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by

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0. Introduction

Recently, certain class of rational elliptic surfaces have become relevant because they provide examples of topological manifolds which are all homeomorphic but not diffeomorphic.

The aim of this note is to explain an example of this type of surfaces, the so called Dolgachev surfaces, and to compute their groups of biholomorphic automorphisms.

I want to thank Hans Schreyer for pointing out a mistake in my talk during the workshop, and for telling me about the paper by Miranda and Persson (See ref. [6]), where one can find a more general description of the automorphims of elliptic rational surfaces.

1. General results on elliptic surfaces.

A compact complex surface S is elliptic if there exists a curve C and a holomorphic map $f:S \dashrightarrow C$ such that the generic fibre $f^{-1}(z)$ is an elliptic curve for all but a finite number of $z \in C$. Elliptic surfaces have been thouroghly studied by Kodaira [5]. The points $\{a_1, \ldots, a_k\} \subset C$ where the fibre is not a regular ellitic curve correspond to the singular fibres. The map f is a regular map on all the other values.

The structure of the singular fibres is well known. In case they are divisors of multiplicity one, they are called simple, otherwise they are multiple singular fibres.

From now on we will assume that the elliptic surface S has no excepcional curve in a fibre, that is, S is relatively minimal.

Let us assume that S has no multiple fibres, Kodaira proves ([5] sections 8 to 10) that there exists an elliptic surface B, fibred over the same base curve C, such that the map B-->C has a section. The fibres of S are the same as the fibres of B. And we think of S as obtained by "glueing" the fibres of B in a twisted way. This surface B is characterized by having a section, and is called "the basic member" of a family to which S belongs. In this form we can escribe the topological invariants of S in terms of those of B ([5] Sect. 12), and more importantly, the automorphisms of S are described also (see Vila Freyer [7]).

We have an exact sequence:

$$0 \to G \to \mathrm{Aut}(S) \to H \to 0$$

where G corresponds to a subgroup of holomorphic sections of the fibration B-->C, and H is a subgroup of automorphisms of C. If the surface S has multiple singular fibres, then it can be obtained by a series of logarithmic transforms of another elliptic surface S'. The surface S' will have at most simple singular fibres. A logarithmic transform is a way of replacing a fibre of multiplicity one by a fibre of multiplicity greater than one. And it is a reversible process. (See [1]). So that the Automorphisms of S and the Automorphisms of S' are related [7].

2. Rational elliptic Surfaces.

Take $\mathbb{P}^2\mathbb{C}$ and suppose we have two cubic curves defined respectively by F_0 and F_∞ which intersect transversely in 9 points: $\{x_i\}$ for i=1, ..., 9.

We get a pencil of cubic curves $\{ \lambda F_0 + \mu F_\infty = 0 \ : \ (\lambda,\mu) \in \mathbb{P}^1\mathbb{C} \}$ passing thru the nine points x_i. This pencil induces a fibration:

$$\mathbb{P}^2\mathbb{C} - \{ x_i : i=1, \ldots, 9 \} \to \mathbb{P}^1\mathbb{C}$$

so that blowing up the nine points we get a new surface $X = \mathbb{P}^2\{x_1,\ldots,x_9\}$ and an elliptic fibration over $\mathbb{P}^1\mathbb{C}$. (Where we denote the blow-up of $\mathbb{P}^2\mathbb{C}$ at the nine points $\{x_i\}$ by $\mathbb{P}^2\{x_1,\ldots,x_9\}$). The elliptic fibration of X is $f:X \dashrightarrow C$.

The canonical bundle of X can be computed [2], and the canonical divisor is $K_X = -3\rho^* H + E_1 + \ldots + E_9$. Where H corresponds to the hyperplane divisor of $\mathbb{P}^2\mathbb{C}$ and the divisors E_i are the exceptional divisors obtained from the blow-up at x_i. Also, by construction,

$$f^{-1}(z) = 3\rho^* H - E_1 - \ldots - E_9$$

is a generic fibre F. This implies that $K_X = -F$. We have also the following properties (see Morgan and Freedman [2]):

i) X has not any multiple fibre.

ii) X is simply connected.

iii) The birational invariants of X are the irregularity $q(S) = h^0(S,\Omega^1) = 0$, and the geometric genus $p_g(S) = h^0(S,\Omega^2) = 12$. And more importantly,

iv) If we choose a pencil of elliptic curves such that it is a Lefschetz pencil (See Griffiths and Harris [3] Pag. 509) that is, it is a pencil of generically irreducible curves, such that each singular curve has only one singularity of type double point). Then it is easy to see that we can compute the number of singular curves from the Euler number of S and the intersection number of a generic fibre ([3] pag. 509). We obtain that X has 12 singular fibres. Each one of them is a rational curve with a double point and no other singularity.

v) X is algebraic and there are sections $\sigma:\mathbb{P}^1\mathbb{C} \dashrightarrow X$. For this it is enough to check that the exceptional curves E_i intersect the generic fibre F at only one point:

$$E_i F = - E_i K_X$$

and since the genus $\pi(E_i)$ of E_i is zero. Using the formula for the virtual genus we have:

$$0 = \pi(E_i) = (E_i^2 + E_i K_X)/2 + 1,$$

and we know that $E_i^2 = -1$. So that $E_i F = 1$.

Conversely any non singular section $\sigma:\mathbb{P}^1\mathbb{C} \dashrightarrow X$ is an exceptional curve in X, as can be seen by doing the same computation as before.

Hartshorne notes ([4], Remark 5.8.1) that the blow up of $\mathbb{P}^2\mathbb{C}$ at 9 or more points may have an infinite number of exceptional curves. In fact Miranda and Persson in [6] classify all the elliptic rational surfaces with no multiple singular fibres that have only a finite number of exceptional curves. They all will have less that 4 singular fibres (See Theorem 4.1 in [6]). From this we obtain:

Theorem: The group of holomorphic automorphisms of X is discrete and infinite.

Since the sections of $\mathbb{P}^1\mathbb{C}$-->X form a discrete group. And the automorphisms on the base curve that preserve the elliptic fibration has to keep invariant 12 points based on the singular fibres.

3. Dolgachev Surfaces.

Given the elliptic fibration $f:X \dashrightarrow \mathbb{P}^1\mathbb{C}$, the Dolgachev surfaces are constructed by replacing two non-singular fibres by the same fibres but considered as divisors with multiplicities p and q respectively, with p and q two positive integers relatively prime between each other. That is, we performa a logarithmic transform at two non-singular fibres. We denote the new surface by $X_{p,q}$. (For details on the construction see Friedman and Morgan [3]).

By the results in the previous section, we have:

Theorem: The group of automorphisms of the surfaces $X_{p,q}$ respect the elliptic fibration, and is an infinite, descrete group.

And the reason as before is that the automorphisms on the sections of X will induce automorphisms on $X_{p,q}$.

4. References.

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