Then, $u_k \in L^p(\Omega)$ and $u_k \longrightarrow u$ in $L^p_{\ell oc}(\Omega)$. This shows that $L^p(\Omega)$ is dense in $L^p_{\ell oc}(\Omega)$. Since $L^p(\Omega) \hookrightarrow L^p_{\ell oc}(\Omega)$, and $C^\infty_c(\Omega)$ is dense in $L^p(\Omega)$, $1 \le p < \infty$, the result follows.

Let $u \in L^{\infty}_{\ell oc}(\Omega)$, be such that $u(x) = 1$ on A, $u(x) = -1$ on B, where $m_n(A) > 0$, $m_n(B) > 0$ and $\Omega = A \cup B$. It is not difficult to see that u cannot be approximated by elements in $C^{\infty}_{c}(\Omega)$. Thus, $C^{\infty}_{c}(\Omega)$ is not dense in $L^{\infty}_{\ell oc}(\Omega)$.

In order to study the dual of the space $L^{p}_{\ell oc}(\Omega)$, we introduce the following definitions.

Let $u \in L^{1}_{\ell oc}(\Omega)$. We say that u has *compact support* in Ω, if there exists $K \subset\subset \Omega$, such that $u = 0$ a.e. on $\Omega \setminus K$.
For $1 \le q \le \infty$, we let

$$L^{q}_{c}(\Omega) = \{\ u \in L^{q}(\Omega) : u \text{ has compact support in } \Omega\ \}.$$

Fix $1 \le p \le \infty$, and let q be the conjugate exponent of p. Given $v \in L^{q}_{c}(\Omega)$, choose $K \subset\subset \Omega$ in such a way that $v = 0$ a.e. on $\Omega \setminus K$. Next, define the linear functional $\Lambda v : L^{p}_{\ell oc}(\Omega) \longrightarrow \mathbb{K}$:

$$\langle u, \Lambda v\rangle = \int_{\Omega} uv\ dx = \int_{K} uv\ dx\ , \qquad u \in L^{p}_{\ell oc}(\Omega).$$

Then $\Lambda v \in (L^{p}_{\ell oc}(\Omega))'$.

PROPOSITION A.5. If $1 \le p < \infty$, then the linear correspondence

$$\Lambda : L^{q}_{c}(\Omega) \longrightarrow (L^{p}_{\ell oc}(\Omega))',$$

is one-to-one and onto.

PROOF. Suppose that $v \in L^{q}_{c}(\Omega)$ and $\Lambda v = 0$. Then,

$$\int_{\Omega} uv\ dx = 0, \qquad v \in L^{p}_{\ell oc}(\Omega).$$

Since $C^{\infty}_{c}(\Omega) \subset L^{p}_{\ell oc}(\Omega)$, from du Bois-Reymond lemma (Adams [1], p.59) it follows that $u = 0$. Therefore, Λ is one-to-one.

Now, take $\lambda \in (L^p_{\ell oc}(\Omega))'$. From the characterization of the topology in $L^p_{\ell oc}(\Omega)$ by means of the seminorms $\|\cdot\|_{p,U_k}$ (see, (A.5)), it follows that there exist an open set $\omega \subseteq \Omega$ and a positive constant C, such that $\bar{\omega} \subset\subset \Omega$ and

$$|\langle u,\lambda\rangle| \le C\|u\|_{p,\omega}, \qquad u \in L^p_{\ell oc}(\Omega). \tag{A.15}$$

Given $w \in L^p(\omega)$, we define its extension $\tilde{w} \in L^p(\Omega)$ as $\tilde{w} = w$ on ω and $\tilde{w} = 0$ on $\Omega \setminus \omega$. Consider the linear functional $\mu : L^p(\omega) \longrightarrow \mathbb{K}$, given by

$$\langle w,\mu\rangle = \langle\tilde{w},\lambda\rangle, \qquad w \in L^p(\omega). \tag{A.16}$$

From (A.15), we have $\mu \in (L^p(\omega))'$. Hence, there is a $v \in L^q(\omega)$ such that

$$\langle w,\mu\rangle = \int_\omega wv\, dx, \qquad w \in L^p(\omega). \tag{A.17}$$

For $u \in L^p_{\ell oc}(\Omega)$, define $u_\omega = u$ on ω, $u_\omega = 0$ on $\Omega \setminus \omega$; and observe that $u_\omega \in L^p_{\ell oc}(\Omega)$. From (A.15), (A.16) and (A.17) we obtain

$$\langle u,\lambda\rangle = \langle u_\omega,\lambda\rangle = \langle u_\omega,\mu\rangle = \int_\omega uv\, dx = \int_\Omega u\tilde{v}\, dx \quad , \ u \in L^p_{\ell oc}(\Omega),$$

and $\lambda = \Lambda\tilde{v}$. Therefore, Λ is onto.

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Banach space. In this case, we define the spaces $W^{-m}(X;\mathcal{L})$, which allow us to obtain a family of spaces with continuous inclusions

$$\dots \hookrightarrow W_0^m(X;\mathcal{L}) \hookrightarrow \dots \hookrightarrow X \hookrightarrow \dots \hookrightarrow W^{-m}(Y;\mathcal{L}^*) \hookrightarrow \dots$$

When X is reflexive, Φ is dense in each of these spaces.

In Chapter IV, we study the Hilbert-Sobolev spaces, that is, the case when X is a Hilbert space. In Section 7 we define the gradient, divergence and Laplace operators, and obtain their basic properties. In Section 8 we study the corresponding Dirichlet problem, and its relation with Friedrichs' inequality. The same is done, in Section 9, for the Neumann problem and Poincaré inequality, following some ideas of Deny and Lions in [3].

Finally, in the Appendix, we give a detailed account of those properties of the spaces $L^p_{\ell oc}(\Omega)$ utilized along our exposition, mainly in the illustrative examples, which we believe some are well known, but we were unable to find them in the literature.

$$(u,\Delta_{\mathcal{L}}v) = (grad_{\mathcal{L}}u, grad_{\mathcal{L}}v), \qquad u \in W^1(H;\mathcal{L}).,\tag{9.1.1}$$

Then, it is clear that $\mathcal{N}$ is a vector space, and that $\Phi \subseteq \mathcal{N}$. On $\mathcal{N}$ we consider the inner product

$$(v,w)_{\mathcal{N}} = (v,w)_{\mathcal{L},1} + (\Delta_{\mathcal{L}}v, \Delta_{\mathcal{L}}w).\tag{9.1.2}$$

PROPOSITION 9.1.1. $\mathcal{N}$ is a Hilbert space. Also we have:

(i) $\mathcal{N} \hookrightarrow W^1(H;\mathcal{L})$.

(ii) $\Delta_{\mathcal{L}} : \mathcal{N} \longrightarrow H$, is continuous.

PROOF. Let $D(\Delta_{\mathcal{L}})$ be the subspace consisting of all $u \in W^1(H;\mathcal{L})$ such that $\Delta_{\mathcal{L}}u \in H$. According with Remark 1.2.1, let X be the space induced on $W^1(H;\mathcal{L})$ by the operator $\Delta_{\mathcal{L}} : D(\Delta_{\mathcal{L}}) \subseteq W^1(H;\mathcal{L}) \longrightarrow H$. Being $W^1(H;\mathcal{L})$ and H Hilbert spaces, from Corollary 1.2.5 and Remark 1.2.1, X will be a Hilbert space if $\Delta_{\mathcal{L}}$ is closed.

So, let us consider $\{v_k\} \subseteq D(\Delta_{\mathcal{L}})$, $v \in W^1(H;\mathcal{L})$, and $w \in H$ such that

$$v_k \longrightarrow v \text{ in } W^1(H;\mathcal{L}) \text{ and } \Delta_{\mathcal{L}}v_k \longrightarrow w \text{ in } H.\tag{9.1.3}$$

We have then

$$(u,\Delta_{\mathcal{L}}v_k) = (grad_{\mathcal{L}}u, grad_{\mathcal{L}}v_k), \qquad u \in W^1(H;\mathcal{L}).$$

If we let $k \longrightarrow \infty$, from (9.1.3) we obtain

$$(u,w) = (grad_{\mathcal{L}}u, grad_{\mathcal{L}}v), \qquad u \in W^1(H;\mathcal{L}).$$

In particular, from (7.2.9) we have $v \in D(\Delta_{\mathcal{L}})$ and $\Delta_{\mathcal{L}}v = w$. Therefore, X is a Hilbert space.

Now, for every $u \in W^1(H;\mathcal{L})$, the linear functional

$$v \longrightarrow (\Delta_{\mathcal{L}}v,u) - (grad_{\mathcal{L}}v, grad_{\mathcal{L}}u), \qquad v \in X,$$

is continuous. From this it follows that $\mathcal{N}$ is a closed subspace of X. The other assertions are clear.

Let $u \in L^{\infty}_{\ell oc}(\Omega)$, be such that $u(x) = 1$ on A, $u(x) = -1$ on B, where $m_n(A) > 0$, $m_n(B) > 0$ and $\Omega = A \cup B$. It is not difficult to see that u cannot be approximated by elements in $C^{\infty}_c(\Omega)$. Thus, $C^{\infty}_c(\Omega)$ is not dense in $L^{\infty}_{\ell oc}(\Omega)$.

In order to study the dual of the space $L^{p}_{\ell oc}(\Omega)$, we introduce the following definitions.

Let $u \in L^{1}_{\ell oc}(\Omega)$. We say that u has *compact support* in Ω, if there exists $K \subset\subset \Omega$, such that $u = 0$ a.e. on $\Omega \setminus K$.

For $1 \le q \le \infty$, we let

$$L^{q}_{c}(\Omega) = \{ u \in L^{q}(\Omega) : u \text{ has compact support in } \Omega \}.$$

Fix $1 \le p \le \infty$, and let q be the conjugate exponent of p. Given $v \in L^{q}_{c}(\Omega)$, choose $K \subset\subset \Omega$ in such a way that $v = 0$ a.e. on $\Omega \setminus K$. Next, define the linear functional $\Lambda v : L^{p}_{\ell oc}(\Omega) \longrightarrow \mathbb{K}$:

$$\langle u, \Lambda v \rangle = \int_{\Omega} uv \, dx = \int_{K} uv \, dx , \qquad u \in L^{p}_{\ell oc}(\Omega).$$

Then $\Lambda v \in (L^{p}_{\ell oc}(\Omega))'$.

PROPOSITION A.5. If $1 \le p < \infty$, then the linear correspondence

$$\Lambda : L^{q}_{c}(\Omega) \longrightarrow (L^{p}_{\ell oc}(\Omega))',$$

is one-to-one and onto.

PROOF. Suppose that $v \in L^{q}_{c}(\Omega)$ and $\Lambda v = 0$. Then,

$$\int_{\Omega} uv \, dx = 0, \qquad v \in L^{p}_{\ell oc}(\Omega).$$

Since $C^{\infty}_{c}(\Omega) \subset L^{p}_{\ell oc}(\Omega)$, from du Bois-Reymond lemma (Adams [1], p.59) it follows that $u = 0$. Therefore, Λ is one-to-one.