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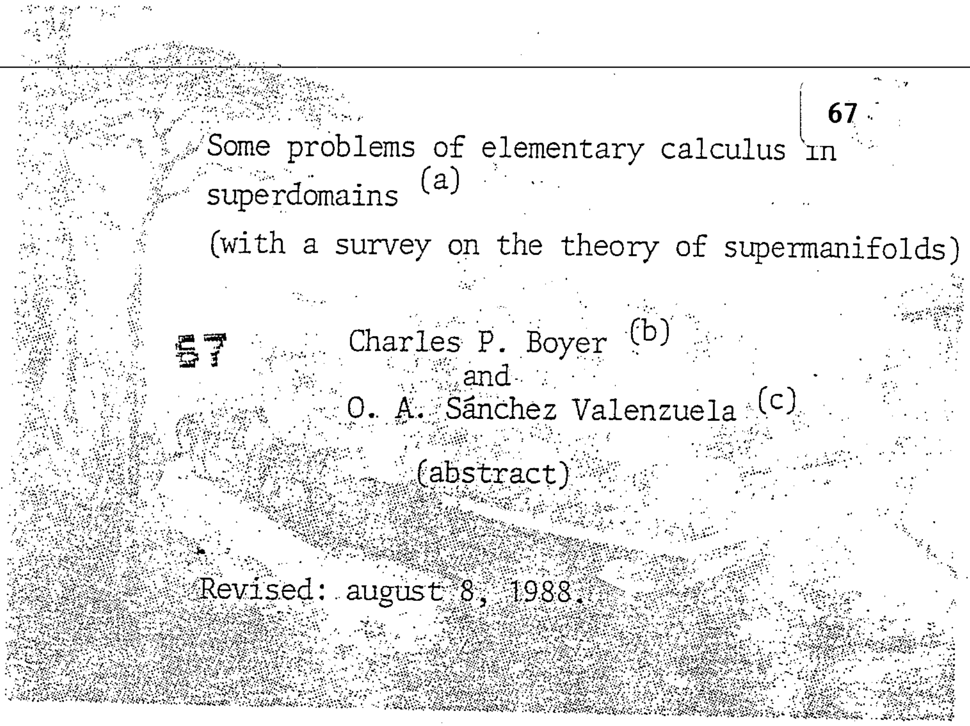

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Some problems of elementary calculus in superdomains [a]

(with a survey on the theory of supermanifolds)

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(abstract)

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Some problems of elementary calculus in superdomains[a]

(with a survey on the theory of supermanifolds)

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(abstract)

It is observed that the fundamental theorem of calculus does not hold in general for real superdomains of even dimension one and nontrivial odd dimension. Similarly, it is observed that the theorem of the rank of elementary calculus does not generalize to supermanifolds unless some modifications are made. After dealing with some concrete examples, it becomes clear the importance of developing some simple algebraic criteria by means of which one can give definite answers so as to know precisely in which cases the conclusions of these theorems hold true. It is suggested that the development of such criteria amounts to a generalization of the De Rham cohomology to include in a nontrivial way the effect of the odd variables. This paper is expository and self-contained; its purpose is to give an elementary and detailed account of these problems.

(a) A talk given by the junior author at the XX National Congress of Mathematics of the Mexican Mathematical Society, while still a fellow of the Instituto de Investigaciones en Matematicas Aplicadas y Sistemas (U.N.A.M.)

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1. Motivation: smooth manifolds

One may approach the theory of C^∞ manifolds the way algebraic geometers do; namely, by defining a *real* C^∞ [1] manifold as a ringed space, (M, C^∞_M), consisting of a topological [2] manifold, M, and a sheaf of R-algebras: the sheaf C^∞_M of differentiable functions on M. Thus, if $U \subset M$ is an open set, small enough so as to introduce a set of local coordinates $\{x^1, x^2, \ldots, x^m\}$, $f \in C^\infty_M(U)$ means that the coordinate representative $\hat{f} : x(U) \subset R^m \longrightarrow R$ of f, is a differentiable map in the sense of calculus; i.e.,

$$\hat{f} = \hat{f}(x^1, x^2, \ldots, x^m) \in C^\infty(x(U)).$$

We recall that the *stalk* of the sheaf C^∞_M at the point $p \in M$, denoted by $C^\infty_{M,p}$, is the set of equivalence classes corresponding to the relation $\{\forall f \in C^\infty_M(U),\ g \in C^\infty_M(V)\}$, $f \sim g \iff (\exists W \subset M$, open; $p \in W)$, such that $\rho^U_W(f) = \rho^V_W(g)$ [3]; in other words, it is the direct limit, $\mathrm{Lim}_{U \ni p} C^\infty_M(U)$, with U ranging over the open subsets containing p. The equivalence class in $C^\infty_{M,p}$ of an element $f \in C^\infty_M(U)$, with $p \in U$, is denoted by f_p; it is called the *germ* of f at the point p. Thus, by definition, the stalk at p is the set of germs at p; that is,

$$C^\infty_{M,p} = \{ f_p \mid f \in C^\infty_M(V),\ p \in V \subset M \}$$

(1) Real analytic, complex, or algebraic manifolds are similarly defined.
(2) Hausdorff, paracompact and with a countable base.
(3) ρ^U_W stands for the restriction morphism $C^\infty(U) \longrightarrow C^\infty(W)$; it is defined whenever $W \subset U$.

This set can be given the structure of an R-algebra by defining the operations of addition, scalar multiplication, and product, pointwise. One notes that $C^{\infty}_{M,p}$ is a local ring; that is, it has only one maximal ideal, $\mathfrak{M}_p$: the one consisting of those germs of functions that vanish at the point p.

Morphisms between C^{∞} manifolds are then defined as morphisms of ringed spaces that preserve, at each point, the unique maximal ideal of the stalk. Thus, a C^{∞} *map*, $\varphi : (M, C^{\infty}_{M}) \longrightarrow (N, C^{\infty}_{N})$, is a pair $(\tilde{\varphi}, \varphi^{\#})$ consisting of a continuous map

$$\tilde{\varphi} : M \longrightarrow N,$$

and a collection $\varphi^{\#} = \{ \varphi^{\#}_{U} : U \subset N, \text{ open} \}$ of morphisms of R-algebras

$$\varphi^{\#}_{U} : C^{\infty}_{N}(U) \longrightarrow C^{\infty}_{M}(\tilde{\varphi}^{-1}(U))$$

satisfying the following two conditions:

(i) for each $p \in \tilde{\varphi}^{-1}(U) \subset M$, with $U \subset N$ open,

$$\varphi^{\#}_{\tilde{\varphi}(p)}(\mathfrak{M}_{\tilde{\varphi}(p)}) \subset \mathfrak{M}_p,$$

where $\varphi^{\#}_{\tilde{\varphi}(p)}(f_{\tilde{\varphi}(p)})$ is defined as

$$\varphi^{\#}_{\tilde{\varphi}(p)}(f_{\tilde{\varphi}(p)}) := (\varphi^{\#}_{U}(f))_p,$$

(ii) for each pair of open sets, $U \subset V$ of N, the morphisms $\varphi^{\#}_{U}$ and $\varphi^{\#}_{V}$ commute with the restriction maps $\rho^{U}_{V} : C^{\infty}_{N}(U) \longrightarrow C^{\infty}_{N}(V)$, and $\rho^{\tilde{\varphi}^{-1}(U)}_{\tilde{\varphi}^{-1}(V)} : C^{\infty}_{M}(\tilde{\varphi}^{-1}(U)) \longrightarrow C^{\infty}_{M}(\tilde{\varphi}^{-1}(V))$; that is,

$$\varphi^{\#}{}_V \circ \rho^U{}_V = \rho^{\varphi\ (U)}{}_{\tilde{\varphi}^{-1}(V)} \circ \varphi^{\#}{}_U .$$

In other words, condition (*ii*) says that the collection $\varphi^{\#}$ defines a *sheaf homomorphism*

$$\varphi^{\#} : C^{\infty}{}_N \longrightarrow \tilde{\varphi}_* C^{\infty}{}_M ;$$

$\tilde{\varphi}_* C^{\infty}{}_M$ being the *direct image sheaf* of the sheaf $C^{\infty}{}_M$ under the continuous map $\tilde{\varphi} : M \longrightarrow N$ (4). On the other hand, (*i*) says that $\varphi^{\#}$ has to be *local* on each stalk.

An important consequence of this definition, when taken together with the fact that there can be no non-trivial **R**-algebra maps from **R** into **R**, is that for each open set $U \subset N$, and each $f \in C^{\infty}{}_N(U)$ (cf., [18]),

$$\varphi^{\#}{}_U (f) = f \circ \tilde{\varphi}$$

that is to say, a differentiable map φ, as defined above, is completely determined by the set of values $\{\tilde{\varphi}(p) : p \in M\}$ of its underlying continuous map.

The tangent and cotangent bundles over a given C^{∞} manifold $(M, C^{\infty}{}_M)$ - and in general, any C^{∞} vector bundle of finite rank over M - are defined within this approach by making them correspond with locally free sheaves of $C^{\infty}{}_M$-modules over M; namely, with $\mathrm{Der}\, C^{\infty}{}_M$ and $\mathrm{Hom}(\mathrm{Der}\, C^{\infty}{}_M , C^{\infty}{}_M)$, respectively. The question

(4) This is the sheaf over N defined by means of the assignment $U \longmapsto (\tilde{\varphi}_* C^{\infty}{}_M)(U) = C^{\infty}{}_M(\tilde{\varphi}^{-1}(U))$, for each open subset $U \subset N$.

then arises so as to find

(i) new topological manifolds, TM and T^*M, respectively, equipped with structural sheaves that makes them into C^∞ manifolds,

and

(ii) differentiable maps (actually, submersions),

$$\pi_{TM} : TM \longrightarrow M \qquad \text{and} \qquad \pi_{T^*M} : T^*M \longrightarrow M,$$

in such a way that the corresponding sheaves of local sections of these maps become isomorphic to the sheaves $\mathrm{Der}\, C^\infty_M$, and $\mathrm{Hom}\,(\mathrm{Der}\, C^\infty_M, C^\infty_M)$, respectively. But there is a general and well known construction that produces a C^∞ vector bundle over M (in the geometric sense) out of a locally free sheaf of C^∞_M-modules over M (cf., [19] or [20]). The main idea consists of relating the free C^∞_M-modules obtained over any two overlaping open sets, say U and W, by means of an invetrible matrix with entries in $C^\infty_M(U \cap W)$. The collection of matrices obtained this way, for all the possible pairs (U, W) with non-empty intersection, represent the transition functions for a vector bundle. It turns out that the sheaf of sections of this bundle is naturally isomorphic to the locally free sheaf of C^∞_M-modules over M one started with.

In analizing this construction one realizes that the crucial steps are provided, first, by the existence of a natural correspondence,

$$C^\infty_M(U) \longleftrightarrow \text{set of } C^\infty \text{ maps } (U, C^\infty_M|_U) \longrightarrow (\mathbb{R}, C^\infty_{\mathbb{R}})$$

in the sense that each $f \in C^\infty{}_M(U)$ defines a unique C^∞ map, $f = (\tilde{f}, f^\#)$ from the open submanifold $(U, C^\infty{}_M|_U)$ into the *very special manifold* $(\mathbf{R}, C^\infty{}_{\mathbf{R}})$. Second, the $C^\infty{}_M$-module operations on a direct sum of the form $C^\infty{}_M(U) \oplus C^\infty{}_M(U) \oplus \cdots \oplus C^\infty{}_M(U)$ are defined componentwise and therefore, the ultimate point is to be able to define them in $C^\infty{}_M(U)$; there, however, the definitions are straightforward, for we can simply use the ring structure of $\mathbf{R}$ to define, for any two maps $\tilde{f}, \tilde{g} : U \longrightarrow \mathbf{R}$, the maps $\tilde{f} + \tilde{g} : U \longrightarrow \mathbf{R}$ and $\tilde{f}\tilde{g} : U \longrightarrow \mathbf{R}$, by letting,

$$(\forall\, p \in U) \qquad (\tilde{f} + \tilde{g})(p) = \tilde{f}(p) + \tilde{g}(p) \qquad \text{and} \qquad (\tilde{f}\tilde{g})(p) = \tilde{f}(p)\tilde{g}(p).$$

For example, when we apply the construction to the locally free sheaves of $C^\infty{}_M$-modules $\mathrm{Der}\, C^\infty{}_M$ and $\mathrm{Hom}(\mathrm{Der}\, C^\infty{}_M, C^\infty{}_M)$, the topological manifolds TM and T^*M have both the same dimension; namely, twice the dimension of M.

The exterior algebra bundle of M, $\wedge T^*M$, is constructed from the exterior algebra sheaf $\wedge \mathrm{Hom}(\mathrm{Der}\, C^\infty{}_M, C^\infty{}_M)$ viewed as a locally free sheaf of $C^\infty{}_M$-modules over M. It is naturally decomposed into the direct sum

$$\wedge \mathrm{Hom}(\mathrm{Der}\, C^\infty{}_M, C^\infty{}_M) = \bigoplus_k \wedge^k \mathrm{Hom}(\mathrm{Der}\, C^\infty{}_M, C^\infty{}_M)$$

which in turn; yields the Whitney sum of vector bundles of the various exterior powers of $\wedge T^*M$; that is,

$$\wedge T^*M = \bigoplus_k \wedge^k T^*M \qquad \text{(Whitney sum).}$$

The sheaf of sections of this bundle - usually denoted by $U \mapsto \Omega(U)$, instead of $U \mapsto \Gamma(U, \wedge T^*M)$; $U \subset M$ open - gets decomposed into

$$\Omega(U) = \bigoplus_k \Omega^k(U)$$

The elements of $\Omega^k(U)$ are called the differential k-forms over U. One notes that if the bundle T^*M is trivial over the open set U, then,

$$\Omega^k(U) \simeq C^\infty{}_M(U) \otimes \wedge^k[e_1, e_2, \ldots, e_{\dim M}],$$

where $[e_1, e_2, \ldots, e_{\dim M}]$ denotes the (dimM)-dimensional vector space over **R** generated by the linearly independent set $\{e_1, e_2, \ldots, e_{\dim M}\}$.

The submodules $\Omega^k(U)$ together define the De Rham complex of the manifold M, as the sequence,

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{\dim M}(U) \longrightarrow 0$$

given in terms of the operator of exterior differentiation. We recall that the operator d is completely characterized by the following properties (see, for example, [5]):

(i) $df = \sum_i \partial_{x^i}(f)\, dx^i, \quad \forall f \in \Omega^0(U) \simeq C^\infty(U).$

(ii) $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta), \quad \forall \omega \in \Omega^k(U),\ \eta \in \Omega(U).$

(iii) $d \circ d = 0.$

Due to the third property, one has,

$$(\forall k \geqq 1) \qquad \mathrm{Im}\, d\big|_{\Omega^{k-1}(U)} \subset \mathrm{Ker}\, d\big|_{\Omega^{k}(U)}.$$

Therefore, the cohomology of the De Rham complex gets defined as,

$$H^k(U) = \mathrm{Ker}\, d\big|_{\Omega^{k}(U)} / \mathrm{Im}\, d\big|_{\Omega^{k-1}(U)}.$$

It is well known that the special manifold $(\mathbf{R}, C^\infty_{\mathbf{R}})$ has trivial cohomology and since the De Rham complex of $\mathbf{R}$ terminates at $\Omega^1(\mathbf{R})$, $\mathrm{Ker}\, d\big|_{\Omega^1(\mathbf{R})} = \Omega^1(\mathbf{R})$. It then follows that any 1-form on $\mathbf{R}$ can be *integrated*. The technical device behind this assertion is, of course, the fundamental theorem of calculus. In fact, given the 1-form

$$\omega = f\, dx; \quad f \in C^\infty(\mathbf{R}),$$

there is a 0-form $g \in C^\infty(\mathbf{R})$, such that, $dg = \omega$; namely,

$$g : x \mapsto \int_a^x f(s)\, ds.$$

This situation, in which any 1-form over the special manifold $\mathbf{R}$ can be integrated is to be contrasted to what occurs in the theory of supermanifolds.

2. About the prefix *super*

Before going into the theory of supermanifolds, we would like to say a few words concerning the terminology used in the subject. It is now a standard convention to let *super* mean $\mathbf{Z}_2$-*graded* (cf., [2]). Thus, for example, a supervector space $\mathbf{V}$ (over the real field $\mathbf{R}$, say) is an ordinary real vector space $\mathbf{V}$, together with a prescribed direct sum decomposition

$$\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{V}_1$$

Elements of $\mathbf{V}_\mu$ are called *homogeneous of degree* μ (also called *even* if $\mu=0$ and *odd* if $\mu=1$) and the degree of a homogeneous element $v \in \mathbf{V}$ is denoted by $|v|$. It is understood that the map $v \mapsto |v|$ is defined only on the disjoint union of the sets $\mathbf{V}_0$ and $\mathbf{V}_1$ and takes its values in the ring $\mathbf{Z}_2$.

If $\mathbf{V}$ and $\mathbf{W}$ are two given supervector spaces, the ordinary vector space $\mathrm{Hom}(\mathbf{V},\mathbf{W})$ of linear maps from $\mathbf{V}$ into $\mathbf{W}$ can be naturally graded over $\mathbf{Z}_2$ as follows:

$$\mathrm{Hom}(\mathbf{V},\mathbf{W}) = \mathrm{Hom}(\mathbf{V},\mathbf{W})_0 \oplus \mathrm{Hom}(\mathbf{V},\mathbf{W})_1$$

where,

$$\mathrm{Hom}(\mathbf{V},\mathbf{W})_\mu = \{ f \in \mathrm{Hom}(\mathbf{V},\mathbf{W}) \mid f(\mathbf{V}_\nu) \subset \mathbf{W}_{\nu+\mu} ;\ \nu \in \mathbf{Z}_2 \}.$$

Thus, $\mathrm{Hom}(\mathbf{V},\mathbf{W})$ becomes a supervector space itself. The maps from $\mathrm{Hom}(\mathbf{V},\mathbf{W})_0$ (that is, the *even maps*) are of special importance themselves: they preserve the gradation. In considering a category whose objects are supervector spaces, the morphisms are forced to be the elements of $\mathrm{Hom}(\mathbf{V},\mathbf{W})_0$ (cf., [6]).

Just as there is a natural way of Z_2-grading Hom(V,W) in terms of the Z_2-gradings of V and W, there is also a natural Z_2-gradation in the tensor product $V\otimes W$ of two supervector spaces; namely,

$$V\otimes W = (V\otimes W)_0 \oplus (V\otimes W)_1$$

where,

$$(V\otimes W)_\lambda = \bigoplus_{\mu+\nu=\lambda} V_\mu \otimes W_\nu$$

Thus, if $v\in V$ and $w\in W$ are homogeneous, $v\otimes w$ is homogeneous, and $|v\otimes w| = |v| + |w|$.

An *associative* R-*superalgebra*[5] A is a real supervector space $A = A_0 \oplus A_1$, together with a distinguished element[6], $1_A \in A_0$, and a distinguished morphism $\pi \in \mathrm{Hom}(A\otimes A, A)_0$, such that

$$(\forall a \in A), \qquad \pi(1_A \otimes a) = a = \pi(a\otimes 1_A)$$

and

$$\pi\circ(\pi\otimes id) = \pi\circ(id\otimes\pi)$$

As usual, $\pi(a\otimes b)$ is denoted by ab. It is then clear that, $|ab| = |a| + |b|$.[7]

An associative superalgebra A is called *supercommutative* if and only if [8]

$$(\forall a, b \in A, \text{ homogeneous}), \qquad ab = (-1)^{|a||b|} ba$$

(5) k-superalgebras, for any field k, are similarly defined.

(6) Let us recall that, to give a distinguished element, $1_A \in A_0$, is the same as to give a distinguished algebra morphism (see §3 below), $\kappa \in \mathrm{Hom}(R, A)$; the relation is the following: $(\forall\lambda\in R)$, $\kappa(\lambda) = \lambda 1_A$.

(7) The typical example of an associative superalgebra is End V; the morphism π is just composition and the gradation is the one of Hom(V,V) given above.

(8) The typical example of a supercommutative superalgebra is the exterior algebra, $\wedge U$, of an ordinary vector space U, relative to the Z_2-gradation $(\wedge U)_0 = \oplus_k \wedge^{2k} U$, $(\wedge U)_1 = \oplus_k \wedge^{2k+1} U$.

If A and B are two given associative superalgebras, a *morphism* $\Phi: A \longrightarrow B$ between them is an element $\Phi \in \mathrm{Hom}(A,B)_0$, such that

$$\Phi(1_A) = 1_B$$

and

$$\Phi(ab) = \Phi(a)\Phi(b).$$

The *tensor product of two superalgebras*, A and B, is their tensor product $A \otimes B$ as supervector spaces, endowed with the superalgebra structure given by letting

$$1_{A\otimes B} = 1_A \otimes 1_B$$

and [9]

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1 a_2)\otimes(b_1 b_2),$$

for all homogeneous $b_1 \in B$ and $a_2 \in A$, and extending the definition by bilinearity.

Let A be a supercommutative superalgebra and let V be a supervector space. To give a (left) A-*module structure* on V is to specify a superalgebra morphism $\Psi: A \longrightarrow \mathrm{End}\, V$. The element $\Psi(a)v$ is usually denoted by av, for all $a \in A$ and $v \in V$.

It should be clear by now how to proceed with further definitions and concepts in *linear superalgebra*. We shall refer the reader to [17] for details.

(9) This is another example of the so called Quillen's rule [9] (see definition of supercommutativity above; see also [4] and [6]): *when something of degree p moves past something of degree q, the sign $(-1)^{pq}$ appears.*

3. Review of Supermanifolds

Various forms can be found in the literature of getting at the notion of supermanifold, but the approach that seems to be more popular among mathematicians is the one that goes within the spirit of algebraic geometry. Thus, a *real* supermanifold is basically defined as a ringed space (M, A_M) consisting of a topological manifold, M, and a sheaf of supercommutative R-superalgebras, A_M, defined over it. The various conditions imposed on A_M yield the various definitions found in the literature (cf., [10] and references therein).

Thus, for example, the approach we have followed in previous works ([14], [15], and [16]) is the one of Leites and Manin (cf.,[4],[7]), which is similar to Kostant's original version [3] but sensibly less general[10]. Manin defines in [7] a *real smooth supermanifold* as a ringed space (M, A_M) as above with the following conditions imposed on A_M:

(10) According to Kostant, an (m,n)-dimensional supermanifold is a pair (M, A_M) consisting of an ordinary m-dimensional C^∞ manifold M, and a sheaf A_M of supercommutative superalgebras, such that,

(*i*) for each non-empty open subset $U \subset M$, there is defined a superalgebra homomorphism $A_M(U) \ni f \mapsto \tilde{f} \in C^\infty{}_M(U)$ that commutes with restrictions, and

(*ii*) each open subset U of M can be covered by open neighborhoods U_i $(i \in I)$, such that,

(*ii*.1) $\exists$ a subalgebra $C(U_i) \subset (A_M(U_i))_0$ (called a *function factor* of $A_M(U_i)$), such that the map $C(U_i) \ni f \mapsto \tilde{f} \in C^\infty{}_M(U_i)$, is an isomorphism, and

(*ii*.2) $\exists$ odd elements $s_1{}^{(i)}, s_2{}^{(i)}, \ldots, s_n{}^{(i)} \in (A_M(U_i))_1$, such that, $s_1{}^{(i)} s_2{}^{(i)} \cdots s_n{}^{(i)} \neq 0$, and if $D(U_i)$ denotes the subsuperalgebra of $A_M(U_i)$ generated by them, the map $C(U_i) \otimes D(U_i) \ni f \otimes w \mapsto fw \in A_M(U_i)$ is an isomorphism of superalgebras.

The U_i's are then called A_M-*splitting neighborhoods* of odd dimension n, and $C(U_i)$ and $D(U_i)$ are said to be a pair of *splitting factors for* A_M over U_i.

Now, Kostant asserts in his proposition 2.4.2 that if U is an A_M-splitting neighborhood with $(C(U), D(U))$ a given pair of splitting factors, and if V is an open subset contained in U, there exists a unique function factor $C(V)$

(footnote, next page)

Let $J_M = ((A_M)_1)$ be the sheaf of ideals generated by the odd subsheaf $(A_M)_1$ over M(11). Then, on the one hand, we obtain a sheaf of commutative algebras over M, $Gr^0 A_M := A_M / J_M$, and a sheaf epimorphism

$$\Delta : A_M \longrightarrow Gr^0 A_M$$

defined by the canonical projection onto the quotient. On the other hand, we may consider the J_M-adic filtration of A_M defined by,

$$A_M = J_M{}^0 \supset J_M{}^1 \supset J_M{}^2 \supset \cdots J_M{}^k \supset \cdots$$

and form the corresponding sheaf of graded algebras associated with it:

$$Gr\, A_M = \bigoplus_{k \geqq 0} Gr^k A_M\, ; \qquad Gr^k A_M := J_M{}^k / J_M{}^{k+1}$$

of $A_M(V)$, such that, $\rho^U{}_V(C(U)) \subset C(V)$; furthermore, the setting $D(V) = \rho^U{}_V(D(U))$ yields a commutative diagram of superalgebra morphisms of the form:

$$\begin{array}{ccc} C(U) \otimes D(U) & \longrightarrow & A_M(U) \\ \rho^U{}_V \otimes \rho^U{}_V \downarrow & & \downarrow \rho^U{}_V \\ C(V) \otimes D(V) & \longrightarrow & A_M(V) \end{array}$$

However, it does not seem to follow from Kostant's definitions that this commutative diagram *factors* so as to yield a commutative diagram of the form

$$\begin{array}{ccccc} C(U) \otimes D(U) & \longrightarrow & & \longrightarrow & A_M(U) \\ \downarrow & \searrow & C^\infty(U) \otimes \Lambda[n] & \nearrow & \downarrow \\ C(V) \otimes D(V) & \longrightarrow & \vdots\downarrow & \longrightarrow & A_M(V) \\ & \searrow & C^\infty(V) \otimes \Lambda[n] & \nearrow & \end{array}$$

where the new vertical dotted arrow is the restriction map of the sheaf $C^\infty \otimes \Lambda[n]$. Note that if it does, Kostant's definition is the same as that used in [4] and [7]. We shall leave here as an open question if it is possible at all, to give an example of a supermanifold in this sense of Kostant that is not a supermanifold in the sense of [4] and [7]. (We are indebted to S. Gitler for helpful discussions regarding this particular point).

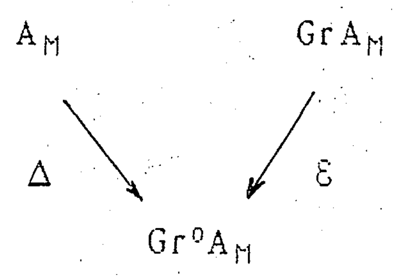
(11) That is, over an open subset $U \subset M$, $A_M(U) = A_M(U)_0 \oplus A_M(U)_1$, so $J_M(U)$ is the ideal generated by $A_M(U)_1$

Under the usual definitions of addition and multiplication performed on germs, each $Gr^k A_M$ becomes a sheaf of $Gr^0 A_M$-modules over M. In fact, when viewed as a sheaf of $Gr^0 A_M$-algebras, $Gr\,A_M$ is generated by $Gr^1 A_M$ Furthermore, it has the structure of a sheaf of *augmented* $Gr^0 A_M$-algebras over M, with augmentation map given by the sheaf morphism

$$\varepsilon : Gr\,A_M \longrightarrow Gr^0 A_M$$

defined by the projection of $Gr\,A_M$ onto the direct summand $Gr^0 A_M$. Moreover, since $A_M(U)$ is supercommutative, $Gr\,A_M$ is in fact a homomorphic image of the sheaf of $Gr^0 A_M$-algebras over M, $\wedge_{Gr^0 A} Gr^1 A_M$. It is a straightforward matter to check that if the filtration is finite (i.e., if there is some k, such that $J_M{}^k = 0$), then $Gr\,A_M$ is actually isomorphic to $\wedge_{Gr^0 A} Gr^1 A_M$.

Thus, when we are given a supermanifold (M, A_M), we always have the following morphisms of sheaves defined:



In these terms, the defining conditions for a real smooth supermanifold are:

(i) For each $x \in M$, the stalk $A_{M,x}$ is a local super-ring.

(ii) The sheaf $Gr^0 A_M$ is isomorphic to the sheaf $C^\infty{}_M$ of real smooth functions over M.

(iii) $Gr^1 A_M$ is a locally free sheaf of $Gr^0 A_M$-modules of finite rank over M (and the rank is called the *odd dimension* of the supermanifold).

(iv) For each point $x \in M$ there is an open neighborhood U of x and an isomorphism of sheaves of supercommutative superalgebras over U,

$$\varphi_U : A_M|_U \longrightarrow Gr\, A_M|_U$$

such that, $\mathcal{E} \circ \varphi_U = \Delta$.

A supermanifold *morphism* from (M, A_M) into (N, A_N), is a pair $\Phi = (\tilde{\varphi}, \varphi^{\#})$ consisting of a continuous map

$$\tilde{\varphi} : M \longrightarrow N$$

and a sheaf homomorphism

$$\varphi^{\#} : A_N \longrightarrow \tilde{\varphi}_* A_M$$

which is local on each stalk.

It is a well known fact (c.f., [3], and [4]) that a supermanifold morphism is completely determined by the superalgebra morphism that the sheaf homomorphism gives rise to; that is, by

$$\varphi^{\#} : A_N(N) \longrightarrow A_M(\tilde{\varphi}^{-1}(N))$$

Note, in particular, that every supermanifold comes equipped with the supermanifold morphism

$$\Delta : (M, C^{\infty}{}_M) \longrightarrow (M, A_M)$$

uniquely determined by the canonical projection

$$A_M(U) \longrightarrow (A_M / J_M)(U) \simeq C^{\infty}{}_M(U) \quad ; \; f \longmapsto \tilde{f}$$

This morphism is useful in evaluating any $f \in A_M(U)$ on the *points* of M, in the sense that for a given $p \in M$, there is also a supermanifold morphism

$$\Delta_p : (\{*\}, \mathbf{R}) \longrightarrow (M, A_M),$$

the object $(\{*\}, \mathbf{R})$ being the supermanifold consisting of a single point and the constant sheaf **R**, the reals, over it. Δ_p is defined by,

$$(\forall f \in A_M(U)) \qquad \Delta_p{}^{\#} f = \tilde{f}(p).$$

Note that $(\{*\}, \mathbf{R})$ is a terminal object; the unique morphism from any supermanifold into it is the *constant morphism*

$$C_{(M, A_M)} : (M, A_M) \longrightarrow (\{*\}, \mathbf{R}).$$

determined by the unique superalgebra morphism $\mathbf{R} \longrightarrow A_M(M)$

$$(\forall \lambda \in \mathbf{R}) \qquad C_{(M, A_M)}{}^{\#} \lambda = \lambda 1_{A_M(M)}$$

i.e , the one that comes with the definition of any superalgebra over **R**.

4. Specific differences and analogies with smooth manifolds

In order to compare with the theory of C^∞ manifolds, note that if $U \subset M$ is an open set, small enough so as to find a definite isomorphism $\varphi_U : A_M|_U \to \mathrm{Gr}\, A_M|_U$, then,

$$A_M(U) \simeq C^\infty{}_M(U) \otimes \wedge[\zeta^1, \zeta^2, \ldots, \zeta^n],$$

where $\{\zeta^1, \zeta^2, \ldots, \zeta^n\}$ is a set of free $C^\infty{}_M(U)$-generators of $\mathrm{Gr}^1 A_M$ over U; they are an example of what is called a system of *odd (local) coordinates* (i.e., only defined over U). Thus, once an isomorphism φ_U is given, any *superfunction*, i.e., any element $f \in A_M(U)$, can be written uniquely in the form

$$f = \tilde{f} + \sum f_\mu \zeta^\mu + \sum f_{\mu\nu} \zeta^\mu \zeta^\nu + \sum f_{\mu\nu\sigma} \zeta^\mu \zeta^\nu \zeta^\sigma + \cdots + f_{12\cdots n} \zeta^1 \zeta^2 \cdots \zeta^n$$

with $\tilde{f}, f_\mu, f_{\mu\nu}, f_{\mu\nu\sigma}, \ldots, f_{12\cdots n} \in C^\infty{}_M(U)$. Thus, superfunctions over U look exactly as sections over U of the exterior algebra bundle of a vector bundle. As we shall shortly see, however, this does not mean that supermanifolds are just exterior algebra bundles of C^∞ vector bundles. Let us only pause here to note that if U as above is furthermore a coordinate neighborhood in the usual sense, local coordinates $\{x^1, x^2, \ldots, x^m\}$ may be introduced in U, and the collection $\{x^1, x^2, \ldots, x^m; \zeta^1, \zeta^2, \ldots, \zeta^n\}$ becomes an example of what is called a system of local (super)coordinates over $U \subset M$ for the supermanifold (M, A_M). In this context, the collection $\{x^1, x^2, \ldots, x^m\}$ is refered to as a set of *even coordinates* and one says that (M, A_M) is an (m,n)-dimensional supermanifold.

Now, one of the most important points to bear in mind in the theory of supermanifolds is that even though the structural sheaf A_M may be locally identified with the sheaf of sections of the exterior algebra bundle of some vector bundle over M, the morphisms on the supermanifold need not be morphisms of vector bundles. Thus, for example, an automorphism of $(U, A_M|_U)$ is not required to come from any map of $C^\infty{}_M(U)$-modules. All what is demanded is that the map

$$\varphi_U{}^{\#} : A_M(U) \longrightarrow A_M(U),$$

that defines it, be a morphism of superalgebras. This means that if $\{x^1, x^2, \ldots, x^m; \zeta^1, \zeta^2, \ldots, \zeta^n\}$ is a system of supercoordinates of the special kind considered above, we shall be able to write

$$\varphi_U{}^{\#} x^i = f^i + \sum f^i{}_{\mu\nu}\, \zeta^\mu \zeta^\nu + J_M(U)^4$$

and

$$\varphi_U{}^{\#} \zeta^\lambda = \sum f^\lambda{}_\mu\, \zeta^\mu + \sum f^\lambda{}_{\mu\nu\sigma}\, \zeta^\mu \zeta^\nu \zeta^\sigma + J_M(U)^5$$

and in general, the functions $f^i{}_{\mu\nu}$, $f^\lambda{}_{\mu\nu\sigma}$, etc. do not have to vanish. Therefore, the category of supermanifolds admits, in principle, more general morphisms than vector bundle maps. This observation makes it clear that a more general definition of a coordinate system is needed. The one accepted within this approach is the original definition of Kostant [3] (see also [4]):

A supercoordinate system for the supermanifold (M, A_M) over the open neighborhood $U \subset M$, consists of a collection $\{f^1, f^2, \ldots, f^m\}$ of even superfunctions (i.e., $f^i \in (A_M(U))_0$), together with a collection $\{\zeta^1, \zeta^2, \ldots, \zeta^n\}$ of odd superfunctions (i.e., $\zeta^\mu \in (A_M(U))_1$), such that,

(i) the collection of C^∞ functions on U, $\{\tilde{f}^1, \tilde{f}^2, \ldots, \tilde{f}^m\}$, forms a coordinate system (in the usual sense) for the open set $U \subset M$, and,

(ii) the collection $\{\zeta^1, \zeta^2, \ldots, \zeta^n\}$ is maximal among all collections of odd superfunctions with the property that $\zeta^1 \zeta^2 \cdots \zeta^n \neq 0$.

Vector bundles within the category of supermanifolds may be approached in exactly the same way as in the C^∞ case; the only technical detail that has to be taken care of is to realize that the rôle of the manifold $(\mathbf{R}, C^\infty{}_{\mathbf{R}})$ is now taken by the (1,1)-dimensional supermanifold $\mathbf{R}^{1|1} = (\mathbf{R}, C^\infty{}_{\mathbf{R}} \otimes \wedge[\zeta])$, as it was emphasized in [12], [13] and [14].

Thus, one may prove that the sheaves $\mathrm{Der}\, A_M = (\mathrm{Der}\, A_M)_0 \oplus (\mathrm{Der}\, A_M)_1$ and $\mathrm{Hom}\,(\mathrm{Der}\, A_M, A_M) = (\mathrm{Hom}\,(\mathrm{Der}\, A_M, A_M))_0 \oplus (\mathrm{Hom}\,(\mathrm{Der}\, A_M, A_M))_1$, where,

$$(\mathrm{Der}\, A_M)_\mu = \{\text{sheaf morphisms } X : A_M \longrightarrow A_M \mid \forall f, g \in A_M(U),\ f \text{ homogeneous},\ X(fg) = (Xf)g + (-1)^{\mu|f|} f\, Xg\}$$

and

$$(\mathrm{Hom}\,(\mathrm{Der}\, A_M, A_M))_\mu = \{\text{sheaf morphisms } \theta : \mathrm{Der}\, A_M \longrightarrow A_M \mid \theta((\mathrm{Der}\, A_M)_\nu) \subset (A_M)_{\nu+\mu}\}$$

are locally free sheaves of A_M-modules over M of rank (even dim M, odd dim M) $= (m,n)$ (c.f., [3] or [4]). One may also produce two supermanifolds, (STM, STA_M) and (ST^*M, ST^*A_M), each of dimension $(2m+n, 2n+m)$, together with supermanifold morphisms

$$\pi_{STM} : (STM, STA_M) \longrightarrow (M, A_M)$$

and

$$\pi_{ST^*M} : (ST^*M, ST^*A_M) \longrightarrow (M, A_M),$$

so that the sheaves defined by assigning the sets

$$\Gamma(STM) = \{\text{supermanifold morphisms } \sigma : (U, A_M|_U) \longrightarrow (STM, STA_M) \mid \pi_{STM} \circ \sigma = id\}$$

and

$$\Gamma(ST^*M) = \{\text{supermanifold morphisms } \sigma : (U, A_M|_U) \longrightarrow (ST^*M, ST^*A_M) \mid \pi_{ST^*M} \circ \sigma = id\}$$

to each open subset $U \subset M$, become isomorphic to $\mathrm{Der}\, A_M$ and $\mathrm{Hom}(\mathrm{Der}\, A_M, A_M)$, respectively. The procedure for doing this mimics the one followed in the C^∞ case(12). This time, however, it is crucial to realize that the supermanifold $\mathbf{R}^{1|1}$ can be endowed with the structure of an abstract ring in the category of supermanifolds (c.f., [1] for definitions and examples); that is, that one may define supermanifold morphisms

$$\sigma : \mathbf{R}^{1|1} \times \mathbf{R}^{1|1} \longrightarrow \mathbf{R}^{1|1} \qquad \text{and} \qquad \mu : \mathbf{R}^{1|1} \times \mathbf{R}^{1|1} \longrightarrow \mathbf{R}^{1|1}$$

called supersum and supermultiplication, that allow the standard construction to go through (see [14] and for applications and further results on linearity and bilinearity, see [15]).

(12) As can be seen from the definitions, C^∞-manifolds occur as special cases of supermanifolds. Morphisms between smooth manifolds are special cases of supermanifold morphisms, too. Thus, the category of C^∞-manifolds gets subsumed as a full subcategory of the category of C^∞-supermanifolds; namely, as the one defined by those objects having odd dimension equal to zero. But now, if we are given any such supermanifold, say (M, C^∞_M), we can apply our general construction to produce (STM, STC^∞_M) and $(ST^*M, ST^*C^\infty_M)$ which are supermanifolds of nontrivial odd dimension: they are $(2m, m)$-dimensional. Then, the zero section (a notion that makes good sense in the theory of supervector bundles, as can be deduced from the foundations layed out in [14]), defines an embedding of the original smooth manifold into any of these supermanifolds.

The construction of the De Rham complex of a supermanifold follows essentially the same steps as in the C^∞ case. An important difference, however, is that the presence of odd coordinates prevents the $\Omega^i(M, A_M)$'s from being the zero module at some stage. The reason is that the sheaf of sections $\Omega^i(M, A_M)$ looks locally like

$$\Omega^i(U, A_M|_U) \simeq A_M(U) \otimes \Lambda^i[\{dx^1, dx^2, \ldots, dx^m\} \oplus \{d\zeta^1, d\zeta^2, \ldots, d\zeta^n\}],$$

where $\Lambda^i[\{dx^1, \ldots, dx^m\} \oplus \{d\zeta^1, \ldots, d\zeta^n\}]$ denotes the supervector space corresponding to the $i^{\underline{th}}$ exterior power of the (m,n)-dimensional supervector space spanned by the even generators $\{dx^1, \ldots, dx^m\}$ and the odd generators $\{d\zeta^1, \ldots, d\zeta^n\}$. Let us recall that the general definition of the Z_2-graded exterior algebra $\Lambda(V_0 \oplus V_1)$ associated to the Z_2-graded vector space $V = V_0 \oplus V_1$ is given by (c.f., footnote (9)),

$$\Lambda(V_0 \oplus V_1) = \bigotimes(V_0 \oplus V_1) / \text{Ideal generated}\{x \otimes y + (-1)^{|x||y|} y \otimes x \,|\, x, y \in V \text{ homogeneous}\}$$

Then, one proves that

$$\Lambda^i(V_0 \oplus V_1) = \bigoplus_{i=j+k} \Lambda^j(V_0) \otimes S^k(V_1)$$

and therefore,

$$\Omega^i(U, A_M|_U) \simeq A_M(U) \otimes \Big\{ \bigoplus_{i=j+k} \Lambda^j(\{dx^a\}) \otimes S^k(\{d\zeta^b\}) \Big\}.$$

Since, odd dim $M > 1$ implies, $\dim_{\mathbf{R}} S^k(\{d\zeta^b\}) > 1$ for all $k \neq 0$, it follows that in general, $\Omega^i(U, A_M|_U) \neq \{0\}$ for all $i \in \mathbb{N}$. Let us point out that the modules $\Omega^i(U, A_M|_U)$ have a Z_2-grading, too; in fact, we may write,

$$\Omega^i(U, A_M|_U) \simeq (\Omega^i(U, A_M|_U))_0 \oplus (\Omega^i(U, A_M|_U))_1$$

where,

$$(\Omega^i(U, A_M|_U))_\lambda = \bigoplus_{\lambda=\mu+\nu} \left\{ (A_M(U))_\mu \otimes \left\{ \bigoplus_{i=j+k} \Lambda^j(\{dx^a\}) \otimes S^k(\{d\zeta^b\}) \right\}_\nu \right\}$$

and

$$\left\{ \bigoplus_{i=j+k} \Lambda^j(\{dx^a\}) \otimes S^k(\{d\zeta^b\}) \right\}_\nu = \bigoplus_{i=j+k} \left\{ \Lambda^j(\{dx^a\}) \otimes S^k(\{d\zeta^b\}) \right\}_\nu$$

$$= \bigoplus_{i=j+k} \left\{ \bigoplus_{\nu=\rho+\sigma} \left\{ (\Lambda^j(\{dx^a\}))_\rho \otimes (S^k(\{d\zeta^b\}))_\sigma \right\} \right\}$$

The exterior derivative for superforms can be characterized as in the C^∞ case by means of the following properties (c.f., [3] and [4]; however, one must insist this time in using the right A_M-supermodule structure, as emphisized by Kostant):

(*i*) $\forall f \in \Omega^0(U, A_M|_U) \simeq A_M(U)$, $\quad df = \sum_a dx^a \, \partial_{x^a}(f) + \sum_b dx^b \, \partial_{\zeta^b}(f)$.

(*ii*) $\forall \omega \in (\Omega^j(U, A_M|_U))_\mu$, $\eta \in \Omega(U, A_M|_U)$, $\quad d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^j \omega \wedge d(\eta)$

(*iii*) $d^2 = d \circ d = 0$.

where, ∂_{ζ^b} is an odd derivation; that is, that for all homogeneous $f \in A_M(U)$, and all $h \in A_M(U)$,

$$\partial_{\zeta^b}(fh) = \partial_{\zeta^b}(f)h + (-1)^{|f|} f \, \partial_{\zeta^b}(h)$$

Again, the third property allows us to define cohomology in the usual way for the complex obtained from the Ω^j's. To give an example, let us look at the De Rham complex of the special supermanifold $\mathbf{R}^{1|1} = (\mathbf{R}, C^\infty_{\mathbf{R}} \otimes \wedge[\zeta])$. The reader will have no trouble in convincing himself that,

$$\Omega^{k+1}(U, A_{\mathbf{R}}|_U) \simeq dx(d\zeta)^k \otimes A_{\mathbf{R}}(U) \oplus (d\zeta)^{k+1} \otimes A_{\mathbf{R}}(U)$$

One notes that in general, a given 1-superform, say $\omega = (dx)f + (d\zeta)g$, with $f, g \in A_{\mathbf{R}}(U)$, is not the differential of a 0-superform (i.e., a superfunction, say $F \in A_{\mathbf{R}}(U)$). In fact, if we write $F = a + b\zeta$, with $a, b \in C^\infty(U)$, then,

$$dF = (dx)(a' + b'\zeta) + (d\zeta)b$$

Hence, the 1-superforms $\eta = (d\zeta)c\zeta$, with $c \in C^\infty(U)$, can never be exterior derivatives of superfunctions. Nevertheless, talking at the level of cohomology, we can easily prove that $H^{k+1}(\mathbf{R}^{1|1}) \simeq \{0\}$. Indeed, the most general $(k+1)$-superform can be written as

$$\omega = dx(d\zeta)^k(a + b\zeta) + (d\zeta)^{k+1}(\alpha + \beta\zeta)$$

with $a, b, \alpha, \beta \in C^\infty(\mathbf{R})$. Then, its exterior derivative is given by

$$d\omega = dx(d\zeta)^{k+1}\{(\alpha' + (-1)^{k+1}b) + \beta'\zeta\} + (-1)^{k+1}(d\zeta)^{k+2}\beta$$

Hence, $\omega \in \operatorname{Ker} d|_{\Omega^{k+1}(\mathbf{R}^{1|1})}$ if and only if $\alpha' = (-1)^k b$ and $\beta = 0$. That is, if and only if,

$$\omega = dx(d\zeta)^k(a + (-1)^k\alpha'\zeta) + (d\zeta)^{k+1}\alpha$$

In such a case we can certainly find a k-superform, say

$$\eta = dx(d\zeta)^{k-1}(A + B\zeta) + (d\zeta)^k(\Theta + \Gamma\zeta)$$

such that, $d\eta = \omega$. In fact, we have

$$d\eta = dx(d\zeta)^k\{(\Theta' + (-1)^k B) + \Gamma'\zeta\} + (-1)^k(d\zeta)^{k+1}\Gamma$$

so, we simply put $\Gamma = (-1)^k\alpha$, $\Theta' + (-1)^k B = a$, and A arbitrary. Thus, any closed superform on $\mathbf{R}^{1|1}$ is exact.

What is at issue here is a general fact already pointed out by Kostant in his pioneering work [3] and explained in full detail in [18].[13] The cohomology of superforms on any supermanifold is isomorphic to the De Rham cohomology (of ordinary differential forms) of its underlying smooth manifold. In certain sense this is no surprise at all, since the De Rham cohomology gives nothing else than the topological (Čech) cohomology and the approach to supermanifolds we are following here does not change the underlying manifold. It would be desirable then to develop new criteria that allow us to uncover phenomena such as the fact that not every 1-superform on $\mathbf{R}^{1|1}$ is the exterior differential of a superfunction there. From the work done in [8] and [11] we may expect such criteria to be cohomological in nature; furthermore, it would come as a pleasent resource into the theory to be able to detect other peculiar properties of supermanifolds through the calculation of some appropriate cohomology supergroups[14].

(13) We are indebted to H. Boseck for bringing to our attention the work of T. Schmitt.

(14) For example, in the theory of complex manifolds, the different complex structures on a given manifold are parameterized by the elements of certain cohomology groups of holomorphic sections of a suitable vector bundle defined on it (The junior author would like to thank R. Vila for illuminating this and other points in supemanifold theory through helpful discussions of known examples in complex manifolds).

Another problem, where the development of some additional algebraic criteria is needed, is the supermanifold counterpart of the theorem of the rank of elementary calculus. For the sake of comparison, let us review first the situation in calculus. suppose that we are given a C^∞-map

$$\Phi = (\tilde{\varphi}, \varphi^*) : (R^m, C^\infty{}_{R^m}) \longrightarrow (R^p, C^\infty{}_{R^p})$$

and introduce local coordinates, say $\{x^1, x^2, \ldots, x^m\}$ on the open set $U \subset R^m$ and $\{y^1, y^2, \ldots, y^m\}$ on the open set $V \subset R^p$, with $\tilde{\varphi}^{-1}(V) \subset U$, so as to express each $\varphi^* y^i$ as a differentiable function of the coordinates $\{x^1, x^2, \ldots, x^m\}$. The jacobian matrix of Φ is the $p \times m$ matrix [15]

$$J\Phi = \begin{pmatrix} \dfrac{\partial \varphi^* y^1}{\partial x^1} & \dfrac{\partial \varphi^* y^1}{\partial x^2} & \cdots & \dfrac{\partial \varphi^* y^1}{\partial x^m} \\ \dfrac{\partial \varphi^* y^2}{\partial x^1} & \dfrac{\partial \varphi^* y^2}{\partial x^2} & \cdots & \dfrac{\partial \varphi^* y^2}{\partial x^m} \\ \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial \varphi^* y^p}{\partial x^1} & \dfrac{\partial \varphi^* y^p}{\partial x^2} & \cdots & \dfrac{\partial \varphi^* y^p}{\partial x^m} \end{pmatrix}$$

The problem of the theorem of the rank may be roughly stated as follows: can we find some change of local coordinates in $U \subset R^m$ and $V \subset R^p$, say

$$\alpha : U' \subset R^m \longrightarrow U \subset R^m \qquad \text{and} \qquad \beta : V \subset R^p \longrightarrow V' \subset R^p,$$

in such a way that the morphism $\beta \circ \Phi \circ \alpha$ has a jacobian of the simplest possible

(15) It is clearly a matrix with entries in $C^\infty{}_{R^m}(U)$. One notes that not every matrix $M \in (C^\infty{}_{R^m}(U))^{p \times m}$ can be the jacobian matrix of a morphism. Clearly, a necessary condition for $M = (M_{ij})$ to be the jacobian of some C^∞ map $R^m \longrightarrow R^p$, is that the p 1-forms defined by $\omega_i = \sum_j M_{ij}\, dx^j$ be all closed: that is, $d\omega_i = 0$, for all $i = 1, \ldots, p$. This condition is by no means sufficient either: but, if it turns out that on the open set U, $d\omega_i = 0 \Rightarrow \omega_i = d\eta_i$, with $\eta_i \in C^\infty{}_{R^m}(U)$, (a question answered by cohomology!), then the matrix M is indeed the jacobian matrix of the morphism defined by the equations $\varphi^* \tilde{y}^i = \eta_i$.

form? Since $J\beta(u): R^p \longrightarrow R^p$ and $J\alpha(v): R^m \longrightarrow R^m$ are linear isomorphisms for all $u \in U'$ and $v \in V$, and the rank of a given matrix must be an invariant under isomorphisms, we seek for α and β such that, at some point $x_0 \in U$,

$$J(\beta \circ \Phi \circ \alpha)(\alpha^{-1}(x_0)) = \begin{pmatrix} 1_{k \times k} & 0 \\ * & 0 \end{pmatrix}$$

where k is the rank of the matrix $J\Phi$ at $x_0 \in U$. As a remarkable fact, the theorem of the rank asserts that this form of the jacobian is actually achieved throughout some open neighborhood $\tilde{U} \subset U$ of x_0, provided the rank of $J\Phi$ is exactly k on $\tilde{U}$. Moreover, when $k = p \leqq m$ at the single point x_0, one may conclude that the same property prevails throughout an open neighborhood of it, in which case, the morphism Φ is easily seen to be an immersion there.

One may approach the problem algebraically; that is, one may ask whether or not the given matrix $J\Phi$ with coefficients in the ring $C^\infty{}_{\mathbf{R}^m}(U)$ may be brought to such a triangular form. In fact, one notes that the elementary operations on matrices with entries in $C^\infty{}_{\mathbf{R}^m}(U)$ correspond to well determined coordinate changes. In these terms, the theorem of the rank of elementary calculus asserts that any jacobian matrix can be so triangularized.

This situation is to be contrasted to what occurs in the realm of superdomains. There, we have the notion of jacobian too (cf., [4]). It is constructed in exactly the same way as above; namely, assume that we are given a superdomain morphism

$$\Phi = (\tilde{\varphi}, \varphi^*) : (R^m, C^\infty{}_{\mathbf{R}^m} \otimes \wedge[n]) \longrightarrow (R^p, C^\infty{}_{\mathbf{R}^p} \otimes \wedge[q])$$

and assume that local coordinates have been introduced, $\{x^1, \dots, x^m; \zeta^1, \dots, \zeta^n\}$ on the open set $U \subset \mathbf{R}^m$ and $\{y^1, \dots, y^p; \xi^1, \dots, \xi^q\}$ on the open set $V \subset \mathbf{R}^p$, with $\tilde{\varphi}^{-1}(V) \subset U$, so as to write $\varphi^* y^j$ as an even superfunction of the coordinates $\{x^1, \dots, x^m; \zeta^1, \dots, \zeta^n\}$ and $\varphi^* \xi^\mu$ as an odd superfunction of them. Then, the *superjacobian*, $J\Phi$, of the morphism Φ, is the $(p+q) \times (m+n)$ *even matrix*(16),

$$J\Phi = \begin{pmatrix} \dfrac{\partial \varphi^* y^1}{\partial x^1} & \cdots & \dfrac{\partial \varphi^* y^1}{\partial x^m} & \dfrac{\partial \varphi^* y^1}{\partial \zeta^1} & \cdots & \dfrac{\partial \varphi^* y^1}{\partial \zeta^n} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial \varphi^* y^p}{\partial x^1} & \cdots & \dfrac{\partial \varphi^* y^p}{\partial x^m} & \dfrac{\partial \varphi^* y^p}{\partial \zeta^1} & \cdots & \dfrac{\partial \varphi^* y^p}{\partial \zeta^n} \\ \dfrac{\partial \varphi^* \xi^1}{\partial x^1} & \cdots & \dfrac{\partial \varphi^* \xi^1}{\partial x^m} & \dfrac{\partial \varphi^* \xi^1}{\partial \zeta^1} & \cdots & \dfrac{\partial \varphi^* \xi^1}{\partial \zeta^n} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \dfrac{\partial \varphi^* \xi^q}{\partial x^1} & \cdots & \dfrac{\partial \varphi^* \xi^q}{\partial x^m} & \dfrac{\partial \varphi^* \xi^q}{\partial \zeta^1} & \cdots & \dfrac{\partial \varphi^* \xi^q}{\partial \zeta^n} \end{pmatrix}$$

In order to give a meaning to the notion of rank of this matrix at a given point, say $x_0 \in U$, one must *evaluate* it at x_0. But now, evaluation of any superfunction of $C^\infty{}_{\mathbf{R}^m}(U) \otimes \Lambda[n]$ at some given point x_0 (or more generally, of any matrix of superfunctions) has to be understood in the sense explained in §3; that is, by projecting onto the algebra $C^\infty{}_{\mathbf{R}^m}(U)$ first via $f \mapsto \tilde{f}$, and evaluating at x_0 in the usual sense afterwards. Since the antidiagonal blocks of this matrix have entries in the ideal generated by $(C^\infty{}_{\mathbf{R}^m}(U) \otimes \Lambda[n])_1$, they both project onto blocks filled

(16) It is a matrix with entries in $C^\infty{}_{\mathbf{R}^m}(U) \otimes \Lambda[n]$. Since the ∂_{x^i} are even derivations of this superalgebra, $\partial_{x^i}(\varphi^* y^j) \in (C^\infty{}_{\mathbf{R}^m}(U) \otimes \Lambda[n])_0 = C^\infty{}_{\mathbf{R}^m}(U) \otimes (\Lambda[n])_0$ and $\partial_{x^i}(\varphi^* \xi^\nu) \in (C^\infty{}_{\mathbf{R}^m}(U) \otimes \Lambda[n])_1 = C^\infty{}_{\mathbf{R}^m}(U) \otimes (\Lambda[n])_1$. Similarly, as the ∂_{ζ^μ} are odd derivations, $\partial_{\zeta^\mu}(\varphi^* y^j) \in C^\infty{}_{\mathbf{R}^m}(U) \otimes (\Lambda[n])_1$, while $\partial_{\zeta^\mu}(\varphi^* \zeta^\nu) \in C^\infty{}_{\mathbf{R}^m}(U) \otimes (\Lambda[n])_0$ Thus, the $p \times m$ and the $q \times n$ diagonal blocks have even entries, and the remainig antidiagonal blocks have them odd. This is, by definition, an even matrix.

in with zeroes. Hence, upon evaluation of $J\Phi$ at $x_0 \in U$, we are only left with a block-diagonal matrix. The rank of $J\Phi$ at x_0 is then by definition the pair (r,s) if the rank of the block $((\partial_{x^i} \varphi^* y^j)^{\sim}(x_0))$ is r and the rank of the block $((\partial_{\xi^\mu} \varphi^* \zeta^\nu)^{\sim}(x_0))$ is s.

It is still true in the theory of supermanifolds that the elementary operations performed on columns and rows of $J\Phi$ correspond to multiplication from the right and from the left by the superjacobian of well determined supercoordinate changes on the domain and codomain, respectively. However, it is possible to have a superdomain morphism Φ for which the rank of $J\Phi$ is everywhere (r,s), but no supercoordinate changes on domain and codomain (α and β, respectively) can ever be found so as to bring $J(\beta \circ \Phi \circ \alpha)$ to the form

$$\begin{pmatrix} 1_{r\times r} & 0 & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & 1_{s\times s} & 0 \\ * & 0 & * & 0 \end{pmatrix} \qquad (*)$$

as a matrix with entries in $C^\infty_{\mathbf{R}^m}(U) \otimes \Lambda[n]$. The reason now is that matrices with coefficients in a ring where nilpotents exist, are not in general triangularizable in this way. They are always triangularizable in the following way (cf. [1]) and (19) below

$$\begin{pmatrix} 1_{r\times r} & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 1_{s\times s} & 0 \\ 0 & * & 0 & * \end{pmatrix}, \qquad (**)$$

but this form is useless for the most interesting theoretical purposes; for example, not being able to produce zeroes in the remaining starred blocks amounts to not having found appropriate coordinate changes in domain and codomain so as to make Φ look like the *standard local embedding*,

$$\varphi^* y^j = x^j \quad \text{if} \quad 1 \leqq j \leqq r \quad ; \quad \varphi^* y^j = 0 \quad \text{if} \quad r+1 \leqq j \leqq p$$
$$\varphi^* \zeta^\nu = \xi^\nu \quad \text{if} \quad 1 \leqq \nu \leqq s \quad ; \quad \varphi^* \zeta^\nu = 0 \quad \text{if} \quad s+1 \leqq \nu \leqq q$$

This phenomenon was first observed by Leites in [4]; what he did there was to go around the problem by defining *constant rank*, not by the property of having the same rank at all points of a given neighborhood, but by the property of being diagonalizable(17) throughout that neighborhood. Although these notions coincide in calculus, they need not coincide in supermanifold theory as the following example will now show(18).

Let us consider the superdomain $\mathbf{R}^{2|2} = (\mathbf{R}^2, \mathcal{R}^{2|2})$ with its standard coordinate system $\{x, y, \xi, \zeta\}$. Suppose we are given the supermanifold morphism

$$\Phi : \mathbf{R}^{2|2} \longrightarrow \mathbf{R}^{2|2}$$

specified in terms of the given coordinates by means of

$$\Phi^* x = f + g\,\xi\zeta; \quad f,g \in C^\infty(\mathbf{R}^2) \quad ; \quad \Phi^* \xi = a\,\xi + b\,\zeta\,; \quad a,b \in C^\infty(\mathbf{R}^2)$$
$$\Phi^* y = h + k\,\xi\zeta; \quad h,k \in C^\infty(\mathbf{R}^2) \quad ; \quad \Phi^* \zeta = c\,\xi + d\,\zeta\,; \quad c,d \in C^\infty(\mathbf{R}^2).$$

We shall further assume that,

(i) h is a constant; that is, $\partial_x h = \partial_y h = 0$.

(ii) $d = 0$, identically.

(17) It is easy to see that once a matrix has been brought to the form (*), elementary operations performed on rows and columns make it possible to even bring it to the diagonal form $\mathbf{diag}\{1_{r\times r}, 0, 1_{s\times s}, 0\}$; on the other hand, it is impossible to achieve such a diagonal form from (**).

(18) It is worth our while to note that an example has been given in [4] of a non-diagonalizable matrix with coefficients in a superalgebra of the form $C^\infty_{\mathbf{R}^m}(U) \otimes \Lambda[n]$. Unfortunately, such a matrix is not the superjacobian of any morphism (see our footnote (14) above); hence it does not illustrates the phenomenon.

(*iii*) $\partial_x f$ is nowhere zero, but $\partial_y f = 0$, identically.

(*iv*) a is nowhere zero.

Under these assumptions, the superjacobian matrix of Φ is:

$$\begin{pmatrix} \partial_x f + \partial_x g\, \xi\zeta & \partial_y g\, \xi\zeta & g\,\zeta & -g\,\xi \\ \partial_x k\, \xi\zeta & \partial_y k\, \xi\zeta & k\,\zeta & -k\,\xi \\ (\partial_x a)\xi + (\partial_x b)\zeta & (\partial_y a)\xi + (\partial_y b)\zeta & a & b \\ (\partial_x c)\xi & (\partial_y c)\xi & c & 0 \end{pmatrix}$$

Its rank is clearly (1,1) at all points. On the other hand, it is a straightforward matter to check that this matrix is elementary equivalent to(19):

$$\begin{pmatrix} \partial_x f + \partial_x g\, \xi\zeta & 0 & 0 & 0 \\ 0 & (\partial_y k \mp k a^{-1} \partial_y a)\xi\zeta & 0 & -k(\xi + a^{-1} b\, \zeta) \\ 0 & 0 & d_{11} & 0 \\ 0 & (\partial_y c - c a^{-1} \partial_y a)\xi - (a^{-1} c\, \partial_y b)\zeta & 0 & -a^{-1} b c \mp g\, (\partial_x f)^{-1} \partial_x (a^{-1} b c)\xi\zeta \end{pmatrix}$$

(19) It is easy to see what the general result should be: assume that we are given an even matrix with the following block decomposition

$$\begin{pmatrix} A_{11} & A_{12} & \beta_{11} & \beta_{12} \\ A_{21} & A_{22} & \beta_{21} & \beta_{22} \\ \gamma_{11} & \gamma_{12} & D_{11} & D_{12} \\ \gamma_{21} & \gamma_{22} & D_{21} & D_{22} \end{pmatrix}$$

The A and D blocks have even entries, the β and γ blocks have odd entries, and only A_{11} and D_{11} are invertible; that is, the rest of the blocks have purely nilpotent entries. Then, by elementary operations, this is equivalent to

$$\begin{pmatrix} A_{11} & 0 & 0 & 0 \\ 0 & X_{11} & 0 & X_{12} \\ 0 & 0 & d_{11} & 0 \\ 0 & X_{21} & 0 & X_{22} \end{pmatrix}$$

and the X_{ij}'s are explicitly given in terms of the original entries as follows:

$$X_{11} = (A_{22} - A_{21} A_{11}^{-1} A_{12}) \pm (\beta_{21} - A_{21} A_{11}^{-1} \beta_{11})\, d_{11}^{-1} (\gamma_{12} - \gamma_{11} A_{11}^{-1} A_{12})$$
$$X_{12} = (\beta_{22} - A_{21} A_{11}^{-1} \beta_{12}) - (\beta_{21} - A_{21} A_{11}^{-1} \beta_{11})\, d_{11}^{-1} (D_{12} \pm \gamma_{11} A_{11}^{-1} \beta_{12})$$
$$X_{21} = (\gamma_{22} - \gamma_{21} A_{11}^{-1} A_{12}) - (D_{21} \pm \gamma_{21} A_{11}^{-1} \beta_{11})\, d_{11}^{-1} (\gamma_{12} - \gamma_{11} A_{11}^{-1} A_{12})$$
$$X_{22} = (D_{22} \pm \gamma_{21} A_{11}^{-1} \beta_{12}) - (D_{21} \pm \gamma_{21} A_{11}^{-1} \beta_{11})\, d_{11}^{-1} (D_{12} \pm \gamma_{11} A_{11}^{-1} \beta_{12}).$$

We should warn the reader: upper signs result from matrix multiplication according to the rules of linear superalgebra for left modules (as explained in [17]), whereas lower signs result from the usual matrix multiplication.

The point is that *unless*

$$k = 0 \ , \quad b = 0 \quad \text{and} \quad c = a\,e^{\psi(x)},$$

for some smooth function ψ depending only on x, *the superjacobian matrix of* Φ *will not be diagonalizable.* That is, if any of these conditions is not satisfied, then the morphism Φ will not have constant rank in the sense of [4].

What is certainly desirable here is to have algebraic (and presumably simpler) criteria that can give the answer at once about the diagonalizability of the given superjacobian (or any matrix for this matter) and to have certain means so as to measure the obstructions for this to be possible in general. Again, it seems that such criteria would have to be of a cohomological nature.

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